

Article

Saddlepoint Inference for Nonlinear Statistics from Inverse Gaussian Models: Applications to Clinical, Engineering, and Environmental Data

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Abstract

This paper applies established saddlepoint approximation techniques to nonlinear statistics arising from inverse Gaussian models. In particular, we consider the product of independent inverse Gaussian random variables and the ratio of weighted linear combinations of inverse Gaussian random variables, for which exact distributions are generally unavailable in closed form. For the product statistic, a logarithmic transformation converts the problem into one involving the cumulant generating function of a sum of log-transformed variables. This cumulant generating function is expressed in terms of fractional moments including modified Bessel functions of the second kind. For the ratio statistic, the event including the ratio is reformulated in terms of a linear statistic, which enables the use of saddlepoint density and Lugannani-Rice distribution approximations. The proposed formulation accommodates heterogeneous model parameters, overlapping numerator and denominator components, and flexible coefficient structures subject to positivity of the denominator. Simulation studies show that the proposed approximations provide accurate results across a range of sample sizes, skewness regimes, and parameter configurations. Furthermore, simulation results indicate that the saddlepoint approximation is more accurate than the normal approximation. Sensitivity analysis confirms that the proposed approximations are reasonably stable under moderate inverse Gaussian parameter misspecification. Three real data applications including clinical illness scores, engineering repair times, and environmental runoff measurements illustrate the practical usefulness of the approach. Overall, the results indicate that the saddlepoint approximation provides an accurate and computationally efficient tool for inference on nonlinear statistics from inverse Gaussian models when exact distributions are not available.

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1. Introduction

The inverse Gaussian (IG) distribution is a cornerstone of probability and statistical theory, owing to its well-established mathematical structure and broad range of applications. It is based on a solid statistical and physical foundation that arises as the

first-passage time distribution of Brownian motion with positive drift, as detailed in Seshadri [1]. This distribution is an ideal choice for reliability and survival analysis because it is positive, flexible, and right-skewed. This makes it natural for modelling lifetimes, degradation times, and failure times when failure occurs after degradation crosses a threshold. This is clearly demonstrated in Wiener/IG process degradation and accelerated life models [2,3]. In cognitive neuroscience, it is used to describe reaction times or decision making [4,5]. The importance of the IG distribution in queuing theory and operations research is demonstrated through diffusion approximations, which describe workloads and occupation periods as first-passage times for Brownian processes [6]. In engineering and physics, the IG directly models threshold-crossing times in transport and detection problems due to its first-passage origin [7]. Despite the extensive research interest in the IG distribution, accurate distributional results remain limited to a narrow class of linear and quadratic functions. While key references such as Chhikara and Folks [8], Seshadri [1], Whitmore [9], Tian and Wilding [10], Krishnamoorthy and Tian [11], Peña and Jauch [12], and Chakraborty and Jana [13] discussed the distributions of sums, sample means, quadratic functions, and ratios associated with the sample mean, they lack closed-form solutions for more complex nonlinear functions, such as the product of independent random variables (RVs) following IG distribution, and ratios involving their linear combinations. In degradation and reliability studies, products of positive measurements may summarize multiplicative accumulation or geometric-mean behavior, while ratios of weighted sums may compare degradation, repair burden, or operating stress across systems or time periods. Because these statistics do not retain the algebraic structure of the IG distribution family, they fall outside the scope of classical convolution and transformation techniques, rendering exact inference infeasible. In light of these challenges, the saddlepoint approximation (SPA) emerges as a robust analytical alternative. It leverages explicit moment generating function to provide highly accurate approximations for the density and distribution functions. This methodology is particularly effective for IG models, for which the moment generating function is available, even though the associated nonlinear statistics remain analytically challenging.

SPA techniques provide a powerful analytical tool for obtaining highly accurate distributional approximations. By exploiting the cumulant generating function (CGF) of the considered statistic, saddlepoint methods are known to deliver near exact approximations even for small and moderate sample sizes. The SPA literature begins with the foundations laid by Daniels [14]. The distribution function (DF) approximation was introduced by Lugannani and Rice [15], and the SPA for estimating equations was developed by Daniels [16], as well as general developments by Barndorff-Nielsen and Cox [17], Reid [18], Jensen [19], Kolassa [20], Butler [21], and Abd-Elfattah [22]. Recently, Jensen [23] discussed accuracy and limitations in high-dimensional regimes. Arevalillo [24] proposed an empirical approximation to quantiles using the Lugannani–Rice SPA method. Joutard [25] examined the numerical behavior of certain SPA methods near endpoints of support. Several recent papers illustrate the practical use of the SPA for non-parametric tests and randomized designs. A saddlepoint formula was developed for the weighted log-rank tests under Wei’s urn design by Abd El-Raheem and Abd-Elfattah [26]. Abd El-Raheem et al. [27] introduced the SPA of the non-parametric tests for panel count and current status data. Abd El-Raheem and Hosny [28,29] approximated the p -value of certain non-parametric tests. Abd El-Raheem and Hosny [30] also addressed SPA for random sums and linear combinations of independent exponentiated exponential RVs.

The current paper applies established SPA techniques to obtain accurate density and distributional approximations for product and ratio statistics under IG assumptions. The contribution is not intended as a new development in saddlepoint theory itself, but rather as

a systematic IG-specific formulation, implementation, and numerical assessment of classical SPAs for nonlinear statistics whose exact distributions are generally unavailable. Although saddlepoint methods have been applied to products and ratios in other distributional settings, such as gamma models [22], the IG case has distinctive analytical features. For the product statistic, the logarithmic transformation leads naturally to fractional moments, or equivalently Mellin-type moments, involving modified Bessel functions of the second kind; these moments yield a tractable CGF and facilitate both density and distribution approximations. For the ratio statistic, weighted linear combinations of IG variables yield a CGF with a square-root structure and parameter-dependent admissible domains. The significant contributions of this study are therefore application oriented and may be summarized as follows. First, the paper provides a systematic saddlepoint treatment of product and ratio statistics under IG models, while explicitly relying on established saddlepoint theory. Second, the proposed formulation accommodates heterogeneous IG parameters, overlapping numerator and denominator components, and flexible coefficient structures, including negative coefficients where admissible, subject to the essential requirement that the denominator remains positive. Third, a comprehensive simulation study is conducted to examine the finite-sample accuracy of the SPA compared with the traditional normal approximation (NA) under challenging scenarios, including heterogeneous, highly skewed, and overlapping component regimes. Furthermore, the simulation study provides numerical stability checks and sensitivity analysis under parameter misspecification. Fourth, the practical usefulness of the method is demonstrated through clinical, engineering, and environmental data examples. To the best of our knowledge, this is the first systematic saddlepoint treatment of such product and ratio statistics for IG models with these levels of flexibility.

For ease of reference, the main notation and abbreviations used throughout the article are summarized in Table 1.

Table 1. Main notation and abbreviations used throughout the article.

Symbol/Abbreviation	Meaning
IG	Inverse Gaussian
RV	Random variable
RVs	Random variables
SPA	Saddlepoint approximation
NA	Normal approximation
CGF	Cumulant generating function
DF	Distribution function
K–S	Kolmogorov–Smirnov
MAE	Mean absolute error
MAEs	Mean absolute errors
MC	Monte Carlo
U_j	The j th inverse Gaussian random variable
θ_j	Mean parameter of the j th inverse Gaussian random variable
ξ_j	Shape parameter of the j th inverse Gaussian random variable
α_j, β_j	Known coefficients in the numerator and denominator linear combinations
Q	Product statistic, $Q = \prod_{j=1}^m U_j$
S	Log-transformed product statistic, $S = \log Q$
q	A positive realization or threshold of the product statistic Q
s	A realization or threshold of the log-product statistic S
$\Psi(t)$	Cumulant generating function of $S = \log Q$
$\tilde{t}_q$	Saddlepoint satisfying $\Psi'(\tilde{t}_q) = \log q$
$\hat{f}_Q(q)$	Saddlepoint approximation for the density of Q at q
$\hat{F}_Q(q)$	Saddlepoint approximation for the distribution function of Q at q

Table 1. Cont.

Symbol/Abbreviation	Meaning
$\hat{f}_{Q,N}(q)$	Normal approximation for the density of Q at q
$\hat{F}_{Q,N}(q)$	Normal approximation for the distribution function of Q at q
A	Numerator weighted sum, $A = \sum_{j=1}^m \alpha_j U_j$
B	Denominator weighted sum, $B = \sum_{j=k}^n \beta_j U_j$
R	Ratio statistic, $R = A/B$
r	A realization or threshold of the ratio statistic R
H_r	Linear statistic associated with the ratio, $H_r = A - rB$
$\Lambda_r(s)$	Cumulant generating function of H_r
$\hat{s}_r$	Saddlepoint satisfying $\Lambda'_r(\hat{s}_r) = 0$
$\hat{f}_R(r)$	Saddlepoint approximation for the density of R at r
$\hat{F}_R(r)$	Saddlepoint approximation for the distribution function of R at r
$\hat{f}_{R,N}(r)$	Normal approximation for the density of R at r
$\hat{F}_{R,N}(r)$	Normal approximation for the distribution function of R at r
$K_\nu(x)$	Modified Bessel function of the second kind of order ν
ν	Order parameter of the modified Bessel function
$\Phi(\cdot)$	Standard normal distribution function
$\phi(\cdot)$	Standard normal density function
ω_q, v_q	Lugannani–Rice quantities for the product statistic
ω_r, v_r	Lugannani–Rice quantities for the ratio statistic

The remainder of the article is organized as follows. Section 2 discusses the approximations for the density and distribution functions of the product statistic. Section 3 presents the approximations for the density and distribution functions of the ratio statistic. Section 4 describes the simulation study and assesses the numerical performance of the proposed approximation for the product statistic. Section 5 presents the simulation study and evaluates the numerical accuracy of the suggested approximation for the ratio statistic. Section 6 compares the computational costs of the SPA, NA, and MC procedures for representative product and ratio settings. Section 7 provides a sensitivity analysis of model misspecification on SPA performance. Section 8 illustrates the real data applications of the considered statistics. The paper concludes with a summary of observations, limitations, and suggestions for further extensions.

2. Product of IG Variables

In this section, we introduce the saddlepoint density and distribution functions for the product of m IG variables. Let U_j , for $j = 1, 2, \dots, m$, be independent $\text{IG}(\theta_j, \xi_j)$ RVs. The quantity of interest is the product statistic

$$Q = \prod_{j=1}^m U_j,$$

which frequently arises in reliability systems, multiplicative risk models, and cumulative degradation processes. For analytical convenience, define

$$S = \log Q = \sum_{j=1}^m \log U_j.$$

Then the DF of the product statistic satisfies

$$F_Q(q) = \Pr(Q \leq q) = \Pr(S \leq \log q) = F_S(\log q), \quad q > 0,$$

and the density is related by the standard change of variables identity;

$$f_Q(q) = \frac{1}{q} f_S(\log q), \quad q > 0.$$

The saddlepoint approach is applied to the statistic S . Its moment generating function can be written as

$$M_S(t) = \mathbb{E}(e^{tS}) = \mathbb{E}\left(\prod_{j=1}^m U_j^t\right) = \prod_{j=1}^m \mathbb{E}(U_j^t),$$

where $\mathbb{E}(U_j^t)$ denotes the Mellin transform of U_j . For $U_j \sim IG(\theta_j, \xi_j)$, the Mellin transform admits the following closed-form expression:

$$\mathbb{E}(U_j^t) = \theta_j^t \frac{K_{t-\frac{1}{2}}\left(\frac{\xi_j}{\theta_j}\right)}{K_{-\frac{1}{2}}\left(\frac{\xi_j}{\theta_j}\right)},$$

where $K_\nu(\cdot)$ is the modified Bessel function of the second kind. A derivation of the Mellin transformation of U_j is provided in Appendix A.

Consequently, the CGF of S is

$$\Psi(t) = \log M_S(t) = \sum_{j=1}^m \left[t \log \theta_j + \log K_{t-\frac{1}{2}}\left(\frac{\xi_j}{\theta_j}\right) - \log K_{-\frac{1}{2}}\left(\frac{\xi_j}{\theta_j}\right) \right].$$

The domain of $\Psi(t)$ is the whole real line, because the IG Mellin transform $\mathbb{E}(U_j^t)$ is finite for all $t \in \mathbb{R}$. Thus,

$$\mathcal{D}_\Psi = \{t \in \mathbb{R} : \Psi(t) < \infty\} = \mathbb{R}.$$

Let $x_j = \xi_j/\theta_j$ and set $\nu = t - \frac{1}{2}$. Since the dependence on t enters through the order ν of $K_\nu(x)$, we use derivatives with respect to the order.

Then the first two derivatives of $\Psi(t)$ are

$$\Psi'(t) = \sum_{j=1}^m \left[\log \theta_j + \frac{\partial}{\partial \nu} \log K_\nu(x_j) \right], \quad \Psi''(t) = \sum_{j=1}^m \frac{\partial^2}{\partial \nu^2} \log K_\nu(x_j).$$

Closed-form expressions for derivatives of $K_\nu(x)$ with respect to the order ν are not available in a simple form suitable for direct implementation. Therefore, these derivatives can be evaluated using stable numerical differentiation or specialized numerical routines. In the numerical implementation, derivatives with respect to the order ν are computed using central finite differences applied to $\log K_\nu(x)$, rather than to $K_\nu(x)$ itself. This logarithmic implementation reduces overflow and underflow effects and improves stability when x is small or when the IG distribution is highly skewed. Several step sizes are examined to verify that the resulting SPAs are insensitive to moderate changes in the differentiation step; see Table 5.

For a given $s \in \mathbb{R}$, the saddlepoint $\tilde{t} = \tilde{t}(s)$ is defined as the solution to the equation $\Psi'(\tilde{t}) = s$. Since $\Psi(t)$ is the CGF of $S = \log Q$, it is convex on $\mathbb{R}$. Provided that the variables are non-degenerate, $\Psi(t)$ is strictly convex and $\Psi'(t)$ is strictly increasing. Consequently, for any s in the interior of the support of S , the saddlepoint equation $\Psi'(\tilde{t}) = s$ has a unique real solution.

In our application, we evaluate the distribution of S at $s = \log q$ when approximating $F_Q(q) = F_S(\log q)$, and at the same s when approximating the density via $f_Q(q)$.

2.1. Saddlepoint Density for the Product Statistic

The SPA for the density of the statistic S at s is

$$\hat{f}_S(s) = \frac{1}{\sqrt{2\pi \Psi''(\tilde{t})}} \exp(\Psi(\tilde{t}) - \tilde{t}s), \quad \text{where } \Psi'(\tilde{t}) = s.$$

Using $s = \log q$ and the Jacobian transformation $f_Q(q) = q^{-1}f_S(\log q)$, the corresponding approximation is

$$\hat{f}_Q(q) = \frac{1}{q\sqrt{2\pi \Psi''(\tilde{t}_q)}} \exp(\Psi(\tilde{t}_q) - \tilde{t}_q \log q), \quad q > 0, \quad (1)$$

where $\tilde{t}_q$ solves $\Psi'(\tilde{t}_q) = \log q$.

2.2. Saddlepoint Distribution for the Product Statistic

A highly accurate approximation for $F_S(s)$ is provided by the Lugannani–Rice formula. Let $\tilde{t}$ be the saddlepoint satisfying $\Psi'(\tilde{t}) = s$, and define

$$\omega = \text{sgn}(\tilde{t})\sqrt{2(\tilde{t}s - \Psi(\tilde{t}))}, \quad \text{and} \quad v = \tilde{t}\sqrt{\Psi''(\tilde{t})}.$$

Then the Lugannani–Rice approximation for $F_S(s)$ is

$$\hat{F}_S(s) = \Phi(\omega) + \phi(\omega) \left(\frac{1}{\omega} - \frac{1}{v} \right), \quad \text{for } s \neq \mathbb{E}(S),$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard normal distribution and density functions, respectively. Therefore, for the product statistic Q we have

$$\hat{F}_Q(q) = \hat{F}_S(\log q) = \Phi(\omega_q) + \phi(\omega_q) \left(\frac{1}{\omega_q} - \frac{1}{v_q} \right), \quad \text{for } q \neq \mathbb{E}(Q), \quad (2)$$

with $\tilde{t}_q$ defined by $\Psi'(\tilde{t}_q) = \log q$, and

$$\omega_q = \text{sgn}(\tilde{t}_q)\sqrt{2(\tilde{t}_q \log q - \Psi(\tilde{t}_q))}, \quad \text{and} \quad v_q = \tilde{t}_q\sqrt{\Psi''(\tilde{t}_q)}.$$

When the saddlepoint $\tilde{t}_q = 0$, both Lugannani–Rice quantities ω_q and v_q are equal to zero, and the term $(1/\omega_q - 1/v_q)$ is undefined. In this case, the Lugannani–Rice approximation is evaluated using its limiting form at the mean,

$$\hat{F}_S(\Psi'(0)) = \frac{1}{2} + \frac{\Psi'''(0)}{6\sqrt{2\pi}\{\Psi''(0)\}^{3/2}}.$$

It should be noted that since $S = \log Q$, the lower endpoint $Q \downarrow 0$ corresponds to $S \downarrow -\infty$. Therefore, extremely small product values may lead to more demanding numerical saddlepoint calculations, especially under highly skewed IG models. The simulation study in Section 4 includes lower-tail probabilities to assess this issue empirically. Although the results indicate good accuracy at the reported probability levels, caution is warranted for probabilities much smaller than those considered in this article.

2.3. Normal Approximation for the Product Statistic

For comparison with the SPA, we also consider a conventional NA for the product statistic. Since $Q = \prod_{j=1}^m U_j$ is strictly positive and typically skewed, a direct NA for Q is not appropriate. Therefore, we apply the NA to the log-transformed

statistic $S = \log Q = \sum_{j=1}^m \log U_j$, which is the same transformed statistic used in the saddlepoint formulation.

Let $\mu_S = \mathbb{E}(S)$ and $\sigma_S^2 = \text{Var}(S)$ denote the mean and variance of the statistic S . Then $\mu_S = \Psi'(0)$ and $\sigma_S^2 = \Psi''(0)$. Hence, the NA assumes

$$S \approx N(\Psi'(0), \Psi''(0)).$$

Consequently, for $q > 0$,

$$F_Q(q) = \Pr(Q \leq q) = \Pr(S \leq \log q)$$

is approximated by

$$\hat{F}_{Q,N}(q) = \Phi\left(\frac{\log q - \Psi'(0)}{\sqrt{\Psi''(0)}}\right), \quad q > 0, \quad (3)$$

where the subscript N denotes the NA.

The corresponding NA for the density of Q follows from the change of variables $S = \log Q$. Since

$$f_Q(q) = \frac{1}{q} f_S(\log q), \quad q > 0,$$

and

$$f_S(s) \approx \frac{1}{\sqrt{\Psi''(0)}} \phi\left(\frac{s - \Psi'(0)}{\sqrt{\Psi''(0)}}\right),$$

we obtain

$$\hat{f}_{Q,N}(q) = \frac{1}{q\sqrt{\Psi''(0)}} \phi\left(\frac{\log q - \Psi'(0)}{\sqrt{\Psi''(0)}}\right), \quad q > 0.$$

3. Ratio of Linear Combinations of IG Variables

Let $U_j, j = 1, \dots, L$, be independent IG RVs,

$$U_j \sim IG(\theta_j, \xi_j), \quad \theta_j > 0, \xi_j > 0,$$

where θ_j denotes the mean parameter and ξ_j is the shape parameter.

Consider two weighted sums constructed from U_j :

$$A = \sum_{j=1}^m \alpha_j U_j, \quad \text{and} \quad B = \sum_{j=k}^n \beta_j U_j,$$

where α_j and β_j are known real coefficients which may be negative. We focus particularly on the case $1 \leq k < m$, in which the numerator and denominator may share components and are therefore not independent. Hence, the statistic of interest is the ratio,

$$R = \frac{A}{B},$$

which covers a broad class of practically relevant quantities, including efficiency measures, reliability indices, and normalized summaries of lifetime.

Throughout the ratio analysis, we assume that the denominator B is strictly positive almost surely. This condition is automatically satisfied when $\beta_j \geq 0$ for all j and at least one $\beta_j > 0$, because IG RVs are positive. If mixed-sign denominator coefficients are used, denominator positivity is not automatic and must be verified analytically or imposed as a model constraint. In the numerical studies and real data illustrations reported

below, the denominator weights are chosen to be positive, ensuring that the transformation $R \leq r \iff A - rB \leq 0$ is valid.

For notational convenience in the derivations below, we extend the coefficient sequences to the common index set $j = 1, \dots, L$, where $L = \max(m, n)$, by adopting the convention that $\alpha_j = 0$ for $j > m$ and $\beta_j = 0$ for $j < k$ or $j > n$. Under this zero extension convention, the numerator and denominator may be written uniformly as

$$A = \sum_{j=1}^L \alpha_j U_j, \quad B = \sum_{j=1}^L \beta_j U_j,$$

and consequently

$$H_r = A - rB = \sum_{j=1}^L (\alpha_j - r\beta_j) U_j,$$

where r is a realization of the ratio statistic R . Then the DF of R satisfies

$$\Pr(R \leq r) = \Pr(H_r \leq 0).$$

This representation reduces inference for the ratio R to inference on the transformed linear statistic H_r . Equivalently, the ratio R may be viewed as the root of the estimating equation $A - rB = 0$. Therefore, the present ratio approximation is an application of the classical saddlepoint approach for estimating equations, developed by Daniels [16], specialized here to weighted linear combinations of independent IG variables with possible overlap between numerator and denominator components.

3.1. Cumulant Generating Function for the Ratio Statistic

Let $\Lambda(s, t)$ be the joint CGF of $A = \sum_{j=1}^L \alpha_j U_j$ and $B = \sum_{j=1}^L \beta_j U_j$. Then

$$\Lambda(s, t) = \sum_{j=1}^L \frac{\xi_j}{\theta_j} \left[1 - \left(1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j s + \beta_j t) \right)^{1/2} \right],$$

defined on the set

$$\{(s, t) : 1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j s + \beta_j t) > 0, \text{ for all } \alpha_j \text{ and } \beta_j, j = 1, \dots, L\}.$$

Let $\Lambda_r(s)$ denote the CGF of H_r . Then using independence and the CGF of the IG distribution, we obtain

$$\Lambda_r(s) = \sum_{j=1}^L \frac{\xi_j}{\theta_j} \left[1 - \left(1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j - r\beta_j) s \right)^{1/2} \right].$$

Let $c_j(r) = \alpha_j - r\beta_j$. The admissible domain of $\Lambda_r(s)$ is determined by

$$1 - \frac{2\theta_j^2}{\xi_j} c_j(r) s > 0, \quad j = 1, \dots, L.$$

Therefore, the admissible domain of $\Lambda_r(s)$ is

$$\mathcal{D}_r = \left(\max_{c_j(r) < 0} \frac{\xi_j}{2\theta_j^2 c_j(r)}, \min_{c_j(r) > 0} \frac{\xi_j}{2\theta_j^2 c_j(r)} \right),$$

where the lower endpoint is $-\infty$ if no $c_j(r) < 0$, and the upper endpoint is $+\infty$ if no $c_j(r) > 0$.

The first two derivatives of $\Lambda_r(s)$ are

$$\Lambda'_r(s) = \sum_{j=1}^L (\alpha_j - r\beta_j) \frac{\theta_j}{\left(1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j - r\beta_j)s\right)^{1/2}},$$

$$\Lambda''_r(s) = \sum_{j=1}^L (\alpha_j - r\beta_j)^2 \frac{\theta_j^3}{\xi_j \left(1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j - r\beta_j)s\right)^{3/2}}.$$

3.2. Saddlepoint Density for the Ratio Statistic

The SPA for the density of the statistic R is then expressed in (4) as

$$\hat{f}_R(r) = \frac{\exp(\Lambda_r(\hat{s}_r)) G_r(\hat{s}_r)}{\sqrt{2\pi \Lambda''_r(\hat{s}_r)}}, \quad (4)$$

where the Jacobian-type quantity $G_r(\hat{s}_r)$ is given by

$$G_r(\hat{s}_r) = \frac{\partial}{\partial t} \Lambda(s, t) \Big|_{t=-rs} = \sum_{j=1}^L \beta_j \frac{\theta_j}{\left(1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j - r\beta_j)\hat{s}_r\right)^{1/2}},$$

and $\hat{s}_r$ is the solution of the saddlepoint equation $\Lambda'_r(\hat{s}_r) = 0$. For more details about the density approximation in (4); see Section 12.1.1 in Butler [21].

The saddlepoint search is restricted to the interior of $\mathcal{D}_r$. The saddlepoint $\hat{s}_r$ exists if 0 belongs to the range of $\Lambda'_r(s)$ over the interior of $\mathcal{D}_r$. Since $\Lambda_r(s)$ is a CGF, it is convex on this domain. If H_r is non-degenerate, then $\Lambda_r(s)$ is strictly convex and $\Lambda'_r(s)$ is strictly increasing. Therefore, $\Lambda'_r(\hat{s}_r) = 0$ has at most one real solution. In numerical implementation, root-finding is performed only after verifying that $\Lambda'_r(s)$ changes sign over the admissible interval.

3.3. Saddlepoint Distribution for the Ratio Statistic

Since $\Pr(R \leq r) = \Pr(H_r \leq 0)$, the Lugannani–Rice formula yields

$$\hat{F}_R(r) = \Phi(\omega_r) + \phi(\omega_r) \left(\frac{1}{\omega_r} - \frac{1}{v_r} \right), \quad \text{for } r \neq \mathbb{E}(R), \quad (5)$$

where

$$\omega_r = \text{sgn}(\hat{s}_r) \sqrt{-2\Lambda_r(\hat{s}_r)} = \text{sgn}(\hat{s}_r) \left(-2 \sum_{j=1}^L \frac{\xi_j}{\theta_j} \left[1 - \left(1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j - r\beta_j)\hat{s}_r \right)^{1/2} \right] \right)^{1/2},$$

and

$$v_r = \hat{s}_r \sqrt{\Lambda''_r(\hat{s}_r)} = \hat{s}_r \left(\sum_{j=1}^L (\alpha_j - r\beta_j)^2 \frac{\theta_j^3}{\xi_j \left(1 - \frac{2\theta_j^2}{\xi_j} (\alpha_j - r\beta_j)\hat{s}_r \right)^{3/2}} \right)^{1/2}.$$

3.4. Normal Approximation for the Ratio Statistic

The NA is applied to the transformation statistic, $H_r = A - rB$, used in the SPA formulation, under the assumption $B > 0$.

Define $\mu_A = \mathbb{E}(A)$, $\mu_B = \mathbb{E}(B)$, $\sigma_A^2 = \text{Var}(A)$, $\sigma_B^2 = \text{Var}(B)$, and $\sigma_{AB} = \text{Cov}(A, B)$, where these quantities denote the means, variances, and covariance of A and B , respectively. Then

$$\mu_H(r) = \mathbb{E}(H_r) = \mu_A - r\mu_B,$$

and

$$\sigma_H^2(r) = \text{Var}(H_r) = \sigma_A^2 + r^2\sigma_B^2 - 2r\sigma_{AB}.$$

Thus,

$$H_r \approx N(\mu_H(r), \sigma_H^2(r)).$$

Consequently,

$$F_R(r) = \Pr(R \leq r) = \Pr(H_r \leq 0)$$

is approximated by

$$\hat{F}_{R,N}(r) = \Phi\left(\frac{-\mu_H(r)}{\sigma_H(r)}\right) = \Phi\left(\frac{r\mu_B - \mu_A}{\sqrt{\sigma_A^2 + r^2\sigma_B^2 - 2r\sigma_{AB}}}\right). \quad (6)$$

For independent IG RVs, $U_j \sim IG(\theta_j, \xi_j)$, we have

$$\mathbb{E}(U_j) = \theta_j, \quad \text{Var}(U_j) = \frac{\theta_j^3}{\xi_j}.$$

Therefore,

$$\begin{aligned} \mu_A &= \sum_{j=1}^L \alpha_j \theta_j, & \mu_B &= \sum_{j=1}^L \beta_j \theta_j, \\ \sigma_A^2 &= \sum_{j=1}^L \alpha_j^2 \frac{\theta_j^3}{\xi_j}, & \sigma_B^2 &= \sum_{j=1}^L \beta_j^2 \frac{\theta_j^3}{\xi_j}, \end{aligned}$$

and

$$\sigma_{AB} = \sum_{j=1}^L \alpha_j \beta_j \frac{\theta_j^3}{\xi_j}.$$

The corresponding NA for the density of R is obtained by differentiating the approximate distribution function with respect to r . Therefore, the normal density approximation for the ratio statistic is

$$\hat{f}_{R,N}(r) = \phi\left(\frac{r\mu_B - \mu_A}{\sigma_H(r)}\right) \frac{\mu_B \sigma_H^2(r) - (r\mu_B - \mu_A)(r\sigma_B^2 - \sigma_{AB})}{\sigma_H^3(r)}.$$

4. Simulation Study for the Product Statistic

To examine the robustness of the SPA for the product statistic, several IG parameter configurations are considered. In all scenarios below, the components are assumed to be independent and identically distributed according to $IG(\theta, \xi)$, with both parameters fixed across indices.

- Case 1: $IG_1 = IG(1, 5)$. This is a moderately dispersed IG case.
- Case 2: $IG_2 = IG(1, 10)$. This has the same mean as IG_1 but with lower variability and less asymmetry.

- Case 3: $IG_3 = IG(4, 0.5)$. This is an extremely heavy-tailed and highly right-skewed case.
- Case 4: $IG_4 = IG(6, 2)$. This is a large mean, high variance IG distribution with substantial right tail behavior.
- Case 5: $IG_5 = IG(0.4, 6)$. This distribution is tightly concentrated near a small positive value and is only mildly right skewed.
- Case 6: $IG_6 = IG(0.9, 3.5)$. This is a moderately dispersed case with noticeable right skewness.

The six IG models are selected to represent a wide range of distributional shapes. Since θ determines the mean and ζ controls the concentration, these parameter combinations generate distributions with small, moderate, and large means, as well as low, moderate, and high dispersion. In particular, IG_1 and IG_2 isolate the effect of increasing the shape parameter at a fixed mean, IG_3 and IG_4 represent highly skewed and strongly dispersed cases, and IG_5 and IG_6 provide relatively concentrated distributions with small to moderate means. Thus, the selected settings are useful for assessing behavior across both regular and extreme IG regimes.

To examine the influence of non-identical components, we consider smoothly varying parameter sequences across $j = 1, \dots, m$. Two heterogeneous configurations are used as follows:

- Case 7: $IG_7 = IG(0.8 + 0.4 \frac{j-1}{m-1}, 10)$, $j = 1, \dots, m$. The mean parameters increase linearly with the index j ,

$$\theta_j = 0.8 + 0.4 \frac{j-1}{m-1}, \quad j = 1, \dots, m,$$

while the shape is kept fixed:

$$\zeta_j = 10, \quad j = 1, \dots, m.$$

- Case 8: $IG_8 = IG(0.8 + 0.4 \frac{j-1}{m-1}, 5 + 10 \frac{j-1}{m-1})$, $j = 1, \dots, m$. The mean parameters are as in IG_7 ,

$$\theta_j = 0.8 + 0.4 \frac{j-1}{m-1}, \quad j = 1, \dots, m,$$

and the shape parameters also vary linearly:

$$\zeta_j = 5 + 10 \frac{j-1}{m-1}, \quad j = 1, \dots, m.$$

The eight parameter settings described above are used to generate samples of size $m = 5, 20$, and 50 from the IG distribution. For each sample size m and each parameter scenario, the SPA and NA are assessed at the probability levels

$$\alpha = \mathcal{A} = \{0.01, 0.05, 0.10, 0.50, 0.90, 0.95, 0.99\}.$$

Let q_α denote the α -quantile of Q under the given scenario. For each reference quantile q_α , the SPA and NA for the DF of Q are evaluated using Equations (2) and (3), yielding $\hat{F}_Q(q_\alpha)$ and $\hat{F}_{Q,N}(q_\alpha)$, respectively. To obtain stable reference values, q_α is calculated from 10^6 simulated replicates of Q . As we described in Section 2, the calculation of the SPA for the product statistic Q depends on the central finite differences of the modified Bessel function of the second kind. Throughout this simulation study, a finite-difference step size of $h = 10^{-4}$ is used. Tables 2–4 report $\hat{F}_Q(q_\alpha)$ and $\hat{F}_{Q,N}(q_\alpha)$ for the three sample sizes

$m = 5, 20$, and 50 , respectively. In Tables 2–4, the entry with the smaller absolute error relative to the nominal probability level α is shown in bold.

Table 2. $\hat{F}_Q(q_\alpha)$ and $\hat{F}_{Q,N}(q_\alpha)$ for the product of 5 IG variates under the eight scenarios.

α	Meth.	IG_1	IG_2	IG_3	IG_4	IG_5	IG_6	IG_7	IG_8
0.01	SPA	0.00999	0.00992	0.00992	0.00993	0.00986	0.00995	0.01001	0.00990
	NA	0.01045	0.01019	0.01754	0.01443	0.01003	0.01063	0.01034	0.01034
0.05	SPA	0.05027	0.05013	0.04995	0.05000	0.05024	0.05014	0.05036	0.05028
	NA	0.05040	0.05014	0.05869	0.05497	0.04972	0.05029	0.04998	0.05030
0.10	SPA	0.10033	0.10033	0.09992	0.10024	0.10030	0.10027	0.10026	0.10039
	NA	0.10057	0.09983	0.10411	0.10197	0.09936	0.09992	0.10017	0.10005
0.50	SPA	0.49906	0.49895	0.49998	0.50069	0.49866	0.49922	0.49943	0.49915
	NA	0.49940	0.49872	0.48275	0.48948	0.49997	0.49782	0.49983	0.49974
0.90	SPA	0.90004	0.90005	0.90006	0.90032	0.90019	0.90010	0.90012	0.89995
	NA	0.90085	0.90001	0.90580	0.90470	0.89969	0.90011	0.90052	0.90021
0.95	SPA	0.95003	0.95003	0.95031	0.95030	0.95021	0.95009	0.95021	0.94993
	NA	0.95057	0.95015	0.95703	0.95517	0.94986	0.95048	0.95057	0.95005
0.99	SPA	0.99000	0.99001	0.99009	0.99009	0.98997	0.98998	0.98996	0.99000
	NA	0.98999	0.99000	0.99357	0.99231	0.99005	0.99005	0.99014	0.98989

Table 3. $\hat{F}_Q(q_\alpha)$ and $\hat{F}_{Q,N}(q_\alpha)$ for the product of 20 IG variates under the eight scenarios.

α	Meth.	IG_1	IG_2	IG_3	IG_4	IG_5	IG_6	IG_7	IG_8
0.01	SPA	0.01009	0.01013	0.00983	0.00982	0.01018	0.01008	0.01015	0.01010
	NA	0.01027	0.01000	0.01311	0.01170	0.01001	0.01019	0.01003	0.01017
0.05	SPA	0.05027	0.05013	0.04943	0.04967	0.04990	0.05019	0.05004	0.05005
	NA	0.05005	0.05004	0.05374	0.05211	0.05004	0.05021	0.05006	0.05005
0.10	SPA	0.10005	0.09989	0.09936	0.09929	0.09973	0.10013	0.09977	0.09997
	NA	0.09969	0.10034	0.10222	0.10111	0.10008	0.10028	0.10013	0.10007
0.50	SPA	0.50007	0.50029	0.49969	0.50029	0.50096	0.49996	0.50060	0.50060
	NA	0.49954	0.49997	0.49189	0.49440	0.50031	0.50001	0.49954	0.50001
0.90	SPA	0.89998	0.90009	0.90034	0.90019	0.90023	0.90006	0.89992	0.90024
	NA	0.90022	0.89971	0.90298	0.90187	0.89989	0.90091	0.90035	0.90036
0.95	SPA	0.95003	0.94996	0.95012	0.95005	0.95021	0.95000	0.94994	0.95005
	NA	0.95047	0.95005	0.95400	0.95244	0.94986	0.95049	0.95024	0.95014
0.99	SPA	0.99003	0.98993	0.98995	0.98988	0.98996	0.99009	0.98991	0.99000
	NA	0.99005	0.99012	0.99212	0.99133	0.99014	0.99014	0.99003	0.99002

Table 4. $\hat{F}_Q(q_\alpha)$ and $\hat{F}_{Q,N}(q_\alpha)$ for the product of 50 IG variates under the eight scenarios.

α	Meth.	IG_1	IG_2	IG_3	IG_4	IG_5	IG_6	IG_7	IG_8
0.01	SPA	0.00993	0.00999	0.00987	0.00990	0.00992	0.01000	0.00996	0.01003
	NA	0.01011	0.01018	0.01182	0.01119	0.00997	0.01016	0.01009	0.01003
0.05	SPA	0.04989	0.05000	0.04988	0.05002	0.04998	0.04989	0.05008	0.04981
	NA	0.05022	0.05012	0.05278	0.05144	0.04971	0.04982	0.04980	0.04977
0.10	SPA	0.09996	0.10002	0.09979	0.09993	0.09940	0.09989	0.09998	0.10001
	NA	0.10005	0.09979	0.10175	0.10100	0.09977	0.09972	0.09979	0.09984
0.50	SPA	0.49999	0.50036	0.50021	0.50047	0.50224	0.49955	0.50024	0.49944
	NA	0.49940	0.49978	0.49508	0.49618	0.50024	0.49957	0.49880	0.49964
0.90	SPA	0.90014	0.90022	0.90045	0.90003	0.90042	0.90012	0.90018	0.90035
	NA	0.90018	0.90008	0.90156	0.90020	0.89998	0.90015	0.89945	0.89946
0.95	SPA	0.95029	0.95001	0.95041	0.94993	0.95006	0.95014	0.94997	0.95017
	NA	0.95040	0.95000	0.95210	0.95092	0.95026	0.94998	0.94933	0.94991
0.99	SPA	0.98994	0.98998	0.99007	0.99000	0.98999	0.99002	0.98995	0.98988
	NA	0.99011	0.99011	0.99116	0.99074	0.99004	0.99024	0.98996	0.98994

Tables 2–4 demonstrate that the SPA for the DF of the product statistic is highly accurate across all considered sample sizes and parameter settings. In nearly all cases, the approximated levels $\hat{F}_Q(q_\alpha)$ are extremely close to the nominal probabilities α , indicating excellent calibration of the approximation. The tail probabilities, particularly at $\alpha = 0.01$ and $\alpha = 0.99$, are reproduced with remarkable accuracy, with errors typically of order 10^{-4} .

or smaller. The largest deviations are generally observed at the central probability level $\alpha = 0.50$, which appears to be the most challenging region for approximation. Among the three tables, the best overall performance is observed for $m = 20$, where the maximum absolute error is only 0.00096. For $m = 5$ and 50, the corresponding maximum absolute errors are 0.00134 and 0.00224, respectively, all of which remain very small. Overall, the results indicate that the SPA is robust to changes in dimension, dispersion, skewness, and parameter heterogeneity, and therefore provides a reliable and accurate tool for approximating the DF of the product of independent IG variables. Moreover, Tables 2–4 compare the SPA with the corresponding NA for the product statistic under the eight IG scenarios. The SPA gives more consistently accurate results than the NA across the three sample sizes and probability levels. The superiority of the SPA is particularly clear in the highly skewed and heavy-tailed settings, such as IG_3 and IG_4 , where the NA often shows noticeable deviations from the nominal probabilities, especially in the tails. In the more concentrated scenarios, such as IG_2 and IG_5 , the NA is sometimes competitive and occasionally slightly closer to the nominal level. Nevertheless, the SPA remains highly accurate across all scenarios and sample sizes, showing greater robustness to skewness, heterogeneity, and changes in dimension.

Figures 1 and 2 provide a visual comparison between SPA and NA for the product statistic. Figure 1 displays the mean absolute error (MAE) of the SPA and the NA by sample sizes. It shows that the SPA generally has a smaller MAE than the NA across the considered sample sizes. To further compare the two approximations, we also examine the paired absolute error plots in Figure 2. For each simulation setting, the absolute error of the NA is plotted against the corresponding absolute error of the SPA. The diagonal line represents equal accuracy. In Figure 2, points below this line indicate cases where the SPA has a smaller error than the NA. The concentration of points below the diagonal provides direct visual evidence that the SPA is generally more accurate.

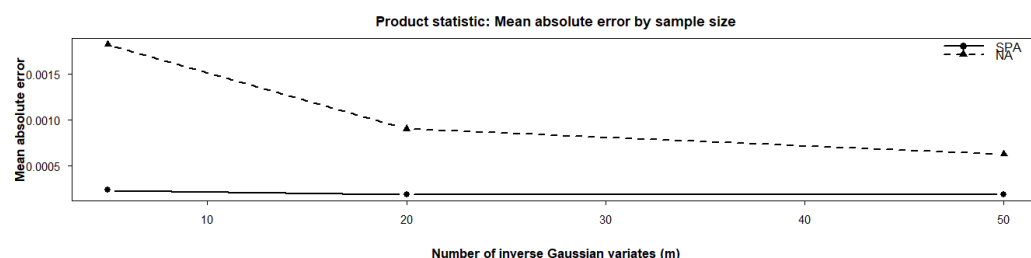


Figure 1. MAE of the SPA and NA for the product statistic, averaged over all eight IG scenarios and probability levels.

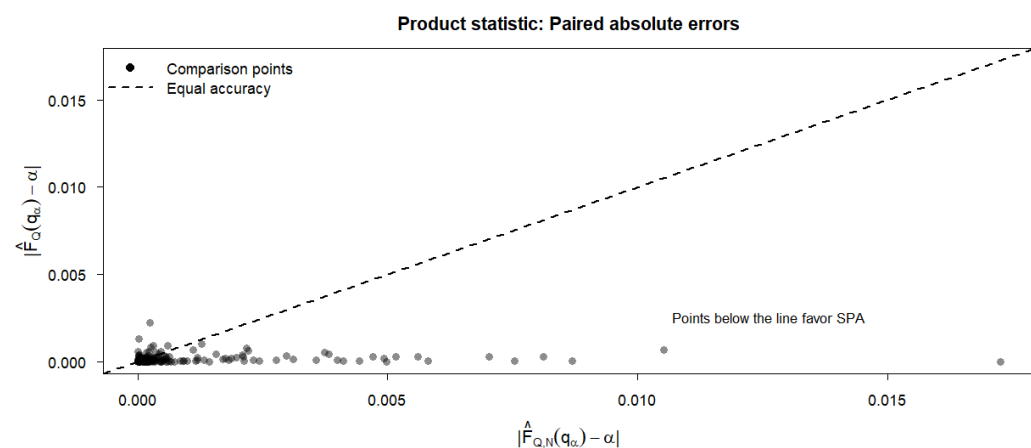


Figure 2. Paired absolute errors for the product statistic.

To assess the sensitivity of the approximation of the product to numerical differentiation, the derivatives of $\log K_\nu(x)$ with respect to ν are recomputed using several step sizes of central difference, including $h = 10^{-3}, 10^{-4}, 10^{-5}$. As an example, we apply this sensitivity analysis to two IG cases, IG_1 and IG_7 . Table 5 displays the saddlepoint probabilities for the two cases IG_1 and IG_7 with step sizes of the central difference of $h = 10^{-5}$ and $h = 10^{-3}$. The corresponding results for $h = 10^{-4}$ are displayed in Tables 2–4. From Table 5, we note that the saddlepoint probabilities are practically unchanged from the number of decimal places reported in Tables 2–4, except in a few cases (at the central probability level) where very small numerical differences are observed. This numerical sensitivity analysis supports the stability of the finite-difference implementation used in the simulation study.

Table 5. $\hat{F}_Q(q_\alpha)$ for the product of m IG variates under the IG_1 and IG_7 scenarios calculated with a finite-difference step size of $h = 10^{-5}$ and 10^{-3} .

Case	h	m	α						
			0.01	0.05	0.10	0.50	0.90	0.95	0.99
IG_1	10^{-5}	5	0.00999	0.05027	0.10033	0.49899	0.90004	0.95003	0.99000
		20	0.01009	0.05027	0.10006	0.50175	0.89998	0.95003	0.99003
		50	0.00993	0.04980	0.09996	0.48800	0.90014	0.95029	0.98994
IG_1	10^{-3}	5	0.00999	0.05027	0.10033	0.49906	0.90004	0.95003	0.99000
		20	0.01009	0.05027	0.10005	0.49798	0.89998	0.95003	0.99003
		50	0.00993	0.04989	0.09996	0.49322	0.90014	0.95029	0.98994
IG_7	10^{-5}	5	0.01001	0.05036	0.10026	0.49451	0.90012	0.95022	0.98996
		20	0.01015	0.05004	0.09977	0.50217	0.89992	0.94994	0.98991
		50	0.00996	0.05008	0.09998	0.48706	0.90018	0.94997	0.98995
IG_7	10^{-3}	5	0.01001	0.05036	0.10026	0.49936	0.90012	0.95021	0.98996
		20	0.01015	0.05004	0.09977	0.50037	0.89992	0.94994	0.98991
		50	0.00996	0.05008	0.09998	0.49764	0.90018	0.94997	0.98995

5. Simulation Study for the Ratio Statistic

Let $U_1, \dots, U_L$ be independent IG variables

$$U_j \sim IG(\theta_j, \xi_j), \quad j = 1, \dots, L.$$

To assess the accuracy of the SPA for the ratio statistic under a range of distributional regimes, we consider the following three parameter scenarios.

- **Scenario A** (moderate dispersion): $\theta_j = 1.2$ and $\xi_j = 12$, $j = 1, \dots, L$. This scenario serves as a baseline setting with moderate skewness and stable curvature of the CGF.
- **Scenario B** (high dispersion / heavy right tail): $\theta_j = 0.9$ and $\xi_j = 3.5$, $j = 1, \dots, L$. This configuration induces heavier tails and greater skewness, thereby providing a stringent test of tail accuracy for the SPA.
- **Scenario C** (small mean regime): $\theta_j = 0.4$ and $\xi_j = 6$, $j = 1, \dots, L$. This scenario is included to assess the numerical stability in the presence of small-valued observations.

5.1. Weight Designs

The ratio statistic is defined as

$$R = \frac{\sum_{j=1}^m \alpha_j U_j}{\sum_{j=k}^n \beta_j U_j},$$

with α_j and β_j allowed to take negative values. To explore the impact of coefficient patterns on approximation accuracy, we use three different weight designs.

- **Weight design** W_1 : $\alpha_j = j^{-0.6}$, $j = 1, \dots, m$ and $\beta_j = (n - j + k)^{-0.6}$, $j = k, \dots, n$.
- **Weight design** W_2 : $\alpha_j = j^{-1}$, $j = 1, \dots, m$ and $\beta_j = (n - j + k)^{-1}$, $j = k, \dots, n$.
- **Weight design** W_3 : $\alpha_j = (m - j + 1)^{-1}$, $j = 1, \dots, m$ and $\beta_j = j^{-1}$, $j = k, \dots, n$.

5.2. Indexing Schemes for Constructing the Ratio

Generate $U_1, \dots, U_L$ from the IG distribution. The numerator is formed from indices $1, \dots, m$ while the denominator uses indices $k, \dots, n$. To represent small, moderate, and large configurations with varying degrees of overlap, we consider the following indexing schemes:

- T_1 (**small with overlap**): $(k, m, n) = (3, 12, 16)$. The numerator and denominator share indices $3, \dots, 12$.
- T_2 (**denominator substantially longer**): $(k, m, n) = (1, 18, 30)$. Here, the denominator includes all numerator terms and additional observations, leading to strong overlap and increased denominator stability.
- T_3 (**large, moderate overlap**): $(k, m, n) = (6, 25, 50)$. This scheme reflects a larger-sample environment with partial overlap.
- T_4 (**tail overlap**): $(k, m, n) = (10, 25, 35)$. Overlap occurs primarily near the upper end of the numerator index set.

5.3. Simulation Protocol

This subsection outlines the Monte Carlo procedure used to evaluate the finite-sample performance of the proposed SPA for the ratio statistic. For a fixed combination of an IG parameter scenario (A–C), a coefficient design (W_1 – W_3), and an index configuration (k, m, n) , a random sample $U_1, \dots, U_L$ is generated from $IG(\theta_j, \xi_j)$ according to the specified scenario. Using the generated sample, the numerator and denominator are formed as

$$A = \sum_{j=1}^m \alpha_j U_j, \quad B = \sum_{j=k}^n \beta_j U_j,$$

and the ratio $R = A/B$ is computed.

For each scenario, a large Monte Carlo run of size 10^6 is conducted to approximate the distribution of R . Based on these values, reference quantiles r_α are obtained for probability levels

$$\alpha \in \mathcal{A} = \{0.01, 0.05, 0.10, 0.50, 0.90, 0.95, 0.99\}.$$

These quantiles serve as a high-precision benchmark against which SPA and NA are assessed. For each reference quantile r_α , the SPA and NA for the DF of R are evaluated using Equations (5) and (6), which yields $\hat{F}_R(r_\alpha)$ and $\hat{F}_{R,N}(r_\alpha)$, respectively. Tables 6–8 report the SPA and NA for the three IG scenarios (A–C), respectively. In Tables 6–8, the smaller absolute error of the SPA and the NA relative to the nominal probability level is highlighted in bold.

The numerical results in Tables 6–8 show that the SPA for the DF of the ratio statistic is highly accurate across all three IG scenarios. The proximity of the approximated values $\hat{F}_R(r_\alpha)$ to the nominal probabilities α indicates excellent calibration. The absolute errors are frequently zero at the reported precision, while the largest absolute errors are 0.0022, 0.0047, and 0.0019 for Scenarios A, B, and C, respectively. The best performance is observed in Scenarios A and C, corresponding to the more concentrated IG settings. The SPA is most accurate in the tails ($\alpha = 0.01, 0.99$), with the largest errors consistently occurring at the median ($\alpha = 0.50$). These findings confirm the robustness of the SPA across diverse IG parameter regimes. Furthermore, the results in Tables 6–8 indicate that the SPA clearly

outperforms the NA in the three IG scenarios. Among the 252 comparisons reported in these tables, the SPA gives the smaller absolute error in 244 cases, whereas the NA gives the smaller error in only 7 cases, with one tie. The overall MAE of the SPA is approximately 0.00066, compared with 0.00915 for the NA. Thus, the SPA provides substantially more accurate and reliable approximations for the ratio statistic across the considered scenarios.

Table 6. $\hat{F}_R(r_\alpha)$ and $\hat{F}_{R,N}(r_\alpha)$ under Scenario A, IG(1.2, 12).

α	Meth.	T_1			T_2			T_3			T_4		
		W_1	W_2	W_3	W_1	W_2	W_3	W_1	W_2	W_3	W_1	W_2	W_3
0.01	SPA	0.0100	0.0099	0.0100	0.0100	0.0098	0.0098	0.0100	0.0101	0.0099	0.0099	0.0101	0.0100
	NA	0.0147	0.0186	0.0213	0.0118	0.0137	0.0149	0.0152	0.0206	0.0243	0.0143	0.0202	0.0238
0.05	SPA	0.0499	0.0502	0.0500	0.0498	0.0497	0.0496	0.0498	0.0501	0.0498	0.0500	0.0504	0.0501
	NA	0.0582	0.0630	0.0666	0.0535	0.0569	0.0589	0.0580	0.0649	0.0688	0.0571	0.0641	0.0686
0.10	SPA	0.0997	0.0999	0.0997	0.0992	0.0992	0.0995	0.0999	0.1005	0.1002	0.1000	0.1005	0.1005
	NA	0.1081	0.1113	0.1138	0.1038	0.1073	0.1091	0.1061	0.1116	0.1147	0.1057	0.1111	0.1144
0.50	SPA	0.5003	0.5007	0.5009	0.4993	0.4991	0.4999	0.5005	0.5015	0.5011	0.5003	0.5022	0.5016
	NA	0.4900	0.4828	0.4790	0.4969	0.4961	0.4938	0.4865	0.4763	0.4687	0.4895	0.4781	0.4709
0.90	SPA	0.9005	0.9008	0.9008	0.9001	0.9002	0.9005	0.9002	0.9006	0.9009	0.9000	0.9007	0.9009
	NA	0.8979	0.8984	0.8988	0.8973	0.8944	0.8947	0.9012	0.9021	0.9030	0.9003	0.9012	0.9030
0.95	SPA	0.9502	0.9505	0.9505	0.9498	0.9500	0.9503	0.9501	0.9506	0.9505	0.9501	0.9506	0.9505
	NA	0.9511	0.9524	0.9536	0.9483	0.9464	0.9473	0.9539	0.9568	0.9595	0.9525	0.9558	0.9585
0.99	SPA	0.9899	0.9900	0.9900	0.9901	0.9900	0.9900	0.9899	0.9901	0.9901	0.9899	0.9900	0.9901
	NA	0.9917	0.9925	0.9930	0.9897	0.9889	0.9897	0.9931	0.9948	0.9960	0.9920	0.9943	0.9954

Table 7. $\hat{F}_R(r_\alpha)$ and $\hat{F}_{R,N}(r_\alpha)$ under Scenario B, IG(0.9, 3.5).

α	Meth.	T_1			T_2			T_3			T_4		
		W_1	W_2	W_3	W_1	W_2	W_3	W_1	W_2	W_3	W_1	W_2	W_3
0.01	SPA	0.0095	0.0097	0.0095	0.0096	0.0094	0.0094	0.0098	0.0101	0.0100	0.0097	0.0101	0.0098
	NA	0.0192	0.0276	0.0334	0.0147	0.0184	0.0210	0.0193	0.0296	0.0370	0.0179	0.0289	0.0353
0.05	SPA	0.0489	0.0493	0.0490	0.0489	0.0482	0.0482	0.0495	0.0504	0.0504	0.0493	0.0503	0.0505
	NA	0.0657	0.0757	0.0807	0.0584	0.0651	0.0691	0.0636	0.0758	0.0838	0.0625	0.0751	0.0825
0.10	SPA	0.0983	0.0995	0.0992	0.0988	0.0976	0.0972	0.0990	0.1004	0.1009	0.0990	0.1008	0.1010
	NA	0.1160	0.1234	0.1269	0.1088	0.1167	0.1200	0.1114	0.1211	0.1272	0.1105	0.1210	0.1269
0.50	SPA	0.5006	0.5029	0.5034	0.5001	0.4999	0.5004	0.5006	0.5041	0.5045	0.5001	0.5047	0.5045
	NA	0.4859	0.4760	0.4693	0.4960	0.4947	0.4892	0.4799	0.4653	0.4552	0.4840	0.4681	0.4572
0.90	SPA	0.9021	0.9030	0.9033	0.9019	0.9025	0.9030	0.9017	0.9036	0.9037	0.9010	0.9033	0.9033
	NA	0.8924	0.8925	0.8918	0.8924	0.8862	0.8859	0.8999	0.8994	0.8996	0.8984	0.8988	0.8984
0.95	SPA	0.9515	0.9519	0.9520	0.9516	0.9517	0.9516	0.9514	0.9524	0.9522	0.9508	0.9523	0.9518
	NA	0.9477	0.9490	0.9496	0.9449	0.9397	0.9408	0.9546	0.9581	0.9608	0.9523	0.9561	0.9588
0.99	SPA	0.9903	0.9903	0.9905	0.9905	0.9904	0.9906	0.9904	0.9905	0.9906	0.9904	0.9906	0.9905
	NA	0.9909	0.9916	0.9921	0.9883	0.9858	0.9869	0.9939	0.9957	0.9971	0.9923	0.9950	0.9963

Table 8. $\hat{F}_R(r_\alpha)$ and $\hat{F}_{R,N}(r_\alpha)$ under Scenario C, IG(0.4, 6).

α	Meth.	T_1			T_2			T_3			T_4		
		W_1	W_2	W_3	W_1	W_2	W_3	W_1	W_2	W_3	W_1	W_2	W_3
0.01	SPA	0.0100	0.0100	0.0098	0.0100	0.0098	0.0100	0.0099	0.0101	0.0098	0.0099	0.0101	0.0099
	NA	0.0136	0.0169	0.0188	0.0114	0.0124	0.0134	0.0139	0.0184	0.0209	0.0136	0.0176	0.0204
0.05	SPA	0.0498	0.0502	0.0497	0.0498	0.0497	0.0499	0.0500	0.0502	0.0503	0.0501	0.0503	0.0500
	NA	0.0558	0.0602	0.0626	0.0528	0.0548	0.0565	0.0560	0.0615	0.0651	0.0552	0.0611	0.0644
0.10	SPA	0.0998	0.0997	0.0996	0.0994	0.0993	0.0995	0.1002	0.1004	0.1004	0.1003	0.1004	0.1004
	NA	0.1055	0.1086	0.1104	0.1032	0.1051	0.1069	0.1048	0.1091	0.1114	0.1040	0.1083	0.1110
0.50	SPA	0.5005	0.5005	0.4995	0.4992	0.4992	0.5002	0.4999	0.5010	0.5006	0.5005	0.5019	0.5006
	NA	0.4919	0.4856	0.4829	0.4970	0.4973	0.4944	0.4895	0.4804	0.4746	0.4910	0.4809	0.4756
0.90	SPA	0.9003	0.9004	0.9004	0.8999	0.9000	0.9006	0.9002	0.9004	0.9006	0.9000	0.9004	0.9003
	NA	0.8991	0.8994	0.8996	0.8979	0.8966	0.8968	0.9018	0.9027	0.9033	0.9007	0.9018	0.9023
0.95	SPA	0.9501	0.9503	0.9504	0.9497	0.9497	0.9501	0.9501	0.9502	0.9503	0.9500	0.9503	0.9502
	NA	0.9514	0.9524	0.9537	0.9488	0.9480	0.9486	0.9536	0.9566	0.9582	0.9524	0.9555	0.9570
0.99	SPA	0.9899	0.9899	0.9900	0.9901	0.9900	0.9900	0.9900	0.9900	0.9900	0.9898	0.9899	0.9901
	NA	0.9916	0.9924	0.9930	0.9898	0.9895	0.9900	0.9925	0.9943	0.9953	0.9918	0.9938	0.9947

Figures 3 and 4 provide a visual comparison between the SPA and the NA for the ratio statistic. Figure 3 compares the MAE of the SPA and NA for the ratio statistic. It shows that the SPA has substantially a smaller MAE than the NA across all three IG scenarios, with the largest advantage appearing in the more skewed settings. Figure 4 shows a paired absolute error comparison between the SPA and NA for the ratio statistic. Each point corresponds to one combination of scenario, coefficient design, indexing scheme, and probability level. Points below the diagonal line indicate cases where the SPA has a smaller absolute error than the NA. The concentration of points below the diagonal confirms that the SPA is generally more accurate for the ratio statistic.

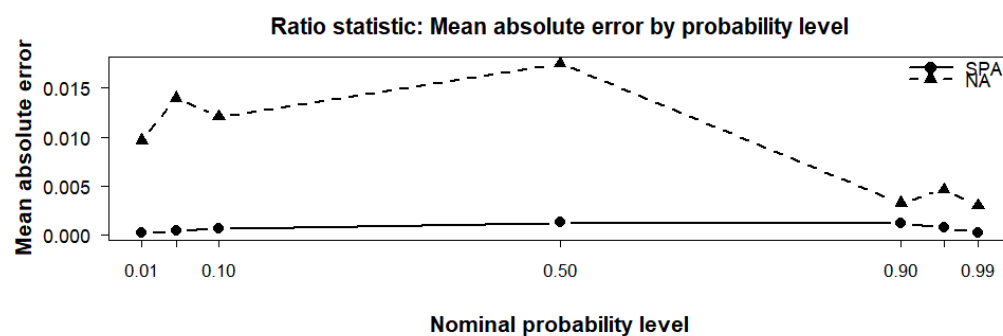


Figure 3. MAE of the SPA and NA for the ratio statistic, averaged over all IG scenarios, coefficient designs, and indexing schemes at each nominal probability level.

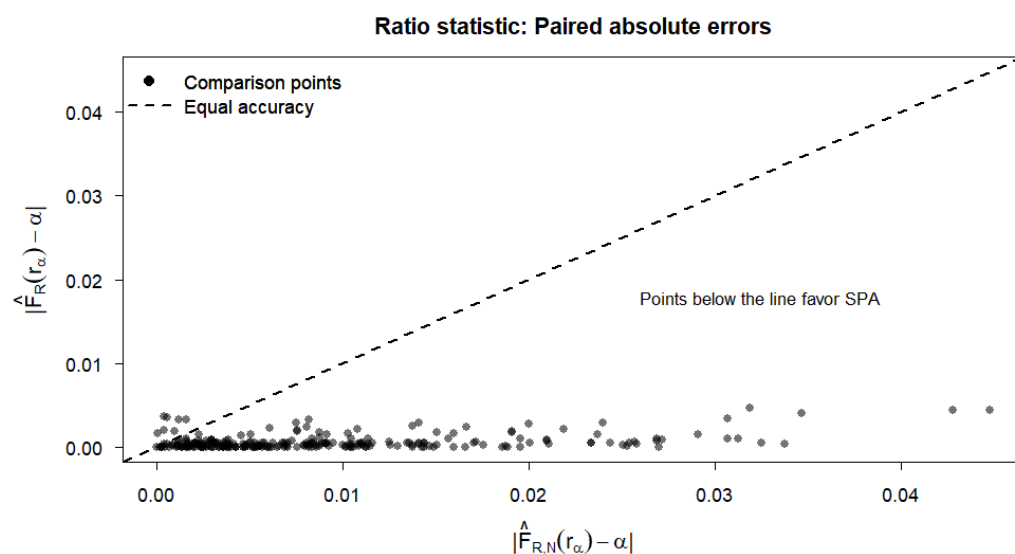


Figure 4. Paired absolute errors for the ratio statistic.

6. Computational-Time Comparison

In addition to approximation accuracy, we assess the computational cost of the NA, SPA, and MC simulation using representative product and ratio settings. For the product statistic, we consider $m = 10$ independent variables from $IG(1, 5)$ and evaluate the methods at the corresponding 0.95 reference quantile. For the ratio statistic, we consider Scenario A, $IG(1.2, 12)$, with indexing scheme $T_1 = (3, 12, 16)$ and weight design W_1 , at the 0.95 reference quantile. The analytical methods are evaluated repeatedly, and the reported values are median elapsed times per probability evaluation. For the MC method, the reference quantile generation time and probability estimation time are recorded separately. The total MC time is the sum of these two components, where independent simulation samples are used for quantile generation and probability estimation. All computations are performed

in R version 4.5.3 on a computer equipped with an Intel Core i7-4510U processor and 16 GB of DDR3 RAM, using a single processing core.

Table 9 shows that the NA has the lowest computational cost, as expected, because it requires only the first two cumulants of the transformed statistics. The SPA requires additional computation for numerical root-finding and evaluation of the Lugannani–Rice correction. For the product statistic, it also requires numerical differentiation with respect to the order of the modified Bessel function. Despite this additional cost, the SPA remains substantially faster than the complete MC benchmark. For the representative product setting, the SPA is approximately 168 times faster than MC simulation with 10^6 replications. These results clarify the computational role of the SPA. The SPA is not faster than the NA; rather, it offers a favorable accuracy–computation trade-off. The simulation results show that the SPA is generally substantially more accurate than the NA, especially in highly skewed settings and away from central probability levels, while the timing results show that it requires only a small fraction of the computation needed for a high-precision MC benchmark.

Table 9. Median computation times, in seconds, for representative product and ratio settings.

Statistic	NA	SPA	MC (10^6)
Product statistic	0.0519	0.2938	49.3907
Ratio statistic	0.0070	0.0686	114.8516

7. Sensitivity to Model Misspecification

To assess the robustness of the SPA to model misspecification, we perform a parameter perturbation sensitivity analysis.

Let θ and ξ denote the baseline IG parameters. We consider perturbed parameters of the form $\theta^* = c_\theta \theta$ and $\xi^* = c_\xi \xi$, where $c_\theta, c_\xi \in \{0.8, 1.0, 1.2\}$. The case $c_\theta = c_\xi = 1$ corresponds to the correctly specified model, whereas the remaining cases represent moderate underestimation or overestimation of the mean and shape parameters. For each perturbation pair (c_θ, c_ξ) , the SPA is evaluated using the perturbed parameters, while the Monte Carlo reference quantiles are obtained under the baseline model. The approximation error is measured by the absolute error $|\hat{F}_Q(q_\alpha; \theta^*, \xi^*) - \alpha|$, for the product statistic, and similarly by $|\hat{F}_R(r_\alpha; \theta^*, \xi^*) - \alpha|$, for the ratio statistic. The reported values are the MAEs over the probability levels $\alpha = 0.01, 0.05, 0.10, 0.50, 0.90, 0.95, 0.99$.

For the product statistic, we consider the representative case

$$Q = \prod_{j=1}^{10} U_j, \quad U_j \sim IG(1, 5).$$

Table 10 reports the MAEs of the SPA at different values of c_θ and c_ξ . From Table 10, we note that the smallest error occurs at the correctly specified model, namely $c_\theta = c_\xi = 1$, where the MAE is 0.0002. This confirms the high accuracy of the SPA under correct parameter specification. As the perturbation factors move away from one, the error increases, as expected. The results also indicate that the SPA for the product is more sensitive to misspecification of the mean parameter θ than to misspecification of the shape parameter ξ . In particular, when $c_\theta = 1$, perturbing c_ξ to 0.8 or 1.2 gives relatively moderate errors. However, when $c_\theta = 0.8$ or $c_\theta = 1.2$, the MAEs increase substantially. This suggests that accurate estimation of the mean parameter is especially important for reliable approximation of the product distribution.

Table 10. MAEs of the SPA for the product statistic over the considered probability levels.

c_{ξ}	c_{θ}		
	0.8	1.0	1.2
0.8	0.2639	0.0243	0.1631
1.0	0.2422	0.0002	0.1912
1.2	0.2254	0.0162	0.2112

For the ratio statistic, we consider an illustrative case from Scenario A, $U_j \sim IG(1.2, 12)$, with indexing scheme $T_1 = (k, m, n) = (3, 12, 16)$ and weight design W_1 .

Table 11 shows the MAEs of the SPA at different values of c_{θ} and c_{ξ} for the ratio statistic. Unlike the product statistic, the smallest errors occur along the diagonal cases $c_{\theta} = c_{\xi}$, namely $(0.8, 0.8)$, $(1.0, 1.0)$, and $(1.2, 1.2)$. This behavior is expected because the ratio statistic is invariant under a common proportional scaling of the underlying IG RVs. Therefore, when both parameters are perturbed proportionally, the distribution of the ratio remains nearly unchanged. In contrast, off-diagonal perturbations, where c_{θ} and c_{ξ} differ, lead to larger errors. The largest error occurs when $c_{\theta} = 1.2$ and $c_{\xi} = 0.8$, where the MAE is 0.0290. Overall, these results indicate that the SPA for the ratio statistic is highly stable under common-scale parameter perturbations and remains reasonably robust under moderate misspecification, although accuracy decreases when the relative parameter structure is substantially distorted.

Table 11. MAEs of the SPA for the ratio statistic over the considered probability levels.

c_{ξ}	c_{θ}		
	0.8	1.0	1.2
0.8	0.0002	0.0152	0.0290
1.0	0.0131	0.0002	0.0123
1.2	0.0221	0.0109	0.0002

The sensitivity analysis shows that the SPA is highly accurate under correct parameter specification and remains stable under certain moderate perturbations. For the product statistic, the approximation is more affected by misspecification of the mean parameter θ , whereas for the ratio statistic, the approximation is relatively insensitive to common proportional perturbations of θ and ξ . These findings are consistent with the structure of the two statistics: the product statistic is directly affected by changes in scale, while the ratio statistic is largely invariant to common scaling of all underlying variables. Therefore, the SPA is robust in practical settings with mild parameter uncertainty, although substantial misspecification may increase the approximation error.

8. Real Data Examples

8.1. Example 1

As a clinical real data illustration, we consider the psychiatric dataset discussed by Kleinbaum and Kupper [31] and Amin et al. [32], in which newly hospitalized patients were evaluated at baseline using two subjective indices, namely an older index labeled ‘degree of mental retardation’ in the original source, and the degree of distrust of doctors, and were then assessed using a degree of illness score after six months. For the purposes of the present paper, we focus on the univariate response variable, namely the 53 six-month illness scores: 44, 25, 10, 28, 25, 72, 45, 25, 12, 24, 46, 8, 15, 28, 26, 27, 4, 14, 21, 22, 60, 10, 60, 12, 28, 39, 14, 8, 11, 7, 23, 16, 26, 8, 11, 12, 50, 9, 13, 22, 23, 31, 20, 66, 9, 12, 21, 13, 10, 4, 18, 10, 7.

Assuming the model $IG(\theta, \xi)$, the maximum-likelihood estimates are $\hat{\theta} = 22.71698$ and $\hat{\xi} = 38.99089$. The corresponding Kolmogorov–Smirnov (K-S) distance for the fitted

model is $D_n = 0.08479$ with p -value = 0.507. These results indicate that the IG model provides a reasonable fit to the illness scores data.

Let $U_1, U_2, \dots$ denote future independent six-month illness scores from the fitted model $IG(\hat{\theta}, \hat{\xi})$.

8.1.1. Product Statistic

Consider the product

$$Q_5 = \prod_{j=1}^5 U_j.$$

The probability $\Pr(Q_5 \leq q)$ can be approximated at different values of q using the proposed Lugannani–Rice SPA in Equation (2). Table 12 represents the SPA and NA of $\Pr(Q_5 \leq q)$ when $q = 10^5, 20^5, 30^5$; these thresholds correspond to geometric-mean conditions $Q_5^{1/5} \leq 10, Q_5^{1/5} \leq 20, Q_5^{1/5} \leq 30$, respectively. Furthermore, Monte Carlo (MC) reference probabilities for these quantities are computed from 5×10^6 simulated samples under the fitted IG model, and the results are shown in Table 12.

Table 12. SPA, MC, and NA for the probabilities of the product statistic in the clinical illness score example.

Probability	SPA	MC	NA	$ SPA - MC $	$ NA - MC $
$\Pr(Q_5 \leq 10^5)$	0.02560	0.025632	0.027201	0.000032	0.001569
$\Pr(Q_5 \leq 20^5)$	0.63901	0.638777	0.637543	0.000233	0.001234
$\Pr(Q_5 \leq 30^5)$	0.95243	0.952417	0.953812	0.000013	0.001395

Hence, from Table 12, we note that there is approximately a 63.9% probability that the geometric mean of the next five illness scores does not exceed 20.

8.1.2. Ratio Statistic

Consider the ratio

$$R_8 = \frac{U_1 + U_2 + U_3 + U_4}{U_5 + U_6 + U_7 + U_8},$$

which compares total illness burden across two future four-patient groups.

The probability $\Pr(R_8 \leq r) = \Pr(H_r \leq 0)$ is approximated by applying the proposed Lugannani–Rice SPA and NA in Equations (5) and (6). Table 13 displays the SPA, NA, and MC approximation to $\Pr(R_8 \leq r)$ at different values of r .

Table 13. SPA, MC, and NA for the probabilities of the ratio statistic in the clinical illness score example.

Probability	SPA	MC	NA	$ SPA - MC $	$ NA - MC $
$\Pr(R_8 \leq 0.8)$	0.33545	0.335354	0.341193	0.000096	0.005839
$\Pr(R_8 \leq 1.0)$	0.50000	0.499905	0.500000	0.000095	0.000095
$\Pr(R_8 \leq 1.25)$	0.66455	0.664416	0.658807	0.000134	0.005610
$\Pr(R_8 \leq 1.5)$	0.78021	0.780148	0.766300	0.000062	0.013848

From Table 13, we note that, for example, the probability that one future four-patient group has total six-month illness burden no greater than 150% of another is approximately 0.780.

From Tables 12 and 13, the SPA probabilities are consistently closer to the MC estimates than the NA probabilities.

8.2. Example 2

To illustrate the practical relevance of the SPA methods, we consider the active repair times (in hours) for an airborne communication transceiver, a classical IG example reported

by Chhikara and Folks [33]. The observed repair times are 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

Fitting the model $IG(\theta, \xi)$ to the 46 repair times yields the parameter estimates $\hat{\theta} = 3.606522$ and $\hat{\xi} = 1.658853$. The corresponding K-S distance for the fitted model is $D_n = 0.0682$, with a large p -value = 0.9092, indicating that the IG model provides a reasonable fit.

Under this fitted model, let $U_1, U_2, \dots$ denote future independent repair times from $IG(\hat{\theta}, \hat{\xi})$.

8.2.1. Product Statistic

Assume the product

$$Q_3 = U_1 U_2 U_3.$$

The probability $\Pr(Q_3 \leq q)$ is approximated using the SPA, NA, and MC methods at $q = 8, 27, 64$; these thresholds, which correspond to the geometric-mean conditions, $Q_3^{1/3} \leq 2$, $Q_3^{1/3} \leq 3$, $Q_3^{1/3} \leq 4$, respectively. The results of the approximate probabilities for $\Pr(Q_3 \leq q)$ are displayed in Table 14.

Table 14. SPA, MC, and NA for the probabilities of the product statistic in the repair-time example.

Probability	SPA	MC	NA	$ SPA - MC $	$ NA - MC $
$\Pr(Q_3 \leq 8)$	0.53640	0.536933	0.525971	0.000533	0.010962
$\Pr(Q_3 \leq 27)$	0.76100	0.761235	0.760397	0.000235	0.000838
$\Pr(Q_3 \leq 64)$	0.87310	0.873263	0.877665	0.000163	0.004402

From Table 14, we observe that there is an approximate probability of 53.6% that the geometric mean of the next three repair times does not exceed 2 hours.

8.2.2. Ratio Statistic

Assume the ratio

$$R_6 = \frac{U_1 + U_2 + U_3}{U_4 + U_5 + U_6},$$

which compares the total repair burden across two future maintenance periods. The probability $\Pr(R_6 \leq r)$ is approximated using the SPA, NA, and MC methods. The results of such probabilities at different values of r are presented in Table 15.

Table 15. SPA, MC, and NA for the probabilities of the ratio statistic in the repair-time example.

Probability	SPA	MC	NA	$ SPA - MC $	$ NA - MC $
$\Pr(R_6 \leq 0.8)$	0.41660	0.417512	0.427221	0.000912	0.009709
$\Pr(R_6 \leq 1.0)$	0.50000	0.500073	0.500000	0.000073	0.000073
$\Pr(R_6 \leq 1.25)$	0.58340	0.582417	0.572779	0.000983	0.009638
$\Pr(R_6 \leq 1.5)$	0.64910	0.647593	0.627711	0.001507	0.019881

From a degradation and reliability perspective, the probabilities in Table 15 quantify the relative repair or degradation burden expected across two future maintenance periods, thereby providing useful information for maintenance planning and system reliability assessment.

From Table 15, we observe that the probability that one three-repair block requires no more total repair time than another is approximately 0.50, while the probability that it requires at most 125% of the other block is about 0.583.

Across both Tables 14 and 15, SPA consistently outperforms NA by providing a much closer approximation to the reference MC values.

8.3. Example 3

As an empirical example, we analyze runoff measurements collected at Jug bridge, Maryland, originally reported by Folks and Chhikara [34] as their third hydrological case study. The dataset consists of 25 observed runoff values: 0.17, 0.19, 0.23, 0.33, 0.39, 0.39, 0.40, 0.45, 0.52, 0.56, 0.59, 0.64, 0.66, 0.70, 0.76, 0.77, 0.78, 0.95, 0.97, 1.02, 1.12, 1.24, 1.59, 1.74, and 2.92. Fitting the model $IG(\theta, \xi)$ to the 25 runoff amounts yields the parameter estimates $\hat{\theta} = 0.8032$ and $\hat{\xi} = 1.4397$. The corresponding K-S distance for the fitted model is $D_n = 0.0792$, with p -value = 0.9645, indicating that the IG model provides a reasonable fit.

Let $U_1, U_2, \dots$ denote future independent runoff amounts from the fitted model $IG(\hat{\theta}, \hat{\xi})$.

8.3.1. Product Statistic

Let the product of five future runoff amounts be

$$Q_5 = U_1 U_2 U_3 U_4 U_5.$$

Using the SPA, NA, and MC methods, the probability $\Pr(Q_5 \leq q)$ is approximated at the thresholds $q = 0.03125, 1$, and 7.59375 . These different values of q correspond to geometric-mean conditions $Q_5^{1/5} \leq 0.5$, $Q_5^{1/5} \leq 1.0$, $Q_5^{1/5} \leq 1.5$, respectively. The resulting approximations are presented in Table 16.

Table 16. SPA, MC, and NA for the probabilities of the product statistic in the Jug Bridge runoff example.

Probability	SPA	MC	NA	$ SPA - MC $	$ NA - MC $
$\Pr(Q_5 \leq 0.03125)$	0.20570	0.205858	0.203883	0.000158	0.001975
$\Pr(Q_5 \leq 1)$	0.93020	0.930275	0.931666	0.000075	0.001391
$\Pr(Q_5 \leq 7.59375)$	0.99770	0.997673	0.997767	0.000027	0.000094

From Table 16, there is approximately a 99.7% probability that the geometric mean of the next five runoff amounts does not exceed 1.5.

8.3.2. Ratio Statistic

Let the ratio statistic take the form

$$R_8 = \frac{U_1 + U_2 + U_3 + U_4}{U_5 + U_6 + U_7 + U_8},$$

which compares total runoff across two future four-event periods. The probability $\Pr(R_8 \leq r)$ is evaluated via SPA, NA, and MC methods for different thresholds of r , and the estimation results are summarized in Table 17.

Table 17. SPA, MC, and NA for the probabilities of the ratio statistic in the Jug Bridge runoff example.

Probability	SPA	MC	NA	$ SPA - MC $	$ NA - MC $
$\Pr(R_8 \leq 0.8)$	0.33220	0.332183	0.337908	0.000017	0.005725
$\Pr(R_8 \leq 1.0)$	0.50000	0.499968	0.500000	0.000032	0.000032
$\Pr(R_8 \leq 1.25)$	0.66780	0.667981	0.662092	0.000181	0.005889
$\Pr(R_8 \leq 1.5)$	0.78490	0.785017	0.771152	0.000117	0.013865

Thus, from Table 17, the probability that runoff in one four-event period does not exceed 80% of that in another period is approximately 0.332.

From Tables 16 and 17, we note that the absolute error of the SPA is substantially smaller than that of the NA. Consequently, the SPA provides more accurate probability estimates.

9. Conclusions

This article applies established SPA methods for the density and distribution functions of product and ratio statistics arising from IG models. By exploiting appropriate transformations and explicit CGFs, accurate approximations for both density and distribution functions are obtained. Simulation results indicate that the suggested techniques perform well across diverse IG settings, including highly skewed, heterogeneous, and overlapping component settings. The results also demonstrate that the SPA is more accurate than the NA. The real data examples involving clinical illness scores, engineering repair times, and environmental runoff measurements illustrate the practical usefulness of the proposed approach. These applications show that the proposed SPAs can be used by practitioners to evaluate geometric-mean quantities, cumulative burden measures, and relative comparison indices without relying only on large-scale MC simulation. In general, the results show that SPA is a suitable and precise tool for inference on nonlinear statistics when exact distributional results are unavailable. The sensitivity analysis further shows that the SPA remains reasonably robust to moderate model misspecification, although accurate specification of the mean parameter is particularly important for product statistic. Despite the good numerical performance observed in the simulation studies and real data examples, the proposed approach has some limitations, including the need for numerical differentiation of modified Bessel functions with respect to their order, the requirement that the denominator in the ratio statistic remains positive, and the need to verify the existence of a real saddlepoint. Future work may extend the proposed framework to dependent and multivariate IG models, regression settings, censored or truncated observations, and other IG related distributions.

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Appendix A. Derivation of the Mellin Transform of an IG Random Variable

Let $U \sim IG(\theta, \xi)$, with density

$$f_U(u) = \left(\frac{\xi}{2\pi u^3} \right)^{1/2} \exp \left\{ -\frac{\xi(u - \theta)^2}{2\theta^2 u} \right\}, \quad u > 0.$$

For real t ,

$$\mathbb{E}(U^t) = \int_0^\infty u^t f_U(u) du.$$

After expanding the quadratic term in the exponent, the integral takes the form

$$\mathbb{E}(U^t) = C \int_0^\infty u^{t-3/2} \exp\left\{-au - \frac{b}{u}\right\} du,$$

where $a = \xi/(2\theta^2)$, $b = \xi/2$, and $C = \left(\frac{\xi}{2\pi}\right)^{1/2} \exp\left(\frac{\xi}{\theta}\right)$. Using the standard integral identity [35] [Formula 3.471.9]

$$\int_0^\infty x^{\nu-1} \exp\left\{-ax - \frac{b}{x}\right\} dx = 2\left(\frac{b}{a}\right)^{\nu/2} K_\nu(2\sqrt{ab}), \quad a > 0, b > 0,$$

with $\nu = t - \frac{1}{2}$, $a = \xi/(2\theta^2)$, and $b = \xi/2$, we obtain

$$\mathbb{E}(U^t) = \theta^t \frac{K_{t-1/2}(\xi/\theta)}{K_{-1/2}(\xi/\theta)}.$$

References

1. Seshadri, V. The Inverse Gaussian Distribution: Statistical Theory and Applications. In *Lecture Notes in Statistics*; Springer: Berlin/Heidelberg, Germany, 1999; Volume 137.
2. Whitmore, G.A. Estimating degradation by a Wiener diffusion process subject to measurement error. *Lifetime Data Anal.* **1995**, *1*, 307–319.
3. Park, C.; Padgett, W.J. Accelerated degradation models for failure time data based on Wiener and inverse Gaussian processes. *J. Qual. Technol.* **2005**, *37*, 13–26.
4. Bogacz, R.; Brown, E.; Moehlis, J.; Holmes, P.; Cohen, J.D. The physics of optimal decision making: A formal analysis of models of performance in two-alternative forced-choice tasks. *Psychol. Rev.* **2006**, *113*, 700–765.
5. Ratcliff, R.; McKoon, G. The diffusion decision model: Theory and data for two-choice decision tasks. *Neural Comput.* **2008**, *20*, 873–922.
6. Asmussen, S. Applied Probability and Queues. In *Stochastic Modelling and Applied Probability*, 2nd ed.; Springer: New York, NY, USA, 2003; Volume 51.
7. Cox, D.R.; Miller, H.D. *The Theory of Stochastic Processes*; Chapman and Hall: London, UK, 1965.
8. Chhikara, R.; Folks, J.L. *The Inverse Gaussian Distribution: Theory, Methodology, and Applications*; CRC Press: Boca Raton, FL, USA, 2024.
9. Whitmore, G.A. Inverse Gaussian ratio estimation. *J. R. Stat. Soc. Ser. C Appl. Stat.* **1986**, *35*, 8–15.
10. Tian, L.; Wilding, G.E. Confidence intervals of the ratio of means of two independent inverse Gaussian distributions. *J. Stat. Plan. Inference* **2005**, *133*, 381–386.
11. Krishnamoorthy, K.; Tian, L. Inferences on the difference and ratio of the means of two inverse Gaussian distributions. *J. Stat. Plan. Inference* **2008**, *138*, 2082–2089.
12. Peña, V.; Jauch, M. Properties of the generalized inverse Gaussian with applications to Monte Carlo simulation and distribution function evaluation. *Stat. Probab. Lett.* **2025**, *220*, 110359.
13. Chakraborty, A.; Jana, N. Bayes estimation of ratio of scale-like parameters for inverse Gaussian distributions and applications to classification. *Comput. Stat.* **2025**, *40*, 2249–2276.
14. Daniels, H.E. Saddlepoint approximations in statistics. *Ann. Math. Stat.* **1954**, *25*, 631–650.
15. Lugannani, R.; Rice, S. Saddlepoint Approximation for the Distribution of the Sum of Independent Random Variables. *Adv. Appl. Probab.* **1980**, *12*, 475–490.
16. Daniels, H.E. Saddlepoint Approximations for Estimating Equations. *Biometrika* **1983**, *70*, 89–96.
17. Barndorff-Nielsen, O.; Cox, D.R. Edgeworth and saddle-point approximations with statistical applications. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **1979**, *41*, 279–299.
18. Reid, N. Saddlepoint Methods and Statistical Inference. *Stat. Sci.* **1988**, *3*, 213–238.
19. Jensen, J.L. *Saddlepoint Approximations*; Oxford University Press: Oxford, UK, 1995.
20. Kolassa, J.E. Series Approximation Methods in Statistics In *Lecture Notes in Statistics*, 3rd ed.; Springer: New York, NY, USA, 2006; Volume 88.
21. Butler, R.W. *Saddlepoint Approximations with Applications*; Cambridge University Press: Cambridge, UK 2007.
22. Abd-Elfattah, E.F. Densities and distribution functions of the ratio of two linear functions and the product of Gamma variates: A saddlepoint approach. *Commun. Stat.-Theory Methods* **2017**, *46*, 7327–7336.

23. Jensen, J.L. On the Use of Saddlepoint Approximations in High Dimensional Inference. *Sankhya A* **2021**, *83*, 379–392.
24. Arevalillo, J.M. On the Empirical Approximation to Quantiles from Lugannani–Rice Saddlepoint Formula. *Stat. Probab. Lett.* **2024**, *209*, 110105.
25. Joutard, C. Numerical Comparison of Some Saddlepoint Approximations Near the Endpoints of the Support. *J. Stat. Theory Pract.* **2024**, *18*, 46.
26. Abd El-Raheem, A.M.; Abd-Elfattah, E.F. Weighted log-rank tests for censored data under Wei’s urn design: Saddlepoint approximation and confidence intervals. *J. Biopharm. Stat.* **2023**, *36*, 1–10.
27. Abd El-Raheem, A.M.; Hosny, M.; Abd-Elfattah, E.F. Statistical inference of the class of nonparametric tests for the panel count and current status data from the perspective of the saddlepoint approximation. *J. Math.* **2023**, *2023*, 9111653.
28. Abd El-Raheem, A.M.; Hosny, M. Saddlepoint p-values for a class of nonparametric tests for the current status and panel count data under generalized permuted block design. *AIMS Math.* **2023**, *8*, 18866–18880.
29. Abd El-Raheem, A.M.; Hosny, M. Saddlepoint approximation for the p-values of some distribution-free tests. *AIMS Math.* **2025**, *10*, 2602–2618.
30. Abd El-Raheem, A.M.; Hosny, M. On the Distribution of the Random Sum and Linear Combination of Independent Exponentiated Exponential Random Variables. *Symmetry* **2025**, *17*, 200.
31. Kleinbaum, D.G.; Kupper, L.L. *Applied Regression Analysis and Other Multivariable Methods*; Duxbury Press: North Scituate, RI, USA, 1978.
32. Amin, M.; Muhammad, A.U.; Qasim, M. Diagnostic techniques for the inverse Gaussian regression model. *Commun. Stat.-Theory Methods* **2022**, *51*, 2552–2564.
33. Chhikara, R.; Folks, J.L. The inverse Gaussian distribution as a lifetime model. *Technometrics* **1977**, *19*, 461–468.
34. Folks, J.L.; Chhikara, R. The inverse Gaussian distribution and its statistical application—A review. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **1978**, *40*, 263–275.
35. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, 8th ed.; Academic Press: Amsterdam, 2015.

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