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Transposed Second-Order Symmetry and Second-Order Marginality: An Exact Axiomatic Classification of Shapley-Type Solutions

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Abstract

We study the relationship between mixed second differences of a transferable-utility cooperative game and payoff changes caused by player deletion. Standard second-order symmetry compares two entries in a fixed recipient row of the deletion-effect matrix. We introduce its transposed counterpart, transposed second-order symmetry, which compares two recipients in a fixed deleted-player column. The Shapley value satisfies this property, and the associated payoff comparison admits a positive-kernel representation in terms of mixed second differences. We prove that efficiency, second-order marginality, and transposed second-order symmetry do not by themselves characterize the Shapley value. Instead, they characterize exactly a centered-bias family obtained by adding a player-specific constant and subtracting its average on the active player set; the parameter is unique up to a common additive constant. Adding the zero-game property removes this affine freedom and yields the Shapley value, while normalization only on two-player zero games is already sufficient. We also establish the logical independence of the four axioms in the normalized characterization and clarify the relation between transposed second-order symmetry, standard second-order symmetry, and balanced contributions.

Keywords: cooperative game theory; transferable-utility game; Shapley value; axiomatic characterization; second-order marginality; transposed second-order symmetry; deletion effects

MSC: 91A12

1. Introduction

The Shapley value is one of the central one-point solutions for cooperative games with transferable utility. Its original characterization combines efficiency, symmetry, the null-player property, and additivity [1]. Later characterizations isolate different aspects of the same allocation rule. Young [2] replaces additivity by marginality (or strong monotonicity), thereby making a player's payoff depend only on that player's marginal contributions. Myerson [3] takes a relational approach: the balanced-contributions property requires the payoff effect of deleting player i on player j to equal the reverse effect of deleting j on i .

The interaction between productivity and relational axioms has generated a substantial literature. Differential marginality and its weakenings compare payoff changes across games [4–6], while disjoint productivity provides a complementary restriction on player interactions [7]. Restricted forms of balanced contributions compare selected pairs of



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players [8,9]. Weighted Shapley values arise naturally when symmetry or balance requirements are weakened [10–12].

Casajus [13] introduced a second-order version of Young’s approach. A player’s second-order productivity with respect to another player is represented by a mixed second difference of the coalition function. The corresponding second-order payoff is the change in the latter player’s payoff when the former player leaves. Efficiency, second-order symmetry, and second-order marginality characterize the Shapley value. The framework has subsequently been adapted to the Owen value and the Banzhaf value [14,15]; a second-order average productivity approach to the solidarity value has also been developed [16].

This paper focuses on an orientation distinct from that of standard second-order symmetry. For a solution φ , write

$$R_{ij}^{\varphi}(v) := \varphi_i(v) - \varphi_i(v_{-j}),$$

so that rows of $R^{\varphi}(v)$ are payoff recipients and columns are deleted players. Standard second-order symmetry compares $R_{ki}^{\varphi}(v)$ and $R_{kj}^{\varphi}(v)$: it keeps the recipient k fixed. By contrast, the transposed comparison considers $R_{ik}^{\varphi}(v)$ and $R_{jk}^{\varphi}(v)$, thereby keeping the deleted player k fixed. We refer to this as the transposed orientation; the corresponding formal property, called *transposed second-order symmetry*, is stated in Axiom 7.

The transposition creates a nontrivial normalization issue. Efficiency, second-order marginality, and transposed second-order symmetry do not by themselves characterize the Shapley value. For an arbitrary player-indexed map $a : \mathcal{U} \rightarrow \mathbb{R}$, the rule

$$\varphi_i^a(v) = \text{Sh}_i(v) + a_i - \bar{a}_N, \quad \bar{a}_N := \frac{1}{|N|} \sum_{\ell \in N} a_{\ell},$$

satisfies all three properties. The added term preserves efficiency because it sums to zero; its deletion effect depends on the deleted player but not on the game or the recipient. Hence neither second-order marginality nor the column comparison can detect it.

Our main result shows that this is the only obstruction. More precisely:

- (i) We give Möbius and positive-kernel formulas for transposed comparisons under the Shapley value;
- (ii) We prove that efficiency, second-order marginality, and transposed second-order symmetry characterize exactly the centered-bias family $\{\varphi^a : a \in \mathbb{R}^{\mathcal{U}}\}$;
- (iii) We obtain the Shapley value by imposing the zero-game property, and show that zero normalization on two-player games already suffices;
- (iv) We establish the logical independence of the four axioms in the normalized characterization;
- (v) We relate the new property to standard second-order symmetry through Myerson’s balanced-contributions condition.

The classification result is useful beyond the uniqueness corollary. It identifies precisely what information is and is not contained in a family of second-order deletion comparisons. In particular, it separates interaction allocation from player-specific baseline transfers without assuming linearity, continuity, or anonymity.

The paper is organized as follows. Section 2 introduces the notation and recalls the relevant axioms. Section 3 develops transposed second-order symmetry and its monotone version. Section 4 proves the normalized Shapley characterization. Section 5 gives the exact classification without normalization. Section 6 proves the independence of the axioms and compares the axiom systems. Section 7 concludes.

2. Games, Mixed Differences, and Deletion Effects

Let $\mathcal{U}$ be a countably infinite universe of players, and let

$$\mathcal{N} := \{N \subseteq \mathcal{U} : 1 \leq |N| < \infty\}.$$

For $N \in \mathcal{N}$, a transferable-utility game on N is a function $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$. The vector space of games on N is denoted by $\mathbb{V}(N)$, and $\mathbb{V} := \bigcup_{N \in \mathcal{N}} \mathbb{V}(N)$. For $T \subseteq N$, the restriction of v to T is denoted by $v|_T$. If $i \in N$, we write $v_{-i} := v|_{N \setminus \{i\}}$. The zero game on N is denoted by $\mathbf{0}_N$.

For every nonempty $T \subseteq N$, the unanimity game u_T^N is defined by

$$u_T^N(S) = \begin{cases} 1, & T \subseteq S, \\ 0, & T \not\subseteq S. \end{cases}$$

Every $v \in \mathbb{V}(N)$ has the unique representation

$$v = \sum_{\emptyset \neq T \subseteq N} \lambda_T(v) u_T^N, \quad (1)$$

where the Harsanyi dividends [17] are

$$\lambda_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S). \quad (2)$$

A solution on $\mathbb{V}$ is a mapping φ that assigns a vector $\varphi(v) \in \mathbb{R}^N$ to every $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$. The Shapley value is

$$\text{Sh}_i(v) = \sum_{\substack{\emptyset \neq T \subseteq N \\ i \in T}} \frac{\lambda_T(v)}{|T|} = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S \cup \{i\}) - v(S)). \quad (3)$$

2.1. First- and Second-Order Productivity

Players $i, j \in N$ are symmetric in v if

$$v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$$

for every $S \subseteq N \setminus \{i, j\}$. For distinct $i, j \in N$, define the mixed second difference

$$\Delta_{ij}^v(S) := v(S \cup \{i, j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S), \quad S \subseteq N \setminus \{i, j\}. \quad (4)$$

It measures how the presence of i changes the marginal contribution of j to S . Algebraically,

$$\Delta_{ij}^v(S) = \Delta_{ji}^v(S). \quad (5)$$

For pairwise distinct $i, j, k \in N$, players i and j are second-order symmetric with respect to k in v if

$$\Delta_{ik}^v(S) = \Delta_{jk}^v(S) \quad \text{for every } S \subseteq N \setminus \{i, j, k\}. \quad (6)$$

Note that first-order symmetry of i and j implies (6) for every $k \notin \{i, j\}$.

2.2. Deletion Effects and Axioms

For a solution φ , define the deletion-effect matrix $R^\varphi(v)$ off the diagonal by

$$R_{ij}^\varphi(v) := \varphi_i(v) - \varphi_i(v_{-j}), \quad i, j \in N, \quad i \neq j. \quad (7)$$

Thus $R_{ij}^\varphi(v)$ is the effect of player j on the payoff of recipient i , measured by deleting j . We use the following standard properties.

Axiom 1 (Efficiency, E). For every $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$,

$$\sum_{i \in N} \varphi_i(v) = v(N).$$

Axiom 2 (Zero-game property, Z). For every $N \in \mathcal{N}$ and $i \in N$,

$$\varphi_i(\mathbf{0}_N) = 0.$$

Axiom 3 (Symmetry, S). For every $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$, if i and j are symmetric in v , then $\varphi_i(v) = \varphi_j(v)$.

Axiom 4 (Marginality, M). For every $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $i \in N$, if

$$v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$$

for every $S \subseteq N \setminus \{i\}$, then $\varphi_i(v) = \varphi_i(w)$.

Young's theorem states that E, S, and M characterize the Shapley value [2]. Myerson's relational condition is

$$\text{BC: } R_{ij}^\varphi(v) = R_{ji}^\varphi(v) \quad \text{for every } N \in \mathcal{N}, v \in \mathbb{V}(N), \text{ and all distinct } i, j \in N,$$

and E together with BC also characterizes the Shapley value [3].

The following second-order properties are due to Casajus [13].

Axiom 5 (Second-order symmetry, 2S). For every $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and pairwise distinct $i, j, k \in N$, if i and j are second-order symmetric with respect to k in v , then

$$R_{ki}^\varphi(v) = R_{kj}^\varphi(v).$$

Axiom 6 (Second-order marginality, 2M). For every $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and distinct $i, j \in N$, if

$$\Delta_{ij}^v(S) = \Delta_{ij}^w(S) \quad \text{for every } S \subseteq N \setminus \{i, j\},$$

then

$$R_{ji}^\varphi(v) = R_{ji}^\varphi(w).$$

Because (5) holds and the axiom is quantified over all ordered pairs, replacing the conclusion in 2M by $R_{ij}^\varphi(v) = R_{ij}^\varphi(w)$ does not produce a different property: it is 2M with i and j interchanged.

We shall use the following established implication, proved as Proposition 6 in [13].

Proposition 1. Second-order marginality implies marginality.

3. Transposed Second-Order Symmetry

Standard 2S holds the payoff recipient fixed and compares two deleted players. We now transpose these roles.

The premise remains the standard relation in (6): players i and j are second-order symmetric with respect to k . The adjective *transposed* refers only to the orientation of the

payoff-effect conclusion: k is held fixed as the deleted player, while the effects on recipients i and j are compared.

Axiom 7 (Transposed second-order symmetry, $2S^\top$). For every $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and pairwise distinct $i, j, k \in N$, if i and j are second-order symmetric with respect to k in v , then

$$R_{ik}^\varphi(v) = R_{jk}^\varphi(v). \quad (8)$$

In matrix language, $2S$ compares two entries in row k , whereas $2S^\top$ compares two entries in column k :

property	premise	conclusion
$2S$	$\Delta_{ik}^v = \Delta_{jk}^v$	$R_{ki}^\varphi(v) = R_{kj}^\varphi(v)$
$2S^\top$	$\Delta_{ik}^v = \Delta_{jk}^v$	$R_{ik}^\varphi(v) = R_{jk}^\varphi(v)$
BC	none	$R_{ij}^\varphi(v) = R_{ji}^\varphi(v)$

The equalities in the premise are restricted to coalitions in $N \setminus \{i, j, k\}$ as in (6).

3.1. Möbius Representation

Lemma 1. Let $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and let $i, j, k \in N$ be pairwise distinct. Set $L = N \setminus \{i, j, k\}$. Then

$$\Delta_{ik}^v(S) = \Delta_{jk}^v(S) \quad \text{for every } S \subseteq L$$

if and only if

$$\lambda_{T \cup \{i, k\}}(v) = \lambda_{T \cup \{j, k\}}(v) \quad \text{for every } T \subseteq L. \quad (9)$$

Proof. The Harsanyi representation gives

$$\Delta_{ik}^v(S) = \sum_{T \subseteq S} \lambda_{T \cup \{i, k\}}(v), \quad \Delta_{jk}^v(S) = \sum_{T \subseteq S} \lambda_{T \cup \{j, k\}}(v).$$

Equality of the two cumulative sums for all $S \subseteq L$ is equivalent, by Möbius inversion on 2^L , to equality of their summands for all $T \subseteq L$. $\square$

Proposition 2. The Shapley value satisfies transposed second-order symmetry.

Proof. For distinct $i, j, k \in N$, put $L = N \setminus \{i, j, k\}$. From (3),

$$R_{ik}^{\text{Sh}}(v) = \sum_{T \subseteq L} \left(\frac{\lambda_{T \cup \{i, k\}}(v)}{|T| + 2} + \frac{\lambda_{T \cup \{i, j, k\}}(v)}{|T| + 3} \right), \quad (10)$$

$$R_{jk}^{\text{Sh}}(v) = \sum_{T \subseteq L} \left(\frac{\lambda_{T \cup \{j, k\}}(v)}{|T| + 2} + \frac{\lambda_{T \cup \{i, j, k\}}(v)}{|T| + 3} \right). \quad (11)$$

Under the premise of $2S^\top$, Lemma 1 identifies the first dividends in the two sums; the dividends containing all three players already coincide. Hence (8) follows. $\square$

Proposition 3. On the class of solutions satisfying balanced contributions, second-order symmetry and transposed second-order symmetry are equivalent.

Proof. Under BC,

$$R_{ki}^\varphi(v) = R_{ik}^\varphi(v) \quad \text{and} \quad R_{kj}^\varphi(v) = R_{jk}^\varphi(v).$$

Thus, under the common premise $\Delta_{ik}^v = \Delta_{jk}^v$, the conclusions of $2S$ and $2S^\top$ are identical after applying BC. $\square$

3.2. A Monotone Form

The equality axiom has a natural order-preserving strengthening.

Axiom 8 (Transposed second-order monotonicity, $2\mathbf{Mo}^\top$). For every $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and pairwise distinct $i, j, k \in N$, if

$$\Delta_{ik}^v(S) \geq \Delta_{jk}^v(S) \quad \text{for every } S \subseteq N \setminus \{i, j, k\},$$

then

$$R_{ik}^\varphi(v) \geq R_{jk}^\varphi(v).$$

As usual, applying the inequality in both directions shows that $2\mathbf{Mo}^\top$ implies $2\mathbf{S}^\top$. For the Shapley value, the monotone property follows from a positive-kernel formula.

Proposition 4. Let $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and let $i, j, k \in N$ be pairwise distinct. Set $L = N \setminus \{i, j, k\}$ and define

$$g(S) := \Delta_{ik}^v(S) - \Delta_{jk}^v(S), \quad S \subseteq L.$$

Then

$$R_{ik}^{\text{Sh}}(v) - R_{jk}^{\text{Sh}}(v) = \sum_{S \subseteq L} \frac{(|S| + 1)! (|N| - |S| - 3)!}{(|N| - 1)!} g(S). \quad (12)$$

Consequently, the Shapley value satisfies $2\mathbf{Mo}^\top$.

Proof. Set

$$q(T) = \lambda_{T \cup \{i, k\}}(v) - \lambda_{T \cup \{j, k\}}(v), \quad T \subseteq L.$$

Equations (10) and (11) give

$$R_{ik}^{\text{Sh}}(v) - R_{jk}^{\text{Sh}}(v) = \sum_{T \subseteq L} \frac{q(T)}{|T| + 2}.$$

Moreover, $g(S) = \sum_{T \subseteq S} q(T)$. Möbius inversion and a change in the order of summation show that the coefficient of $g(S)$ is

$$\sum_{T: S \subseteq T \subseteq L} \frac{(-1)^{|T| - |S|}}{|T| + 2}.$$

Writing $m = |L|$ and $s = |S|$, this coefficient equals

$$\sum_{r=0}^{m-s} \binom{m-s}{r} \frac{(-1)^r}{s+r+2} = \int_0^1 x^{s+1} (1-x)^{m-s} dx = \frac{(s+1)!(m-s)!}{(m+2)!}.$$

Since $m = |N| - 3$, this is precisely the coefficient in (12), and it is strictly positive. $\square$

3.3. A Three-Player Illustration

Example 1. Let $N = \{1, 2, 3\}$ and suppose that the nonzero Harsanyi dividends are

$$\lambda_{\{1\}} = 2, \quad \lambda_{\{2\}} = 1, \quad \lambda_{\{1,2\}} = 6, \quad \lambda_{\{1,3\}} = \lambda_{\{2,3\}} = 4, \quad \lambda_N = 3.$$

Then players 1 and 2 are second-order symmetric with respect to player 3, and

$$\text{Sh}(v) = (8, 7, 5).$$

The off-diagonal part of the deletion-effect matrix is

$$R^{\text{Sh}}(v) = \begin{pmatrix} - & 4 & 3 \\ 4 & - & 3 \\ 3 & 3 & - \end{pmatrix}.$$

The equality in column 3 is $2S^\top$; the equality in row 3 is $2S$. The symmetry of the whole off-diagonal matrix is BC .

Now choose a map $a : \mathfrak{U} \rightarrow \mathbb{R}$ with $a_1 = 3$, $a_2 = 0$, and $a_3 = -3$; its values outside N are arbitrary. Define $\varphi_i^a(v) = \text{Sh}_i(v) + a_i - \bar{a}_N$. Then $\varphi^a(v) = (11, 7, 2)$. Direct calculation gives

$$R_{13}^{\varphi^a}(v) = R_{23}^{\varphi^a}(v) = \frac{9}{2},$$

so the transposed equality holds in this game. In contrast,

$$R_{31}^{\varphi^a}(v) = \frac{3}{2} \quad \text{and} \quad R_{32}^{\varphi^a}(v) = 3,$$

so standard $2S$ fails. This example also anticipates the exact family identified in Section 5.

4. The Normalized Characterization

We first prove the Shapley characterization under zero-game normalization. The only delicate point is the two-player case: $2S^\top$ is vacuous on a fixed two-player set, so the countably infinite universe is used to introduce a third player.

Lemma 2 (Two-player unanimity games). *Suppose that φ satisfies E , Z , $2M$, and $2S^\top$. For every distinct $i, k \in \mathfrak{U}$ and every $c \in \mathbb{R}$,*

$$\varphi_i(cu_{\{i,k\}}) = \varphi_k(cu_{\{i,k\}}) = \frac{c}{2}. \quad (13)$$

Proof. Fix three distinct players i, j, k and let $M = \{i, j, k\}$. Consider

$$w = c u_{\{i,k\}}^M + c u_{\{j,k\}}^M.$$

Players i and j are second-order symmetric with respect to k . Since $w_{-k} = \mathbf{0}_{\{i,j\}}$, Z and $2S^\top$ imply

$$\varphi_i(w) = \varphi_j(w). \quad (14)$$

No dividend of w contains both i and j , so $\Delta_{ij}^w = \Delta_{ij}^{\mathbf{0}_M} = 0$. Applying $2M$ with deleted player j and recipient i , and then using Z , gives

$$\varphi_i(w) - \varphi_i(w_{-j}) = \varphi_i(\mathbf{0}_M) - \varphi_i(\mathbf{0}_{M \setminus \{j\}}) = 0.$$

Therefore

$$\varphi_i(w) = \varphi_i(cu_{\{i,k\}}).$$

Similarly,

$$\varphi_j(w) = \varphi_j(cu_{\{j,k\}}).$$

Together with (14), this shows that, for fixed k , the payoff of the other player in a two-player unanimity game is independent of that player's identity. Denote this common payoff by $b_k(c)$. Efficiency on $\{i, k\}$ gives

$$b_k(c) + b_i(c) = c \quad \text{for all distinct } i, k.$$

Using a third player shows that all $b_i(c)$ are equal, and hence each equals $c/2$. $\square$

Proposition 5. Every solution satisfying E, Z, 2M, and $2S^\top$ satisfies symmetry.

Proof. We argue by induction on $n = |N|$. The one-player case is immediate. Let $N = \{i, j\}$ and suppose that i and j are symmetric in v . Put

$$\alpha = v(\{i\}) = v(\{j\}), \quad c = v(N) - 2\alpha.$$

The games v and cu_N^N have the same mixed second difference. By 2M, one-player efficiency, Z, and Lemma 2,

$$\varphi_i(v) - \alpha = \varphi_i(cu_N^N) - \varphi_i(\mathbf{0}_{\{i\}}) = \frac{c}{2}.$$

The same argument applies to j , so $\varphi_i(v) = \varphi_j(v)$.

Now let $n \geq 3$ and assume the claim for smaller player sets. Let i and j be symmetric in $v \in \mathbb{V}(N)$, and choose $k \in N \setminus \{i, j\}$. The players i and j remain symmetric in v_{-k} , so the induction hypothesis gives

$$\varphi_i(v_{-k}) = \varphi_j(v_{-k}).$$

First-order symmetry of i and j also implies that they are second-order symmetric with respect to k . By $2S^\top$,

$$\varphi_i(v) - \varphi_i(v_{-k}) = \varphi_j(v) - \varphi_j(v_{-k}).$$

Combining the two equalities yields $\varphi_i(v) = \varphi_j(v)$. $\square$

Theorem 1 (Normalized transposed-symmetry characterization). The Shapley value is the unique solution on $\mathbb{V}$ satisfying efficiency, the zero-game property, second-order marginality, and transposed second-order symmetry.

Proof. The Shapley value satisfies E and Z. It satisfies 2M by Proposition 5 in [13], and it satisfies $2S^\top$ by Proposition 2.

Conversely, let φ satisfy the four properties. Proposition 1 gives M, and Proposition 5 gives S. Young's characterization by E, M, and S [2] therefore yields $\varphi = \text{Sh}$. $\square$

Corollary 1. The Shapley value is the unique solution satisfying E, Z, 2M, and $2Mo^\top$.

Proof. The Shapley value satisfies $2Mo^\top$ by Proposition 4. Since $2Mo^\top$ implies $2S^\top$, uniqueness follows from Theorem 1. $\square$

5. Exact Classification Without Normalization

The zero-game property in Theorem 1 removes an exact affine degree of freedom. For a map $a : \mathcal{U} \rightarrow \mathbb{R}$, write

$$\bar{a}_N = \frac{1}{|N|} \sum_{\ell \in N} a_\ell$$

and define the *centered-bias Shapley solution*

$$\text{Sh}_i^a(v) := \text{Sh}_i(v) + a_i - \bar{a}_N, \quad v \in \mathbb{V}(N), \quad i \in N. \quad (15)$$

Lemma 3 (Zero-game cocycle). *Let $b_i(N) \in \mathbb{R}$ be given for every $N \in \mathcal{N}$ and $i \in N$. Assume*

$$\sum_{i \in N} b_i(N) = 0, \quad (16)$$

$$b_i(N) - b_i(N \setminus \{k\}) = b_j(N) - b_j(N \setminus \{k\}) \quad (17)$$

whenever $i, j, k \in N$ are pairwise distinct. Then there exists a map $a : \mathcal{U} \rightarrow \mathbb{R}$ such that

$$b_i(N) = a_i - \bar{a}_N \quad \text{for every } N \in \mathcal{N}, i \in N. \quad (18)$$

The map a is unique up to addition of a common constant.

Proof. By (16), $b_i(\{i\}) = 0$. For distinct i, j , put

$$p_{ij} := b_i(\{i, j\}) = -b_j(\{i, j\});$$

thus $p_{ji} = -p_{ij}$. Apply (17) to a triple $\{i, j, k\}$. Deleting k gives

$$b_i(\{i, j, k\}) - b_j(\{i, j, k\}) = 2p_{ij}.$$

The analogous equalities for the other pairs imply

$$p_{ik} = p_{ij} + p_{jk}.$$

Fix $o \in \mathcal{U}$, set $a_o = 0$, and set $a_i = 2p_{io}$ for $i \neq o$. Equation (5) gives

$$2p_{ij} = a_i - a_j \quad \text{for all distinct } i, j.$$

For a set N containing i and j , repeatedly apply (17) to delete players other than i and j . This yields

$$b_i(N) - b_j(N) = b_i(\{i, j\}) - b_j(\{i, j\}) = a_i - a_j.$$

Together with (16), these pairwise differences determine $b_i(N)$ and give (18).

If a' gives the same family b , then applying (18) to every two-player set shows $a_i - a_j = a'_i - a'_j$ for all i, j . Hence $a' - a$ is constant. $\square$

Theorem 2 (Exact classification). *A solution φ on $\mathbb{V}$ satisfies E, 2M, and $2S^\top$ if and only if there exists $a : \mathcal{U} \rightarrow \mathbb{R}$ such that*

$$\varphi_i(v) = \text{Sh}_i(v) + a_i - \bar{a}_N \quad \text{for every } N \in \mathcal{N}, v \in \mathbb{V}(N), i \in N. \quad (19)$$

The parameter a is unique modulo a common additive constant.

Proof. First suppose that φ satisfies E, 2M, and $2S^\top$, and put

$$b_i(N) := \varphi_i(\mathbf{0}_N).$$

Efficiency on zero games gives (16). Since all mixed second differences of a zero game vanish, $2S^\top$ applied to $\mathbf{0}_N$ gives (17). Lemma 3 therefore yields a map a with $b_i(N) = a_i - \bar{a}_N$.

Define

$$\psi_i(v) := \varphi_i(v) - b_i(N), \quad v \in \mathbb{V}(N).$$

The solution ψ satisfies E because the $b_i(N)$ sum to zero, and it satisfies Z by construction. It retains 2M because

$$R_{ji}^\psi(v) = R_{ji}^\varphi(v) - (b_j(N) - b_j(N \setminus \{i\})),$$

where the subtracted term is independent of the game. It also retains $2S^\top$: the two bias differences subtracted from $R_{ik}^\varphi(v)$ and $R_{jk}^\varphi(v)$ are equal by (17). Theorem 1 gives $\psi = \text{Sh}$, which proves (19). Uniqueness modulo constants follows from Lemma 3.

Conversely, fix $a : \mathcal{U} \rightarrow \mathbb{R}$ and let $\varphi = \text{Sh}^a$. Efficiency is immediate because

$$\sum_{i \in N} (a_i - \bar{a}_N) = 0.$$

For distinct $i, k \in N$,

$$R_{ik}^{\text{Sh}^a}(v) = R_{ik}^{\text{Sh}}(v) + \bar{a}_{N \setminus \{k\}} - \bar{a}_N. \quad (20)$$

The additional term is independent of v , so 2M is inherited from the Shapley value. It is also independent of the recipient i , so $2S^\top$ is inherited from Proposition 2. $\square$

Corollary 2 (Two-player normalization is sufficient). *In Theorem 1, the zero-game property can be replaced by the strictly weaker condition*

$$Z_2: \quad \varphi_i(\mathbf{0}_{\{i,j\}}) = 0 \quad \text{for all distinct } i, j \in \mathcal{U}.$$

Proof. By Theorem 2,

$$\varphi_i(\mathbf{0}_{\{i,j\}}) = \frac{a_i - a_j}{2}.$$

Thus Z_2 forces $a_i = a_j$ for all players. The centered term in (19) then vanishes on every player set. $\square$

Corollary 3 (Equality and order preservation). *Within the class of efficient solutions satisfying 2M, transposed second-order symmetry and transposed second-order monotonicity are equivalent.*

Proof. The implication $2Mo^\top \Rightarrow 2S^\top$ is immediate. For the converse, Theorem 2 shows that every solution satisfying E, 2M, and $2S^\top$ is of the form Sh^a . Equation (20) shows that the centered term cancels from every transposed comparison. Proposition 4 then gives $2Mo^\top$. $\square$

Remark 1 (Interpretation of the affine freedom). *The term $a_i - \bar{a}_N$ is a player-specific baseline transfer, recentered so that the active population receives no net transfer. When player k leaves, the induced change in this transfer is $\bar{a}_{N \setminus \{k\}} - \bar{a}_N$ for every remaining recipient. It is therefore invisible to both game-to-game comparisons in 2M and recipient comparisons in column k under $2S^\top$. The zero-game property removes exactly this undetectable component.*

6. Logical Independence and Comparison

Theorem 3. *The four properties E, Z, 2M, and $2S^\top$ in Theorem 1 are logically independent.*

Proof. We give one solution for each omitted property.

E cannot be omitted. The zero solution, $\varphi_i(v) = 0$ for all games and players, satisfies Z, 2M, and $2S^\top$, but it is not efficient.

2M cannot be omitted. The equal-division solution

$$\text{ED}_i(v) := \frac{v(N)}{|N|}$$

satisfies E and Z. For a fixed deleted player k ,

$$R_{ik}^{\text{ED}}(v) = \frac{v(N)}{|N|} - \frac{v(N \setminus \{k\})}{|N| - 1},$$

which is independent of the recipient i ; hence ED satisfies $2S^\top$. It fails 2M. On $N = \{i, j\}$, the games $v = \mathbf{0}_N$ and $w = u_{\{i\}}^N$ have identical mixed second differences, but

$$R_{ji}^{\text{ED}}(v) = 0 \quad \text{and} \quad R_{ji}^{\text{ED}}(w) = \frac{1}{2}.$$

$2S^\top$ cannot be omitted. Fix positive player weights $p = (p_i)_{i \in \mathcal{U}}$ that are not all equal, and define the weighted dividend solution

$$W_i^p(v) := \sum_{\substack{\emptyset \neq T \subseteq N \\ i \in T}} \frac{p_i}{\sum_{\ell \in T} p_\ell} \lambda_T(v). \quad (21)$$

It satisfies E and Z. Moreover,

$$R_{ji}^{W^p}(v) = \sum_{T \supseteq \{i, j\}} \frac{p_j}{\sum_{\ell \in T} p_\ell} \lambda_T(v),$$

and

$$\Delta_{ij}^v(S) = \sum_{Q \subseteq S} \lambda_{Q \cup \{i, j\}}(v).$$

Thus Möbius inversion shows that equality of the mixed-difference profiles is equivalent to equality of all dividends containing $\{i, j\}$, and hence W^p satisfies 2M. Choose pairwise distinct i, j, k with $p_i \neq p_j$ and let

$$v = u_{\{i, k\}}^{\{i, j, k\}} + u_{\{j, k\}}^{\{i, j, k\}}.$$

Then i and j are second-order symmetric with respect to k , but

$$R_{ik}^{W^p}(v) = \frac{p_i}{p_i + p_k} \neq \frac{p_j}{p_j + p_k} = R_{jk}^{W^p}(v).$$

Z cannot be omitted. Choose a nonconstant $a : \mathcal{U} \rightarrow \mathbb{R}$. By Theorem 2, Sh^a satisfies E, 2M, and $2S^\top$. On a two-player zero game,

$$\text{Sh}_i^a(\mathbf{0}_{\{i, j\}}) = \frac{a_i - a_j}{2},$$

so Z fails for some pair. $\square$

Comparison with the Standard Second-Order Characterization

The two characterizations use the same notions of efficiency and second-order marginality but impose different orientations of symmetry. Casajus's system E+2M+2S directly removes player-specific zero-game biases: efficiency and $2S$ already imply first-order symmetry, as shown in Proposition 4 of [14]. By contrast, $2S^\top$ preserves the centered-bias family, so a normalization condition is necessary.

This distinction can be summarized as follows:

axiom system	identified class	extra normalization
$E + 2M + 2S$	$\{Sh\}$	none
$E + 2M + 2S^T$	$\{Sh^a : a \in \mathbb{R}^U / \mathbb{R}1\}$	Z_2 (or Z)

Under balanced contributions, Proposition 3 makes $2S$ and $2S^T$ equivalent. Without BC, the transposition is substantive; Example 1 provides an efficient solution satisfying $2M$ and $2S^T$, but not $2S$.

7. Conclusions

Transposed second-order symmetry compares how the deletion of one player changes the payoffs of two recipients whose second-order productivities relative to that player are identical. It is the column counterpart of standard second-order symmetry in the deletion-effect matrix. The Shapley value satisfies both orientations, and its transposed comparisons are weighted sums, with strictly positive coefficients, of differences in second-order productivity.

The principal structural conclusion is that transposed comparisons leave a precisely describable affine freedom. Efficiency, second-order marginality, and transposed second-order symmetry identify the Shapley value up to a centered player-specific bias, and no other deviations are possible. Zero normalization on two-player games removes that bias and recovers the Shapley value. The result therefore distinguishes two roles often combined in axiomatic characterizations: allocation of coalition interactions and normalization of player-specific baseline transfers.

One possible application lies in finite, coalition-based representations of evolutionary games. Recent work shows that the evolution of cooperation in multiplayer public-goods games depends on collective-reputation criteria, while in structured donation games asymmetric responses to different cooperative partners can change the long-run strategy composition [18,19]. These models use evolutionary payoff dynamics rather than Shapley allocation, so the present theorems do not apply to them directly. If, however, expected group performance at a fixed population state is used to define coalition worth, mixed second differences measure how one participant changes another's marginal productivity, and deletion effects record how removing a participant changes the allocations of those who remain. In such a representation, transposed second-order symmetry is an equity requirement for recipients with identical second-order productivity relative to the deleted participant. The classification theorem then shows that this comparison does not by itself identify the allocation: player-specific baseline offsets remain unless a zero-game normalization is imposed. Constructing and analyzing such cooperative representations of evolutionary public-goods and donation games is left for future work.

Several extensions require separate analysis. On restricted player domains, the cocycle argument may leave one bias parameter per connected component of the comparison graph. For coalition-structure games, a transposed condition can be imposed both within and across components, but the corresponding normalization space need not be one-dimensional. These questions are not consequences of the present theorem and are left for future work.

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