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Existence of Solutions for Quasilinear Schrödinger Equations Under Non-Coercive Conditions

Wenkai Li, Xinya Zhang, Yi Zhen and Lili Wan *

School of Mathematics and Physics, Southwest University of Science and Technology, Mianyang 621010, China; 15882579304@163.com (W.L.); 17311883189@163.com (X.Z.)

* Correspondence: 15882872311@163.com

Abstract

In this paper, we study the existence of solutions for a class of quasilinear Schrödinger equations under general potential conditions. The quasilinear term causing the functional to be ill-defined in $H^1(\mathbb{R}^N)$ is addressed via a dual approach involving a change of variables. To overcome the lack of coercivity in the potential V , a class of weighted function spaces is introduced, and their compact embedding properties are established. Moreover, the boundedness of the Cerami sequence is obtained with new conditions on the nonlinearity, which can be unbounded in the variable x . Recent results from the literature are improved and extended.

Keywords: quasilinear Schrödinger equation; variational method; non-coercive condition

MSC: 35B38; 35G99

1. Introduction

In this paper, we consider the following quasilinear Schrödinger equations:

$$-\Delta u + V(x)u - \Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = h(x, u), \quad x \in \mathbb{R}^N, \quad (1)$$

where $\frac{1}{2} < \alpha \leq 1$, $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

Researchers have extensively studied Equation (1) over the past two decades due to its strong physical background. Equation (1) is referred to as the so-called modified nonlinear Schrödinger equation. For the case $\alpha = 1$, solutions of Equation (1) are related to solitary wave solutions of the equation

$$i\partial_t z = -\Delta z + W(x)z - m(|z|^2)z - \kappa\Delta\phi(|z|^2)\phi'(|z|^2)z, \quad (2)$$

where $z : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $W : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential, $m, \phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ are suitable functions, and $\kappa \in \mathbb{R}$. It can be used to describe physical phenomena corresponding to various types of κ and ϕ . For example, if $\kappa = 0$, Equation (2) becomes a semilinear Schrödinger equation, which has been widely investigated, we refer the readers to reference [1–3]. Different forms of the nonlinear function $\phi(s)$ correspond to distinct physical phenomena. The linear case, $\phi(s) = s$, models the time evolution of a wave function in a superfluid film and was investigated by Kurihara [4]. The case $\phi(s) = 1 + s$, however, describes the self-channeling of a high-power ultrashort laser in matter (see reference [5]). For more physical applications, we refer to reference [6–13] and references therein.



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Recently, many researchers have focused on Equations (1) and (2) via variational methods. In reference [14], the multiplicity of solution for the following problem:

$$-\Delta u + V(x)u - \Delta(u^2)u = g(x, u), \quad x \in \mathbb{R}^N,$$

was studied via the Nehari method, where g and V are periodic in $x_1, \dots, x_N$. The function g is odd in u , subcritical and satisfies a monotonicity condition. In reference [15], Zhang et al. used the Lagrange multiplier method to prove the existence of the ground state solutions. Adachi and Watanabe in reference [16] discussed the uniqueness of the ground state solutions for the following quasilinear Schrödinger equations:

$$-\Delta u + \lambda u - \kappa \Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = |u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

via a dual approach. In [17], Luo et al. focused on the infinitely many negative energy solutions for the following quasilinear Schrödinger equation involving concave and convex nonlinearities:

$$-\Delta u + V(x)u - \Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = \lambda g(x, u) + \mu h(x, u), \quad x \in \mathbb{R}^N,$$

via a new method and dual fountain theorem, where $N \geq 3$, $\frac{3}{4} < \alpha \leq 1$, λ and μ are parameters, $g, h \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and the potential $V(x)$ satisfies the following assumption:

(V₁') $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x)$, and for each $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

With the advancement of research, the potential $V(x)$ has attracted significant attention from researchers. In reference [18,19], researchers discussed the indefinite potentials; finally, they obtained an unbounded sequence of solutions and obtained the existence of a nontrivial solution for the equation. Recently, Ma et al. [20] investigated the existence of solutions for a class of quasilinear Schrödinger equations with subcube growth nonlinear terms g and the noncompact potential V . They considered the following conditions:

(V₂') $V \in C^1(\mathbb{R}^3, \mathbb{R})$;

(V₃') $V \geq m > 0$, $\lim_{|x| \rightarrow \infty} V(x) = V_\infty < \infty$.

Moreover, Yang et al. [11] focused on the existence of a least energy nodal solution for a quasilinear Schrödinger equation. They assumed that the potential V is radially symmetric and satisfies

(V₄') $V(x) = V(|x|)$ and $V \in L^\infty(\mathbb{R}^N)$, or

(V₅') $\lim_{|x| \rightarrow \infty} V(x) = \infty$.

In reference [21], Zhang et al. devoted themselves to studying the existence and multiplicity of sign-changing solutions of the quasilinear Schrödinger equation by considering the potential condition:

(V₆') $V \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $V(x) = V(|x|)$ and $\inf_{x \in \mathbb{R}^3} V(x) > 0$.

The common feature of the aforementioned works is that the potential $V(x)$ is required to be coercive or bounded at infinity. To our knowledge, there are few results on the quasilinear Schrödinger Equation (1) involving noncoercive conditions with $\alpha \neq 1$. It is well known that the noncoercive conditions will lead to the lack of global compactness, and it is difficult to verify the (PS) or (C) condition. Inspired by the above-mentioned works, this paper is aimed at considering the existence of nontrivial weak solutions of the quasilinear Schrödinger Equation (1) under the noncoercive conditions, and there is no growth of the potential function at infinity.

2. Main Results and Preliminaries

In order to reduce the statements for main results, we assume $h(x, u) = a(x)g(u)$ and list the assumptions as follows:

$$(V) \quad 0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x);$$

We use the following notation: If $\Omega \subset \mathbb{R}^N$ is open and $2 \leq s < 2^* = 2N/(N-2)$, we set

$$v_s(\Omega) = \begin{cases} \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + V(x)|u|^2) dx}{\left(\int_{\Omega} |u|^s dx\right)^{2/s}} & \text{if } \Omega \neq \emptyset, \\ \infty & \text{if } \Omega = \emptyset. \end{cases}$$

now we shall consider the following assumption:

$$(V_1) \quad \lim_{R \rightarrow \infty} v_s(\mathbb{R}^N \setminus \bar{B}_R) = \infty \text{ for some } s \in [2, 2^*), \text{ where } \bar{B}_R \text{ is the closed ball with center at origin and radius } R;$$

$$(g_0) \quad \text{there exist constants } \gamma > 1, c_0, R_0 > 0 \text{ such that } a(x) \leq c_0 \left(1 + (V(x))^{\frac{1}{\gamma}}\right), \forall |x| \geq R_0, \\ \text{where } a \in C(\mathbb{R}^N, \mathbb{R}^+) \text{ with } a(x) \geq 1;$$

$$(g_1) \quad \text{there exists } 4\alpha < p \leq 2^* + 2\alpha - \frac{4\alpha}{\gamma(N-2)} \text{ such that } \lim_{|t| \rightarrow +\infty} \frac{g(t)}{|t|^{p-1}} < +\infty;$$

$$(g_2) \quad G(0) = 0 \text{ and } g(t) = o(|t|) \text{ as } |t| \rightarrow 0, \text{ where } G(t) = \int_0^t g(s) ds;$$

$$(g_3) \quad \text{there exists } \nu > 4\alpha \text{ such that } \tilde{G}(t) := \frac{1}{\nu} t g(t) - G(t) \geq 0 \text{ for all } t \in \mathbb{R};$$

$$(g_4) \quad \text{there exists } d > 0 \text{ such that } G(t) \geq -d|t|^{4\alpha} \text{ for all } t \in \mathbb{R};$$

$$(g_5) \quad \frac{G(t)}{|t|^{4\alpha}} \rightarrow +\infty \text{ as } |t| \rightarrow \infty.$$

In this article, our main result is the following theorem.

Theorem 1. Assume that conditions (V), (V₁) and (g₀)–(g₅) hold; then Equation (1) has a nontrivial weak solution. Furthermore, if $g(t)$ is odd in t , then Equation (1) has a sequence $\{u_n\} \subset E$ of solutions such that $\|u_n\|_E \rightarrow \infty$ and $J(u_n) \rightarrow +\infty$.

Remark 1. According to Corollary 1.4 in reference [22], our condition (V₁) incorporates the coercive condition (V'₅), the non-coercive condition (V'₁) and the non-coercive condition:

$$(V'_7) \quad \text{The function } [V(x)]^{-1} \text{ belongs to } L^1(\mathbb{R}^N).$$

For example, it follows from (V'₁) that

$$\lim_{R \rightarrow \infty} \text{meas}(\Omega_M \cap (\mathbb{R}^N \setminus \bar{B}_R)) = 0,$$

where $\Omega_M := \{x \in \mathbb{R}^N : V(x) \leq M\}$. Then we can derive that condition (V₁) holds by means of contradiction.

Remark 2. There exist functions that satisfy all assumptions in Theorem 1. For example, let $V(x) = |x \sin|x|| + 1$, $a(x) = 1 + [|x \sin|x|| + 1]^{\frac{1}{8}}$ and $G(t) = |t|^5$ with $N = 3$, $\alpha = 1$, $p = 6$, $\nu = 5$, $\gamma = 8$. It is worth noting that recent literature concerning the existence of solutions for quasilinear Schrödinger equations under non-coercive conditions (see references [17,20,23–25]) requires that the nonlinear term be bounded with respect to the variable x , which acts as a crucial premise to verify the boundedness of a Cerami or (PS) sequence. However, we permit the nonlinearity to be unbounded in x , which makes our conclusions distinct from the existing references.

The proof of Theorem 1 presents several major challenges. Firstly, the presence of the quasilinear term makes the functional associated with Equation (1) not well defined in $H^1(\mathbb{R}^N)$. Inspired by Wu's work [23], we employ a dual approach by changing variables.

Secondly, since there are no coercive assumptions on V , it is difficult to prove the convergence of the Cerami sequence. To overcome this problem, we introduce a class of weighted function spaces and prove that they possess the property of compact embedding. Thirdly, the nonlinear term can be unbounded with respect to the variable x ; we need to prove that the energy functional still satisfies condition (C) and the geometric structure required by the Mountain Pass Lemma. In the following, we elaborate on how we overcome them one by one.

In this paper, we set

$$H = H^1(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N) \right\}$$

with the inner product

$$\langle u, v \rangle_H = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv) dx$$

and the norm

$$\|u\| = \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \right]^{\frac{1}{2}}.$$

Let

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx < +\infty \right\}$$

with the inner product

$$\langle u, v \rangle_E = \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v + V(x)uv] dx$$

and the norm

$$\|u\|_E = \langle u, u \rangle_E^{\frac{1}{2}}.$$

Then both H and E are Hilbert spaces. By the continuity of the embedding

$$E \rightarrow L^s(\mathbb{R}^N), \quad \forall s \in [2, 2^*],$$

there exist constants $r_s > 0$, such that

$$\|u\|_s \leq r_s \|u\|_E, \quad \forall u \in E, \quad (3)$$

where we denote by $\|\cdot\|_s$ the norm of $L^s(\mathbb{R}^N)$. Moreover, let $J : E \rightarrow \mathbb{R}$ be the energy functional of (1) defined by

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx + \frac{1}{4\alpha} \int_{\mathbb{R}^N} |\nabla (|u|^{2\alpha})|^2 dx - \int_{\mathbb{R}^N} a(x)G(u) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx + \alpha \int_{\mathbb{R}^N} |u|^{2(2\alpha-1)} |\nabla u|^2 dx - \int_{\mathbb{R}^N} a(x)G(u) dx. \end{aligned}$$

We remark that the nonlinear functional $\int_{\mathbb{R}^N} |u|^{2(2\alpha-1)} |\nabla u|^2 dx$ is not defined on all of $H^1(\mathbb{R}^N)$ except for $N = 1$. Therefore it is difficult to handle $J(u)$ directly. We adopt the dual approach used in reference [23]. As in reference [23], we can define f to be a solution of the following ODE:

$$f'(t) = \frac{1}{\sqrt{1 + 2\alpha|f(t)|^{2(2\alpha-1)}}} \quad \text{on } t \in [0, +\infty), f(0) = 0,$$

and

$$f(-t) = -f(t) \quad \text{on } t \in [0, +\infty).$$

After the change of variables, we have the following functional

$$I(v) := J(f(v)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) dx - \int_{\mathbb{R}^N} a(x) G(f(v)) dx,$$

which is well defined in E under our assumptions. Moreover, the critical points of the functional I correspond to the weak solutions of the following equation

$$-\Delta v = \frac{1}{\sqrt{1 + 2\alpha |f(v)|^{2(2\alpha-1)}}} [a(x)g(f(v)) - V(x)f(v)] \text{ in } \mathbb{R}^N,$$

which is called a dual variational formulation of Equation (1). In the following, we always denote by C any suitable positive constant.

Lemma 1 ([23]). *The function $f(t)$ enjoys the following properties:*

- (f₁) f is uniquely determined, of class $C^\infty(\mathbb{R}, \mathbb{R})$, and invertible.
- (f₂) $0 < f'(t) \leq 1$, $\forall t \in \mathbb{R}$.
- (f₃) $|f(t)| \leq |t|$, $\forall t \in \mathbb{R}$.
- (f₄) $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 1$.
- (f₅) $\lim_{t \rightarrow \infty} \frac{|f(t)|^{2\alpha}}{|t|} = \sqrt{2\alpha}$.
- (f₆) $\frac{f(t)}{2} \leq \alpha t f'(t) \leq \alpha f(t)$, $\forall t \geq 0$; $\alpha f(t) \leq \alpha t f'(t) \leq \frac{f(t)}{2}$, $\forall t \leq 0$.
- (f₇) $|f(t)|^{2\alpha} \leq \sqrt{2\alpha} |t|$, $\forall t \in \mathbb{R}$.
- (f₈) The function $f^2(t)$ is strictly convex.
- (f₉) There exists a positive constant $\theta > 0$ such that

$$|f(t)| \geq \begin{cases} \theta |t|, & |t| \leq 1, \\ \theta |t|^{\frac{1}{2\alpha}}, & |t| \geq 1. \end{cases}$$

- (f₁₀) There exist positive constants c_1 and c_2 such that

$$|t| \leq c_1 |f(t)| + c_2 |f(t)|^{2\alpha}, \forall t \in \mathbb{R}.$$

- (f₁₁) $|f(t)|^{2\alpha-1} f'(t) \leq \frac{1}{\sqrt{2\alpha}}$, $\forall t \in \mathbb{R}$.
- (f₁₂) $f(t)$ is odd, $f^2(t)$ is even.
- (f₁₃) For each $\xi > 0$, there exists a positive constant $C(\xi)$ such that

$$|f(\xi t)|^{2\alpha} \leq C(\xi) |f(t)|^{2\alpha}.$$

- (f₁₄) The function $f(t) f'(t) t^{-1}$ is strictly decreasing for $t > 0$.
- (f₁₅) The function $f^m(t) f'(t) t^{-1}$ is strictly increasing for $m \geq 4\alpha - 1$ and $t > 0$.

Let $L_a^s(\mathbb{R}^N)$ be the weighted space of measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ under the norm

$$\|u\|_{L_a^s} = \left(\int_{\mathbb{R}^N} a(x) |u(x)|^s dx \right)^{\frac{1}{s}}.$$

The conclusion of the following lemma is inspired by reference [22]. We present the detailed derivation for reader's convenience.

Lemma 2. *Under the assumptions of (V) , (V_1) and (g_0) , E is contionly embedded into $L_a^s(\mathbb{R}^N)$ for $2 \leq s \leq 2^* - \frac{4}{\gamma(N-2)}$ and compactly embedded into $L_a^s(\mathbb{R}^N)$ for $2 \leq s < 2^* - \frac{4}{\gamma(N-2)}$.*

Proof. We first prove the continuing of the embedding. It is sufficient to prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} a(x)|u|^s dx < C\|u\|^s, \quad \forall s \in [2, 2^* - \frac{4}{\gamma(N-2)}]. \quad (4)$$

First, for $R \geq R_0$, we have

$$\int_{\mathbb{R}^N} a(x)|u|^s dx = \int_{B_R} a(x)|u|^s dx + \int_{\mathbb{R}^N \setminus B_R} a(x)|u|^s dx.$$

By (g₀) we get

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)|u|^s dx &\leq \|a\|_{L^\infty(B_R)} \int_{B_R} |u|^s dx + c_0 \int_{\mathbb{R}^N \setminus B_R} \left(1 + (V(x))^{\frac{1}{\gamma}}\right) |u|^s dx \\ &= \|a\|_{L^\infty(B_R)} \int_{B_R} |u|^s dx + c_0 \int_{\mathbb{R}^N \setminus B_R} (V(x))^{\frac{1}{\gamma}} |u|^s dx + c_0 \int_{\mathbb{R}^N \setminus B_R} |u|^s dx \\ &< (c_0 + \|a\|_{L^\infty(B_R)}) \|u\|_s^s + c_0 \int_{\mathbb{R}^N \setminus B_R} (V(x))^{\frac{1}{\gamma}} |u|^s dx. \end{aligned} \quad (5)$$

Note that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} (V(x))^{\frac{1}{\gamma}} |u|^s dx &= \int_{\mathbb{R}^N \setminus B_R} (V(x)|u|^2)^{\frac{1}{\gamma}} |u|^{s-\frac{2}{\gamma}} dx \\ &\leq \left(\int_{\mathbb{R}^N \setminus B_R} (V(x)|u|^2) dx \right)^{\frac{1}{\gamma}} \left(\int_{\mathbb{R}^N \setminus B_R} (|u|^{s-\frac{2}{\gamma}})^{\frac{\gamma}{\gamma-1}} dx \right)^{\frac{\gamma-1}{\gamma}} \\ &\leq \|u\|_E^{\frac{2}{\gamma}} \|u\|_{\frac{s\gamma-2}{\gamma-1}}^{\frac{s\gamma-2}{\gamma}}, \end{aligned} \quad (6)$$

substituting (5) into (6) we obtain

$$\int_{\mathbb{R}^N} a(x)|u|^s dx \leq (c_0 + \|a\|_{L^\infty(B_R)}) \|u\|_s^s + c_0 \|u\|_E^{\frac{2}{\gamma}} \|u\|_{\frac{s\gamma-2}{\gamma-1}}^{\frac{s\gamma-2}{\gamma}}. \quad (7)$$

We claim that

$$2 \leq \frac{s\gamma-2}{\gamma-1} \leq 2^*.$$

Indeed, we have

$$\begin{aligned} s &\leq 2^* - \frac{4}{\gamma(N-2)} \\ &= 2^* + \frac{-4 + 2N - 2N}{\gamma(N-2)} \\ &= 2^* + \frac{2N-4}{\gamma(N-2)} - \frac{2N}{\gamma(N-2)} \\ &= 2^* + \frac{2}{\gamma} - \frac{2^*}{\gamma} \\ &= 2^* \left(1 - \frac{1}{\gamma}\right) + \frac{2}{\gamma}, \end{aligned}$$

thus we can get

$$\frac{s\gamma-2}{\gamma-1} \leq 2^*.$$

On the other hand, for $2 \leq s$, one has

$$2 = 2 \frac{\gamma-1}{\gamma-1} = \frac{2\gamma-2}{\gamma-1} \leq \frac{s\gamma-2}{\gamma-1}.$$

Consequently, by Equation (3) we can get there exists $C > 0$ such that

$$\|u\|_s^s \leq r_s^s \|u\|_E^s, \text{ and } \|u\|_{\frac{s\gamma-2}{\gamma-1}}^{\frac{s\gamma-2}{\gamma}} \leq r_s^s \|u\|_E^{\frac{s\gamma-2}{\gamma}}, \quad (8)$$

which combine Equations (3) and (8) implies that

$$\int_{\mathbb{R}^N} a(x) |u|^s dx \leq C \|u\|_s^s.$$

Next, we prove the compactness of the embedding. Let $\Omega \subset \mathbb{R}^N$ be open, by interpolation inequality for $2 \leq t < s < 2^*$ there exists a number $\theta_1 \in (0, 1)$, depending only on N, s and t , such that

$$\|u\|_s \leq \|u\|_t^{\theta_1} \|u\|_{2^*}^{(1-\theta_1)}, \quad \forall u \in W_0^{1,2}(\Omega).$$

By Gagliardo–Nirenberg inequality,

$$\|u\|_s \leq C(s, t, N) \|u\|_t^{\theta_1} \|\nabla u\|_2^{(1-\theta_1)}.$$

Thus, we obtain

$$\|u\|_s^2 \leq C(s, t, N) \|u\|_t^{2\theta_1} \|u\|^{2(1-\theta_1)}.$$

Consequently,

$$\begin{aligned} v_s(\Omega) &= \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\|u\|_E^2}{\|u\|_s^2} \\ &\geq \frac{1}{C(s, t, N)} \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\|u\|_E^2}{\|u\|_t^{2\theta_1} \|u\|^{2(1-\theta_1)}} \\ &= \frac{1}{C(s, t, N)} \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\|u\|_E^{2\theta_1}}{\|u\|_t^{2\theta_1}} \\ &= \frac{1}{C(s, t, N)} \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \left(\frac{\|u\|_E^2}{\|u\|_t^2} \right)^{\theta_1} \\ &= \frac{1}{C(s, t, N)} (v_t(\Omega))^{\theta_1}. \end{aligned} \quad (9)$$

The proof of the following inequality follows by a similar argument, if $2 \leq s < t < 2^*$, there exists $\theta_2 \in (0, 1)$ such that

$$v_s(\Omega) \geq \frac{1}{C(s, t, N)} (v_t(\Omega))^{\theta_2}. \quad (10)$$

Consequently, it follows (9) and (10)

$$\lim_{R \rightarrow \infty} v_t(\mathbb{R}^N \setminus B_R) = \infty \text{ implies } \lim_{R \rightarrow \infty} v_s(\mathbb{R}^N \setminus B_R) = \infty, \quad (11)$$

whenever $s > t$ or $t > s$.

Let $(u_k) \subset E$ be a bounded sequence, up to a subsequence, we may assume that $u_k \rightharpoonup u$ weakly in E .

Assertion 1 $u_k \rightarrow u$ strongly in $L^s(\mathbb{R}^N)$ for $2 \leq s \leq 2^* - \frac{4}{\gamma(N-2)}$. Indeed, let $\varphi \in C^\infty(\mathbb{R}^N, [0, 1])$ be a cut-off function such that $\|\nabla \varphi\|_\infty \leq 1$ and

$$\varphi(x) = \begin{cases} 0 & \text{if } |x| \leq R, \\ 1 & \text{if } |x| > R + 1. \end{cases}$$

Thus,

$$\begin{aligned}\|u_k - u\|_s &= \|(1 - \varphi)(u_k - u) + \varphi(u_k - u)\|_s \\ &\leq \|(1 - \varphi)(u_k - u)\|_s + \|\varphi(u_k - u)\|_s \\ &= \|(1 - \varphi)(u_k - u)\|_{L^s(B_{R+1})} + \|\varphi(u_k - u)\|_{L^s(\mathbb{R}^N \setminus B_R)}.\end{aligned}\quad (12)$$

Since $2^* - \frac{4}{\gamma(N-2)} < 2^*$, then $H^1(B_{R+1})$ is compactly embedded into $L^s(B_{R+1})$ for $2 \leq s \leq 2^* - \frac{4}{\gamma(N-2)}$. Thus, up to a subsequence,

$$\|(1 - \varphi)(u_k - u)\|_{L^s(B_{R+1})} \rightarrow 0. \quad (13)$$

Now, by the definition of $v_s(\mathbb{R}^N \setminus B_R)$, it follows

$$\|\varphi(u_k - u)\|_{L^s(\mathbb{R}^N \setminus B_R)}^2 \leq \frac{\|\varphi(u_k - u)\|_E^2}{v_s(\mathbb{R}^N \setminus B_R)}. \quad (14)$$

Note that

$$\begin{aligned}\|\varphi(u_k - u)\|_E^2 &= \int_{\mathbb{R}^N} |\nabla \varphi(u_k - u)|^2 + |\varphi(\nabla(u_k - u))|^2 + V(x)|\varphi(u_k - u)|^2 dx \\ &\leq \int_{\mathbb{R}^N} |u_k - u|^2 dx + \int_{\mathbb{R}^N} |\nabla(u_k - u)|^2 + V(x)|u_k - u|^2 dx \\ &\leq \|u_k - u\|_2^2 + \|u_k - u\|_E^2.\end{aligned}$$

Thus, we can obtain

$$\|\varphi(u_k - u)\|_E^2 \leq C,$$

which together with (11), (14), (V_1) , implies

$$\|\varphi(u_k - u)\|_{L^s(\mathbb{R}^N \setminus B_R)} \rightarrow 0 \text{ as } R \rightarrow +\infty. \quad (15)$$

Finally, we conclude from Equations (12)–(14) that Assertion 1 holds.

Assertion 2 $u_k \rightarrow u$ strongly in $L_a^s(\mathbb{R}^N)$ for $2 \leq s < 2^* - \frac{4}{\gamma(N-2)}$. By (5) and $2 \leq (s\gamma - 2)/(\gamma - 1) < 2^*$, we have

$$\int_{\mathbb{R}^N} a(x)|u_k - u|^s dx \leq (c_0 + \|a\|_{L^\infty(B_R)})\|u_k - u\|_s^s + c_0\|u_k - u\|_E^{\frac{2}{\gamma}}\|u_k - u\|_{\frac{s\gamma-1}{\gamma-1}}^{\frac{s\gamma-2}{\gamma-1}}.$$

Thus, using Assertion 1 it follows that

$$\int_{\mathbb{R}^N} a(x)|u_k - u|^s dx \rightarrow 0.$$

This completes the proof of Lemma 2. $\square$

Lemma 3. If assumptions (V) , (g_0) – (g_2) hold, then the functional I is well defined on E , and $I \in C^1(E, \mathbb{R})$.

Proof. For any $\varepsilon > 0$, by (g_1) and (g_2) , there exists $C(\varepsilon) > 0$ such that

$$|g(f(v))| \leq \varepsilon|f(v)| + C(\varepsilon)|f(v)|^{p-1},$$

and

$$|G(f(v))| \leq \frac{\varepsilon}{2}|f(v)|^2 + \frac{C(\varepsilon)}{p}|f(v)|^p. \quad (16)$$

Set $\lambda = \frac{4\alpha(2\alpha 2^* - p)}{p(2\alpha 2^* - 4\alpha)}$. It follows from (f_7) and (g_0) that

$$\begin{aligned}
 \int_{\mathbb{R}^N} a(x) |f(v)|^p dx &\leq \int_{\mathbb{R}^N} a(x)^{\frac{p}{4\alpha}} |f(v)|^{\lambda p} |f(v)|^{(1-\lambda)p} dx \\
 &\leq \left(\int_{\mathbb{R}^N} a(x) |f(v)|^{4\alpha} dx \right)^{\frac{\lambda p}{4\alpha}} \left(\int_{\mathbb{R}^N} a(x) |f(v)|^{4\alpha} dx \right)^{\frac{(1-\lambda)p}{4\alpha}} \\
 &\leq C \left(\int_{\mathbb{R}^N} a(x) v^2 dx \right)^{\frac{\lambda p}{4\alpha}} \left(\int_{\mathbb{R}^N} a(x) v^2 dx \right)^{\frac{(1-\lambda)p}{4\alpha}} \\
 &\leq C \|v\|_{L_a^2}^{2 \cdot \frac{\lambda p}{4\alpha} + 2 \cdot \frac{(1-\lambda)p}{4\alpha}} \\
 &= C \|v\|_E^{\frac{p}{2\alpha}},
 \end{aligned} \tag{17}$$

where $2 < \frac{p}{2\alpha}$ because of $4\alpha < p$. Therefore, for any $\varepsilon > 0$, by Equations (16) and (17), one has

$$\begin{aligned}
 I(v) &= J(f(v)) \\
 &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) dx - \int_{\mathbb{R}^N} a(x) G(f(v)) dx \\
 &\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) v^2 dx - \int_{\mathbb{R}^N} a(x) G(f(v)) dx \\
 &\leq \frac{1}{2} \|v\|_E^2 - \int_{\mathbb{R}^N} a(x) G(f(v)) dx \\
 &\leq \frac{1}{2} \|v\|_E^2 + \int_{\mathbb{R}^N} a(x) \left[\frac{\varepsilon}{2} |f(v)|^2 + \frac{C(\varepsilon)}{p} |f(v)|^p \right] dx \\
 &\leq \frac{1}{2} \|v\|_E^2 + \int_{\mathbb{R}^N} a(x) \left[\frac{\varepsilon}{2} |v|^2 + \frac{C(\varepsilon)}{p} \|v\|_E^{\frac{p}{2\alpha}} \right] dx \\
 &\leq \frac{1}{2} \|v\|_E^2 + \frac{\varepsilon}{2} \|v\|_{L_a^2}^2 + \frac{C(\varepsilon)}{p} C \|v\|_E^{\frac{p}{2\alpha}} \\
 &\leq \frac{1}{2} \|v\|_E^2 + \frac{C\varepsilon}{2} \|v\|_E^2 + \frac{C(\varepsilon)}{p} C \|v\|_E^{\frac{p}{2\alpha}} \\
 &< +\infty, \quad \forall v \in E.
 \end{aligned}$$

It is trivial to prove that I is continuously differentiable on E , we omit it here. This completes the proof. $\square$

Lemma 4. If assumptions (V) and (g_0) – (g_2) hold, we have

$$\int_{\mathbb{R}^N} a(x) [g(f(v_n)) f'(v_n) - g(f(v)) f'(v)] (v_n - v) dx = o(1), \tag{18}$$

whenever $v_n \rightharpoonup v$ in E .

Proof. Let $v_n \rightharpoonup v$ in E , then there exists a constant $M_1 > 0$ such that $\|u_n\|_E \leq M_1$ and $\|u\|_E \leq M_1$. By Equation (16), (f_2) , (f_3) , (f_7) and (f_{11}) , we can deduce that

$$\begin{aligned}
 &\int_{\mathbb{R}^N} a(x) [g(f(v_n)) f'(v_n) - g(f(v)) f'(v)] (v_n - v) dx \\
 &\leq \int_{\mathbb{R}^N} a(x) \left[\left(\varepsilon |f(v_n)| + C(\varepsilon) |f(v_n)|^{p-1} \right) |f'(v_n)| + \left(\varepsilon |f(v)| + C(\varepsilon) |f(v)|^{p-1} \right) |f'(v)| \right] |v_n - v| dx \\
 &\leq \int_{\mathbb{R}^N} a(x) \left[\left(\varepsilon |v_n| + C(\varepsilon) |f(v_n)|^{p-2\alpha} |f(v_n)|^{2\alpha-1} \right) |f'(v_n)| \right. \\
 &\quad \left. + \left(\varepsilon |f(v)| + C(\varepsilon) |f(v)|^{p-1} \right) |f'(v)| \right] |v_n - v| dx \\
 &\leq \int_{\mathbb{R}^N} a(x) \left[\left(\varepsilon |v_n| + C(\varepsilon) (\sqrt{2\alpha})^{\frac{p-4\alpha}{2\alpha}} |v_n|^{\frac{p-2\alpha}{2\alpha}} \right) + \left(\varepsilon |v| + C(\varepsilon) (\sqrt{2\alpha})^{\frac{p-4\alpha}{2\alpha}} |v|^{\frac{p-2\alpha}{2\alpha}} \right) \right] |v_n - v| dx.
 \end{aligned} \tag{19}$$

By Hölder's inequality we get

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)|v_n||v_n - v|dx &\leq \left(\int_{\mathbb{R}^N} a(x)|v_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} a(x)|v_n - v|^2 dx \right)^{\frac{1}{2}} \\ &\leq \|v_n\|_{L_a^2} \cdot \|v_n - v\|_{L_a^2}, \end{aligned} \quad (20)$$

and according to the same method, we can obtain

$$\int_{\mathbb{R}^N} a(x)|v||v_n - v|dx \leq \|v\|_{L_a^2} \cdot \|v_n - v\|_{L_a^2}. \quad (21)$$

Also, we have

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)|v_n|^{\frac{p-2\alpha}{2\alpha}}|v_n - v|dx &\leq \left(\int_{\mathbb{R}^N} a(x)|v_n|^{\frac{p-2\alpha}{\alpha}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} a(x)|v_n - v|^2 dx \right)^{\frac{1}{2}} \\ &\leq \|v_n\|_{L_a^{\frac{p-2\alpha}{\alpha}}} \cdot \|v_n - v\|_{L_a^2}, \end{aligned} \quad (22)$$

where $\frac{p-2\alpha}{\alpha} \leq 2^* - \frac{4}{\gamma(N-2)}$. Following the same approach, we can get

$$\int_{\mathbb{R}^N} a(x)|v|^{\frac{p-2\alpha}{2\alpha}}|v_n - v|dx \leq \|v\|_{L_a^{\frac{p-2\alpha}{\alpha}}} \cdot \|v_n - v\|_{L_a^2}. \quad (23)$$

Then from Equations (19)–(23) and Lemma 2 we can deduce that Equation (18) is held. This completes the proof of Lemma 4. $\square$

Lemma 5 ([26]). *Let E be a real Banach space and $J : E \rightarrow \mathbb{R}$ be a C^1 -smooth functional and satisfy the (C) condition, that is, $\{u_n\}$ has a convergent subsequence in $H^1(\mathbb{R}^N)$ whenever $\{J(u_n)\}$ is bounded and $\|J'(u_n)\|_{E^*}(1 + \|u_n\|_E) \rightarrow 0$ as $n \rightarrow \infty$. If J satisfies the following conditions*

- (i) $J(0) = 0$;
- (ii) There exist constants $\varrho, \alpha > 0$ such that $J|_{\partial B_\varrho(0)} \geq \alpha$;
- (iii) There exists $e \in E \setminus \bar{B}_\varrho(0)$ such that $J(e) \leq 0$.

$B_\varrho(0)$ is an open ball in E of radius ϱ centred at 0, then J possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}.$$

3. Proof of Theorem 1

Lemma 6. *Under the assumptions of Theorem 1, I satisfies the (C) condition.*

Proof. Let $\{v_n\} \subset E$ be a Cerami sequence of I , i.e., $\{I(v_n)\}$ is bounded and $(1 + \|v_n\|_E)I'(v_n) \rightarrow 0$ in E^* . We divide the proof into two steps.

Step 1: We prove that the sequence $\{v_n\}$ is bounded. Arguing indirectly, we assume that $\|v_n\|_E \rightarrow +\infty$ as $n \rightarrow \infty$. We define

$$A_n^2 := \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x)f^2(v_n) \right] dx.$$

Then it follows from Lemma 7 in reference [17] that $A_n^2 \rightarrow \infty$ as $n \rightarrow \infty$. Since $\{v_n\}$ is Cerami sequence of I , then by (g_3) we can get

$$\begin{aligned}
 o_n(1) &= \frac{I(v_n) - \frac{1}{v} \left\langle I'(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle}{A_n^2} = \frac{1}{A_n^2} \left[\int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{1 + 4\alpha^2 f^{4\alpha-2}(v_n)}{v(1 + 2\alpha f^{4\alpha-2}(v_n))} \right) |\nabla v_n|^2 dx \right. \\
 &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{1}{v} \right) V(x) f^2(v_n) dx \\
 &\quad \left. + \int_{\mathbb{R}^N} a(x) \left(\frac{1}{v} g(f(v_n)) f(v_n) - G(f(v_n)) \right) dx \right] \\
 &\geq \frac{1}{A_n^2} \left[\int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{2\alpha}{v} \right) |\nabla v_n|^2 dx \right. \\
 &\quad \left. + \int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{2\alpha}{v} \right) V(x) f^2(v_n) dx + \int_{\mathbb{R}^N} a(x) \tilde{G}(f(v_n)) dx \right] \\
 &= \left(\frac{1}{2} - \frac{2\alpha}{v} \right) \frac{A_n^2}{A_n^2} + \frac{1}{A_n^2} \int_{\mathbb{R}^N} a(x) \tilde{G}(f(v_n)) dx \\
 &\geq \left(\frac{1}{2} - \frac{2\alpha}{v} \right),
 \end{aligned}$$

which is a contradiction. Hence $\|v_n\|_E$ is bounded.

Step 2: We prove that the sequence $\{v_n\}$ possesses a convergent subsequence in E . Indeed, by the boundedness of $\{v_n\}$ and the compactness of the embedding $E \rightarrow L_a^2$, up to a subsequence, one has $v_n \rightharpoonup v$ in E , $v_n \rightarrow v$ in L_a^2 , and $v_n(x) \rightarrow v(x)$ a.e. on $\mathbb{R}^N$. Hence, as in the proof of Lemma 3.10 and Lemma 3.11 in reference [14], we may prove that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 dx + \int_{\mathbb{R}^N} V(x) [f(v_n) f'(v_n) - f(v) f'(v)] (v_n - v) dx \geq C \|v_n - v\|_E^2. \quad (24)$$

By (24) and (20), one has

$$\begin{aligned}
 o_n(1) &= \langle I'(v_n) - I'(v), v_n - v \rangle \\
 &= \int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 dx + \int_{\mathbb{R}^N} V(x) [f(v_n) f'(v_n) - f(v) f'(v)] (v_n - v) dx \\
 &\quad - \int_{\mathbb{R}^N} a(x) [g(f(v_n)) f'(v_n) - g(f(v)) f'(v)] (v_n - v) dx \\
 &\geq C \|v_n - v\|_E^2 + o_n(1).
 \end{aligned}$$

Hence, $v_n \rightarrow v$ in E . This completes the proof of Step 2. Summing up the above steps, we conclude that I satisfies the (C) condition. The proof is completed. $\square$

Lemma 7. Under the assumptions of Theorem 1, I has a mountain pass geometry.

Proof. For small $0 < \rho \ll 1$, define

$$S_\rho = \{v \in E : \|v\|_E = \rho\}.$$

Next, we claim that there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) f^2(v)) dx \geq C_1 \|v\|_E^2, \quad \forall v \in S_\rho. \quad (25)$$

Otherwise, there exists a sequence $\{v_n\} \subset S_\rho$ such that

$$A_n^2 := \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)f^2(v_n)) dx < \frac{1}{n} \|v_n\|_E^2. \quad (26)$$

Then by (26), one has $\frac{A_n^2}{\|v_n\|_E^2} \rightarrow 0$ as $n \rightarrow +\infty$. Set $w_n = \frac{v_n}{\|v_n\|_E}$ and $g_n = \frac{f^2(v_n)}{\|v_n\|_E^2}$. Then we have

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(x)g_n(x) dx \rightarrow 0.$$

Therefore,

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx \rightarrow 0, \quad \int_{\mathbb{R}^N} V(x)g_n(x) dx \rightarrow 0, \quad \int_{\mathbb{R}^N} V(x)w_n^2 dx \rightarrow 1.$$

We now demonstrate that for each $\varepsilon > 0$, there exists $C_2 > 0$ (independent of n) such that

$$\text{meas}(\Omega_n) < \varepsilon, \text{ and } \Omega_n := \{x \in \mathbb{R}^N : |v_n(x)| \geq C_2\},$$

where “meas” denotes that the Lebesgue measure. If this assertion is false, there exist $\varepsilon_0 > 0$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that for all $k \in \mathbb{N}$,

$$\text{meas}(\{x \in \mathbb{R}^N : |v_{n_k}(x)| \geq k\}) \geq \varepsilon_0 \geq 0.$$

Set $\Omega_{n_k} := \{x \in \mathbb{R}^N : |v_{n_k}(x)| \geq k\}$. By (f_3) , (f_9) and (V) ,

$$\rho = \|v_{n_k}\|_E^2 \geq A_{n_k}^2 \geq \int_{\mathbb{R}^N} V(x)f^2(v_{n_k}) dx \geq \int_{\Omega_{n_k}} V(x)f^2(v_{n_k}) dx \geq Ck^{\frac{1}{a}}\varepsilon_0 \rightarrow +\infty$$

as $k \rightarrow \infty$, which is a contradiction. Hence the assertion holds. For $|v_n(x)| \leq C_2$, properties (f_9) and (f_{13}) imply that

$$\frac{C}{C_2^2} v_n^2(x) \leq f^2\left(\frac{1}{C_2} v_n(x)\right) \leq C_3 f^2(v_n(x)),$$

from which we deduce the existence of the constant $C_4 > 0$ such that

$$\int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 dx \leq C_4 \int_{\mathbb{R}^N} V(x)g_n(x) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By the absolute continuity of the integral, there exists $\varepsilon > 0$ such that whenever $\Omega_n \subset \mathbb{R}^N$ and $\text{meas}(\Omega_n) < \varepsilon$, we have

$$\int_{\Omega} V(x)w_n^2 dx < \frac{1}{2}.$$

For this ε , we get

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)w_n^2 dx &= \int_{\Omega_n} V(x)w_n^2 dx + \int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 dx \\ &\leq \frac{1}{2} + \int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 dx \\ &\rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which contradicts the fact that $\int_{\mathbb{R}^N} V(x)w_n^2 dx \rightarrow 1$. Consequently, we show that Equation (25) holds. For $v \in S_\rho$, by Equations (16), (17) and (25) and (f_3) , we have

$$\begin{aligned} I(v) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v) dx - \int_{\mathbb{R}^N} a(x)G(f(v)) dx \\ &\geq \frac{C_1}{2} \|v\|_E^2 - \int_{\mathbb{R}^N} a(x)G(f(v)) dx \\ &\geq \frac{C_1}{2} \|v\|_E^2 - \frac{\varepsilon}{2} \int_{\mathbb{R}^N} a(x)|v|^2 dx - \frac{C(\varepsilon)}{p} \int_{\mathbb{R}^N} a(x)|f(v)|^p dx \\ &\geq \frac{C_1}{2} \|v\|_E^2 - \frac{\varepsilon}{2} \|v\|_{L_a^2}^2 - \frac{C(\varepsilon)}{p} C \int_{\mathbb{R}^N} |f(v)|^p dx \\ &\geq \frac{C_1}{2} \|v\|_E^2 - \frac{C\varepsilon}{2} \|v\|_E^2 - \frac{C(\varepsilon)}{p} C\rho^{\frac{p}{2\alpha}} \\ &= \frac{C_1}{2} \rho^2 - \frac{C\varepsilon}{2} \rho^2 - \frac{C(\varepsilon)}{p} C\rho^{\frac{p}{2\alpha}} := \delta > 0, \end{aligned}$$

for small $\varepsilon > 0$ and $\rho > 0$.

Next we prove that there exists v_0 such that $I(v_0) < 0$ and $\|v_0\|_E > \rho$. We define $a_1 = \inf_{|x| \leq 1} a(x) > 0$ and $a_2 = \sup_{|x| \leq 1} a(x) > 0$. Let

$$v_1 \in C_0^\infty(\{|x| \leq 1\}, \mathbb{R}) \quad \text{and} \quad \|v_1\|_E = 1.$$

Set

$$\beta > \frac{1}{2a_1\theta^{4\alpha} \int_{|x| \leq 1} |v_1|^2 dx}.$$

By (g_5) , there exists $\xi > 0$ and $|t| \geq \xi$ such that

$$G(t) \geq \beta|t|^{4\alpha},$$

then, by (g_4) we can deduce that

$$G(t) \geq \beta(|t|^{4\alpha} - \xi^{4\alpha}) - d\xi^{4\alpha}, \quad |t| \geq \xi. \quad (27)$$

By (f_9) and (27), for any η large enough, we have

$$G(f(\eta v_1)) \geq \beta|f(\eta v_1)|^{4\alpha} - (\beta + d)\xi^{4\alpha} \geq \beta\theta^{4\alpha}\eta^2|v_1|^2 - (\beta + d)\xi^{4\alpha}. \quad (28)$$

Hence, by (f_3) and (28) one has

$$\begin{aligned} I(\eta v_1) &= \frac{1}{2} \int_{|x| \leq 1} |\nabla(\eta v_1)|^2 dx + \frac{1}{2} \int_{|x| \leq 1} V(x)f^2(\eta v_1) dx - \int_{|x| \leq 1} a(x)G(f(\eta v_1)) dx \\ &\leq \frac{1}{2} \int_{|x| \leq 1} |\nabla(\eta v_1)|^2 dx + \frac{1}{2} \int_{|x| \leq 1} V(x)\eta^2|v_1|^2 dx - \int_{|x| \leq 1} a(x)G(f(\eta v_1)) dx \\ &\leq \frac{\eta^2}{2} \|v_1\|_E^2 - \int_{|x| \leq 1} a(x)G(f(\eta v_1)) dx \\ &= \frac{\eta^2}{2} - \int_{|x| \leq 1} a(x)G(f(\eta v_1)) dx \\ &\leq \frac{\eta^2}{2} - \int_{|x| \leq 1} a(x)\beta\theta^{4\alpha}\eta^2|v_1|^2 dx + \int_{|x| \leq 1} a(x)(\beta + d)\xi^{4\alpha} dx \\ &\leq \frac{\eta^2}{2} - \int_{|x| \leq 1} a_1\beta\theta^{4\alpha}\eta^2|v_1|^2 dx + \int_{|x| \leq 1} a_2(\beta + d)\xi^{4\alpha} dx \\ &\leq \left(\frac{1}{2} - a_1\beta\theta^{4\alpha} \int_{|x| \leq 1} |v_1|^2 dx \right) \eta^2 + a_2(\beta + d)\xi^{4\alpha} V_N, \end{aligned}$$

where V_N denotes the volume of the unit ball. It is obvious that $I(\eta v_1) \rightarrow -\infty$ as $\eta \rightarrow +\infty$. Therefore, there exists $\eta_1 > \rho$ such that $I(\eta_1 v_1) < 0$. Let $v_0 = \eta_1 v_1$; we can obtain that $I(v_0) < 0$, which completes the proof of Lemma 7. $\square$

Proof of Theorem 1. By Lemmas 6 and 7, we can see that I satisfies the (C) condition and has a mountain-pass geometry. According to Lemma 5, there exists a nontrivial critical point $v \in E$ for the functional I . Consequently, $u = f(v)$ constitutes a nontrivial weak solution to the original Equation (1). This completes the proof of Theorem 1. $\square$

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