

Article

Coincidence Point Results for g -Interpolative Kannan and g -Interpolative Ćirić–Reich–Rus Contractions with Applications to Fredholm Integral Equations

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Abstract

In this paper, we investigate the existence of coincidence points for g -interpolative Kannan and g -interpolative Ćirić–Reich–Rus contraction mappings in the framework of rectangular m -metric spaces. By exploiting the geometric structure of rectangular m -metric spaces together with the concept of g -interpolation, we establish different coincidence point results under suitable interpolative exponent conditions. For g -interpolative Kannan contractions, the analysis is carried out by considering the three cases when the sum of the interpolative exponents is equal to, less than, and greater than one. In the case of g -interpolative Ćirić–Reich–Rus contractions, the results are obtained for the cases in which the sum of the interpolative exponents is less than one and greater than one. The established results significantly generalize and unify a wide range of results in the literature. Since metric spaces, rectangular metric spaces, partial metric spaces, rectangular partial metric space, and m -metric spaces are all special cases of rectangular m -metric spaces, the obtained results immediately yield the corresponding conclusions in these settings. Moreover, by choosing the mapping g to be the identity mapping, the proposed contractive conditions reduce to the classical interpolative Kannan and interpolative Ćirić–Reich–Rus contractions, thereby recovering different known results as special cases. To demonstrate the applicability of the developed theory, illustrative examples are presented for mappings defined on both real and complex domains, showing that the proposed framework is not limited to real-valued settings. Finally, as an application of the main results, we establish the existence of solutions for a class of Fredholm integral equations, thereby illustrating the effectiveness and broad applicability of the proposed coincidence point results.

Keywords: g -interpolative Kannan contractions; g -interpolative Ćirić–Reich–Rus contractions; coincidence point; rectangular m -metric; Fredholm integral equations

MSC: 47H10; 54H25; 54E35



Academic Editor: Salvatore Sessa

Received: 29 July 2026

Revised: 23 August 2026

Accepted: 25 August 2026

Published: 28 August 2026

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1. Introduction

Fixed-point theory is one of the fundamental branches of modern mathematics, with deep connections to analysis, topology, operator theory, and nonlinear functional analysis. It provides powerful tools for establishing the existence and uniqueness of solutions to a wide range of mathematical problems. Fixed-point methods have significant applications

in differential equations, integral equations, optimization, dynamical systems, game theory, economics, engineering, computer science, and machine learning. Owing to its broad applicability and strong theoretical foundation, fixed-point theory continues to play a vital role in both pure and applied mathematics.

The foundation of metric fixed-point theory was laid by Banach in 1922 through his celebrated Banach contraction principle [1]. This landmark result opened new avenues for research and has since become one of the cornerstones of nonlinear analysis. Since then, fixed-point theory has developed extensively in various directions and has found numerous applications. Broadly speaking, research in fixed-point theory has evolved along two major directions. The first focuses on establishing fixed-point theorems for new classes of contractive mappings, while the second is concerned with extending these results to more general metric and generalized metric spaces.

Regarding the first direction, in 1968, Kannan [2] introduced a new type of contraction, now known as the Kannan contraction, and established a corresponding fixed-point theorem. Unlike the Banach contraction, a Kannan contraction is not necessarily continuous. Later, Karapinar [3] introduced the concept of an interpolative Kannan contraction and proved the corresponding fixed-point theorem. This result was subsequently generalized by Gaba et al. [4,5], who established the theorem for the case in which the sum of the interpolative exponents is less than one. Later, Errai et al. [6] extended this result to the case where the sum of the interpolative exponents is greater than one. Furthermore, in [7], Karapinar et al. introduced the interpolative Ćirić–Reich–Rus contraction and proved the corresponding fixed-point theorem in the setting of partial metric spaces.

Moreover, in addition to the existence of fixed points, the study of coincidence points is of considerable importance, as it generalizes the concept of fixed points within a broader framework. In this direction, Errai et al. [6] introduced the notion of a g -interpolative Kannan contraction in the setting of b -metric spaces and established the existence of coincidence points for such mappings.

On the other hand, another active area of research in fixed-point theory concerns the development of distance spaces. Over the years, numerous generalizations of the classical metric space have been proposed to broaden the applicability of fixed-point techniques. Among the most well known of these are rectangular metric spaces, partial metric spaces, m -metric spaces, partial rectangular metric spaces, and rectangular m -metric spaces. The definitions and fundamental properties of these spaces can be found in [8–12], together with the references cited therein. Among these generalized structures, rectangular m -metric spaces provide the most comprehensive framework, since each of the aforementioned spaces can be viewed as a special case of a rectangular m -metric space, for better understanding see Figure 1. Moreover, to facilitate a better understanding of these structures, we present various metrics, including those constructed from numerical values, in the examples section of the paper.

The results concerning interpolative contractions have been extensively investigated for various classes under different metric structures; see, for example, [3–7,13]. Nevertheless, the existence of coincidence points for such contractions, particularly within the framework of rectangular m -metric spaces, has not received any attention. This reveals a notable gap in the existing literature. To address this gap, the present study investigates the existence of coincidence points for interpolative contractions in rectangular m -metric spaces, thereby extending the existing theory to a more general metric structure.

In this paper, our primary objective is to establish coincidence point results in the setting of rectangular m -metric spaces. To achieve this, we investigate two classes of generalized interpolative contractions: g -interpolative Kannan contractions and g -interpolative Ćirić–Reich–Rus contractions. These mappings generalize the corresponding interpola-

tive contractions available in the existing literature. Since rectangular m -metric spaces encompass several well-known generalized distance spaces as special cases, the results obtained in this work extend and unify many previously established results. To illustrate the applicability of our theoretical findings, we present several examples. In addition, applications of the main results are provided.

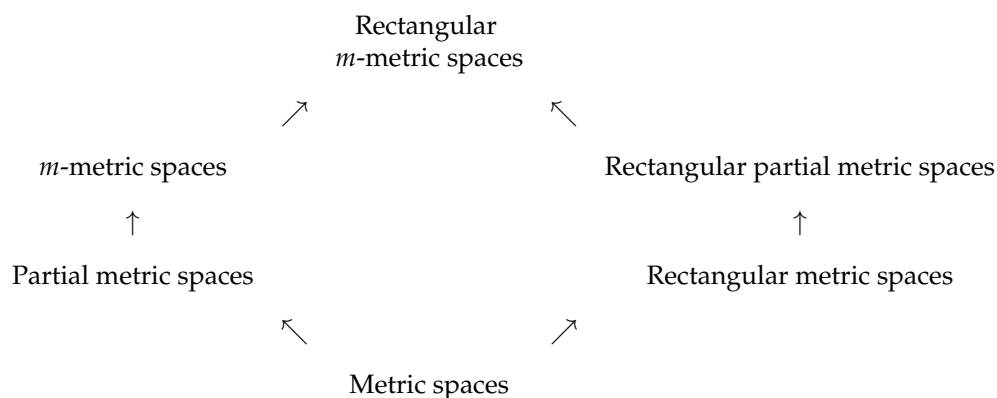


Figure 1. Hierarchy of generalized metric spaces.

The manuscript is organized as follows. Section 1 presents a brief literature review on the development of various contractive mappings and generalized metric spaces. Section 2 introduces the preliminary concepts and fundamental results related to interpolative contractions and generalized metric spaces. Section 3 contains the main results of the paper and is divided into two subsections: the first establishes coincidence point results for g -interpolative Kannan contractions, while the second is devoted to g -interpolative Ćirić–Reich–Rus contractions. Section 4 provides illustrative examples corresponding to all the established results. Section 5 presents applications of the main theorems, and finally, Section 6 concludes the paper.

Notation. Throughout this paper, MS denotes a metric space. We also use the abbreviations FP (fixed point), CP (coincidence point), KC (Kannan contraction), and CRRC (Ćirić–Reich–Rus contraction).

2. Preliminaries

In this section, we present the necessary preliminary concepts and results from the literature. We begin with the contraction introduced by Kannan [2], which, unlike the Banach contraction, does not require continuity and is defined as follows.

Definition 1 ([2]). Suppose $\mathcal{U}$ is an MS. A function Y from $\mathcal{U}$ to $\mathcal{U}$ is called a KC if $\exists c \in \left(0, \frac{1}{2}\right)$ such that

$$\delta(Y\sigma, Y\nu) \leq c(\delta(\sigma, Y\sigma) + \delta(\nu, Y\nu)) \quad \text{for all } \sigma, \nu \in \mathcal{U}.$$

Theorem 1 ([2]). If $(\mathcal{U}, \delta)$ is a complete MS, then every KC on $\mathcal{U}$ has a unique FP.

In 2018, Karapınar [3] introduced an interpolative generalization of the KC mapping and established the corresponding FP theorem in complete MSs.

Definition 2 ([3]). Suppose $(\mathcal{U}, \delta)$ is an MS. A function $Y : \mathcal{U} \rightarrow \mathcal{U}$ is called an interpolative KC if $\exists c \in [0, 1)$ and $\alpha \in (0, 1]$ such that

$$\delta(Y\sigma, Y\nu) \leq c(\delta(\sigma, Y\sigma)^\alpha \delta(\nu, Y\nu)^{1-\alpha}),$$

$\forall \sigma, \nu \in \mathcal{U}$ with $\sigma \neq Y\sigma$ and $\nu \neq Y\nu$.

Theorem 2 ([3]). Suppose $(\mathcal{U}, \delta)$ is a complete MS. Suppose $Y : \mathcal{U} \rightarrow \mathcal{U}$ is an interpolative KC mapping. Then, Y has an FP.

We now turn our attention to CPs and present a result concerning their existence. To this end, we first recall the definition of a CP.

Definition 3. Suppose $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are two self-mappings. A point $\sigma \in \mathcal{U}$ is said to be a CP of Y and g if $Y\sigma = g\sigma$.

Definition 4 ([6]). Suppose $(\mathcal{U}, \delta, s)$ is a b-MS. If $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are two self-mappings on $\mathcal{U}$. We call Y a g -interpolative KC if $\exists c \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\delta(Y\sigma, Y\nu) \leq c[\delta(g\sigma, Y\sigma)]^\alpha [\delta(g\nu, Y\nu)]^{1-\alpha}, \quad (1)$$

$\forall \sigma, \nu \in \mathcal{U}$ with $Y\sigma \neq g\sigma$ and $Y\nu \neq g\nu$.

Theorem 3 ([6]). Suppose $(\mathcal{U}, \delta, s)$ is a b-complete b-MS and Y is a g -interpolative KC. Also suppose that $Y\mathcal{U} \subseteq g\mathcal{U}$ and $g\mathcal{U}$ is closed. If $\exists c, \alpha \in (0, 1)$ such that (1) holds, then Y and g have a CP in $\mathcal{U}$.

Furthermore, keeping in mind that a rectangular m -MS is a generalization of an m -MS, and in turn, an m -MS is a generalization of a partial MS, we include definition of partial MS for the sake of convenience.

Definition 5 ([10]). A function $\delta_p : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}_+$ is called a partial metric if $\forall \sigma, \nu, z \in \mathcal{U}$; the following properties hold:

- (p1) $\delta_p(\sigma, \nu) = \delta_p(\sigma, \sigma) = \delta_p(\nu, \nu) \iff \sigma = \nu$.
- (p2) $\delta_p(\sigma, \sigma) \leq \delta_p(\sigma, \nu)$.
- (p3) $\delta_p(\sigma, \nu) = \delta_p(\nu, \sigma)$.
- (p4) $\delta_p(\sigma, \nu) \leq \delta_p(\sigma, z) + \delta_p(z, \nu) - \delta_p(z, z)$.

If δ_p is a partial metric on a set $\mathcal{U}$, then the pair $(\mathcal{U}, \delta_p)$ is called a partial MS.

Karapinar et al. [7] proposed interpolative Ćirić–Reich–Rus-type contractions in these spaces and established the relevant theorem of existence of FPs for this class of mappings.

Definition 6 ([7]). Suppose $(\mathcal{U}, \delta_p)$ is a partial MS. $Y : \mathcal{U} \rightarrow \mathcal{U}$ is called an interpolative CRRC mapping if $\exists$ constants $c \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, such that

$$\delta_p(Y\sigma, Y\nu) \leq c[\delta_p(\sigma, \nu)]^\alpha [\delta_p(\sigma, Y\sigma)]^\beta [\delta_p(\nu, Y\nu)]^{1-\alpha-\beta},$$

$\forall \sigma, \nu \in \mathcal{U}$ such that $\sigma \neq Y\sigma$ and $\nu \neq Y\nu$.

Theorem 4 ([7]). Suppose $(\mathcal{U}, \delta_p)$ is a partial MS. Assume $Y : \mathcal{U} \rightarrow \mathcal{U}$ is an interpolative CRRC mapping. Then, $\exists \sigma \in \mathcal{U}$ such that $\sigma = Y\sigma$.

Definition 7 ([8]). Assume that $\mathcal{U} \neq \emptyset$. A function $\delta_{\mathcal{M}} : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}_+$ is called an m -metric if for all $\sigma, \nu, z \in \mathcal{U}$ the following conditions are satisfied:

- (m1) $\delta_{\mathcal{M}}(\sigma, \nu) = \delta_{\mathcal{M}}(\sigma, \sigma) = \delta_{\mathcal{M}}(\nu, \nu) \iff \sigma = \nu$
- (m2) $m_{\sigma, \nu} \leq \delta_{\mathcal{M}}(\sigma, \nu)$ where $m_{\sigma, \nu} := \min\{\delta_{\mathcal{M}}(\sigma, \sigma), \delta_{\mathcal{M}}(\nu, \nu)\}$
- (m3) $\delta_{\mathcal{M}}(\sigma, \nu) = \delta_{\mathcal{M}}(\nu, \sigma)$
- (m4) $\delta_{\mathcal{M}}(\sigma, \nu) - m_{\sigma, \nu} \leq \delta_{\mathcal{M}}(\sigma, z) - m_{\sigma, z} + \delta_{\mathcal{M}}(z, \nu) - m_{z, \nu}$.

If $\delta_{\mathcal{M}}$ is an m -metric on a set $\mathcal{U}$, then the pair $(\mathcal{U}, \delta_{\mathcal{M}})$ is said to be an m -MS.

Definition 8 ([11]). Suppose $\mathcal{U}$ is a non-empty set and $\delta_r : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ is a function. If $\forall \sigma, \nu \in \mathcal{U}$, the following conditions are satisfied:

$$(RM_1) \quad \delta_r(\sigma, \nu) = m_{r_{\sigma, \nu}} = M_{r_{\sigma, \nu}} \iff \sigma = \nu$$

$$(RM_2) \quad m_{r_{\sigma, \nu}} \leq \delta_r(\sigma, \nu)$$

$$(RM_3) \quad \delta_r(\sigma, \nu) = \delta_r(\nu, \sigma)$$

$$(RM_4) \quad \delta_r(\sigma, \nu) - m_{r_{\sigma, \nu}} \leq \delta_r(\sigma, u) - m_{r_{\sigma, u}} + \delta_r(u, v) - m_{r_{u, v}} + \delta_r(v, \nu) - m_{r_{v, \nu}},$$

where $m_{r_{\sigma, \nu}} := \min\{\delta_r(\sigma, \sigma), \delta_r(\nu, \nu)\}$, $M_{r_{\sigma, \nu}} := \max\{\delta_r(\sigma, \sigma), \delta_r(\nu, \nu)\}$ and $u, v \in \mathcal{U} \setminus \{\sigma, \nu\}$, then the pair $(\mathcal{U}, \delta_r)$ is called a rectangular m -metric space.

Note that, throughout the text, we use the following notation:

$$m_{r_{\sigma, \nu}} := \min\{\delta_r(\sigma, \sigma), \delta_r(\nu, \nu)\}, \quad M_{r_{\sigma, \nu}} := \max\{\delta_r(\sigma, \sigma), \delta_r(\nu, \nu)\}.$$

Remark 1 ([11]). Suppose $(\mathcal{U}, \delta_r)$ is a rectangular m -MS, then for every $\sigma, \nu \in \mathcal{U}$, we have

$$0 \leq M_{r_{\sigma, \nu}} - m_{r_{\sigma, \nu}} = |\delta_r(\sigma, \sigma) - \delta_r(\nu, \nu)|.$$

Definition 9 ([11]). Assume that $(\mathcal{U}, \delta_r)$ is a rectangular m -MS. Then, the following holds:

1. A sequence $\{\sigma_n\}$ in $\mathcal{U}$ converges to $\sigma \in \mathcal{U}$ if and only if

$$\lim_{n \rightarrow \infty} (\delta_r(\sigma_n, \sigma) - m_{r_{\sigma_n, \sigma}}) = 0.$$

2. We define a sequence $\{\sigma_n\}$ in $\mathcal{U}$ to be a δ_r -Cauchy sequence precisely when

$$\lim_{n, q \rightarrow \infty} (\delta_r(\sigma_n, \sigma_q) - m_{r_{\sigma_n, \sigma_q}}) \quad \text{and} \quad \lim_{n, q \rightarrow \infty} (M_{r_{\sigma_n, \sigma_q}} - m_{r_{\sigma_n, \sigma_q}})$$

exist and are finite.

3. A rectangular m -MS is called δ_r -complete if every δ_r -Cauchy sequence $\{\sigma_n\}$ in $\mathcal{U}$ converges to a point $\sigma \in \mathcal{U}$ such that

$$\lim_{n \rightarrow \infty} (\delta_r(\sigma_n, \sigma) - m_{r_{\sigma_n, \sigma}}) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} (M_{r_{\sigma_n, \sigma}} - m_{r_{\sigma_n, \sigma}}) = 0.$$

Lemma 1 ([11]). Assume that the sequences $\{\sigma_n\}$ and $\{\nu_n\}$ converge to σ and ν , respectively, in the rectangular m -MS $(\mathcal{U}, \delta_r)$. Then, we have

$$\lim_{n \rightarrow \infty} (\delta_r(\sigma_n, \nu_n) - m_{r_{\sigma_n, \nu_n}}) = \delta_r(\sigma, \nu) - m_{r_{\sigma, \nu}}.$$

Lemma 2 ([11]). Suppose that the sequence $\{\sigma_n\}$ converges to σ in a rectangular m -MS. Then, for every $\nu \in \mathcal{U}$ we have

$$\lim_{n \rightarrow \infty} (\delta_r(\sigma_n, \nu) - m_{r_{\sigma_n, \nu}}) = \delta_r(\sigma, \nu) - m_{r_{\sigma, \nu}}.$$

3. Coincidence Points for g -Interpolative Contractions

In this section, we present the CP results for g -interpolative contractions. The discussion is divided into two parts: first, we establish the results for g -interpolative KCs, followed by the results for g -interpolative CRRCs in rectangular m -MSs.

3.1. Coincidence Points for g -Interpolative Kannan Contractions

We establish CP results for g -interpolative KCs in the three cases determined by the sum of the interpolative exponents, namely, $\alpha + \beta = 1$, $\alpha + \beta < 1$, and $\alpha + \beta > 1$.

Let us begin with the definition, where we specify that the required interpolative condition should be satisfied on the following set:

$$\{(\sigma, \nu) \in \mathcal{U} \times \mathcal{U} : Y\sigma \neq g\sigma, Y\nu \neq g\nu, \delta_r(g\sigma, Y\sigma) \neq 0, \delta_r(g\nu, Y\nu) \neq 0\} \subseteq \mathcal{U} \times \mathcal{U}.$$

Definition 10 ($\alpha + \beta = 1$). Suppose $(\mathcal{U}, \delta_r)$ is a rectangular m -MS. Assume $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are two self-mappings. We call the mapping Y a $(g, \alpha + \beta = 1)$ - δ_r -interpolative KC if $\exists$ two constants $c \in [0, 1)$ and $\alpha \in (0, 1)$ satisfying

$$\delta_r(Y\sigma, Y\nu) \leq c[\delta_r(g\sigma, Y\sigma)]^\alpha [\delta_r(g\nu, Y\nu)]^{1-\alpha}$$

$\forall \sigma, \nu \in \mathcal{U}$ with $Y\sigma \neq g\sigma$ and $Y\nu \neq g\nu$ and $\delta_r(g\sigma, Y\sigma) \neq 0, \delta_r(g\nu, Y\nu) \neq 0$.

Theorem 5 ($\alpha + \beta = 1$). Suppose $(\mathcal{U}, \delta_r)$ is a complete rectangular m -MS and Y is a $(g, \alpha + \beta = 1)$ - δ_r -interpolative KC. Also suppose that $Y\mathcal{U}$ is a subset of $g\mathcal{U}$ and $g\mathcal{U}$ is a closed subset of $\mathcal{U}$. Then, $\exists \sigma \in \mathcal{U}$ such that $Y\sigma = g\sigma$.

Proof. Let $\sigma_0 \in \mathcal{U}$, so $Y\sigma_0 \in g\mathcal{U}$. Thus, $\exists \sigma_1 \in \mathcal{U}$ such that $Y\sigma_0 = g\sigma_1$. Now, since $Y\sigma_1 \in g\mathcal{U}$, $\exists \sigma_2 \in \mathcal{U}$ such that $Y\sigma_1 = g\sigma_2$. Continuing in this way, we construct a sequence such that $g\sigma_{n+1} = Y\sigma_n \forall n$. If there is some $n_0 \in \{0, 1, 2, \dots\}$ such that $g\sigma_{n_0} = Y\sigma_{n_0}$, then σ_{n_0} is a CP of Y and g . Assume $g\sigma_n \neq Y\sigma_n \forall n$. By the definition of $(g, \alpha + \beta = 1)$ - δ_r -interpolative KCs, we have

$$\delta_r(g\sigma_{n+2}, g\sigma_{n+1}) = \delta_r(Y\sigma_{n+1}, Y\sigma_n) \leq c[\delta_r(g\sigma_{n+1}, Y\sigma_n)]^\alpha [\delta_r(g\sigma_n, Y\sigma_n)]^{1-\alpha},$$

thus

$$\delta_r(g\sigma_{n+1}, g\sigma_{n+2}) \leq c^{1/1-\alpha} \delta_r(g\sigma_n, g\sigma_{n+1}) \leq c \delta_r(g\sigma_n, g\sigma_{n+1}),$$

because $\frac{1}{1-\alpha} > 1, c \in [0, 1)$ implies $c^{1/1-\alpha} \leq c$. In this way, we obtain $\delta_r(g\sigma_{n+1}, g\sigma_n) \leq c \delta_r(g\sigma_n, g\sigma_{n-1}) \leq \dots \leq c^n \delta_r(g\sigma_0, g\sigma_1)$. So, $\lim_{n \rightarrow \infty} \delta_r(g\sigma_{n+1}, g\sigma_n) \leq \lim_{n \rightarrow \infty} c^n \delta_r(g\sigma_0, g\sigma_1)$ implies that $\lim_{n \rightarrow \infty} \delta_r(g\sigma_{n+1}, g\sigma_n) = 0$ and also $\lim_{n \rightarrow \infty} m_{r_{g\sigma_{n+1}, g\sigma_n}} = 0$. Now, we show that $\{g\sigma_n\}$ is a δ_r -Cauchy sequence. Now, consider the following:

$$\begin{aligned} \delta_r(g\sigma_{n+2}, g\sigma_n) &= \delta_r(Y\sigma_{n+1}, Y\sigma_{n-1}) \leq c[\delta_r(g\sigma_{n+1}, Y\sigma_{n-1})]^\alpha [\delta_r(g\sigma_{n-1}, Y\sigma_{n-1})]^{1-\alpha} \\ &= c[\delta_r(g\sigma_{n+1}, g\sigma_{n+2})]^\alpha [\delta_r(g\sigma_{n-1}, g\sigma_n)]^{1-\alpha} \\ &\leq c[c^{n+1} \delta_r(g\sigma_0, g\sigma_1)]^\alpha [c^{n-1} \delta_r(g\sigma_0, g\sigma_1)]^{1-\alpha} \\ &= c^{2\alpha+n} \delta_r(g\sigma_0, g\sigma_1). \end{aligned}$$

Case A. When m is odd. So, by using the (RM4), we get

$$\begin{aligned} \delta_r(g\sigma_{n+m}, g\sigma_n) - m_{r_{g\sigma_n, g\sigma_{n+m}}} &\leq \delta_r(g\sigma_n, g\sigma_{n+1}) - m_{r_{g\sigma_n, g\sigma_{n+1}}} + \delta_r(g\sigma_{n+1}, g\sigma_{n+2}) \\ &\quad - m_{r_{g\sigma_{n+1}, g\sigma_{n+2}}} + \delta_r(g\sigma_{n+2}, g\sigma_{n+m}) - m_{r_{g\sigma_{n+2}, g\sigma_{n+m}}} \\ &\leq \delta_r(g\sigma_n, g\sigma_{n+1}) + \delta_r(g\sigma_{n+1}, g\sigma_{n+2}) \\ &\quad + (\delta_r(g\sigma_{n+2}, g\sigma_{n+m}) - m_{r_{g\sigma_{n+2}, g\sigma_{n+m}}}). \end{aligned} \quad (2)$$

Similarly, extending the last factor in the above inequality we get the following:

$$\begin{aligned} \delta_r(g\sigma_{n+m}, g\sigma_n) - m_{r_{g\sigma_n, g\sigma_{n+m}}} &\leq \delta_r(g\sigma_n, g\sigma_{n+1}) + \delta_r(g\sigma_{n+1}, g\sigma_{n+2}) \\ &\quad + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) + \dots + \delta_r(g\sigma_{n+m-1}, g\sigma_{n+m}) \\ &\leq (c^n + c^{n+1} + \dots + c^{n+m-1}) \delta_r(g\sigma_0, g\sigma_1). \end{aligned} \quad (3)$$

Case B. When m is even, we get

$$\begin{aligned}\delta_r(g\sigma_{n+m}, g\sigma_n) - m_{r_{g\sigma_n, g\sigma_{n+m}}} &\leq \delta_r(g\sigma_n, g\sigma_{n+2}) + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) \\ &\quad + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) + \dots + \delta_r(g\sigma_{n+m-1}, g\sigma_{n+m}) \\ &\leq (c^n + c^{n+2} + c^{n+3} + \dots + c^{n+m-1})\delta_r(g\sigma_0, g\sigma_1).\end{aligned}\quad (4)$$

Both cases provide us

$$\lim_{n, m \rightarrow \infty} [\delta_r(g\sigma_{n+m}, g\sigma_n) - m_{r_{g\sigma_n, g\sigma_{n+m}}}] = 0, \quad (5)$$

because the right-hand side of (3) and (4) goes to zero when n goes to infinity. In addition, we have $0 \leq M_{r_{g\sigma_{n+m}, g\sigma_n}} - m_{r_{g\sigma_n, g\sigma_{n+m}}} \leq M_{r_{g\sigma_{n+m}, g\sigma_n}}$. We can suppose, without loss of generality, that $M_{r_{g\sigma_{n+m}, g\sigma_n}} = \delta_r(g\sigma_n, g\sigma_n)$; so, we obtain

$$\delta_r(g\sigma_n, g\sigma_n) \leq c\delta_r(g\sigma_{n-1}, Y\sigma_{n-1}) = \delta_r(g\sigma_{n-1}, g\sigma_n) \leq c^{n-1}\delta_r(g\sigma_0, Y\sigma_1). \quad (6)$$

This ensures that

$$\lim_{n, m \rightarrow \infty} [M_{r_{g\sigma_{n+m}, g\sigma_n}} - m_{r_{g\sigma_n, g\sigma_{n+m}}}] = 0. \quad (7)$$

So, by using Definition 9 and relations (5) and (7), we deduce that the sequence $\{g\sigma_n\}$ is a δ_r -Cauchy sequence. Since $(\mathcal{U}, \delta_r)$ is complete, $\{g\sigma_n\}$ converges to $z \in \mathcal{U}$. But $g\mathcal{U}$ is closed and hence $z = g\sigma$ for some $\sigma \in \mathcal{U}$. Also we have

$$\delta_r(Y\sigma_n, Y\sigma) \leq c[\delta_r(g\sigma_n, Y\sigma_n)]^\alpha [\delta_r(g\sigma, Y\sigma)]^{1-\alpha} = c[\delta_r(g\sigma_n, g\sigma_{n+1})]^\alpha [\delta_r(g\sigma, Y\sigma)]^{1-\alpha}, \quad (8)$$

letting $n \rightarrow \infty$ and using the facts that $\lim_{n \rightarrow \infty} \delta_r(g\sigma_n, g\sigma_{n+1}) = 0$ and $\delta_r(z, Y\sigma) < \infty$, we have, $\lim_{n \rightarrow \infty} \delta_r(Y\sigma_n, Y\sigma) = 0$. Moreover, $\lim_{n \rightarrow \infty} m_{r_{Y\sigma_n, Y\sigma}} = 0$. Thus, we obtain

$$\lim_{n \rightarrow \infty} [\delta_r(Y\sigma_n, Y\sigma) - m_{r_{Y\sigma_n, Y\sigma}}] = 0. \quad (9)$$

Thus, $Y\sigma_n$ converges to $Y\sigma$ with respect to the δ_r -metric. Again, $\delta_r(g\sigma_n, Y\sigma_n) = \delta_r(g\sigma_n, g\sigma_{n+1})$ yields $\lim_{n \rightarrow \infty} \delta_r(g\sigma_n, Y\sigma_n) = 0$. Furthermore,

$$\lim_{n \rightarrow \infty} [\delta_r(Y\sigma_n, g\sigma_n) - m_{r_{Y\sigma_n, g\sigma_n}}] = 0, \quad (10)$$

by using Lemma 1, we get that $\delta_r(Y\sigma, g\sigma) - m_{r_{Y\sigma, g\sigma}} = 0$, because $g\sigma_n$ converges to $g\sigma$ and $Y\sigma_n$ converges to $Y\sigma$. Hence,

$$\delta_r(Y\sigma, g\sigma) = m_{r_{Y\sigma, g\sigma}}. \quad (11)$$

Also $\lim_{n \rightarrow \infty} [\delta_r(Y\sigma_n, g\sigma_n) - m_{r_{Y\sigma_n, g\sigma_n}}] = 0 \Rightarrow \lim_{n \rightarrow \infty} [\delta_r(Y\sigma_n, Y\sigma_{n-1}) - m_{r_{Y\sigma_n, g\sigma_n}}] = 0$. Thus, by using Lemma 1, we get that $\delta_r(Y\sigma, Y\sigma) = m_{r_{Y\sigma, g\sigma}}$. Once again, if we use $Y\sigma_n = g\sigma_{n+1}$, we see that $\delta_r(g\sigma, g\sigma) = m_{r_{Y\sigma, g\sigma}}$. Using Remark 1, we get

$$M_{r_{Y\sigma, g\sigma}} - m_{r_{Y\sigma, g\sigma}} = |\delta_r(g\sigma, g\sigma) - \delta_r(Y\sigma, Y\sigma)| = 0. \quad (12)$$

Relations (11) and (12) show that $M_{r_{Y\sigma, g\sigma}} = m_{r_{Y\sigma, g\sigma}} = \delta_r(Y\sigma, g\sigma)$, hence $Y\sigma = g\sigma$. $\square$

Now, let us consider the case $\alpha + \beta < 1$. In this case, we specify that the required interpolative condition should be satisfied on the following set:

$$\{(\sigma, \nu) \in \mathcal{U} \times \mathcal{U} : g\sigma \neq Y\sigma, g\nu \neq Y\nu, \delta_r(g\sigma, Y\sigma) \geq 1, \delta_r(g\nu, Y\nu) \neq 0\} \subseteq \mathcal{U} \times \mathcal{U}.$$

Definition 11 ($\alpha + \beta < 1$). Suppose $(\mathcal{U}, \delta_r)$ is a rectangular m -MS and $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are two self-mappings on $\mathcal{U}$. We say that the mapping Y is a $(g, \alpha + \beta < 1)$ - δ_r -interpolative KC if $\exists c \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that

$$\delta_r(Y\sigma, Y\nu) \leq c[\delta_r(g\sigma, Y\sigma)]^\alpha [\delta_r(g\nu, Y\nu)]^\beta$$

$$\forall \sigma, \nu \in \mathcal{U} \text{ with } g\sigma \neq Y\sigma, g\nu \neq Y\nu, \text{ and } \delta_r(g\sigma, Y\sigma) \geq 1, \delta_r(g\nu, Y\nu) \neq 0.$$

Theorem 6 ($\alpha + \beta < 1$). Let $(\mathcal{U}, \delta_r)$ be a complete rectangular m -MS and let Y be a $(g, \alpha + \beta < 1)$ - δ_r -interpolative KC. Assume $Y\mathcal{U} \subseteq g\mathcal{U}$ and $g\mathcal{U}$ is closed. Then, Y and g have a CP.

Proof. Let $\sigma_0 \in \mathcal{U}$; we construct a sequence $\{\sigma_n\}$ by $g\sigma_{n+1} = Y\sigma_n \forall n$; we assume that $g\sigma_n \neq Y\sigma_n \forall n$, then

$$\delta_r(g\sigma_{n+1}, g\sigma_{n+2}) = \delta_r(Y\sigma_n, Y\sigma_{n+1}) \leq c[\delta_r(g\sigma_n, Y\sigma_n)]^\alpha [\delta_r(g\sigma_{n+1}, Y\sigma_{n+1})]^\beta.$$

Hence,

$$\delta_r(g\sigma_{n+1}, g\sigma_{n+2}) \leq c[\delta_r(g\sigma_{n+1}, g\sigma_n)]^\alpha [\delta_r(g\sigma_{n+1}, g\sigma_{n+2})]^\beta,$$

which provides

$$[\delta_r(g\sigma_{n+1}, g\sigma_{n+2})]^{1-\beta} \leq c[\delta_r(g\sigma_n, g\sigma_{n+1})]^\alpha,$$

since $\alpha < 1 - \beta$ and $\delta_r(g\sigma_n, g\sigma_{n+1}) \geq 1$, thus we have

$$[\delta_r(g\sigma_{n+1}, g\sigma_{n+2})]^{1-\beta} \leq c[\delta_r(g\sigma_n, g\sigma_{n+1})]^{1-\beta}.$$

Consequently,

$$\delta_r(g\sigma_{n+1}, g\sigma_{n+2}) \leq c\delta_r(g\sigma_n, g\sigma_{n+1}),$$

which leads to

$$\delta_r(g\sigma_n, g\sigma_{n+1}) \leq c^n \delta_r(g\sigma_0, g\sigma_1).$$

Moreover,

$$\begin{aligned} \delta_r(g\sigma_n, g\sigma_{n+2}) &= \delta_r(Y\sigma_{n-1}, Y\sigma_{n+1}) \leq c[\delta_r(g\sigma_{n-1}, Y\sigma_{n-1})]^\alpha [\delta_r(g\sigma_{n+1}, Y\sigma_{n+1})]^\beta \\ &\leq c^{n(\alpha+\beta)+1-\alpha+\beta} [\delta_r(g\sigma_0, g\sigma_1)]^{\alpha+\beta}. \end{aligned}$$

Next, by following the similar procedure as in Theorem 5, we obtain the desired result. $\square$

Now, let us consider the case $\alpha + \beta > 1$. In this case, we specify that the required interpolative condition should be satisfied on the following set:

$$\{(\sigma, \nu) \in \mathcal{U} \times \mathcal{U} : g\sigma \neq Y\sigma, g\nu \neq Y\nu, \delta_r(g\sigma, Y\sigma) \neq 0, \delta_r(g\nu, Y\nu) \neq 0\} \subseteq \mathcal{U} \times \mathcal{U}.$$

Definition 12 ($\alpha + \beta > 1$). Suppose $(\mathcal{U}, \delta_r)$ is a rectangular m -MS and $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are two self-mappings on $\mathcal{U}$. We say that the mapping Y is a $(g, \alpha + \beta > 1)$ - δ_r -interpolative KC if $\exists c \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta > 1$ such that

$$\delta_r(Y\sigma, Y\nu) \leq c[\delta_r(g\sigma, Y\sigma)]^\alpha [\delta_r(g\nu, Y\nu)]^\beta$$

$$\forall \sigma, \nu \in \mathcal{U} \text{ with } g\sigma \neq Y\sigma, g\nu \neq Y\nu, \text{ and } \delta_r(g\sigma, Y\sigma) \neq 0, \delta_r(g\nu, Y\nu) \neq 0.$$

Theorem 7 ($\alpha + \beta > 1$). Let $(\mathcal{U}, \delta_r)$ be a complete rectangular m -MS and let Y be a $(g, \alpha + \beta > 1)$ - δ_r -interpolative KC. Assume $Y\mathcal{U} \subseteq g\mathcal{U}$ and $g\mathcal{U}$ is closed. If $\exists \sigma^* \in \mathcal{U}$ such that $\delta_r(g\sigma^*, Y\sigma^*) \leq 1$, then Y and g have a CP.

Proof. We construct a sequence $g\sigma_{n+1} = Y\sigma_n \forall$ non-negative integers n with $g\sigma_n \neq Y\sigma_n \forall n \in \mathbb{N}$. Then,

$$\delta_r(g\sigma_n, g\sigma_{n+1}) = \delta_r(Y\sigma_n, Y\sigma_n) \leq c[\delta_r(g\sigma_0, Y\sigma_0)]^\alpha [\delta_r(g\sigma_1, Y\sigma_1)]^\beta,$$

which implies

$$\delta_r(g\sigma_1, g\sigma_2)^{1-\beta} \leq c[\delta_r(g\sigma_0, g\sigma_1)]^\alpha.$$

Thus,

$$\delta_r(g\sigma_1, g\sigma_2) \leq c^{1/(1-\beta)} [\delta_r(g\sigma_0, g\sigma_1)]^{\alpha/(1-\beta)} \leq c,$$

because $\frac{\alpha}{1-\beta} > 1$ and $\delta_r(g\sigma_0, g\sigma_1) \leq 1$. Similarly,

$$\delta_r(g\sigma_2, g\sigma_3) \leq c^2.$$

By using mathematical induction and the g -interpolative condition, the following holds:

$$\delta_r(g\sigma_n, g\sigma_{n+1}) \leq c^n.$$

Moreover,

$$\delta_r(g\sigma_n, g\sigma_{n+2}) = \delta_r(Y\sigma_{n-1}, Y\sigma_{n+1}) \leq c[\delta_r(g\sigma_{n-1}, Y\sigma_{n-1})]^\alpha [\delta_r(g\sigma_{n+1}, Y\sigma_{n+1})]^\beta,$$

which gives

$$\delta_r(g\sigma_n, g\sigma_{n+2}) \leq [\delta_r(g\sigma_{n-1}, g\sigma_n)]^\alpha [\delta_r(g\sigma_{n+1}, g\sigma_{n+2})]^\beta \leq c^{1-\alpha+\beta+n(\alpha+\beta)}.$$

Next, by following the similar procedure as in Theorem 5, we obtain the desired result. $\square$

3.2. Coincidence Points of g -Interpolative Ćirić–Reich–Rus Contractions

We establish CP results for g -interpolative CRRCs in the two cases determined by the sum of the interpolative exponents, namely, $\alpha + \beta < 1$ and $\alpha + \beta > 1$.

Now, let us consider the case $\alpha + \beta < 1$. In this case, we specify that the required interpolative condition should be satisfied on the following set:

$$\{(\sigma, \nu) \in \mathcal{U} \times \mathcal{U} : g\sigma \neq Y\sigma, g\nu \neq Y\nu, \delta_r(g\sigma, g\nu) > 0, \delta_r(g\sigma, Y\sigma) > 0, \delta_r(g\nu, Y\nu) > 0\}.$$

Definition 13 ($\alpha + \beta < 1$). Assume $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are self-mappings. Y is called a $(g, \alpha + \beta < 1)$ - δ_r -interpolative CRRC mapping if $\exists c \in [0, 1)$, $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, such that

$$\delta_r(Y\sigma, Y\nu) \leq c[\delta_r(g\sigma, g\nu)]^\beta [\delta_r(g\sigma, Y\sigma)]^\alpha [\delta_r(g\nu, Y\nu)]^{1-\alpha-\beta} \quad (13)$$

$$\forall \sigma, \nu \in \mathcal{U}, g\sigma \neq Y\sigma, g\nu \neq Y\nu \text{ and } \delta_r(g\sigma, g\nu), \delta_r(g\sigma, Y\sigma), \delta_r(g\nu, Y\nu) > 0.$$

Theorem 8 ($\alpha + \beta < 1$). Assume $(\mathcal{U}, \delta_r)$ is a complete rectangular m -MS, $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are self-mappings, $Y\mathcal{U} \subseteq g\mathcal{U}$ and $g\mathcal{U}$ is closed. If Y is a $(g, \alpha + \beta < 1)$ - δ_r -interpolative CRRC mapping, then Y and g have a CP.

Proof. Let $\sigma_0 \in \mathcal{U}$, so $Y\sigma_0 \in g\mathcal{U}$, $\exists \sigma_1 \in \mathcal{U}$ such that $Y\sigma_0 = g\sigma_1$. Also $\sigma_1 \in \mathcal{U}$, $Y\sigma_1 \in g\mathcal{U}$, $\exists \sigma_2 \in \mathcal{U}$ such that $Y\sigma_1 = g\sigma_2$. In the same way, we construct a sequence $\{\sigma_n\} \subseteq \mathcal{U}$ such that $Y\sigma_n = g\sigma_{n+1}$. Suppose that for some integer n_0 , we have $g\sigma_{n_0} = g\sigma_{n_0+1}$. In this case, σ_{n_0} is a CP, and the argument is completed. Otherwise, we assume that $g\sigma_n \neq g\sigma_{n+1} \forall n \geq 0$. We begin by deriving an inequality for $\delta_r(g\sigma_n, g\sigma_{n+1})$. Thus, applying (13), we obtain

$$\begin{aligned}\delta_r(Y\sigma_{n-1}, Y\sigma_n) &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_n))^\beta (\delta_r(g\sigma_{n-1}, Y\sigma_{n-1}))^\alpha (\delta_r(g\sigma_n, Y\sigma_n))^{1-\alpha-\beta} \\ &= \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_n))^\beta (\delta_r(g\sigma_{n-1}, g\sigma_n))^\alpha (\delta_r(g\sigma_n, g\sigma_{n+1}))^{1-\alpha-\beta} \\ &= \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_n))^{\alpha+\beta} (\delta_r(g\sigma_n, g\sigma_{n+1}))^{1-\alpha-\beta},\end{aligned}$$

or

$$\begin{aligned}(\delta_r(g\sigma_n, g\sigma_{n+1}))^{\alpha+\beta} &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_n))^{\alpha+\beta} \\ \delta_r(g\sigma_n, g\sigma_{n+1}) &\leq \mathfrak{c}^{\frac{1}{\alpha+\beta}} \delta_r(g\sigma_{n-1}, g\sigma_n) \leq \mathfrak{c} \delta_r(g\sigma_{n-1}, g\sigma_n),\end{aligned}$$

because, $\alpha + \beta < 1$ and $\mathfrak{c} \in [0, 1)$, further, it follows that

$$\delta_r(g\sigma_n, g\sigma_{n+1}) \leq \mathfrak{c}^n \delta_r(g\sigma_0, g\sigma_1). \quad (14)$$

We now establish a similar relationship for $\delta_r(g\sigma_n, g\sigma_{n+2})$, which will be required to demonstrate that $\{g\sigma_n\}$ forms a δ_r -Cauchy sequence. Using (13) and (14), we obtain

$$\begin{aligned}\delta_r(g\sigma_n, g\sigma_{n+2}) &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_{n+1}))^\beta (\delta_r(g\sigma_{n-1}, g\sigma_n))^\alpha (\delta_r(g\sigma_{n+1}, g\sigma_{n+2}))^{1-\alpha-\beta} \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_{n+1}))^\beta (\mathfrak{c}^{n-1} \delta_r(g\sigma_0, g\sigma_1))^\alpha (\mathfrak{c}^{n+1} \delta_r(g\sigma_0, g\sigma_1))^{1-\alpha-\beta} \\ &= \mathfrak{c}^{1+(n-1)\alpha+(n+1)(1-\alpha-\beta)} (\delta_r(g\sigma_{n-1}, g\sigma_{n+1}))^\beta (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta} \\ &= \mathfrak{c}^{1-2\alpha+(n+1)(1-\beta)} (\delta_r(g\sigma_{n-1}, g\sigma_{n+1}))^\beta (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta}.\end{aligned} \quad (15)$$

Again, by (13) and (14), we get the following:

$$\delta_r(g\sigma_{n-1}, g\sigma_{n+1}) \leq \mathfrak{c}^{1-2\alpha+n(1-\beta)} (\delta_r(g\sigma_{n-2}, g\sigma_n))^\beta (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta} \quad (16)$$

$$\delta_r(g\sigma_{n-2}, g\sigma_n) \leq \mathfrak{c}^{1-2\alpha+(n-1)(1-\beta)} (\delta_r(g\sigma_{n-3}, g\sigma_{n-1}))^\beta (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta}. \quad (17)$$

By (15) and (16),

$$\begin{aligned}\delta_r(g\sigma_n, g\sigma_{n+2}) &\leq \mathfrak{c}^{1-2\alpha+(n+1)(1-\beta)} (\mathfrak{c}^{1-2\alpha+n(1-\beta)} (\delta_r(g\sigma_{n-2}, g\sigma_n))^\beta (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta})^\beta \\ &\quad (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta} \\ &= \mathfrak{c}^{1-2\alpha+(n+1)(1-\beta)+\beta(1-2\alpha+n(1-\beta))} (\delta_r(g\sigma_{n-2}, g\sigma_n))^{\beta^2} (\delta_r(g\sigma_0, g\sigma_1))^{(1-\beta)(1+\beta)} \\ &= \mathfrak{c}^{(1-2\alpha)(1+\beta)+((n+1)+n\beta)(1-\beta)} (\delta_r(g\sigma_{n-2}, g\sigma_n))^{\beta^2} (\delta_r(g\sigma_0, g\sigma_1))^{(1-\beta)(1+\beta)}.\end{aligned} \quad (18)$$

By (17) and (18),

$$\begin{aligned}\delta_r(g\sigma_n, g\sigma_{n+2}) &\leq \mathfrak{c}^{(1-2\alpha)(1+\beta+\beta^2)+((n+1)+n\beta+(n-1)\beta^2)(1-\beta)} (\delta_r(g\sigma_{n-3}, g\sigma_{n-1}))^{\beta^3} \\ &\quad (\delta_r(g\sigma_0, g\sigma_1))^{(1-\beta)(1+\beta+\beta^2)}.\end{aligned}$$

Repeating the above steps yields the following:

$$\begin{aligned}\delta_r(g\sigma_n, g\sigma_{n+2}) &\leq \mathfrak{c}^{(1-2\alpha)(1+\beta+\beta^2+\dots+\beta^{n-1})+((n+1)+n\beta+(n-1)\beta^2+\dots+2\beta^{n-1})(1-\beta)} \\ &\quad \left(\delta_r(g\sigma_0, g\sigma_2)\right)^{\beta^n} \left(\delta_r(g\sigma_0, g\sigma_1)\right)^{(1-\beta)(1+\beta+\beta^2+\dots+\beta^{n-1})} \\ &= \mathfrak{c}^{(1-2\alpha)\left(\frac{1-\beta^n}{1-\beta}\right)+((n+1)+n\beta+(n-1)\beta^2+\dots+2\beta^{n-1})(1-\beta)} \left(\delta_r(g\sigma_0, g\sigma_2)\right)^{\beta^n} \left(\delta_r(g\sigma_0, g\sigma_1)\right)^{1-\beta^n} \\ &= \mathfrak{c}^B \left(\delta_r(g\sigma_0, g\sigma_2)\right)^{\beta^n} \left(\delta_r(g\sigma_0, g\sigma_1)\right)^{1-\beta^n},\end{aligned}\quad (19)$$

where

$$B = (1 - 2\alpha) \left(\frac{1 - \beta^n}{1 - \beta} \right) + (1 - \beta) \left((n + 1) + n\beta + (n - 1)\beta^2 + \dots + 2\beta^{n-1} \right).$$

We simplify the expression and get the following:

$$B = 2(1 - \alpha - \beta) \frac{1 - \beta^n}{1 - \beta} + n. \quad (20)$$

Inserting the expression for B into (19),

$$\delta_r(g\sigma_n, g\sigma_{n+2}) \leq \mathfrak{c}^{2(1-\alpha-\beta)\frac{1-\beta^n}{1-\beta}+n} (\delta_r(g\sigma_0, g\sigma_2))^{\beta^n} (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta^n}. \quad (21)$$

To show $\{g\sigma_n\}$ is a δ_r -Cauchy sequence in $\mathcal{U}$, assume n and q are any positive integers. We discuss the following two cases.

Case-1: When q is odd, that is, $q = 2p + 1$ for some $p \in \mathbb{N}$. Rectangular inequality (RM_4) of the δ_r -metric and inequality (14) then imply

$$\begin{aligned}\delta_r(g\sigma_n, g\sigma_{n+q}) - m_{r_{g\sigma_n, g\sigma_{n+q}}} &\leq \delta_r(g\sigma_n, g\sigma_{n+1}) - m_{r_{g\sigma_n, g\sigma_{n+1}}} + \delta_r(g\sigma_{n+1}, g\sigma_{n+2}) \\ &\quad - m_{r_{g\sigma_{n+1}, g\sigma_{n+2}}} + \delta_r(g\sigma_{n+2}, g\sigma_{n+q}) - m_{r_{g\sigma_{n+2}, g\sigma_{n+q}}} \\ &\leq \delta_r(g\sigma_n, g\sigma_{n+1}) + \delta_r(g\sigma_{n+1}, g\sigma_{n+2}) + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) \\ &\quad + \delta_r(g\sigma_{n+3}, g\sigma_{n+4}) + \delta_r(g\sigma_{n+4}, g\sigma_{n+m}) - m_{r_{g\sigma_{n+4}, g\sigma_{n+q}}} \\ &\vdots \\ &\leq \delta_r(g\sigma_n, g\sigma_{n+1}) + \delta_r(g\sigma_{n+1}, g\sigma_{n+2}) + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) \\ &\quad + \dots + \delta_r(g\sigma_{n+2p}, g\sigma_{n+2p-1}) \\ &\leq \mathfrak{c}^n \delta_r(g\sigma_0, g\sigma_1) + \mathfrak{c}^{n+1} \delta_r(g\sigma_0, g\sigma_1) + \mathfrak{c}^{n+2} \delta_r(g\sigma_0, g\sigma_1) \\ &\quad + \dots + \mathfrak{c}^{n+2p-1} \delta_r(g\sigma_0, g\sigma_1) + \mathfrak{c}^{n+2p} \delta_r(g\sigma_0, g\sigma_1) \\ &\leq (\mathfrak{c}^n + \mathfrak{c}^{n+1} + \dots + \mathfrak{c}^{n+2p} + \dots) \delta_r(g\sigma_0, g\sigma_1) \\ &= \frac{\mathfrak{c}^n}{1 - \mathfrak{c}} \delta_r(g\sigma_0, g\sigma_1).\end{aligned}$$

Case-2: Now, assume q is an even natural number, that is, $q = 2p$, for some $p \in \mathbb{N}$, with $p \geq 2$. Then, rectangular inequality (RM_4), (14), and (21) imply

$$\begin{aligned}
& \delta_r(g\sigma_n, g\sigma_{n+q}) - m_{r_{g\sigma_n, g\sigma_{n+q}}} \leq \delta_r(g\sigma_n, g\sigma_{n+2}) - m_{r_{g\sigma_n, g\sigma_{n+2}}} + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) \\
& \quad - m_{r_{g\sigma_{n+2}, g\sigma_{n+3}}} + \delta_r(g\sigma_{n+3}, g\sigma_{n+q}) - m_{r_{g\sigma_{n+3}, g\sigma_{n+q}}} \\
& \leq \delta_r(g\sigma_n, g\sigma_{n+2}) + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) + \delta_r(g\sigma_{n+3}, g\sigma_{n+q}) - m_{r_{g\sigma_{n+3}, g\sigma_{n+q}}} \\
& \quad \vdots \\
& \leq \delta_r(g\sigma_n, g\sigma_{n+2}) + \delta_r(g\sigma_{n+2}, g\sigma_{n+3}) + \dots + \delta_r(g\sigma_{n+2p-1}, g\sigma_{n+2p}) \\
& \leq c^{2(1-\alpha-\beta)\frac{1-\beta^n}{1-\beta}+n} (\delta_r(g\sigma_0, g\sigma_2))^{\beta^n} (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta^n} + \frac{c^{n+2}}{1-c} \delta_r(g\sigma_0, g\sigma_1) \\
& \quad + \dots + \frac{c^{n+2p-1}}{1-c} \delta_r(g\sigma_0, g\sigma_1) \\
& \leq c^{2(1-\alpha-\beta)\frac{1-\beta^n}{1-\beta}+n} (\delta_r(g\sigma_0, g\sigma_2))^{\beta^n} (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta^n} + \frac{c^n}{1-c} \delta_r(g\sigma_0, g\sigma_1).
\end{aligned}$$

So, it is deduced from both the cases that

$$\begin{aligned}
\delta_r(g\sigma_n, g\sigma_{n+q}) - m_{r_{g\sigma_n, g\sigma_{n+q}}} & \leq c^{2(1-\alpha-\beta)\frac{1-\beta^n}{1-\beta}+n} (\delta_r(g\sigma_0, g\sigma_2))^{\beta^n} (\delta_r(g\sigma_0, g\sigma_1))^{1-\beta^n} \\
& \quad + \frac{c^n}{1-c} \delta_r(g\sigma_0, g\sigma_1).
\end{aligned}$$

Applying the limit to both sides yields

$$\lim_{n,q \rightarrow \infty} (\delta_r(g\sigma_n, g\sigma_{n+q}) - m_{r_{g\sigma_n, g\sigma_{n+q}}}) = 0. \quad (22)$$

Furthermore, it is known that

$$0 \leq M_{r_{g\sigma_n, g\sigma_{n+q}}} - m_{r_{g\sigma_n, g\sigma_{n+q}}} \leq M_{r_{g\sigma_n, g\sigma_{n+q}}}.$$

We can suppose, with no loss of generality, that $M_{r_{g\sigma_n, g\sigma_{n+q}}} = \delta_r(g\sigma_{n+q}, g\sigma_{n+q})$, so

$$\begin{aligned}
M_{r_{g\sigma_n, g\sigma_{n+q}}} & = \delta_r(g\sigma_{n+q}, g\sigma_{n+q}) \\
& \leq c(\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q-1}))^\beta (\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q}))^\alpha (\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q}))^{1-\alpha-\beta} \\
& \leq c(\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q-1}))^\beta (c^{n+q-1} \delta_r(g\sigma_0, g\sigma_1))^{1-\beta},
\end{aligned}$$

thus

$$\lim_{n,q \rightarrow \infty} (M_{r_{g\sigma_n, g\sigma_{n+q}}} - m_{r_{g\sigma_n, g\sigma_{n+q}}}) = 0. \quad (23)$$

So, by Definition 9 and relations (22) and (23), it follows that the sequence $\{g\sigma_n\}$ is δ_r -Cauchy. Since $(\mathcal{U}, \delta_r)$ is complete, so $\{g\sigma_n\}$ converges to some element, say $\bar{\sigma}$, in $\mathcal{U}$. But the fact that $g\mathcal{U}$ is closed implies that $\bar{\sigma} = g\sigma^*$ for some $\sigma^* \in \mathcal{U}$. Now, to prove that σ^* is the CP of Y and g , firstly we prove that $\{Y\sigma_n\}$ converges to $Y\sigma^*$. Since by (14), we have

$$\begin{aligned}
0 \leq \delta_r(Y\sigma_n, Y\sigma^*) - m_{r_{Y\sigma_n, Y\sigma^*}} & \leq \delta_r(Y\sigma_n, Y\sigma^*) \\
& \leq c(\delta_r(g\sigma_n, g\sigma^*))^\beta (\delta_r(g\sigma_n, g\sigma_{n+1}))^\alpha (\delta_r(g\sigma^*, Y\sigma^*))^{1-\alpha-\beta} \\
& \leq c(\delta_r(g\sigma_n, g\sigma^*))^\beta (c^n \delta_r(g\sigma_0, g\sigma_1))^\alpha (\delta_r(g\sigma^*, Y\sigma^*))^{1-\alpha-\beta}.
\end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} (\delta_r(Y\sigma_n, Y\sigma^*) - m_{r_{Y\sigma_n, Y\sigma^*}}) = 0.$$

From Definition 9, it follows that $Y\sigma_n \rightarrow Y\sigma^*$. Also we have

$$0 \leq M_{r_{Y\sigma_n, g\sigma_n}} - m_{r_{Y\sigma_n, g\sigma_n}} \leq M_{r_{Y\sigma_n, \sigma_n}}.$$

Thus, without loss of generality, let $M_{r_{Y\sigma_n, \sigma_n}} = \delta_r(g\sigma_n, g\sigma_n)$; thus,

$$\begin{aligned} 0 \leq M_{r_{Y\sigma_n, g\sigma_n}} - m_{r_{Y\sigma_n, g\sigma_n}} &\leq M_{r_{Y\sigma_n, \sigma_n}} = \delta_r(g\sigma_n, g\sigma_n) \\ &\leq c(\delta_r(g\sigma_{n-1}, g\sigma_{n-1}))^\beta (\delta_r(g\sigma_{n-1}, g\sigma_n))^\alpha (\delta_r(g\sigma_{n-1}, g\sigma_n))^{1-\alpha-\beta} \\ &\leq c(\delta_r(g\sigma_{n-1}, g\sigma_{n-1}))^\beta (c^{n-1} \delta_r(g\sigma_0, g\sigma_1))^{1-\beta}. \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow \infty} (M_{r_{Y\sigma_n, g\sigma_n}} - m_{r_{Y\sigma_n, g\sigma_n}}) = 0. \quad (24)$$

As $g\sigma_n \rightarrow g\sigma^*$ and $Y\sigma_n \rightarrow Y\sigma^*$, so relation (24) along with Lemma 1 gives us

$$M_{r_{Y\sigma^*, g\sigma^*}} = m_{r_{Y\sigma^*, g\sigma^*}}. \quad (25)$$

Moreover,

$$\begin{aligned} 0 \leq \delta_r(Y\sigma_n, g\sigma_n) - m_{r_{Y\sigma_n, g\sigma_n}} &\leq \delta_r(Y\sigma_n, g\sigma_n) \\ &\leq c(\delta_r(g\sigma_{n-1}, g\sigma_n))^\beta (\delta_r(g\sigma_n, g\sigma_{n+1}))^\alpha (\delta_r(g\sigma_{n-1}, g\sigma_n))^{1-\alpha-\beta} \\ &\leq c(\delta_r(g\sigma_{n-1}, g\sigma_n))^\beta (c^n \delta_r(g\sigma_0, g\sigma_1))^\alpha (c^{n-1} \delta_r(g\sigma_0, g\sigma_1))^{1-\alpha-\beta}, \end{aligned}$$

or

$$\lim_{n \rightarrow \infty} (\delta_r(Y\sigma_n, g\sigma_n) - m_{r_{Y\sigma_n, g\sigma_n}}) = 0.$$

Now, by Lemma 1, it follows that

$$\delta_r(Y\sigma^*, g\sigma^*) = m_{r_{Y\sigma^*, g\sigma^*}}. \quad (26)$$

By (25) and (26), we get

$$\delta_r(Y\sigma^*, g\sigma^*) = m_{r_{Y\sigma^*, g\sigma^*}} = M_{r_{Y\sigma^*, g\sigma^*}}.$$

Hence, applying (RM_1) , we obtain $Y\sigma^* = g\sigma^*$. Consequently, σ^* is a CP of g and Y . $\square$

Now, let us consider the case $\alpha + \beta > 1$. In this case, we specify that the required interpolative condition should be satisfied on the following set:

$$\{(\sigma, \nu) \in \mathcal{U} \times \mathcal{U} : g\sigma \neq Y\sigma, g\nu \neq Y\nu, \delta_r(g\sigma, g\nu) > 0, \delta_r(g\sigma, Y\sigma) > 0, \delta_r(g\nu, Y\nu) > 0\}.$$

Definition 14. Suppose $(\mathcal{U}, \delta_r)$ is a rectangular m -MS. Suppose $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are two self-mappings. Then, Y is an $(g, \alpha + \beta > 1)$ - δ_r -interpolative CRRC if $\exists c \in [0, 1)$ and $\alpha, \beta \in (0, 1)$, for which $\alpha + \beta > 1$, satisfying

$$\delta_r(Y\sigma, Y\nu) \leq c(\delta_r(g\sigma, g\nu))^\beta (\delta_r(g\sigma, Y\sigma))^\alpha (\delta_r(g\nu, Y\nu))^{\alpha+\beta-1} \quad (27)$$

$$\forall \sigma, \nu \in \mathcal{U}, g\sigma \neq Y\sigma, g\nu \neq Y\nu \text{ and } \delta_r(g\sigma, g\nu), \delta_r(g\sigma, Y\sigma), \delta_r(g\nu, Y\nu) > 0.$$

Theorem 9. Assume $(\mathcal{U}, \delta_r)$ is a complete rectangular m -MS, $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ are self-mappings, $Y\mathcal{U} \subseteq g\mathcal{U}$ and $g\mathcal{U}$ is closed. Suppose $Y : \mathcal{U} \rightarrow \mathcal{U}$ is a $(g, \alpha + \beta > 1)$ - δ_r -interpolative CRRC. If $\exists$ an element $\sigma \in \mathcal{U}$ with the property that $\delta_r(g\sigma, Y\sigma) \leq 1$, then g and Y have a CP inside $\mathcal{U}$.

Proof. Assume $\sigma_0 \in \mathcal{U}$ is a point with the property that $\delta_r(g\sigma_0, Y\sigma_0) \leq 1$. Since $Y\sigma_0 \in g\mathcal{U}$, $\exists \sigma_1 \in \mathcal{U}$ such that $Y\sigma_0 = g\sigma_1$. Also $\sigma_1 \in \mathcal{U}$, $Y\sigma_1 \in g\mathcal{U}$, $\exists \sigma_2 \in \mathcal{U}$ such that $Y\sigma_1 = g\sigma_2$. In the same way, we construct a sequence $\{\sigma_n\} \subseteq \mathcal{U}$ such that $Y\sigma_n = g\sigma_{n+1}$. If $\exists$ an integer n_0 such that $g\sigma_{n_0} = g\sigma_{n_0+1}$, then σ_{n_0} serves as a CP, and the argument is complete. Hence, we suppose that $g\sigma_n \neq g\sigma_{n+1} \forall n \geq 0$. Then, from (27), we obtain

$$\delta_r(g\sigma_1, g\sigma_2) \leq \mathfrak{c}(\delta_r(g\sigma_0, g\sigma_1))^\beta (\delta_r(g\sigma_0, g\sigma_1))^\alpha (\delta_r(g\sigma_1, g\sigma_2))^{\alpha+\beta-1},$$

thus

$$(\delta_r(g\sigma_1, g\sigma_2))^{2-\alpha-\beta} \leq \mathfrak{c}(\delta_r(g\sigma_0, g\sigma_1))^{\alpha+\beta},$$

therefore

$$\delta_r(g\sigma_1, g\sigma_2) \leq \mathfrak{c}^{\frac{1}{2-\alpha-\beta}} (\delta_r(g\sigma_0, g\sigma_1))^{\frac{\alpha+\beta}{2-\alpha-\beta}}, \quad (28)$$

since, $\alpha, \beta \in (0, 1)$, so $\frac{\alpha+\beta}{2-\alpha-\beta} > 1$, thus $\mathfrak{c}^{\frac{1}{2-\alpha-\beta}} \leq \mathfrak{c}$, and $(\delta_r(g\sigma_0, g\sigma_1))^{\frac{\alpha+\beta}{2-\alpha-\beta}} \leq \delta_r(g\sigma_0, g\sigma_1)$. Therefore, (28) becomes

$$\delta_r(g\sigma_1, g\sigma_2) \leq \mathfrak{c}\delta_r(g\sigma_0, g\sigma_1) \leq \mathfrak{c}. \quad (29)$$

Now, again by (27), we get

$$\delta_r(g\sigma_2, g\sigma_3) \leq \mathfrak{c}^2 \delta_r(g\sigma_0, g\sigma_1) \leq \mathfrak{c}^2.$$

Continuing in this way, the following inequality is obtained $\forall n \in \mathbb{N}$:

$$\delta_r(g\sigma_n, g\sigma_{n+1}) \leq \mathfrak{c}^n. \quad (30)$$

Moreover, by (27) and (30), we get

$$\begin{aligned} \delta_r(g\sigma_n, g\sigma_{n+2}) &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_{n+1}))^\beta (\delta_r(g\sigma_{n-1}, g\sigma_n))^\alpha (\delta_r(g\sigma_{n+1}, g\sigma_{n+2}))^{\alpha+\beta-1} \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_{n+1}))^\beta (\mathfrak{c}^{n-1})^\alpha (\mathfrak{c}^{n+1})^{\alpha+\beta-1} \\ &= \mathfrak{c}^{(2\alpha-1)n+(n+1)\beta} (\delta_r(g\sigma_{n-1}, g\sigma_{n+1}))^\beta. \end{aligned} \quad (31)$$

Again, by (27) and (30), we get

$$\begin{aligned} \delta_r(g\sigma_{n-1}, g\sigma_{n+1}) &\leq \mathfrak{c}(\delta_r(g\sigma_{n-2}, g\sigma_n))^\beta (\delta_r(g\sigma_{n-2}, g\sigma_{n-1}))^\alpha (\delta_r(g\sigma_n, g\sigma_{n+1}))^{\alpha+\beta-1} \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n-2}, g\sigma_n))^\beta (\mathfrak{c}^{n-2})^\alpha (\mathfrak{c}^n)^{\alpha+\beta-1} \\ &= \mathfrak{c}^{(2\alpha-1)(n-1)+n\beta} (\delta_r(g\sigma_{n-2}, g\sigma_n))^\beta. \end{aligned} \quad (32)$$

Similarly,

$$\delta_r(g\sigma_{n-2}, g\sigma_n) \leq \mathfrak{c}^{(2\alpha-1)(n-2)+(n-1)\beta} (\delta_r(g\sigma_{n-3}, g\sigma_{n-1}))^\beta. \quad (33)$$

$$\delta_r(g\sigma_{n-3}, g\sigma_{n-1}) \leq \mathfrak{c}^{(2\alpha-1)(n-3)+(n-2)\beta} (\delta_r(g\sigma_{n-4}, g\sigma_{n-2}))^\beta. \quad (34)$$

Now, we use the inequality (32) in (31):

$$\begin{aligned}\delta_r(g\sigma_n, g\sigma_{n+2}) &\leq \mathfrak{c}^{(2\alpha-1)n+(n+1)\beta} \left(\mathfrak{c}^{(2\alpha-1)(n-1)+n\beta} (\delta_r(g\sigma_{n-2}, g\sigma_n))^\beta \right)^\beta \\ &= \mathfrak{c}^{((2\alpha-1)n+(n+1)\beta)+((2\alpha-1)(n-1)+n\beta)\beta} (\delta_r(g\sigma_{n-2}, g\sigma_n))^{\beta^2}.\end{aligned}\quad (35)$$

Substituting inequality (33) into (35) gives

$$\begin{aligned}\delta_r(g\sigma_n, g\sigma_{n+2}) &\leq \mathfrak{c}^{((2\alpha-1)n+(n+1)\beta)+((2\alpha-1)(n-1)+n\beta)\beta} \\ &\quad \left(\mathfrak{c}^{(2\alpha-1)(n-2)+(n-1)\beta} (\delta_r(g\sigma_{n-3}, g\sigma_{n-1}))^\beta \right)^{\beta^2} \\ &= \mathfrak{c}^{((2\alpha-1)n+(n+1)\beta)+((2\alpha-1)(n-1)+n\beta)\beta+((2\alpha-1)(n-2)+(n-1)\beta)\beta^2} \\ &\quad (\delta_r(g\sigma_{n-3}, g\sigma_{n-1}))^{\beta^3}.\end{aligned}\quad (36)$$

Continuing this process repeatedly, we derive the inequality

$$\delta_r(g\sigma_n, g\sigma_{n+2}) \leq \mathfrak{c}^{A_0+A_1\beta+A_2\beta^2+\dots+A_{n-1}\beta^{n-1}} (m(g\sigma_0, g\sigma_2))^{\beta^n}, \quad (37)$$

where

$$A_i = (2\alpha - 1)(n - i) + (n - (i - 1))\beta, \quad i = 0, 1, 2, \dots, n - 1.$$

Further, by simplification, we get

$$\begin{aligned}B &= ((2\alpha - 1)n + (n + 1)\beta) + ((2\alpha - 1)(n - 1) + n\beta)\beta + ((2\alpha - 1)(n - 2) + (n - 1)\beta)\beta^2 \\ &\quad + \dots + ((2\alpha - 1) + 2\beta)\beta^{n-1} \\ &= (2\alpha - 1)((n + n\beta + n\beta^2 + n\beta^3 + \dots + n\beta^{n-1}) - \beta(1 + 2\beta + 3\beta^2 \\ &\quad + 4\beta^3 + \dots + (n - 1)\beta^{n-1})) + \beta((n + 1) + n\beta + (n - 1)\beta^2 + \dots + 2\beta^{n-1}) \\ &= \frac{n(2\alpha + \beta - 1)}{1 - \beta} - \frac{2\beta(1 - \beta^n)(\alpha + \beta - 1)}{(1 - \beta)^2}.\end{aligned}$$

Therefore, it follows that

$$\delta_r(g\sigma_n, g\sigma_{n+2}) \leq \frac{\mathfrak{c}^{\frac{n(2\alpha+\beta-1)}{1-\beta}}}{\mathfrak{c}^{\frac{2\beta(1-\beta^n)(\alpha+\beta-1)}{(1-\beta)^2}}} (\delta_r(g\sigma_0, g\sigma_2))^{\beta^n}. \quad (38)$$

We now establish that $\{g\sigma_n\}$ is a δ_r -Cauchy sequence. The required inequality for

$$\delta_r(g\sigma_n, g\sigma_{n+m}) - m_{r_{g\sigma_n, g\sigma_{n+m}}}$$

is derived analogously to the proof of Theorem 8. The inequalities (30) and (38) are then employed. Consequently, we obtain

$$\delta_r(g\sigma_n, g\sigma_{n+q}) - m_{r_{g\sigma_n, g\sigma_{n+q}}} \leq \frac{\mathfrak{c}^{\frac{n(2\alpha+\beta-1)}{1-\beta}}}{\mathfrak{c}^{\frac{2\beta(1-\beta^n)(\alpha+\beta-1)}{(1-\beta)^2}}} (\delta_r(g\sigma_0, g\sigma_2))^{\beta^n} + \frac{\mathfrak{c}^n}{1 - \mathfrak{c}}.$$

Given that $\mathfrak{c} \in [0, 1)$ and $\beta \in (0, 1)$, we have $\mathfrak{c}^n \rightarrow 0$ and $\beta^n \rightarrow 0$ as $n \rightarrow \infty$. Hence, the right-hand side of the above inequality converges to zero. Also

$$0 \leq M_{r_{g\sigma_n, g\sigma_{n+m}}} - m_{r_{g\sigma_n, g\sigma_{n+m}}} \leq M_{r_{g\sigma_n, g\sigma_{n+m}}}.$$

We may suppose, without loss of generality, that $M_{r_{g\sigma_n, g\sigma_{n+q}}} = \delta_r(g\sigma_{n+q}, g\sigma_{n+q})$, so

$$\begin{aligned} M_{r_{g\sigma_n, g\sigma_{n+q}}} &= \delta_r(g\sigma_{n+q}, g\sigma_{n+q}) \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q-1}))^\beta (\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q}))^\alpha (\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q}))^{\alpha+\beta-1} \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n+q-1}, g\sigma_{n+q-1}))^\beta (\mathfrak{c}^{n+q-1})^{2\alpha+\beta-1}, \end{aligned}$$

when $n, q \rightarrow \infty$, $\mathfrak{c}^{n+q-1} \rightarrow 0$, so

$$\lim_{n, q \rightarrow \infty} (M_{r_{g\sigma_n, g\sigma_{n+q}}} - m_{r_{g\sigma_n, g\sigma_{n+q}}}) = 0.$$

Thus, by Definition 9, the sequence $\{g\sigma_n\}$ is δ_r -Cauchy. Since $(\mathcal{U}, \delta_r)$ is complete, so the sequence $\{g\sigma_n\}$ converges, say $g\sigma_n \rightarrow \bar{\sigma}$. But since $g\mathcal{U}$ is closed, we have $\bar{\sigma} = g\sigma^*$ for some $\sigma^* \in \mathcal{U}$. To establish that σ^* is a CP of Y and g , we first demonstrate that the sequence $\{Y\sigma_n\}$ converges to $Y\sigma^*$. So, we apply the inequality (30) in the following context:

$$\begin{aligned} 0 \leq \delta_r(Y\sigma_n, Y\sigma^*) - m_{r_{Y\sigma_n, Y\sigma^*}} &\leq \delta_r(Y\sigma_n, Y\sigma^*) \\ &\leq \mathfrak{c}(\delta_r(g\sigma_n, g\sigma^*))^\beta (\delta_r(g\sigma_n, g\sigma_{n+1}))^\alpha (\delta_r(g\sigma^*, Y\sigma^*))^{\alpha+\beta-1} \\ &\leq \mathfrak{c}(\delta_r(g\sigma_n, g\sigma^*))^\beta (\mathfrak{c}^n)^\alpha (\delta_r(g\sigma^*, Y\sigma^*))^{\alpha+\beta-1}. \end{aligned}$$

Taking the limit in the above inequality and knowing that $\mathfrak{c}^n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} (\delta_r(Y\sigma_n, Y\sigma^*) - m_{r_{Y\sigma_n, Y\sigma^*}}) = 0.$$

Hence, by Definition 9, the sequence $\{Y\sigma_n\}$ converges to $Y\sigma^*$. Note that

$$0 \leq M_{r_{Y\sigma_n, g\sigma_n}} - m_{r_{Y\sigma_n, g\sigma_n}} \leq M_{r_{Y\sigma_n, g\sigma_n}}.$$

Thus, we may suppose, without restricting generality, that $M_{r_{Y\sigma_n, g\sigma_n}} = \delta_r(g\sigma_n, g\sigma_n)$; it follows that

$$\begin{aligned} 0 \leq M_{r_{Y\sigma_n, g\sigma_n}} - m_{r_{Y\sigma_n, g\sigma_n}} &\leq M_{r_{Y\sigma_n, g\sigma_n}} = \delta_r(g\sigma_n, g\sigma_n) \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_{n-1}))^\beta (\delta_r(g\sigma_{n-1}, g\sigma_n))^\alpha (\delta_r(g\sigma_{n-1}, g\sigma_n))^{\alpha+\beta-1} \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_{n-1}))^\beta (\mathfrak{c}^{n-1})^{2\alpha+\beta-1}. \end{aligned}$$

Applying the limit $n \rightarrow \infty$ to the inequality above yields

$$\lim_{n \rightarrow \infty} (M_{r_{Y\sigma_n, g\sigma_n}} - m_{r_{Y\sigma_n, g\sigma_n}}) = 0. \quad (39)$$

As $g\sigma_n \rightarrow g\sigma^*$ and $Y\sigma_n \rightarrow Y\sigma^*$, so applying Lemma 1 to relation (39) yields $M_{r_{Y\sigma^*, g\sigma^*}} = m_{r_{Y\sigma^*, g\sigma^*}}$. Also

$$\begin{aligned} 0 \leq \delta_r(Y\sigma_n, g\sigma_n) - m_{r_{Y\sigma_n, g\sigma_n}} &\leq \delta_r(Y\sigma_n, g\sigma_n) \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_n))^\beta (\delta_r(g\sigma_n, g\sigma_{n+1}))^\alpha (\delta_r(g\sigma_{n-1}, g\sigma_n))^{\alpha+\beta-1} \\ &\leq \mathfrak{c}(\delta_r(g\sigma_{n-1}, g\sigma_n))^\beta (\mathfrak{c}^n)^\alpha (\mathfrak{c}^{n-1})^{\alpha+\beta-1}. \end{aligned}$$

Applying the limit $n \rightarrow \infty$ to the inequality above yields

$$\lim_{n \rightarrow \infty} (\delta_r(Y\sigma_n, g\sigma_n) - m_{r_{Y\sigma_n, g\sigma_n}}) = 0.$$

By applying Lemma 1, we obtain

$$\delta_r(Y\sigma^*, g\sigma^*) = m_{r_{Y\sigma^*, g\sigma^*}}.$$

It is concluded that

$$\delta_r(Y\sigma^*, g\sigma^*) = m_{r_{Y\sigma^*, g\sigma^*}} = M_{r_{Y\sigma^*, g\sigma^*}}.$$

So, by using (RM_1) , it follows that $Y\sigma^* = g\sigma^*$. Hence, σ^* is a coincidence point of Y and g . The consequences and special cases of the main results are presented in Table 1. $\square$

Table 1. Consequences and special cases of the main results.

Setting/Choice	Consequence
Complete metric structures	Since MSs, rectangular MSs, partial MSs, partial rectangular MSs, and m -MSs are all special cases of rectangular m -MSs, Theorems 5–9 immediately yield the corresponding CP results in each of these complete spaces.
$g = I_{\mathcal{U}}$ in Theorems 5–7	The obtained CP theorems reduce to the interpolative Kannan fixed-point theorems for m -MSs, thereby recovering Theorems 3.2, 3.6, and 3.8 of [13].
$g = I_{\mathcal{U}}$ in Theorems 8 and 9	The obtained CP theorems reduce to the interpolative Ćirić–Reich–Rus fixed-point theorems established in [14].
Overall contribution	Theorems 5–9 provide a unified framework that yields several CP results for various distance structures and generalizes numerous fixed-point results.

4. Examples

Now, we present illustrative examples corresponding to all of the established results. The examples are divided into two parts. In the first part, we provide examples for the g -interpolative KCs, while in the second part, we present examples for the g -interpolative CRRCs. To demonstrate the generality and applicability of the proposed results, the examples are constructed on a variety of underlying structures, including both real and complex domains.

4.1. Examples for g -Interpolative Kannan Contractions

Now, we present illustrative examples for the three cases of the g -interpolative KCs corresponding to $\alpha + \beta = 1$, $\alpha + \beta < 1$, and $\alpha + \beta > 1$.

Example 1 ($\alpha + \beta = 1$). Consider the set $\mathcal{U} = \{a, b, c, d\}$. Define $\delta_r : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ as follows:

$\delta_r(\sigma, \nu)$	a	b	c	d
a	1	2	3	3
b	2	2	4	4
c	3	4	3	8
d	3	4	8	9

It is straightforward to verify that δ_r is rectangular m -metric. Note that

$$\delta_r(d, d) = 9 > \delta_r(d, c) = 8. \quad (40)$$

which contradicts (p2). Hence, δ_r is not a partial metric. Also note that

$$\delta_r(c, d) - m_{r_{c,d}} = 5 > \delta_r(c, a) - m_{r_{c,a}} + \delta_r(a, d) - m_{r_{a,d}} = 4 \quad (41)$$

which contradicts (m4). Hence, δ_r is not an m -metric. Consider the mappings $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ defined by

$$Y = \begin{pmatrix} a & b & c & d \\ a & c & b & a \end{pmatrix}, \quad g = \begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}.$$

Since for $c = \frac{9}{10}$, $\alpha = \frac{1}{2}$, the mapping Y is a $(g, \alpha + \beta = 1)$ - δ_r -interpolative KC; thus, by following the Theorem 5, Y and g have a CP “ a ” because $Y(a) = g(a)$.

Example 2 ($\alpha + \beta < 1$). Let $\mathcal{U} = [1/2, 2]$ and $\delta_r(\sigma, \nu) = \frac{\sigma + \nu}{2}$. Let us define mappings $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ by

$$Y(\sigma) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq \sigma \leq 1 \\ \frac{1}{2}, & \text{if } 1 < \sigma \leq 2, \end{cases} \quad g(\sigma) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq \sigma \leq 1 \\ \sigma^2, & \text{if } 1 < \sigma \leq 2. \end{cases}$$

We show that these functions satisfy the g -interpolative conditions of Theorem 6 for $\alpha = 1/2$, $\beta = 49/100$, and $c = 9/10$. For $\sigma \in (1, 2]$, we have $\delta_r(Y\sigma, Y\nu) = \frac{1/2 + 1/2}{2} = \frac{1}{2}$. On the other hand,

$$c[\delta_r(g\sigma, Y\sigma)]^\alpha [\delta_r(g\nu, Y\nu)]^\beta \geq \frac{9}{10}(3/4)^{1/2}(3/4)^{49/100} \geq 1/2.$$

Thus, by following Theorem 6, the mappings Y and g have CPs that are actually for all $\sigma \in [1/2, 1]$.

Example 3 ($\alpha + \beta > 1$). We consider $\mathcal{U} = [1/2, 2]$ and define

$$\delta_r(\sigma, \nu) = \begin{cases} \sigma, & \sigma = \nu \\ \sigma + \nu, & \sigma \neq \nu. \end{cases}$$

Let us define $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ by

$$Y\sigma = \begin{cases} \sigma, & \sigma \in [1/2, 3/2] \\ 1/\sigma, & \sigma \in (3/2, 2], \end{cases} \quad g\sigma = \begin{cases} \sigma, & \sigma \in [1/2, 3/2] \\ 3/2, & \sigma \in (3/2, 2]. \end{cases}$$

We choose $\alpha = 1/2$, $\beta = 15/16$, and $c = 11/12$. Then, for $\sigma, \nu \in (3/2, 2]$, we have $\delta_r(Y\sigma, Y\nu) = 1/\sigma + 1/\nu \leq 1$. On the other hand,

$$c[\delta_r(Y\sigma, g\sigma)]^\alpha [\delta_r(g\nu, Y\nu)]^\beta \geq (11/12)(\sqrt{2})(2)^{15/16} \geq 1.$$

Thus, the required g -interpolative condition of Theorem 7 holds, thus all the $\sigma \in [1/2, 3/2]$ are CPs of Y and g .

4.2. Examples for g -Interpolative Ćirić–Reich–Rus Contractions

Now, we present illustrative examples for the two cases of the g -interpolative KCs corresponding to $\alpha + \beta < 1$ and $\alpha + \beta > 1$.

Example 4 ($\alpha + \beta < 1$). Consider the set $\mathcal{U} = \{a, b, c, d\}$. Define $\delta_r : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ as follows:

$\delta_r(\sigma, \nu)$	a	b	c	d
a	9	8	4	3
b	8	3	4	3
c	4	4	2	2
d	3	3	2	1

It is straightforward to verify that δ_r is rectangular m -metric. Note that

$$\delta_r(a, a) = 9 > \delta_r(a, d) = 3. \quad (42)$$

which contradicts (p2). Hence, δ_r is not a partial metric. Also note that

$$\delta_r(a, b) - m_{r_{a,b}} = 5 > \delta_r(a, c) - m_{r_{a,c}} + \delta_r(c, b) - m_{r_{c,b}} = 4, \quad (43)$$

which contradicts (m4). Hence, δ_r is not an m -metric. Consider the mappings $Y, g : \mathcal{U} \rightarrow \mathcal{U}$ defined by

$$Y = \begin{pmatrix} a & b & c & d \\ d & c & b & d \end{pmatrix}, \quad g = \begin{pmatrix} a & b & c & d \\ c & b & a & d \end{pmatrix}.$$

Since for $c = \frac{9}{10}$, $\alpha = \beta = \frac{2}{5}$, the mapping Y is a g -interpolative CRRC. Thus, by Theorem 8 it has a CP that is actually at “ d ” because $Y(d) = d = g(d)$.

Example 5 ($\alpha + \beta > 1$). Let $\mathbb{C}$ be the set of all complex numbers and $\mathcal{U}_\theta \subseteq \mathbb{C}$ be as follows:

$$\mathcal{U}_\theta = \{z \in \mathbb{C} : \arg(z) = \theta\} \cup \{0\},$$

here, for $z = \sigma_1 + i\sigma_2$ we have $|z| = \sqrt{\sigma_1^2 + \sigma_2^2}$ and $\theta = \arctan\left(\frac{\sigma_2}{\sigma_1}\right)$; moreover, $\arg(z) = \theta + 2k\pi$ with $k \in \mathbb{Z}$. Consider a rectangular m -metric on $\mathcal{U}_\theta$ that is defined as

$$\delta_r(z, w) = \frac{|z| + |w|}{2}, \quad z, w \in \mathcal{U}_\theta.$$

Further, the mappings $Y, g : \mathcal{U}_\theta \rightarrow \mathcal{U}_\theta$ are defined by

$$Y(z) = \begin{cases} 0, & 0 \leq |z| < 1, \\ -z, & 1 \leq |z| < 2, \\ z, & |z| \geq 2, \end{cases} \quad g(z) = \begin{cases} 0, & |z| = 0, \\ 10i, & 0 < |z| < 1, \\ 2z^3, & |z| \geq 1. \end{cases}$$

Now, for $\alpha = \beta = \frac{3}{4}$, $c = \frac{9}{10}$, we discuss the following cases to verify that Y is a $(g, \alpha + \beta > 1)$ - δ_r -interpolative CRRC mapping.

Case I. If $0 < |z|, |w| < 1$, then $Y(z) = Y(w) = 0$; hence, $\delta_r(Yz, Yw) = 0$, and the inequality holds trivially.

Case II. If $0 < |z| < 1$, $1 \leq |w| < 2$, then $Y(z) = 0$, $Y(w) = -w$, $g(z) = 10i$, $g(w) = 2w^3$. Thus, $\delta_r(Yz, Yw) = \frac{|w|}{2}$, $\delta_r(gz, gw) = 5 + |w|^3$, $\delta_r(gz, Yz) = 5$, and $\delta_r(gw, Yw) = \frac{2|w|^3 + |w|}{2}$. Thus,

$$\begin{aligned} \delta_r(Yz, Yw) &= \frac{|w|}{2} \leq \left(\frac{9}{10}\right) (5 + |w|^3)^{3/4} (5)^{3/4} \left(\frac{2|w|^3 + |w|}{2}\right)^{1/2} \\ &= c(\delta_r(gz, gw))^\beta (\delta_r(gz, Yz))^\alpha \left(\frac{2|w|^3 + |w|}{2}\right)^{\alpha+\beta-1}. \end{aligned}$$

Hence, the required inequality holds.

Case III. If $0 < |z| < 1$, $|w| \geq 2$, then $Y(z) = 0$, $Y(w) = w$, $g(z) = 10i$, $g(w) = 2w^3$. So it is similar to Case II, thus the required inequality holds.

Case IV. If $1 \leq |z|, |w| < 2$, then $Y(z) = -z$, $Y(w) = -w$, $g(z) = 2z^3$, $g(w) = 2w^3$. So, $\delta_r(Yz, Yw) = \frac{|z|+|w|}{2}$, and $\delta_r(gz, gw) = |z|^3 + |w|^3$, $\delta_r(gz, Yz) = \frac{2|z|^3+|z|}{2}$, $\delta_r(gw, Yw) = \frac{2|w|^3+|w|}{2}$. Then, we have

$$\begin{aligned}\delta_r(Yz, Yw) &= \frac{|z|+|w|}{2} \leq \left(\frac{9}{10}\right) \left(|z|^3 + |w|^3\right)^{3/4} \left(\frac{2|z|^3+|z|}{2}\right)^{3/4} \left(\frac{2|w|^3+|w|}{2}\right)^{1/2} \\ &= c(\delta_r(gz, gw))^\beta (\delta_r(gz, Yz))^\alpha (\delta_r(gw, Yw))^{\alpha+\beta-1}.\end{aligned}$$

Case V. If $1 \leq |z| < 2$, $|w| \geq 2$, then all terms behave similarly, with cubic growth in g dominating linear growth in Y ; hence, the inequality holds.

Case VI. If $|z|, |w| \geq 2$, then $Y(z) = z$, $Y(w) = w$, $g(z) = 2z^3$, $g(w) = 2w^3$. Thus, $\delta_r(Yz, Yw) = \frac{|z|+|w|}{2}$, while the right-hand side of the required $(g, \alpha + \beta > 1)$ - δ_r -interpolative condition grows cubically due to g , and since $\alpha + \beta = \frac{3}{2} > 1$, the inequality holds.

Thus, it follows from all the above cases that Y is a $(g, \alpha + \beta > 1)$ - δ_r -interpolative CRRC. Thus, by following Theorem 9, Y and g have a CP which is actually at $z = \mathbf{0} = (0, 0)$.

5. Applications

Now, we present the applications of our main results. In particular, we consider two different cases corresponding to the classes of contractions, namely, KCs and CRRCs. By employing our main results, we establish the existence of solutions to the Fredholm integral equation in each case.

Let us take $\mathcal{U} = C[c, d]$ as the space of all real-valued continuous functions on $[c, d]$, and let us consider the rectangular m -metric $\delta_r(\xi, \eta) = \sup_{t \in [c, d]} (|\xi(t) - \eta(t)| + a)$ for $a \geq 0$. Suppose we want to solve the following Fredholm integral equation:

$$(Y\xi)(t) = f(t) + \int_c^d G(t, s, \xi(s)) ds, \quad t \in [c, d] \quad (44)$$

where $\xi \in \mathcal{U}$ and $G : [c, d] \times [c, d] \times \mathbb{R} \rightarrow \mathbb{R}$. This problem can be reformulated so as to find the CP of the maps Y and g :

$$\begin{aligned}(Y\xi)(t) &= \int_c^d G(t, s, \xi(s)) ds, \quad t \in [c, d] \\ (g\xi)(t) &= \xi(t) - f(t), \quad t \in [c, d].\end{aligned} \quad (45)$$

5.1. Applications of g -Interpolative Kannan Contractions

The following result establishes the existence of a solution to the Fredholm integral Equation (44). The proof essentially relies on our main result, Theorem 5.

Theorem 10. *Let the following conditions hold:*

1. Y and g are self maps on $\mathcal{U}$ such that $Y\mathcal{U} \subseteq g\mathcal{U}$, $g\mathcal{U}$ is closed.
2. For each ξ and η in $\mathcal{U}$, $\exists t^* \in [c, d]$, $\alpha \in (0, 1)$ and $h \in \mathcal{U}$ such that

$$\begin{aligned}|G(t, s, \xi(s)) - G(t, s, \eta(s))| &\leq (|(g\xi)(t^*) - (Y\xi)(t^*)|)^\alpha (|(g\eta)(t^*) - (Y\eta)(t^*)|)^{1-\alpha} |h(s)| \\ &\quad - \frac{a}{d-c}\end{aligned}$$

where the function h is such that it satisfies $\int_c^d |h(s)| ds \leq c$, for some $c \in (0, 1)$.

Then, the Fredholm integral Equation (44) has a solution in $\mathcal{U}$.

Proof. By following the given conditions, we have

$$\begin{aligned}
 |(Y\zeta)(t) - (Y\eta)(t)| &= \left| \int_c^d (G(t, s, \zeta(s)) - G(t, s, \eta(s))) ds \right| \\
 &\leq \int_c^d |(G(t, s, \zeta(s)) - G(t, s, \eta(s)))| ds \\
 &\leq (|(g\zeta)(t^*) - (Y\zeta)(t^*)|)^\alpha (|(g\eta)(t^*) - (Y\eta)(t^*)|)^{1-\alpha} \int_c^d |h(s)| ds \\
 &\quad - \int_c^d \frac{a}{d-c} ds \\
 &\leq (|(g\zeta)(t^*) - (Y\zeta)(t^*)| + a)^\alpha \\
 &\quad (|(g\eta)(t^*) - (Y\eta)(t^*)| + a)^{1-\alpha} \int_c^d |h(s)| ds - a \\
 &\leq c(\delta_r(g\zeta, Y\zeta))^\alpha (\delta_r(g\eta, Y\eta))^{1-\alpha} - a.
 \end{aligned}$$

Thus, we get

$$|(Y\zeta)(t) - (Y\eta)(t)| + a \leq c(\delta_r(g\zeta, Y\zeta))^\alpha (\delta_r(g\eta, Y\eta))^{1-\alpha}.$$

Since the right-hand side of the above inequality is independent of t , so taking the supremum over $t \in [c, d]$ on the left side, we get

$$\delta_r(Y\zeta, Y\eta) \leq c(\delta_r(g\zeta, Y\zeta))^\alpha (\delta_r(g\eta, Y\eta))^{1-\alpha}.$$

Hence, by following Theorem 5, we obtain that the CP of mappings (45) exists, which in turn yields that the solution of the Fredholm integral Equation (44) exists. $\square$

5.2. Applications of g -Interpolative Ćirić–Reich–Rus Contractions

The following result establishes the existence of a solution to the Fredholm integral Equation (44). The proof essentially relies on our result, Theorem 8.

Theorem 11. Let the following conditions hold:

1. Y and g are self maps on $\mathcal{U}$ such that $Y\mathcal{U} \subseteq g\mathcal{U}$, $g\mathcal{U}$ is closed.
2. For each ζ and η in $\mathcal{U}$, $\exists t^* \in [c, d]$, $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ and $h \in \mathcal{U}$, such that

$$\begin{aligned}
 |G(t, s, \zeta(s)) - G(t, s, \eta(s))| &\leq (|(g\zeta)(t^*) - (g\eta)(t^*)|)^\beta (|(g\zeta)(t^*) - (Y\zeta)(t^*)|)^\alpha \\
 &\quad (|(g\eta)(t^*) - (Y\eta)(t^*)|)^{1-\alpha-\beta} |h(s)| - \frac{a}{d-c},
 \end{aligned}$$

where the function h is such that it satisfies $\int_c^d |h(s)| ds \leq c$, for some $c \in (0, 1)$.

Then, the Fredholm integral Equation (44) has a solution in $\mathcal{U}$.

Proof. By following the given conditions, we have

$$\begin{aligned}
 |(Y\zeta)(t) - (Y\eta)(t)| &\leq \int_c^d |(G(t, s, \zeta(s)) - G(t, s, \eta(s)))| ds \\
 &\leq (|(g\zeta)(t^*) - (g\eta)(t^*)|)^\beta (|(g\zeta)(t^*) - (Y\zeta)(t^*)|)^\alpha \\
 &\quad (|(g\eta)(t^*) - (Y\eta)(t^*)|)^{1-\alpha-\beta} \int_c^d |h(s)| ds - \int_c^d \frac{a}{d-c} ds \\
 &\leq c(\delta_r(g\zeta, g\eta))^\beta (\delta_r(g\zeta, Y\zeta))^\alpha (\delta_r(g\eta, Y\eta))^{1-\alpha-\beta} - a.
 \end{aligned}$$

Thus, we get

$$\delta_r(Y\zeta, Y\eta) \leq c(\delta_r(g\zeta, g\eta))^{\beta}(\delta_r(g\zeta, Y\zeta))^{\alpha}(\delta_r(g\eta, Y\eta))^{1-\alpha-\beta}.$$

Hence, by following Theorem 8, it follows that the solution of the Fredholm integral Equation (44) exists. $\square$

Real-Life Applications of Coincidence Points

There are several real-life applications of coincidence points. A simple example is the case of two people deciding on the same meeting time. Suppose that $g(x)$ represents the meeting time suggested by Person A, while $Y(x)$ represents the meeting time suggested by Person B. If, for some x_0 , we have

$$g(x_0) = Y(x_0) = 10 \text{ AM},$$

then x_0 is a coincidence point, since both individuals independently arrive at the same outcome. Similarly, consider two thermometers measuring the temperature of the same environment. If $g(x)$ and $Y(x)$ represent the temperature readings of two different thermometers and, for some x_0 ,

$$g(x_0) = Y(x_0) = 25^{\circ}\text{C},$$

then x_0 is again a coincidence point. These simple examples illustrate that coincidence points naturally arise when two different processes, methods, or decision mechanisms produce the same outcome, which motivates their study in the framework of fixed-point theory.

6. Conclusions

The present work introduces new CP results for g -interpolative KC mappings and g -interpolative CRRC mappings in the setting of rectangular m -MSs. The obtained results extend the existing theory of interpolative contractions by establishing coincidence point criteria under more general contractive conditions in a broader metric framework. Since rectangular m -MSs generalize many different distance structures, the obtained results admit corresponding analogues in each of these settings. Furthermore, by taking the mapping g as the identity mapping, our results immediately reduce to the corresponding fixed-point theorems for interpolative contractions, demonstrating the generality and consistency of the proposed approach.

The validity and applicability of the theoretical results are illustrated through suitable examples, while applications are presented to emphasize the practical significance of the developed CP theorems. Consequently, the results obtained in this paper provide a unified and flexible framework for investigating CP problems in generalized MSs and further enrich the existing literature on interpolative contractive mappings.

Author Contributions: The authors made the following contributions to this work: Conceptualization, S.H.K. and A.R.; methodology, Z.I., A.R. and S.H.K.; formal analysis, Z.I. and A.R.; investigation, Z.I., A.R. and S.H.K.; validation, S.H.K. and A.R.; writing—original draft preparation, Z.I.; writing—review and editing, A.R. and S.H.K.; supervision, S.H.K. and A.R.; visualization, Z.I. and S.H.K. All authors have read and agreed to the published version of the manuscript.

Funding: The funding for this work was arranged by corresponding author Safeer Hussain Khan.

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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