

Article

On Localized Baer-Type Rings via Idempotent Elements Under Structural Extensions

Awn Alqahtani ^{1,2,*}  and Eltiyeb Ali ^{1,3} 

¹ Department of Mathematics, College of Science and Arts, Najran University, Najran 66462, Saudi Arabia

² Science and Engineering Research Center, Najran University, Najran 66462, Saudi Arabia

³ Department of Mathematics, Faculty of Education, University of Khartoum, Khartoum 321, Sudan

* Correspondence: odalqahtani@nu.edu.sa

Abstract

In this paper, we introduce and systematically investigate several classes of localized Baer-type rings associated with a fixed idempotent element e , namely e -Baer, e -quasi-Baer, e - $p.q$ -Baer, e -PP, and e -APP rings. We develop a unified structural framework relating these localized annihilator classes to weak localized zero-product conditions, including weak e -symmetric, weak e -reversible, weak e -reflexive, and weak e -semicommutative properties. We establish the principal implication relations among these classes and, under suitable hypotheses, obtain equivalence results connecting several localized and global conditions. Examples and counterexamples are provided to distinguish the classes and to demonstrate the failure of converse implications in general. We further study the behavior of localized annihilator conditions under various algebraic constructions and infinite operations. In particular, we establish permanence and transfer results for left e -APP rings under matrix, polynomial, Laurent polynomial, monoid, skew monoid, and skew polynomial extensions, as well as under Morita equivalence. For formal power series extensions, where the left APP property is not preserved in general, we obtain two sufficient conditions on the corner ring eRe that ensure preservation of the left e -APP property. We also establish structural equivalences between left e -APP and left e - $p.q$ -Baer rings under suitable finiteness assumptions. These results provide a systematic connection between classical annihilator theory and its localized counterpart and extend the study of localized regularity and Baer-type properties in ring theory.

Keywords: localized Baer-type rings; weak e -regularity; APP-type extensions; Morita invariance

MSC: 16D70; 16N60; 16D90; 16S36; 16S50



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1. Introduction

Throughout this paper, all rings are assumed to be associative with identity unless explicitly stated otherwise. For a ring R , we denote by $M_n(R)$ the ring of $n \times n$ matrices over R , by $T_n(R)$ its subring of upper triangular matrices, by $E(R)$ (respectively, $E(M_n(R))$) the set of all idempotents in R (respectively, $M_n(R)$), and by $Z(R)$ the center of R . Rings characterized by reflexive, reversible, and symmetric properties have long occupied a central position in noncommutative ring theory. Their extensive study is primarily motivated by their fundamental roles in decomposition theory, idempotent analysis, Baer- and Armendariz-type structures, skew polynomial extensions, and matrix ring constructions [1–6]. Historically,

these classical conditions were introduced to control the behavior of annihilator ideals and semiprime structures. A ring R is called reflexive if $aRb = 0$ implies $bRa = 0$, semicommutative if $ab = 0$ implies $aRb = 0$, reversible if $ab = 0$ implies $ba = 0$, and symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$ [7–11].

In recent years, increasing attention has been devoted to localizing these classical global symmetries relative to a fixed idempotent $e \in E(R)$. This localized framework, encompassing e -reflexive, e -reversible, e -symmetric, and e -semicommutative rings, was introduced to capture finer structural information about the corner subring eRe while relaxing global commutativity requirements [12–17]. Explicitly, for a fixed $e \in E(R)$, a ring R is called e -reversible if $ab = 0$ implies $bae = 0$, e -reflexive if $aRb = 0$ implies $bRae = 0$, and e -semicommutative if $ab = 0$ implies $aRbe = 0$. These localized notions satisfy the hierarchy

$$e\text{-symmetric} \implies e\text{-reversible} \implies e\text{-reflexive} \implies e\text{-semicommutative}.$$

a parallel and closely related direction, the study of annihilator conditions has developed significantly since Kaplansky [18] introduced Baer rings to abstract key operator-theoretic properties of AW^* -algebras and von Neumann algebras. Seeking to characterize finite-dimensional algebras over algebraically closed fields that are isomorphic to twisted matrix units semigroup algebras, Clark [19] introduced the notion of quasi-Baer rings. Subsequently, Birkenmeier, Kim, and Park [20] introduced principally quasi-Baer (or left $p.q.$ -Baer) rings, in which the left annihilator of every principal left ideal is generated by an idempotent. The study of annihilator conditions was further extended to the broader class of left APP-rings, in which the left annihilator $\ell_R(Ra)$ to be right s -unital for every $a \in R$ [21–24].

While these classical annihilator conditions and their global extensions have been extensively studied, a systematic and unified framework that connects e -localized symmetries with e -localized annihilator structures, such as e -Baer, left e - $p.q.$ -Baer, right e -PP, and left e -APP rings, remains largely unexplored. Relating global ring-theoretic properties to their corner subring analogues in eRe reveals fundamental structural relationships and sharp boundaries, particularly concerning their behavior under infinite constructions and extension processes. The primary objective of this paper is to address this gap by establishing a comprehensive hierarchy of localized Baer-type conditions, identifying precise structural relationships among these classes, and investigating their stability under various standard ring-theoretic constructions.

The principal contributions of this paper are summarized as follows.

- We introduce and systematically investigate a family of localized annihilator classes associated with an idempotent e , including e -Baer, e -quasi-Baer, e - $p.q.$ -Baer, e -PP, generalized e -PP, and left e -APP rings. This provides a unified localization of the classical Baer-type hierarchy and establishes the implication relations among these classes.
- We develop the structural theory of these localized annihilator classes by establishing new connections with localized zero-product conditions. Under suitable hypotheses, localized Baer-type properties imply various weak localized symmetry conditions and yield equivalence results among several localized symmetry classes. We also construct explicit counterexamples showing that the corresponding converse implications fail in general.
- We establish permanence results for localized annihilator classes under arbitrary direct products and direct sums. The key mechanism is provided by the corner ring eRe . In particular, for an infinite direct sum, the idempotent e has finite support, so that eRe reduces to a finite direct sum of corner rings. This observation allows the localized

theory to overcome the infinite-coordinate obstruction that occurs in the corresponding classical setting.

- We obtain transfer and preservation results for localized annihilator classes under several fundamental ring constructions, including corner rings, matrix rings, upper triangular matrix rings, endomorphism rings, Morita equivalence, polynomial rings, Laurent polynomial rings, monoid rings, skew monoid rings, skew polynomial rings, and formal power series rings. These results extend and localize several classical transfer theorems from the literature.
- In the formal power series setting, where the left APP property is not preserved in general, we establish two distinct sufficient conditions on the corner ring eRe that ensure the transfer of the left e -APP property from R to $R[[x]]$. The first criterion is based on the descending chain condition on both left and right annihilators, while the second is based on the reduced left duo $\aleph_0$ -self-injective structure of eRe . These criteria provide two complementary mechanisms for overcoming the failure of APP-preservation in formal power series extensions.
- We prove structural equivalence theorems connecting localized Baer-type properties with skew and generalized power series constructions. In particular, under appropriate finiteness conditions, we establish the equivalence between left e -APP and left e - $p.q.$ -Baer rings, thereby identifying conditions under which these two localized annihilator properties coincide.
- We provide examples and counterexamples that establish the sharpness of the hypotheses, distinguish the localized annihilator classes introduced in the paper, and demonstrate the failure of converse implications whenever such converses do not hold in general.

The remainder of this paper is organized as follows.

Section 2 presents the necessary preliminaries and introduces the localized Baer-type classes considered throughout the paper. Section 3 develops the main structural framework by establishing relationships between localized annihilator classes and localized zero-product conditions, together with characterization and equivalence results. Section 4 presents examples and counterexamples that distinguish the localized classes, establish the sharpness of the hypotheses, and demonstrate the failure of converse implications. Section 5 investigates the behavior of the localized annihilator classes under various algebraic constructions, including corner rings, matrix and triangular matrix rings, endomorphism rings, polynomial and Laurent polynomial extensions, monoid and skew monoid rings, skew polynomial rings, generalized power series rings, formal power series rings, and Morita equivalence. Particular attention is given to preservation criteria for the left e -APP property in formal power series extensions and to structural equivalences between left e -APP and left e - $p.q.$ -Baer rings under suitable finiteness conditions. Finally, Section 6 concludes with remarks and directions for future research.

2. Preliminaries and Background

This section collects the definitions, notation, and preliminary results used throughout the paper. For clarity of exposition, it is divided into two subsections. The first develops the localized annihilator framework by introducing the classes of e -Baer, e -quasi-Baer, e - $p.q.$ -Baer, e -PP, and left e -APP rings associated with a fixed idempotent $e \in E(R)$, together with their fundamental properties. The second recalls the localized zero-product conditions associated with idempotent elements, which serve as the principal tools for establishing the structural, transfer, and equivalence results developed in the subsequent sections.

2.1. Localized Baer-Type Rings

In this subsection, we develop the localized Baer-type framework by introducing the classes of e -Baer, e -quasi-Baer, e -p.q.-Baer, e -PP, and left e -APP rings. These notions extend the corresponding classical annihilator conditions to a setting determined by a fixed idempotent and provide the structural foundation for the transfer, preservation, and equivalence results established in the subsequent sections.

Definition 1. For a ring R with an idempotent $e \in E(R)$, we say that R is an e -Baer ring if, for every nonempty subset $X \subseteq R$, there exists an idempotent $\phi \in E(eRe)$ such that $\ell_{eRe}(X) = eRe\phi$, where $\ell_{eRe}(X) = \{\lambda \in eRe \mid \lambda x = 0 \text{ for all } x \in X\}$.

Definition 2. Let R be a ring and let $e \in E(R)$. We say that R is an e -quasi-Baer ring if, for every ideal $I \trianglelefteq R$, there exists an idempotent $\phi \in E(eRe)$ such that $\ell_{eRe}(I) = eRe\phi$, where $\ell_{eRe}(I) = \{\lambda \in eRe \mid \lambda a = 0 \text{ for every } a \in I\}$.

Definition 3. Let R be a ring and let $e \in E(R)$. We say that R is an e -p.q.-Baer ring if, for every $b \in R$, there exists an idempotent $\phi \in E(eRe)$ such that $\ell_{eRe}(Rb) = eRe\phi$, where $\ell_{eRe}(Rb) = \{\lambda \in eRe \mid \lambda rb = 0 \text{ for every } r \in R\}$.

Definition 4. Let R be a ring and let $e \in E(R)$. We say that R is an e -PP ring if, for every $b \in R$, there exists an idempotent $\phi \in E(eRe)$ such that $\ell_{eRe}(b) = eRe\phi$, where $\ell_{eRe}(b) = \{\lambda \in eRe \mid \lambda b = 0\}$.

Definition 5. Let R be a ring and let $e \in E(R)$. We say that R is a left e -APP ring if, for every $b \in R$, the localized left annihilator $\ell_{eRe}(Rb) = \{\lambda \in eRe \mid \lambda rb = 0 \text{ for every } r \in R\}$ is a right s -unital ideal of eRe .

Definition 6. A ring R is called a left APP-ring if the left annihilator of every principal left ideal is right s -unital; that is, for each $d \in R$, the left annihilator $l_R(Rd) = \{r \in R \mid rRd = 0\}$ is a right s -unital ideal of R [25].

Definition 7. Let R be a ring and let $e \in E(R)$. The ring R is called a generalized right e -PP ring if, for every $b \in R$, there exist a positive integer $n = n(b)$ and an idempotent $\phi \in E(eRe)$ such that $\ell_{eRe}(b^n) = eRe\phi$, where $\ell_{eRe}(b^n) = \{\lambda \in eRe \mid \lambda b^n = 0\}$.

Baer-type and generalized annihilator conditions have been extensively investigated (see, e.g., [3,10,20,24,26,27]).

2.2. Localized Zero-Product Conditions and Annihilator Properties

This subsection reviews the localized zero-product conditions and annihilator-related notions that form the foundation for the subsequent development of the paper. We first recall the weak localized symmetry-type conditions introduced in the literature, including weak e -symmetric, weak e -reversible, weak e -reflexive, and weak e -semicommutative rings. We then introduce the notions of left and right semicentral idempotents, followed by the definition of e -symmetric rings. These concepts play a central role in establishing the relationships between localized Baer-type conditions and localized zero-product properties developed in the following sections.

Definition 8. Following the terminology of [28], a ring R is called weak e -symmetric for some $e \in E(R)$ if, for all $a, b, c \in R$, the equality $abc = 0$ implies $each e = 0$.

Definition 9. Following [29], a ring R is called weak e -reversible for some $e \in E(R)$ if, for any $a, b \in R$, the condition $ab = 0$ implies $ebae = 0$.

Definition 10. Let R be a ring and $e \in E(R)$. The ring R is called weak e -reflexive if for all $a, b \in R$, $aRb = 0 \Rightarrow ebRae = 0$ [30].

Definition 11. According to [30], a ring R is called weak e -semicommutative for some $e \in E(R)$ if $db = 0$ implies $edRbe = 0$ for all $d, b \in R$.

Definition 12. Following the definition established in [16], an idempotent $e \in E(R)$ is called left semicentral if $de = ede$ for all $d \in R$, which is equivalent to the condition $(1 - e)Re = 0$. Dually, e is designated as right semicentral if $ex = exe$ for all $x \in R$, a condition equivalently expressed by the structural constraint $eR(1 - e) = 0$.

Definition 13. Let R be a ring and $e \in E(R)$. A ring R is called e -symmetric if $abc = 0$ implies $acbe = 0$ for all $a, b, c \in R$ [16].

3. Baer-Type Hierarchies and Unification

This section investigates the structural relationships between the localized Baer-type properties of a ring R and the corresponding properties of the corner ring eRe . The additional condition $eR \subseteq Z(R)$ is fundamental throughout the section: it ensures that eRe is commutative and establishes a precise correspondence between localized annihilators in R and annihilators in eRe . Under this hypothesis, we establish the transfer of the Baer, quasi-Baer, $p.q.$ -Baer, left PP , and left APP properties of R to their localized counterparts. We further relate the left e - APP property to weak e -reflexivity and, in the e -Baer setting, obtain an equivalence between the weak localized symmetry-type properties of R and the corresponding classical properties of eRe . Finally, we describe the behavior of the localized classes under direct products and direct sums, providing a structural basis for the counterexamples to the converse implications considered later.

The relationships among these properties are summarized in Figure 1, which illustrates the implication hierarchy connecting the classical Baer-type conditions, their localized counterparts, and the weak e -regularity properties.

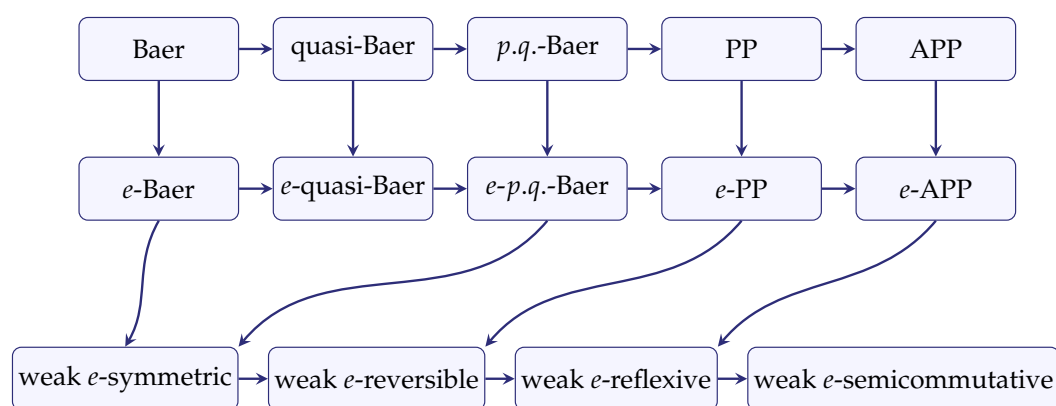


Figure 1. The implication hierarchy among classical Baer-type conditions, their localized counterparts, and weak e -regularity conditions.

We begin with the basic properties of the corner ring $S = eRe$ under the centrality hypothesis.

Lemma 1. Let R be a ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Let $S = eRe$. Then:

- (1) S is a commutative ring with identity e ;

- (2) for every $b \in R$, $\ell_S(Rb) = \ell_S(b) = \ell_S(Sebe)$;
 (3) R is a left e -APP ring if and only if S is a left APP ring.

Proof. (1) Since $S = eRe \subseteq eR \subseteq Z(R)$, every element of S is central in R . Thus S is commutative, and e is its identity.

(2) Let $b \in R$ and $\lambda \in S$. Clearly, $\lambda Rb = 0 \implies \lambda b = 0$.

Conversely, if $\lambda b = 0$, then $\lambda \in Z(R)$ by (1), and hence $\lambda rb = r\lambda b = r(0) = 0$ for all $r \in R$. Thus, $\ell_S(Rb) = \ell_S(b)$. Furthermore, since $\lambda = e\lambda$ and $e \in Z(R)$,

$$\lambda b = \lambda eb = \lambda ebe.$$

As $ebe \in S$ and S is commutative, $\lambda ebe = 0 \iff \lambda S(ebe) = 0$. Therefore, $\ell_S(Rb) = \ell_S(b) = \ell_S(Sebe)$.

(3) Suppose that R is a left e -APP ring. By Definition 5, $\ell_S(Rb)$ is right s -unital for every $b \in R$. If $a \in S$, then, by (2), $\ell_S(Sa) = \ell_S(Ra)$, and hence $\ell_S(Sa)$ is right s -unital. Thus S is a left APP ring.

Conversely, suppose that S is a left APP ring. For every $b \in R$, the ideal $\ell_S(Sebe)$ is right s -unital. By (2), $\ell_S(Rb) = \ell_S(Sebe)$, so $\ell_S(Rb)$ is right s -unital for every $b \in R$. Hence, by Definition 5, R is a left e -APP ring. $\square$

The following results develop the reduction principle established above for localized annihilator conditions. In particular, they establish the transfer of the Baer, quasi-Baer, $p.q$ -Baer, PP , and APP properties to their localized counterparts. In general, none of the corresponding converse implications holds, as demonstrated by the counterexamples in Section 4. These transfer results provide the foundation for the subsequent permanence and equivalence results.

Theorem 1. Let R be an associative ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Then the following implications hold:

- (1) If R is a Baer ring, then R is an e -Baer ring.
- (2) If R is a quasi-Baer ring, then R is an e -quasi-Baer ring.
- (3) If R is a $p.q$ -Baer ring, then R is an e - $p.q$ -Baer ring.
- (4) If R is a left PP -ring, then R is an e - PP ring.
- (5) If R is a left APP-ring, then R is an e -APP ring.

Proof. Let $S = eRe$. By Lemma 1, S is a commutative ring with identity e , and the localized annihilators in R agree with the corresponding annihilators in S . We shall use these facts throughout the proof.

- (1) Baer $\implies$ e -Baer.

Assume that R is a Baer ring. Let $X \subseteq R$. Then there exists an idempotent $p \in E(R)$ such that $\ell_R(X) = Rp$. Since $S = eR$, we have $\ell_S(X) = \ell_R(X) \cap S = Rp \cap eR$. We claim that $Rp \cap eR = eRp$. Indeed, if $x \in Rp \cap eR$, then $x = ex$, and since $x = rp$ for some $r \in R$, $x = ex = erp \in eRp$. The reverse inclusion is immediate. Hence $\ell_S(X) = eRp$. Put $\phi = ep$. Since $e \in Z(R)$ and $e^2 = e$, $p^2 = p$, we have $\phi^2 = (ep)^2 = epep = e^2p^2 = ep = \phi$, so that $\phi \in E(S)$. Moreover, $eRp = eR(ep) = S\phi$. Consequently, $\ell_S(X) = S\phi$, and hence R is an e -Baer ring.

- (2) quasi-Baer $\implies$ e -quasi-Baer.

Assume that R is a quasi-Baer ring, and let $I \trianglelefteq eRe$ be an arbitrary two-sided ideal of eRe . Since $eR \subseteq Z(R)$, we have $e \in Z(R)$. Hence $eR = eRe = Re$, and eRe is a commutative ring with identity e .

We first claim that $I \trianglelefteq R$. Indeed, since $I \subseteq eRe = eR$, for every $r \in R$ and $x \in I$, $rx = r(ex) = (re)x = (er)x \in I$, because $er \in eR = eRe$ and $I \trianglelefteq eRe$. Similarly, $xr = x(er) \in I$, since $er \in eRe$. Thus $I \trianglelefteq R$. Since R is quasi-Baer, there exists an idempotent $p \in E(R)$ such that $\ell_R(I) = Rp$. For $\lambda \in eRe$, we have $\lambda \in \ell_{eRe}(I) \iff \lambda I = 0 \iff \lambda \in \ell_R(I)$. Consequently, $\ell_{eRe}(I) = \ell_R(I) \cap eRe = Rp \cap eRe$. We claim that $Rp \cap eRe = eRp$. If $x \in Rp \cap eRe$, then $x = ex$, and writing $x = rp$ for some $r \in R$, we obtain $x = ex = e(rp) = (er)p \in eRp$.

Conversely, if $x \in eRp$, say $x = erp$, then $x \in Rp$, and $ex = e(erp) = erp = x$. Hence $x \in eRe$, and therefore $Rp \cap eRe = eRp$. Now put $\phi = ep$. Since $e \in Z(R)$ and e, p are idempotents, $\phi^2 = (ep)^2 = epep = e^2p^2 = ep = \phi$. Moreover, $\phi = ep \in eRe$, so $\phi \in E(eRe)$. Finally, $eRp = (eR)(ep) = eRe\phi$. Therefore, $\ell_{eRe}(I) = Rp \cap eRe = eRp = eRe\phi$, $\phi \in E(eRe)$. Thus R is an e -quasi-Baer ring.

(3) $p.q$ -Baer $\implies e$ - $p.q$ -Baer.

Assume that R is a $p.q$ -Baer ring. Let $b \in R$. Then there exists $p \in E(R)$ such that $\ell_R(Rb) = Rp$. Consequently, $\ell_S(Rb) = \ell_R(Rb) \cap S = Rp \cap eR = eRp$. Again, putting $\phi = ep$ gives $\phi \in E(S)$ and $eRp = S\phi$. Hence $\ell_S(Rb) = S\phi$, so R is an e - $p.q$ -Baer ring.

(4) $PP \implies e$ - PP .

Assume that R is a left PP -ring. Let $b \in R$. By the definition of a left PP -ring, there exists an idempotent $p \in E(R)$ such that $\ell_R(b) = Rp$. Therefore, $\ell_S(b) = \ell_R(b) \cap S = Rp \cap eR = eRp$. Let $\phi = ep$. Then $\phi \in E(S)$ and $eRp = S\phi$. Thus $\ell_S(b) = S\phi$, and hence R is an e - PP ring.

(5) $APP \implies e$ - APP .

Assume that R is a left APP ring. By Lemma 1(3), it suffices to show directly that R is a left e - APP ring. Let $b \in R$ and put $L = \ell_S(Rb)$, $S = eRe$. By Lemma 1(2), $L = \ell_S(b) = \ell_R(Rb) \cap S$. Since R is a left APP ring, $\ell_R(Rb)$ is right s -unital. Let $\lambda \in L$. Then $\lambda \in \ell_R(Rb)$, so there exists $u \in \ell_R(Rb)$ such that $\lambda = \lambda u$. Set $v = eu$. Since e is central, $v \in eR = eRe = S$. Moreover, using $\lambda = e\lambda$ and the centrality of e , $\lambda v = \lambda eu = e\lambda u = \lambda u = \lambda$. Finally, for every $r \in R$, $v(rb) = eurb = e(urb) = 0$, because $u \in \ell_R(Rb)$. Hence $v \in \ell_S(Rb) = L$. Thus, for every $\lambda \in L$, there exists $v \in L$ such that $\lambda = \lambda v$. Therefore $L = \ell_S(Rb)$ is right s -unital. Since $b \in R$ was arbitrary, R is a left e - APP ring. $\square$

Proposition 1 ([25] Proposition 2.3). *For any ring R , the following implications hold:*

$$\text{Baer} \implies \text{quasi-Baer} \implies p.q\text{-Baer} \implies PP \implies APP.$$

The classical hierarchy of generalized Baer conditions is well established through the foundational works of Kaplansky [18], Clark [19], Birkenmeier et al. [20], and Zhao and Liu [24]. In the localized setting associated with an idempotent $e \in E(R)$, the corresponding implications require additional care, since the localized annihilators $\ell_{eRe}(Rb)$ and $\ell_{eRe}(b)$ need not coincide in general.

In particular, the equality $\ell_{eRe}(Rb) = \ell_{eRe}(b)$ cannot be deduced merely from the assumption that e is a central idempotent. To avoid this difficulty, we impose the stronger centrality condition $eR \subseteq Z(R)$. This condition implies, in particular, that $e \in Z(R)$ and $eRe \subseteq Z(R)$. More importantly, every element of eRe commutes with every element of R , which yields the annihilator identity $\ell_{eRe}(Rb) = \ell_{eRe}(b)$ for every $b \in R$. This identity is precisely the additional property needed to pass from the e - $p.q$ -Baer condition to the e - PP condition. Under this stronger hypothesis, the classical hierarchy of generalized Baer-type conditions extends to the localized setting as follows.

Theorem 2. Let R be an associative ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Then the following implication chain holds:

$$(1) \text{ } e\text{-Baer} \implies (2) \text{ } e\text{-quasi-Baer} \implies (3) \text{ } e\text{-}p.q\text{-Baer} \implies (4) \text{ } e\text{-PP} \implies (5) \text{ } e\text{-APP}.$$

Proof. Put $S = eRe$. By Lemma 1, S is a commutative ring with identity e , and, for every $b \in R$, $\ell_S(Rb) = \ell_S(b) = \ell_S(Sebe)$. We shall use these identities throughout the proof.

(1) $\implies$ (2). Assume that R is an e -Baer ring, and let $I \trianglelefteq R$. Since I is a nonempty subset of R , Definition 1 yields an idempotent $\phi \in E(S)$ such that $\ell_S(I) = S\phi$. Hence R is an e -quasi-Baer ring.

(2) $\implies$ (3). Assume that R is an e -quasi-Baer ring, and let $b \in R$. Applying Definition 2 to the ideal RbR gives an idempotent $\phi \in E(S)$ such that $\ell_S(RbR) = S\phi$. We claim that $\ell_S(RbR) = \ell_S(Rb)$. Indeed, the inclusion $\ell_S(RbR) \subseteq \ell_S(Rb)$ is immediate. Conversely, if $\lambda \in \ell_S(Rb)$, then, for all $r, s \in R$, $\lambda(rbs) = (\lambda rb)s = 0$. Thus $\lambda \in \ell_S(RbR)$, proving the claim. Consequently, $\ell_S(Rb) = S\phi$, and hence R is an e - $p.q$ -Baer ring.

(3) $\implies$ (4). Assume that R is an e - $p.q$ -Baer ring. For each $b \in R$, Definition 3 gives an idempotent $\phi \in E(S)$ such that $\ell_S(Rb) = S\phi$. By Lemma 1(2), $\ell_S(b) = \ell_S(Rb) = S\phi$. Therefore R is an e -PP ring.

(4) $\implies$ (5). Assume that R is an e -PP ring, and let $b \in R$. By Definition 4, there exists $\phi \in E(S)$ such that $\ell_S(b) = S\phi$. Lemma 1(2) therefore gives $\ell_S(Rb) = \ell_S(b) = S\phi$. For any $\lambda \in \ell_S(Rb)$, write $\lambda = a\phi$ with $a \in S$. Since $\phi^2 = \phi$, $\lambda\phi = a\phi^2 = a\phi = \lambda$. Moreover, $\phi \in S\phi = \ell_S(Rb)$, and hence $\phi rb = 0$ for all $r \in R$. Thus $\ell_S(Rb)$ is right s -unital. Since $b \in R$ was arbitrary, Definition 5 implies that R is a left e -APP ring.

□

In classical ring theory, reduced Baer or $p.q$ -Baer rings are inherently symmetric and reversible. The following results generalize these classical connections (e.g., [31,32]) by showing that localized annihilator conditions induce weak e -symmetry and weak e -reversibility whenever eRe is reduced.

Theorem 3. Let R be a reduced ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. If R is an e -Baer ring, then R is weak e -symmetric.

Proof. Put $S = eRe$. By Lemma 1(1), S is a commutative ring with identity e . Moreover, S is reduced, since R is reduced. We first show that S is a Baer ring. Let $\emptyset \neq X \subseteq S$. Since $X \subseteq R$ and R is an e -Baer ring, Definition 1 gives an idempotent $\phi \in E(S)$ such that $\ell_S(X) = S\phi$. Hence S is a Baer ring. Since every reduced ring is symmetric, S is symmetric.

Now let $a, b, c \in R$ satisfy $abc = 0$. Since $eR \subseteq Z(R)$, $(eae)(ebe)(ece) = eabce = 0$. As S is symmetric, we obtain $(eae)(ece)(ebe) = 0$. Using $e^2 = e$ and the centrality of e , we have $(eae)(ece)(ebe) = eacbe$. Therefore $eacbe = 0$. Thus, R is weak e -symmetric. □

Proposition 2. Let R be an associative ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. If R is an e - $p.q$ -Baer ring and eRe is reduced, then R is weak e -symmetric.

Proof. Put $S = eRe$. By Lemma 1(2), for every $b \in R$, $\ell_S(Rb) = \ell_S(b) = \ell_S(Sebe)$. Since R is an e - $p.q$ -Baer ring, for each $b \in R$ there exists $\phi \in E(S)$ such that $\ell_S(Rb) = S\phi$. Hence $\ell_S(Sebe) = S\phi$, and therefore S is a $p.q$ -Baer ring. By hypothesis, S is reduced, and hence S is symmetric. Now let $x, d, y \in R$ satisfy $xdy = 0$. Since $eR \subseteq Z(R)$, the idempotent e is central, and thus $(exe)(ede)(eye) = exdye = 0$. As S is symmetric, it follows that $(exe)(eye)(ede) = 0$. Using again the centrality of e and $e^2 = e$, we obtain $(exe)(eye)(ede) = exyde$. Consequently, $exyde = 0$. Thus, R is weak e -symmetric. □

Theorem 4. Let R be an associative ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. If R is an e -PP ring and eRe is reduced, then R is weak e -reversible.

Proof. Put $S = eRe$. By Lemma 1(1), S is a commutative ring with identity e . Since S is reduced, every idempotent of S is central.

Let $x, y \in R$ satisfy $xy = 0$. Since $eR \subseteq Z(R)$ and $e^2 = e$, we have $(exe)(eye) = exye = 0$. Hence $exe \in \ell_S(eye)$. As R is an e -PP ring, there exists an idempotent $\phi \in E(S)$ such that $\ell_S(eye) = S\phi$. Thus $exe = s\phi$ for some $s \in S$. Moreover, $\phi \in S\phi = \ell_S(eye)$, and hence $\phi(eye) = 0$. Since $\phi \in Z(S)$, it follows that $(eye)\phi = 0$. Consequently, $eyxe = (eye)(exe) = (eye)s\phi = s(eye)\phi = 0$. Therefore, R is weak e -reversible. $\square$

Proposition 3. Let R be a ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. If R is e -APP, then R is weak e -reflexive.

Proof. Put $S = eRe$. By Lemma 1, S is commutative and $\ell_S(Rb) = \ell_S(b) = \ell_S(Sebe)$ for every $b \in R$. Let $x, y \in R$ satisfy $xRy = 0$. We prove that $eyRxe = 0$. Let $r \in R$. Since $xRy = 0$, we have $xry = 0$. Hence $(exre)(eye) = exreye = exrye = 0$. Thus $exre \in \ell_S(eye)$. By Lemma 1(2), $\ell_S(eye) = \ell_S(Reye)$. Since R is e -APP, this ideal is right s -unital. Hence there exists $\phi \in \ell_S(eye)$ such that $exre = (exre)\phi$. Since $\phi \in \ell_S(eye)$, $\phi(eye) = 0$. Moreover, $\phi eye \in S \subseteq Z(R)$, and therefore

$$eyrxe = (eye)(exre) = (eye)(exre)\phi = (exre)(eye)\phi = 0,$$

since $r \in R$ was arbitrary, $eyRxe = 0$. Thus R is weak e -reflexive. $\square$

Proposition 4. Let R be a ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. If R is a left e -APP ring and eRe is semicommutative, then R is weak e -semicommutative.

Proof. Put $S = eRe$. By Lemma 1(1), S is a commutative ring with identity e , and hence is semicommutative. Let $x, y \in R$ satisfy $xy = 0$. Then $(exe)(eye) = exye = 0$. Since S is semicommutative, it follows that $(exe)S(eye) = 0$. For every $r \in R$, we have $ere \in S$, and therefore $(exe)(ere)(eye) = 0$. Using $eR \subseteq Z(R)$ and $e^2 = e$, we obtain $(exe)(ere)(eye) = exrye$. Thus $exrye = 0$ for every $r \in R$, and hence $exRye = 0$. Therefore, R is weak e -semicommutative. $\square$

Lemma 2. Let R be an e -Baer ring for some $e \in E(R)$. If the corner ring eRe is commutative, then eRe is reduced.

Proof. Put $S = eRe$. By the definition of an e -Baer ring, the corner ring S is a Baer ring. Let $a \in S$ satisfy $a^2 = 0$. Since $a \in \ell_S(a)$ and S is Baer, there exists an idempotent $y \in E(S)$ such that $\ell_S(a) = Sy$. Hence $a = xy$ for some $x \in S$. Therefore, $ay = (xy)y = xy^2 = xy = a$. Moreover, $y \in Sy = \ell_S(a)$, and hence $ya = 0$. Since S is commutative, we have $a = ay = ya = 0$. Thus every square-zero element of S is zero. Consequently, $S = eRe$ is reduced. $\square$

In general, localized symmetry, reversibility, reflexivity, and semicommutativity need not coincide. For ordinary rings, the classical implications

$$\text{reduced} \implies \text{symmetric} \implies \text{reversible} \implies \text{semicommutative}$$

are well known; see, for example, Lambek [33], Birkenmeier et al. [34], and Hashemi and Moussavi [35]. In the localized setting, analogous relationships require additional hypotheses, since the corner ring eRe does not, in general, fully reflect the annihilator

behavior of R . The following result avoids these difficulties by imposing the stronger centrality condition $eR \subseteq Z(R)$. By Lemma 1, the corner ring $S = eRe$ is commutative, and the localized annihilators satisfy $\ell_S(Rb) = \ell_S(b) = \ell_S(Sebe)$, $b \in R$. Moreover, by Lemma 2, if R is e -Baer, then $S = eRe$ is reduced. Thus, under the stronger centrality hypothesis, the classical symmetry-like properties of the corner ring can be transferred to the corresponding localized properties of R .

Theorem 5. *Let R be an e -Baer ring for some $e \in E(R)$, and suppose that $eR \subseteq Z(R)$. Then the following statements are equivalent:*

1. R is weak e -symmetric;
2. R is weak e -reflexive;
3. R is weak e -reversible;
4. R is weak e -semicommutative;
5. eRe is a symmetric ring;
6. eRe is a reflexive ring;
7. eRe is a reversible ring;
8. eRe is a semicommutative ring.

Moreover, the corner ring eRe is commutative and reduced.

Proof. Let $S = eRe$. By Lemma 1(1), S is a commutative ring with identity e , and Lemma 2 shows that S is reduced. Consequently,

$$S \text{ symmetric} \iff S \text{ reflexive} \iff S \text{ reversible} \iff S \text{ semicommutative},$$

it therefore suffices to establish the correspondence between these properties of S and the associated weak e -properties of R .

(1) $\implies$ (2). Assume that R is weak e -symmetric. If $xRy = 0$, then $xy = 0$, and hence $eyxe = 0$. Replacing x by xr , for arbitrary $r \in R$, gives $eyrxe = 0$. Thus $eyRxe = 0$, and so R is weak e -reflexive.

(2) $\implies$ (6). Assume that R is weak e -reflexive. Let $x, y \in S$ satisfy $xSy = 0$. For every $r \in R$, since $x = ex$, $y = ey$, and e is central, we have $xry = x(ere)y = 0$, because $ere \in S$. Hence $xRy = 0$. By weak e -reflexivity, $eyRxe = 0$. In particular, $yx = eyxe = 0$. Since $ySx \subseteq yRx$, it follows that $ySx = 0$. Therefore S is reflexive.

(6) $\implies$ (7). Assume that S is reflexive. Let $x, y \in S$ and $xy = 0$, then $xSy = 0$. Hence $ySx = 0$, and, taking the identity e of S , we obtain $yx = 0$. Thus S is reversible.

(7) $\implies$ (3). Assume that S is reversible. Let $x, y \in R$ satisfy $xy = 0$. Since e is central, $(exe)(eye) = exye = e(xy)e = 0$. By reversibility of S , $(eye)(exe) = eyxe = 0$. Hence R is weak e -reversible.

(3) $\implies$ (8). Assume that R is weak e -reversible. Let $x, y \in S$ satisfy $xy = 0$. Then $yx = eyxe = 0$. Thus S is reversible. Since S is reduced, it is semicommutative. Hence S is semicommutative.

(8) $\implies$ (5). Assume that S is semicommutative. Since S is reduced, the standard implication reduced $\implies$ symmetric yields that S is symmetric.

(5) $\implies$ (1). Assume that S is symmetric. Let $a, b, c \in R$ satisfy $abc = 0$. Since e is central, $(eae)(ebe)(ece) = eabce = 0$. By symmetry of S , $(eae)(ece)(ebe) = eacbe = 0$. Thus R is weak e -symmetric.

□

A well-known result of Liu and Zhao ([25], Corollary 2.11) states that every commutative left APP-ring is reduced. In contrast, Theorem 6 shows that this rigidity need not persist in the localized setting: a commutative e -APP ring need not be reduced, even when

e is a nontrivial idempotent. Thus, the localized notion exhibits behavior that is not forced by the corresponding global APP property.

Theorem 6. *There exists a commutative ring R with a nonzero idempotent $e \in E(R)$ satisfying $eR \subseteq Z(R)$ such that R is a left e -APP ring but is not reduced.*

Proof. Let F be a field and set $R = \left(\frac{F[x]}{\langle x^2 \rangle} \right) \times F$, $e = (0, 1)$. Then R is commutative, $e^2 = e$, and $e \neq 0$. Hence $eR \subseteq Z(R)$. Moreover, $eRe = \{(0, c) : c \in F\} \cong F$. Thus eRe is a left APP ring. By Lemma 1(3), R is a left e -APP ring.

On the other hand, $(x + \langle x^2 \rangle, 0) \neq 0$ and $(x + \langle x^2 \rangle, 0)^2 = 0$. Hence R is not reduced. Therefore, the left e -APP property does not imply reducedness, even under the stronger condition $eR \subseteq Z(R)$. $\square$

Remark 1. Let $\{R_i\}_{i \in I}$ be a family of rings, and let $R = \prod_{i \in I} R_i$ (respectively, $R = \bigoplus_{i \in I} R_i$). For any idempotent $e = (e_i)_{i \in I} \in E(R)$, each $e_i \in E(R_i)$, and the decomposition of e into its component idempotents is unique. Moreover, $eRe \cong \prod_{i \in I} e_i R_i e_i$ for direct products, and $eRe \cong \bigoplus_{i \in I} e_i R_i e_i$ for direct sums. If, in addition, $eR \subseteq Z(R)$, then $e_i R_i \subseteq Z(R_i)$ for every $i \in I$. Thus, under the stronger centrality hypothesis, the localized structure and the associated central corner conditions decompose componentwise. Consequently, the localized properties considered in this paper can be analyzed through the family of corner rings $e_i R_i e_i$, with the centrality condition inherited by each component. This decomposition provides the structural basis for the permanence results established below.

Remark 2. The preceding decomposition has a useful consequence for arbitrary direct sums. Let $S = \bigoplus_{i \in I} R_i$ and $e = (e_i)_{i \in I} \in E(S)$. Since e has finite support, the corner ring satisfies $eSe \cong \bigoplus_{i \in \text{supp}(e)} e_i R_i e_i$, which is a finite direct sum, even when S is an infinite direct sum. If, moreover, $eS \subseteq Z(S)$, then $e_i R_i \subseteq Z(R_i)$ for every $i \in \text{supp}(e)$. Hence, after passing to the corner ring, both the localized structure and the stronger centrality condition involve only finitely many components. In particular, permanence questions for localized annihilator properties under arbitrary direct sums reduce to the corresponding questions for finite direct sums of the corner rings. This observation provides a convenient structural framework for the direct-sum permanence results established in this section.

Following [36], a ring R is called a generalized right PP-ring if, for every $u \in R$, there exists a positive integer $n = n(u)$ such that the right ideal $u^n R$ is projective; equivalently, the right annihilator $r_R(u^n)$ is generated by an idempotent. When R is reduced, this condition coincides with the classical right PP-property by ([36], Lemma 1(iv)). It is known that infinite direct sums of reduced right PP-rings need not be right PP-rings in the non-abelian setting; see [25]. This illustrates the delicate nature of permanence under infinite direct sums.

For the localized left e -APP property, the situation is more tractable under the centrality condition $eR \subseteq Z(R)$. Indeed, if $S = \bigoplus_{i \in I} R_i$ and $e = (e_i)_{i \in I} \in E(S)$, then e has finite support. Hence $eSe \cong \bigoplus_{i \in \text{supp}(e)} e_i R_i e_i$. Moreover, the condition $eS \subseteq Z(S)$ is inherited by the corresponding corner rings. Thus, after passing to eSe , only finitely many active components remain. This finite-support phenomenon, together with the componentwise description of localized annihilators, yields the following permanence result.

Theorem 7. *Let $\{R_i\}_{i \in I}$ be a family of associative rings with identity. Let $R = \prod_{i \in I} R_i$, $e = (e_i)_{i \in I} \in E(R)$, and suppose that $eR \subseteq Z(R)$. Then the following statements hold.*

- (1) R is a left e -APP ring if and only if R_i is a left e_i -APP ring for every $i \in I$.
- (2) R is an e -Baer ring if and only if R_i is an e_i -Baer ring for every $i \in I$.

- (3) Let $S = \bigoplus_{i \in I} R_i$ and let $e = (e_i)_{i \in I} \in E(S)$ satisfy $eS \subseteq Z(S)$. Then S is a left e -APP ring if and only if R_i is a left e_i -APP ring for every $i \in \text{supp}(e)$.

Proof. (1) Since $eRe \cong \prod_{i \in I} e_i R_i e_i$, the centrality assumption $eR \subseteq Z(R)$ implies $e_i R_i \subseteq Z(R_i)$ ($i \in I$). By Lemma 1, for every $i \in I$, R_i is left e_i -APP if and only if $e_i R_i e_i$ is left APP.

Moreover, for $a = (a_i)_{i \in I} \in R$, $\ell_{eRe}(Ra) \prod_{i \in I} \ell_{e_i R_i e_i}(R_i a_i)$. Suppose first that R is left e -APP. Fix $i \in I$ and $a_i \in R_i$. Applying the defining right s -unital condition to the element supported at the i th coordinate shows that $\ell_{e_i R_i e_i}(R_i a_i)$ is right s -unital. Hence $e_i R_i e_i$ is left APP, and therefore R_i is left e_i -APP.

Conversely, suppose that every R_i is left e_i -APP. Then each $L_i = \ell_{e_i R_i e_i}(R_i a_i)$ is right s -unital. Given $x = (x_i)_{i \in I} \in \prod_i L_i$, choose, for each i , an element $u_i \in L_i$ such that $x_i u_i = x_i$. Then $u = (u_i)_{i \in I} \in \prod_i L_i$ and $xu = x$. Thus $\prod_i L_i$ is right s -unital. Hence R is a left e -APP ring.

(2) The decomposition $eRe \cong \prod_{i \in I} e_i R_i e_i$ and the componentwise description of left annihilators give $\ell_{eRe}(X) \prod_{i \in I} \ell_{e_i R_i e_i}(X_i)$ for every nonempty subset $X = \prod_i X_i$ in the corresponding componentwise decomposition. Since idempotents in a direct product are componentwise idempotent, the defining condition for the e -Baer property is equivalent to the corresponding condition in every corner $e_i R_i e_i$. The equivalence with the e_i -Baer property of R_i follows from Lemma 1 and the same componentwise argument.

(3) Put $K = \text{supp}(e) = \{i \in I \mid e_i \neq 0\}$. Since $e \in \bigoplus_{i \in I} R_i$, the set K is finite, and $eSe \cong \bigoplus_{i \in K} e_i R_i e_i$. The assumption $eS \subseteq Z(S)$ implies $e_i R_i \subseteq Z(R_i)$ ($i \in K$).

Suppose that S is left e -APP. For any $k \in K$, consider an element $a \in S$ supported only at the k th coordinate. The corresponding localized annihilator is right s -unital, and its k th component is $\ell_{e_k R_k e_k}(R_k a_k)$. Hence $e_k R_k e_k$ is left APP, and Lemma 1 yields that R_k is left e_k -APP.

Conversely, assume that R_i is left e_i -APP for every $i \in K$. By Lemma 1, each $e_i R_i e_i$ is left APP. Let $a \in S$. Since K is finite, $\ell_{eSe}(Sa) \bigoplus_{i \in K} \ell_{e_i R_i e_i}(R_i a_i)$ is a finite direct sum of right s -unital ideals. Hence it is itself right s -unital. Therefore S is a left e -APP ring. $\square$

Let A be a commutative ring, let B be a unitary subring of A , and let $A_{i=1}^\infty$ be a countable family of isomorphic copies of A . Put $D = \prod_{i=1}^\infty A_i$, and let $R = R(A, B)$ be the subring of D generated by $\bigoplus_{i=1}^\infty A_i$ and the diagonal subring $\bar{B} = \{(b, b, \dots) \mid b \in B\}$ [36].

For every $e = (e_i)_{i=1}^\infty \in E(R)$, the sequence of components stabilizes to an idempotent $e_B \in E(B)$; consequently, $eRe \cong R(e_i A e_{i \geq 1}, e_B B e_B)$.

Theorem 8. Let A be a commutative ring, let B be a unitary subring of A , and let $R = R(A, B)$. Let $e = (e_i)_{i=1}^\infty \in E(R)$ satisfy $e_n = e_{n+1} = \dots = e_B \in E(B)$ for some $n \geq 1$. Then $eR \subseteq Z(R)$, and the following conditions are equivalent:

- (1) R is a left e -APP ring;
- (2) $e_i A e_i$ is a left APP ring for every $i \geq 1$, and $e_B B e_B$ is a left APP ring.

Proof. Since A is commutative, the ring $R = R(A, B)$ is commutative. Hence $eR \subseteq Z(R)$. By Lemma 1, we have R is left e -APP if and only if eRe is left APP. By the stabilization of the components of e , $eRe \cong R(e_i A e_{i \geq 1}, e_B B e_B)$. Each $e_i A e_i$ is commutative, since A is commutative. Therefore, Proposition 3.3 of [25] applies and gives $R(e_i A e_{i \geq 1}, e_B B e_B)$ is left APP if and only if $e_i A e_i$ is left APP for all $i \geq 1$ and $e_B B e_B$ is left APP. Combining these equivalences with the isomorphism above proves the result. $\square$

Remark 3. Theorem 8 is specific to the localized left e -APP property. Its proof reduces the localized condition to the ordinary APP property of the corner ring eRe and then uses the componentwise characterization available for the ring $R(A, B)$. This reduction relies on the fact that the left APP

condition concerns annihilators of individual principal left ideals. The same argument does not directly extend to the stronger localized annihilator classes, such as e -Baer, e -quasi-Baer, and e - $p.q.$ -Baer, whose definitions involve annihilators of arbitrary subsets, ideals, or principal left ideals. These classes therefore require separate permanence arguments.

4. Additional Examples and Counterexamples

The following examples demonstrate that the implications obtained in this section are not reversible, even under the stronger hypothesis $eR \subseteq Z(R)$. Thus, although the centrality of eR strengthens the positive results relating localized properties to their global counterparts, it does not suffice to establish the converses. The examples below illustrate this failure for several Baer-type and weak localized properties.

Remark 4. The examples below use direct product constructions of the form $R = A \times T$, $e = (1, 0)$, where A is chosen to ensure the required localized properties, while T is chosen so that the corresponding global property fails. In particular, when $A = F$ is a field, we have $eR = F \times \{0\} \subseteq Z(R)$, $eRe \cong F$, so the localized property relative to e follows immediately from the corresponding property of F . More specialized examples use the rings introduced in the subsequent lemmas. Overall, these constructions show that the additional assumption $eR \subseteq Z(R)$, although sufficient for the preceding positive results, does not in general make their converses valid.

Example 1. The converse of the implication Baer $\implies e$ -Baer fails even under the additional assumption $eR \subseteq Z(R)$.

Proof. Let F be a field and set $R = F \times \mathbb{Z}_4$, $e = (1, 0) \in E(R)$. Then $eR = eRe = F \times \{0\} \subseteq Z(R)$, $eRe \cong F$. Since F is a Baer ring, and every left annihilator in the field eRe is either 0 or eRe , it follows from Definition 1 that R is an e -Baer ring.

On the other hand, $\mathbb{Z}_4$ is not a Baer ring, since $\ell_{\mathbb{Z}_4}(2) = 2\mathbb{Z}_4$ is not generated by an idempotent. Since a direct product of rings is Baer only if each factor is Baer, $R = F \times \mathbb{Z}_4$ is not a Baer ring. Thus the converse implication fails even under $eR \subseteq Z(R)$. $\square$

Example 2. The converse of quasi-Baer $\implies e$ -quasi-Baer fails even under $eR \subseteq Z(R)$.

Proof. Let F be a field, T a non-quasi-Baer ring, $R = F \times T$, and $e = (1, 0) \in E(R)$. Then $eR = eRe = F \times \{0\} \subseteq Z(R)$ and $eRe \cong F$.

For any ideal $I = I_F \times I_T \trianglelefteq R$, we have $\ell_{eRe}(I) = eRe \cdot e$ if $I_F = 0$, and $\ell_{eRe}(I) = eRe \cdot 0$ if $I_F = F$. Thus $\ell_{eRe}(I)$ is generated by an idempotent in $E(eRe)$, so R is an e -quasi-Baer ring.

Conversely, $R = F \times T$ is quasi-Baer if and only if both factors are. Since T is not, R is not quasi-Baer. $\square$

Example 3. An e - $p.q.$ -Baer ring need not be $p.q.$ -Baer, even if $eR \subseteq Z(R)$.

Proof. Let F be a field, T a non- $p.q.$ -Baer ring, $R = F \times T$, and $e = (1, 0) \in E(R)$. Then $eR = eRe = F \times \{0\} \subseteq Z(R)$ and $eRe \cong F$.

For $b = (f, t) \in R$, we have $\ell_{eRe}(Rb) = eRe \cdot e$ if $f = 0$, and $\ell_{eRe}(Rb) = eRe \cdot 0$ if $f \neq 0$. Thus $\ell_{eRe}(Rb)$ is generated by an idempotent in $E(eRe)$, so R is an e - $p.q.$ -Baer ring.

Conversely, $R = F \times T$ is $p.q.$ -Baer if and only if both factors are. Since T is not, R is not $p.q.$ -Baer. $\square$

Example 4. The converse of left PP $\implies e$ -PP does not hold, even under $eR \subseteq Z(R)$.

Proof. Let F be a field, T a non-left PP ring, $R = F \times T$, and $e = (1, 0) \in E(R)$. Then $eR = eRe = F \times \{0\} \subseteq Z(R)$ and $eRe \cong F$.

For $b = (f, t) \in R$, we have $\ell_{eRe}(b) = eRe \cdot e$ if $f = 0$, and $\ell_{eRe}(b) = eRe \cdot 0$ if $f \neq 0$. Thus $\ell_{eRe}(b)$ is generated by an idempotent in $E(eRe)$, so R is an e -PP ring.

Conversely, $R = F \times T$ is a left PP ring if and only if both factors are. Since T is not, R is not a left PP ring. $\square$

Example 5. There exists a left e -APP ring which is not a left APP-ring, even under the assumption $eR \subseteq Z(R)$.

Proof. Let F be a field, T a non-left APP ring, $R = F \times T$, and $e = (1, 0) \in E(R)$. Then $eR = eRe = F \times \{0\} \subseteq Z(R)$ and $eRe \cong F$.

For $b = (f, t) \in R$, we have $\ell_{eRe}(Rb) = eRe$ if $f = 0$, and $\ell_{eRe}(Rb) = \{0\}$ if $f \neq 0$. In either case, $\ell_{eRe}(Rb)$ is a right s -unital ideal of eRe , so R is a left e -APP ring.

Conversely, $R = F \times T$ is a left APP ring if and only if both factors are. Since T is not, R is not a left APP ring. $\square$

Example 6. There exists a ring R with an idempotent $e \in E(R)$ such that $eR \subseteq Z(R)$, eRe is reduced, and R is weak e -symmetric, but R is not an e - p - q -Baer ring.

Proof. Let S be a commutative reduced ring which is not a p - q -Baer ring, T a nonzero ring, and set $R = S \times T$ with $e = (1, 0)$. Note that $eR = eRe = S \times \{0\} \subseteq Z(R)$, so $eRe \cong S$ is reduced.

To show that R is weak e -symmetric, let $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2) \in R$ satisfy $abc = 0$. Since S is commutative, $a_1b_1c_1 = 0 \implies a_1c_1b_1 = 0$, and hence $each e = (a_1c_1b_1, 0) = 0$. Thus R is weak e -symmetric.

Now, for $b = (b_1, b_2) \in R$, we have $\ell_{eRe}(Rb) = \{(s, 0) \mid sSb_1 = 0\} \cong \ell_S(Sb_1)$. If R were an e - p - q -Baer ring, then by Definition 3, $\ell_S(Sb_1)$ would be generated by an idempotent of S for every $b_1 \in S$, making S a p - q -Baer ring. This contradicts the choice of S , so R is not an e - p - q -Baer ring. $\square$

Lemma 3. Let $S = \mathbb{Z}[x_1, x_2, \dots] / \langle x_i x_j \mid i \neq j \rangle$. Then S is a commutative reduced ring with $E(S) = \{0, 1\}$.

Proof. Commutativity is immediate. Every $u \in S$ has a unique representation $u = n + \sum_{i=1}^m f_i(x_i)$, where $n \in \mathbb{Z}$ and $f_i(x_i) \in x_i \mathbb{Z}[x_i]$. Since $x_i x_j = 0$ for $i \neq j$, distinct nonconstant components multiply to zero.

If $u^k = 0$, comparison of constant terms gives $n = 0$, hence $\sum_i f_i(x_i)^k = 0$. Uniqueness of representation yields $f_i(x_i)^k = 0$ for all i ; since $\mathbb{Z}[x_i]$ is an integral domain, $f_i(x_i) = 0$ for all i . Thus $u = 0$, so S is reduced.

For $e \in E(S)$, write $e = n + \sum_i f_i(x_i)$. From $e^2 = e$ we get $n^2 = n$, hence $n \in \{0, 1\}$. If $n = 0$, then $f_i^2 = f_i$, while if $n = 1$, then $f_i(1 + f_i) = 0$. In either case, $f_i = 0$ in the domain $\mathbb{Z}[x_i]$. Hence $e = 0$ or $e = 1$, and therefore $E(S) = \{0, 1\}$. $\square$

Example 7. There exists a ring R with $e \in E(R)$ such that $eR \subseteq Z(R)$ and eRe is reduced, for which R is weak e -reversible but not a left e -PP ring.

Proof. Let $S = \mathbb{Z}[x_1, x_2, \dots] / \langle x_i x_j \mid i \neq j \rangle$, $T \neq 0$, $R = S \times T$, and $e = (1, 0)$. Note that $eR = eRe = S \times \{0\} \subseteq Z(R)$, so $eRe \cong S$ is reduced with $E(eRe) = \{0, e\}$ by Lemma 3.

To show that R is weak e -reversible, let $a = (a_1, a_2), b = (b_1, b_2) \in R$ with $ab = 0$. Since S is commutative, $a_1 b_1 = 0 \implies b_1 a_1 = 0$, yielding $ebae = (b_1 a_1, 0) = 0$.

Now, for $b = (x_1, 0) \in R$, we have $\ell_{eRe}(b) \cong \ell_S(x_1) = \langle x_2, x_3, \dots \rangle$. Since $\mathbb{Z}[x_1]$ is an integral domain and $E(S) = \{0, 1\}$, $\ell_S(x_1)$ is a non-zero proper ideal containing no

non-trivial idempotents. Thus $\ell_{eRe}(b)$ cannot be generated by an idempotent in $E(eRe)$, so R is not a left e -PP ring. $\square$

Lemma 4. Let $S = \mathbb{Z}_4[x_1, x_2, \dots] / \langle x_i x_j \mid i \neq j \rangle$. Then S is a commutative ring which is not a left APP-ring.

Proof. Clearly, S is commutative. Since $x_i x_1 = 0$ for all $i \geq 2$, we have $\langle x_2, x_3, \dots \rangle \subseteq \ell_S(x_1)$. Conversely, let $f = a + f_1(x_1) + \sum_{i=2}^m f_i(x_i) \in S$, where $a \in \mathbb{Z}_4$ and $f_i(x_i) \in x_i \mathbb{Z}_4[x_i]$. Then $f x_1 = (a + f_1(x_1)) x_1$. Since multiplication by x_1 is injective on $\mathbb{Z}_4[x_1]$, $f x_1 = 0$ forces $a = f_1(x_1) = 0$. Thus, $\ell_S(x_1) = \langle x_2, x_3, \dots \rangle$.

This ideal is not right s -unital: for $x_2 \in \ell_S(x_1)$, any $u \in \langle x_2, x_3, \dots \rangle$ lacks a constant term, making $x_2 u \neq x_2$. Hence, $\ell_S(x_1)$ is not right s -unital, so S is not a left APP-ring. $\square$

Example 8. There exists a ring R with $e \in E(R)$ such that $eR \subseteq Z(R)$ and R is weak e -reflexive, but R is not a left e -APP ring.

Proof. Let $S = \mathbb{Z}_4[x_1, x_2, \dots] / \langle x_i x_j \mid i \neq j \rangle$, $T \neq 0$, $R = S \times T$, and $e = (1, 0)$. Then $eR = eRe = S \times \{0\} \subseteq Z(R)$ and $eRe \cong S$.

To verify weak e -reflexivity, let $a = (a_1, a_2), b = (b_1, b_2) \in R$ with $aRb = 0$. Since S is commutative, $a_1 S b_1 = 0 \implies b_1 S a_1 = 0$, yielding $ebRae = (b_1 S a_1, 0) = 0$. Thus R is weak e -reflexive.

If R were a left e -APP ring, then $eR \subseteq Z(R)$ and Lemma 1(3) would force $eRe \cong S$ to be a left APP-ring, contradicting Lemma 4. Hence R is not a left e -APP ring. $\square$

Example 9. There exists a ring R with an idempotent $e \in E(R)$ such that $eR \subseteq Z(R)$, R is weak e -semicommutative, but R is not a left e -APP ring.

Proof. Let $R = S \times T$ and $e = (1, 0)$ be the ring and idempotent constructed in Example 8. Then R is weak e -reflexive but not a left e -APP ring. Since every weak e -reflexive ring is weak e -semicommutative by ([30], Theorem 1), R is weak e -semicommutative. Hence the result follows. $\square$

5. Applications to Matrix Rings and APP-Type Extensions

This section develops a unified transfer theory for the localized left e -APP property under fundamental ring-theoretic constructions. Building on classical preservation results of Hirano [37], Liu and Zhao [25], and related works, we establish transfer and permanence results for left e -APP rings under matrix and upper triangular matrix rings, corner rings, Morita equivalence, polynomial and Laurent polynomial extensions, monoid and skew monoid rings, and skew polynomial extensions. The formal power series case requires separate treatment, since the left APP property is not preserved in general. We therefore obtain two complementary sufficient conditions on the corner ring eRe that guarantee the transfer of the left e -APP property to $R[[x]]$. Finally, under an appropriate finiteness condition on eRe , we establish the equivalence between the left e -APP and left e -p.q.-Baer properties, identifying a structural setting in which these localized annihilator conditions coincide.

Lemma 5. Let R be an associative ring with identity, let $n \geq 1$ be an integer, and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Put

$$E = \text{diag}(e, e, \dots, e) \in E(M_n(R)).$$

Then E is a central idempotent of $M_n(R)$, and there is a natural ring isomorphism

$$EM_n(R)E \cong M_n(eRe).$$

Proof. Since $e^2 = e$, we have $E^2 = \text{diag}(e^2, \dots, e^2) = E$. Moreover, $eR \subseteq Z(R)$ implies $er = re$ for every $r \in R$. Hence, for every $A = (a_{ij}) \in M_n(R)$,

$$EA = (ea_{ij})_{i,j} = (a_{ij}e)_{i,j} = AE.$$

Thus E is a central idempotent of $M_n(R)$.

Every element of $EM_n(R)E$ has the form $(ea_{ij}e)_{i,j}$, with $ea_{ij}e \in eRe$. Define

$$\Phi : EM_n(R)E \longrightarrow M_n(eRe), \quad (ea_{ij}e)_{i,j} \longmapsto (ea_{ij}e)_{i,j}.$$

The map Φ is clearly bijective, with inverse given by the same entrywise identification. It preserves addition, and for $A = (ea_{ij}e)_{i,j}$ and $B = (eb_{ij}e)_{i,j}$,

$$\begin{aligned} \Phi(AB)_{ij} &= \sum_{k=1}^n (ea_{ik}e)(eb_{kj}e) \\ &= (\Phi(A)\Phi(B))_{ij}. \end{aligned}$$

Hence Φ is a ring isomorphism. Therefore, $EM_n(R)E \cong M_n(eRe)$. $\square$

Lemma 6. Let R be an associative ring with identity, let $n \geq 1$ be an integer, and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Then

$$R \text{ is a left } e\text{-APP ring} \iff M_n(eRe) \text{ is a left APP ring.}$$

Proof. Put $S = eRe$. By Lemma 1(3),

$$R \text{ is a left } e\text{-APP ring} \iff S \text{ is a left APP ring.}$$

Moreover, by ([25], Proposition 3.5), for every unital ring S and every positive integer n , S is a left APP ring $\iff M_n(S)$ is a left APP ring.

Applying this to $S = eRe$ gives

$$eRe \text{ is a left APP ring} \iff M_n(eRe) \text{ is a left APP ring.}$$

Combining the two equivalences yields

$$R \text{ is a left } e\text{-APP ring} \iff M_n(eRe) \text{ is a left APP ring.}$$

$\square$

Theorem 9. Let R be an associative ring with identity, let $n \geq 1$ be an integer, and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Put $E = \text{diag}(e, e, \dots, e) \in E(M_n(R))$. Then R is a left e -APP ring $\iff M_n(R)$ is an E -APP ring.

Proof. By Lemma 5, $EM_n(R)E \cong M_n(eRe)$. Hence, by Definition 5,

$$\begin{aligned} M_n(R) \text{ is an } E\text{-APP ring} &\iff EM_n(R)E \text{ is a left APP ring} \\ &\iff M_n(eRe) \text{ is a left APP ring} \\ &\iff R \text{ is a left } e\text{-APP ring,} \end{aligned}$$

where the last equivalence follows from Lemma 6. $\square$

Remark 5. Let R be an associative ring with identity and let $e \in E(R)$ satisfy the localized centrality condition $eR \subseteq Z(R)$. Since $e \in eR$, this assumption implies that $e \in Z(R)$ is a central idempotent of R . Consequently, $E = \text{diag}(e, e, \dots, e) \in E(T_n(R))$ is a central idempotent of the upper triangular matrix ring $T_n(R)$. The corner subring associated with E satisfies the natural ring isomorphism $ET_n(R)E \cong T_n(eRe)$, induced entrywise by $E(a_{ij})E \mapsto (ea_{ij}e)_{i,j}$. Therefore, under the hypothesis $eR \subseteq Z(R)$, every localized property of $T_n(R)$ defined via the central corner $ET_n(R)E$ including the E -Baer, E -quasi-Baer, E -p.q.-Baer, E -PP, and E -APP conditions reduces naturally to the corresponding classical property of the upper triangular matrix ring $T_n(eRe)$.

The following result shows that the localized left e -APP property is preserved under passage to upper triangular matrix rings. Under the stronger condition $eR \subseteq Z(R)$, the result follows naturally by passing to the corner ring eRe and using the corresponding permanence result for ordinary left APP rings.

Theorem 10. Let R be an associative ring with identity, let n be a positive integer, and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Put $E = \text{diag}(e, e, \dots, e) \in E(T_n(R))$. Then R is an e -APP ring $\iff T_n(R)$ is an E -APP ring.

Proof. Put $S = eRe$. Since $eR \subseteq Z(R)$, we have $e \in Z(R)$. Consequently, E is a central idempotent of $T_n(R)$, and direct matrix multiplication gives

$$ET_n(R)E = \begin{pmatrix} S & S & \cdots & S \\ 0 & S & \cdots & S \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S \end{pmatrix} \cong T_n(S).$$

By Lemma 1(3), R is an e -APP ring $\iff S$ is a left APP ring.

On the other hand, by the definition of the localized property, $T_n(R)$ is an E -APP ring $\iff ET_n(R)E$ is a left APP ring.

Using the above isomorphism and ([25], Proposition 3.6), $ET_n(R)E$ is a left APP ring $\iff T_n(S)$ is a left APP ring $\iff S$ is a left APP ring.

Therefore,

$$\begin{aligned} T_n(R) \text{ is an } E\text{-APP ring} &\iff T_n(S) \text{ is a left APP ring} \\ &\iff S \text{ is a left APP ring} \\ &\iff R \text{ is an } e\text{-APP ring.} \end{aligned}$$

Hence the desired equivalence follows. $\square$

Lemma 7. Let R be an associative ring with identity and let $f \in E(R)$. If R is a left APP ring, then the corner ring fRf is also a left APP ring.

Proof. Put $S = fRf$. Let $a \in S$ and $x \in \ell_S(Sa)$. Since $x = fxf$ and $a = faf$, for every $r \in R$,

$$xra = xfrfa = x(frf)a = 0.$$

Hence $x \in \ell_R(Ra)$. Since R is left APP, $\ell_R(Ra)$ is pure, so there exists $y \in \ell_R(Ra)$ such that $x = xy$. Set $y' = f y f \in S$. Then

$$x = xy = x(fyf) = xy', \quad y'a = (fyf)(faf) = f(ya)f = 0.$$

Thus $y' \in \ell_S(Sa)$ and $x = xy'$. Hence $\ell_S(Sa)$ is pure. Therefore $S = fRf$ is a left APP ring. $\square$

Proposition 5. *Let R be an associative ring with identity and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. If R is a left e -APP ring, then, for every $f \in E(eRe)$, the corner ring fRf is a left APP ring.*

Proof. By Lemma 1(3), R being a left e -APP ring implies that eRe is a left APP ring. Since $f \in E(eRe)$, Lemma 7, applied to the ring eRe , yields that

$$f(eRe)f = fRf$$

is a left APP ring. $\square$

The preceding results establish the stability of the left e -APP property under matrix and triangular matrix extensions. The next theorem shows that this stability is not restricted to these particular constructions. Under the stronger centrality hypothesis, the localized property is compatible with arbitrary Morita equivalences. Thus, the theory is intrinsic to the Morita class rather than to a specific presentation of the ring.

Theorem 11. *Let R be an associative ring with identity, let $e \in E(R)$ satisfy $eR \subseteq Z(R)$, and suppose that R is an e -APP ring. Let P be a finitely generated projective generator as a left R -module, and put $T = \text{End}_R(P)$. Then the endomorphism $E : P \rightarrow P$, $E(p) = ep$, is a central idempotent of T , and T is an E -APP ring. Consequently, under the correspondence of the central idempotents e and E , the localized left e -APP property of R transfers to the corresponding localized left E -APP property of the Morita equivalent ring T .*

Proof. Put $S = eRe$ and $T = \text{End}_R(P)$. Since $eR \subseteq Z(R)$, we have $er = re$ for all $r \in R$. Hence the map $E : P \rightarrow P$, $E(p) = ep$, is an R -endomorphism. Clearly, $E^2 = E$. Moreover, for every $\varphi \in T$ and $p \in P$,

$$(E\varphi)(p) = e\varphi(p) = \varphi(ep) = (\varphi E)(p).$$

Thus E is a central idempotent of T .

By Lemma 1(3), since R is a left e -APP ring, $S = eRe$ is a left APP ring. Since P is finitely generated projective over R , there exist an R -module Q and an integer $m \geq 1$ such that

$$P \oplus Q \cong R^m.$$

Applying $e(-)$ gives

$$eP \oplus eQ \cong eR^m.$$

As $eR = eRe = S$, it follows that eP is a direct summand of the finitely generated free S -module S^m . Hence eP is finitely generated projective over S . Therefore, by ([25], Theorem 3.8), $\text{End}_S(eP)$ is a left APP ring. The natural map

$$ETE \rightarrow \text{End}_S(eP), \quad E\varphi E \mapsto (ep \mapsto e\varphi(p)),$$

is a ring isomorphism. Thus ETE is a left APP ring.

Finally, since E is a central idempotent of T , Lemma 1(3), applied to (T, E) , yields

$$T \text{ is a left } E\text{-APP ring} \iff ETE \text{ is a left APP ring.}$$

Consequently, T is a left E -APP ring. Hence the e -APP property is preserved under Morita equivalence. $\square$

Recall that a monoid S is called a unique product monoid (briefly, a $u.p.$ -monoid) if, for every pair of nonempty finite subsets $A, B \subseteq S$, there exists an element $g \in S$ admitting a unique factorization $g = ab$, where $a \in A$ and $b \in B$.

Unique product monoids form an important class in the theory of monoid rings and include, in particular, ordered monoids, submonoids of free groups, and torsion free nilpotent groups; see, for example, [32,38,39]. Their unique product property provides the combinatorial control needed for coefficient comparisons in monoid extensions. Let R be a ring and let $\alpha : S \rightarrow \text{Aut}(R)$, $s \mapsto \alpha_s$, be a monoid homomorphism. The associated skew monoid ring $R * S$ consists of finite formal sums $\sum_{s \in S} a_s s$, with multiplication induced by the action α . Throughout this subsection, we assume that α satisfies condition $(*)$: for every $a \in R$, the left ideal $\sum_{s \in S} R \alpha_s(a)$ is finitely generated. Transfer results for annihilator conditions in polynomial and monoid extensions constitute an important theme in ring theory. In particular, classical results establish the preservation of various APP and $p.q.$ -Baer-type properties for suitable polynomial and monoid extensions over $u.p.$ -monoids; see [22,37]. The results below extend this transfer framework to the localized setting by considering annihilator properties associated with an idempotent. In the present setting, we impose the stronger condition $eR \subseteq Z(R)$. This hypothesis is compatible with the localized framework developed earlier in the paper. In particular, if $T = eRe$, then Lemma 1(3) identifies the left e -APP property of R with the ordinary left APP property of the corner ring T . Thus, the localized transfer problem can be reduced to the corresponding classical APP problem for T .

The following theorem shows that this reduction is compatible with skew monoid ring extensions. More precisely, under the invariance condition $\alpha_s(e) = e$ and condition $(*)$, the left e -APP property passes from R to the skew monoid ring RS .

Theorem 12. *Let R be an associative ring with identity, let $e \in E(R)$ satisfy $eR \subseteq Z(R)$, and let S be a $u.p.$ -monoid. Let $\alpha : S \rightarrow \text{Aut}(R)$ be a monoid homomorphism such that $\alpha_s(e) = e$ for all $s \in S$, and suppose that condition $(*)$ holds. If R is a left e -APP ring, then the skew monoid ring $R * S$ is a left e -APP ring.*

Proof. Put $T = eRe$. By Lemma 1, T is a commutative ring with identity e , and

$$R \text{ is a left } e\text{-APP ring} \iff T \text{ is a left APP ring.}$$

Hence T is a left APP ring.

Since $\alpha_s(e) = e$ for every $s \in S$, each α_s restricts to an automorphism of T . Moreover, condition $(*)$ for R restricts to condition $(*)$ for T . Therefore, by ([25], Theorem 3.10), the skew monoid ring $T * S$ is a left APP ring.

Since e is central and fixed by every α_s , we have

$$e(R * S)e \cong T * S.$$

Thus $e(R * S)e$ is a left APP ring. By Definition 5, $R * S$ is a left e -APP ring. $\square$

The transfer of regular and generalized Baer-type properties to polynomial extensions has been widely investigated. For instance, Armendariz [31] provided a counterexample demonstrating that polynomial rings over right PP -rings need not inherit the right PP property. Conversely, positive preservation results have been established for left $p.q.$ -Baer rings; specifically, a ring R is left $p.q.$ -Baer if and only if its polynomial ring $R[x]$ (respectively, its Laurent polynomial ring $R[x, x^{-1}]$) is left $p.q.$ -Baer, as shown in ([22], Theorem 2.1) and ([32], Corollary 1.4). Turning to monoid extensions, ([32], Theorem 1.2, Corollary 1.3) proved that, for a $u.p.$ -monoid S , the monoid ring $R[S]$ is left $p.q.$ -Baer

(respectively, a reduced *PP*-ring) if and only if R has the same property. Motivated by these classical extension results, we establish the following analogous result for left *e*-APP rings.

The preceding transfer results for skew monoid rings immediately yield a unified permanence theorem for polynomial, Laurent polynomial, and monoid extensions. The stronger hypothesis $eR \subseteq Z(R)$ ensures that the localized corner eRe lies in the commutative setting required by the preceding results and allows the localized *e*-APP property to be transferred uniformly to these extensions.

Proposition 6. *Let R be a ring and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Let S be a *u.p.*-monoid, and let X be a nonempty set of not necessarily commuting indeterminates. Then the following conditions are equivalent:*

- (1) R is an *e*-APP ring;
- (2) $R[X]$ is an *e*-APP ring;
- (3) $R[x, x^{-1}]$ is an *e*-APP ring;
- (4) $R[S]$ is an *e*-APP ring.

Proof. Put $T = eRe$. By Lemma 1, T is a commutative ring with identity e , and

$$R \text{ is a left } e\text{-APP ring} \iff T \text{ is a left APP ring.} \quad (1)$$

Moreover, since e is central in R , it remains central in each of the extensions under consideration, and hence

$$eR[X]e \cong T[X], \quad eR[x, x^{-1}]e \cong T[x, x^{-1}], \quad eR[S]e \cong T[S]. \quad (2)$$

We first prove (1) $\Rightarrow$ (4). Suppose that R is a left *e*-APP ring. By Equation (1), T is a left APP ring. Consider the trivial action of S on T , given by $\alpha_s = \text{id}_T$ for all $s \in S$. For every $a \in T$,

$$\sum_{s \in S} T\alpha_s(a) = \sum_{s \in S} Ta = Ta,$$

which is a principal left ideal. Thus condition $(*)$ is satisfied. Hence, by Theorem 12, equivalently by ([25], Theorem 3.10), $T[S]$ is a left APP ring. In view of (2) and the defining correspondence in Lemma 1(3), it follows that $R[S]$ is a left *e*-APP ring.

Conversely, suppose that $R[S]$ is a left *e*-APP ring. By Equation (2), $T[S]$ is a left APP ring. Let $b \in T$ and let $a \in \ell_T(Tb)$. Viewing a and b as constant elements of $T[S]$, we have $a \in \ell_{T[S]}(T[S]b)$. Since $T[S]$ is left APP, there exists $g = \sum_{s \in F} a_s s \in \ell_{T[S]}(T[S]b)$ for some finite subset $F \subseteq S$, such that $a = ag$. Let a_0 denote the coefficient of 1_S in g . By comparing the coefficient of 1_S in the equality $a = ag$, we obtain

$$a = aa_0. \quad (3)$$

Furthermore, since $g \in \ell_{T[S]}(T[S]b)$, for every $t \in T$, $g(tb) = 0$. Comparing the coefficient of 1_S gives $a_0 tb = 0$ for every $t \in T$. Thus

$$a_0 \in \ell_T(Tb), \quad (4)$$

it follows from Equations (3) and (4) that, for every $a \in \ell_T(Tb)$, there exists $a_0 \in \ell_T(Tb)$ such that $a = aa_0$. Hence $\ell_T(Tb)$ is right *s*-unital. Since $b \in T$ was arbitrary, T is a left APP ring. By Equation (1), R is a left *e*-APP ring. Therefore, (1) $\iff$ (4).

Finally, let $F(X)$ be the free monoid on X . Since $F(X)$ is a *u.p.*-monoid,

$$R[X] \cong R[F(X)],$$

likewise, the infinite cyclic group $\mathbb{Z}$ is a *u.p.*-monoid and

$$R[x, x^{-1}] \cong R[\mathbb{Z}].$$

Consequently, the equivalence (1) $\iff$ (4) applied to $F(X)$ and $\mathbb{Z}$, respectively, yields (1) $\iff$ (2), (1) $\iff$ (3). Therefore (1) $\iff$ (2) $\iff$ (3) $\iff$ (4), as required. $\square$

It is well known that the left APP property is not, in general, preserved under formal power series extensions. In particular, Nicholson et al. ([40], Example 2.4) constructed a commutative von Neumann regular ring R , and hence a left APP ring, such that $R[[x]]$ is not left APP. Although various additional hypotheses are known to guarantee the preservation of related annihilator properties in formal power series rings, the corresponding question for localized left *e*-APP rings requires a separate analysis. We address this issue by establishing two complementary transfer criteria based on different structural conditions imposed on the corner ring eRe .

Proposition 7. *Let R be a ring and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Let $\alpha \in \text{Aut}(R)$ satisfy $\alpha(e) = e$. Assume that, for every $b \in eRe$, the left ideal $\sum_{i=0}^{\infty} (eRe)\alpha^i(b)$ is finitely generated in eRe . If R is an *e*-APP ring, then the skew polynomial ring $R[x; \alpha]$ is an *e*-APP ring.*

Proof. Put $T = eRe$. By Lemma 1(1), T is a commutative ring with identity e . Since $\alpha(e) = e$, the restriction $\bar{\alpha} = \alpha|_T$ is an automorphism of T . Moreover, $e \in Z(R)$ and $xe = \alpha(e)x = ex$, so e is central in $R[x; \alpha]$. Hence

$$eR[x; \alpha]e \cong T[x; \bar{\alpha}]. \quad (5)$$

Since R is a left *e*-APP ring, Lemma 1(3) implies that T is a left APP ring. Furthermore, for every $b \in T$,

$$\sum_{i=0}^{\infty} T\bar{\alpha}^i(b) = \sum_{i=0}^{\infty} (eRe)\alpha^i(b),$$

which is finitely generated by hypothesis. Thus condition (*) is satisfied for the pair $(T, \bar{\alpha})$. Therefore, by Theorem 12, $T[x; \bar{\alpha}]$ is a left APP ring. In view of Equation (5), $eR[x; \alpha]e$ is a left APP ring. Hence, by Definition 5, $R[x; \alpha]$ is a left *e*-APP ring. $\square$

Lemma 8. *Let S be an associative ring with identity. Assume that S satisfies the descending chain condition on both left and right annihilators. If S is a left APP ring, then the formal power series ring $S[[x]]$ is a left APP ring.*

Proof. Let $f(x) = \sum_{i \geq 0} a_i x^i$ and $g(x) = \sum_{j \geq 0} b_j x^j$ be elements of $S[[x]]$ such that $f(x) \in \ell_{S[[x]]}(S[[x]]g(x))$. Thus $f(x)rg(x) = 0$ for every $r \in S$. Comparing coefficients of x^k , we obtain

$$\sum_{i+j=k} a_i r b_j = 0 \quad (k \geq 0, r \in S). \quad (6)$$

We first prove that $a_i \in \ell_S(Sb_j)$ for all $i, j \geq 0$. We proceed by induction on $i + j$. For $i = j = 0$, Equation (6) gives $a_0 \in \ell_S(Sb_0)$. Since S is left APP, there exists $u \in \ell_S(Sb_0)$ such that $a_0 = a_0 u$. Applying Equation (6) for $k = 1$ with ur in place of r , the term involving b_0 vanishes, and we obtain $a_0 r b_1 = 0$. Substitution in Equation (6) for $k = 1$ then gives $a_1 r b_0 = 0$. Hence both $a_0 \in \ell_S(Sb_1)$ and $a_1 \in \ell_S(Sb_0)$.

Suppose now that the assertion has been established whenever $i + j < k$. Then $a_0, \dots, a_{k-1}$ belong to the left annihilator of the finitely generated left ideal $Sb_0 + \dots + Sb_{k-1}$. By the preceding result on finitely generated left ideals, this annihilator is right *s*-unital. Hence there exists u in this annihilator such that $a_i = a_i u$ for $0 \leq i < k$.

We now apply Equation (6) to the coefficient of x^k , replacing r by ur . Since u annihilates $Sb_0, \dots, Sb_{k-1}$, the terms corresponding successively to $a_krb_0, \dots, a_1rb_{k-1}$ vanish. The remaining term gives $a_0rb_k = 0$. Returning to Equation (6) and repeating the same argument successively for $a_1rb_{k-1}, \dots, a_krb_0$, we obtain $a_i rb_{k-i} = 0$ for every $0 \leq i \leq k$ and every $r \in S$. Thus $a_i \in \ell_S(Sb_{k-i})$ for all $0 \leq i \leq k$, completing the induction. Consequently, $a_i \in \ell_S(Sb_j)$ for all $i, j \geq 0$.

Consider the descending chain

$$\ell_S(Sb_0) \supseteq \ell_S(Sb_0 + Sb_1) \supseteq \ell_S(Sb_0 + Sb_1 + Sb_2) \supseteq \dots$$

By the DCC on left annihilators, this chain stabilizes. Hence, for some m , the annihilator of the left ideal generated by all b_j coincides with $\ell_S(Sb_0 + \dots + Sb_m)$.

Similarly, by the DCC on right annihilators, the chain

$$r_S(a_0) \supseteq r_S(a_0, a_1) \supseteq r_S(a_0, a_1, a_2) \supseteq \dots$$

stabilizes. Thus, for some n , the right annihilator of all the a_i coincides with $r_S(a_0, \dots, a_n)$.

By the assertion proved above, $a_0, \dots, a_n$ belong to $\ell_S(Sb_0 + \dots + Sb_m)$. Since this left ideal is finitely generated and S is left APP, there exists $c \in \ell_S(Sb_0 + \dots + Sb_m)$ such that $a_i = a_i c$ for $0 \leq i \leq n$. Hence $1 - c$ belongs to $r_S(a_0, \dots, a_n)$. By the stabilization of the right annihilator chain, $1 - c$ annihilates every a_i . Therefore $a_i = a_i c$ for all i , and consequently $f(x) = f(x)c$.

Finally, c belongs to the annihilator of the left ideal generated by all b_j . Hence $cSb_j = 0$ for every $j \geq 0$. It follows that $cS[[x]]g(x) = 0$, and therefore $c \in \ell_{S[[x]]}(S[[x]]g(x))$.

Thus, for every $f(x) \in \ell_{S[[x]]}(S[[x]]g(x))$, there exists $c \in \ell_{S[[x]]}(S[[x]]g(x))$ such that $f(x) = f(x)c$. Hence this left annihilator is right s -unital. Since $g(x)$ was arbitrary, $S[[x]]$ is a left APP ring. $\square$

We first establish a transfer criterion based on annihilator chain conditions in the corner ring: if eRe satisfies the descending chain condition on both left and right annihilators, then the e -APP property transfers from R to $R[[x]]$.

Proposition 8. *Let R be a ring and let $e \in E(R)$ satisfy $eR \subseteq Z(R)$. Assume that R is an e -APP ring and that eRe satisfies the descending chain condition on left and right annihilators. Then $R[[x]]$ is an e -APP ring.*

Proof. Put $S = eRe$. By Lemma 1(1) and (3), S is a commutative left APP ring. Moreover, since $eR \subseteq Z(R)$, we have

$$eR[[x]]e = (eRe)[[x]] = S[[x]].$$

By Lemma 8, $S[[x]]$ is a left APP ring. Hence $eR[[x]]e \cong S[[x]]$ is left APP. Therefore, by Definition 5, $R[[x]]$ is an e -APP ring. $\square$

Recall that a ring R is $\aleph_0$ -self-injective if any left R -homomorphism from a countably generated left ideal into R extends to a ring endomorphism given by left multiplication. Additionally, R is called a left duo ring if every left ideal of R is a two-sided ideal; see [41].

Lemma 9. *Let S be a reduced left duo ring which is $\aleph_0$ -self-injective. If S is a left APP ring, then $S[[x]]$ is a left APP ring.*

Proof. This is precisely ([25], Proposition 3.15). $\square$

We next obtain a second transfer criterion based on $\aleph_0$ -self-injectivity and the reduced left duo structure of the corner ring.

Proposition 9. *Let R be an e -APP ring, $e \in E(R)$ satisfy $e \in Z(R)$ and $eR \subseteq Z(R)$. Put $S = eRe$. If S is a reduced left duo $\aleph_0$ -self-injective ring, then the formal power series ring $R[[x]]$ is an e -APP ring.*

Proof. Since R is a left e -APP ring and $e \in Z(R)$, Lemma 1(3) implies that $S = eRe$ is a left APP ring. By hypothesis, S is a reduced left duo $\aleph_0$ -self-injective ring. Thus, by Lemma 9, the formal power series ring $S[[x]]$ is a left APP ring. Since $e \in Z(R)$, we have $eR[[x]]e \cong (eRe)[[x]] = S[[x]]$. Consequently, $eR[[x]]e$ is a left APP ring, which implies by Definition 5 that $R[[x]]$ is an e -APP ring. $\square$

We now identify conditions under which the localized e -APP and e - $p.q.$ -Baer properties coincide. The implication e - $p.q.$ -Baer $\Rightarrow$ e -APP follows from Theorem 2; the converse, however, requires an additional finiteness condition on the localized corner.

Lemma 10 ([25], Proposition 2.7). *Let R be a ring satisfying the ascending chain condition on principal left ideals. Then R is a left APP ring if and only if R is a left $p.q.$ -Baer ring.*

Theorem 13. *Let R be a ring and let $e \in E(R)$ satisfy $e \in Z(R)$ and $eR \subseteq Z(R)$, and put $S = eRe$. Suppose that S satisfies the ascending chain condition on principal left ideals. Then the following conditions are equivalent:*

- (1) R is a left e -APP ring;
- (2) R is a left e - $p.q.$ -Baer ring.

Proof. By Lemma 1(3),

$$R \text{ is a left } e\text{-APP ring} \iff S \text{ is a left APP ring.}$$

Since S satisfies ACC on principal left ideals, Lemma 10 yields

$$S \text{ is a left APP ring} \iff S \text{ is a left } p.q\text{-Baer ring,}$$

where the reverse implication follows from the standard fact that every left $p.q.$ -Baer ring is left APP. It remains to compare the $p.q.$ -Baer property of S with the localized property of R . For $b \in R$ and $s \in S$, since e is central,

$$srb = (se)rb = s(er)eb = s(ere)(ebe),$$

and hence

$$\ell_S(Rb) = \ell_S(Sebe).$$

Thus S is a left $p.q.$ -Baer ring if and only if R is a left e - $p.q.$ -Baer ring. Consequently, R is a left e -APP ring $\iff S$ is a left APP ring $\iff S$ is a left $p.q.$ -Baer ring $\iff R$ is a left e - $p.q.$ -Baer ring. Therefore, (1) and (2) are equivalent. $\square$

6. Conclusions

In this paper, we developed a unified theory of localized Baer-type rings relative to idempotents, extending classical annihilator conditions to a broader localized setting. We introduced and systematically investigated several localized classes, including e -Baer, e -quasi-Baer, e - $p.q.$ -Baer, e -PP, and e -APP rings. We established their basic structural properties, determined the principal implication relations among them, and identified additional

hypotheses under which several localized properties become equivalent. Explicit counterexamples were also constructed to show that the established implications and equivalences cannot, in general, be reversed, even under suitable additional centrality assumptions.

A further aspect of the paper concerns the behavior of localized annihilator properties under standard ring-theoretic constructions. We established preservation and transfer results for matrix and triangular matrix rings, direct products and direct sums, polynomial and skew polynomial extensions, skew monoid rings, generalized power series rings, and Morita equivalence. These results provide a unified framework for transferring localized properties across related ring constructions.

Overall, the results establish a systematic connection between classical Baer-type annihilator conditions and their localized analogues. The framework developed here provides a basis for further investigations of localized annihilator properties in Ore extensions, graded and filtered rings, skew generalized constructions, and related module-theoretic settings, as well as their interactions with Armendariz-type conditions and Morita-theoretic invariants.

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