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Novel Identities and Representations for Euler Polynomials of the Second Kind

Waleed Mohamed Abd-Elhameed ^{1,*}, Naher Mohammed A. Alsafri ², Mohammed Atef Abdallah ³
and Amr Kamel Amin ^{4,5}

¹ Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt

² Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23831, Saudi Arabia; nmalsafri@uj.edu.sa

³ Department of Mathematics, Al-Lith University College, Umm Al-Qura University, Al-Lith 24382, Mecca, Saudi Arabia; maabdallah@uqu.edu.sa

⁴ Department of Mathematics, Adham University College, Umm Al-Qura University, Makkah 28653, Saudi Arabia; akgadelrab@uqu.edu.sa

⁵ Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt

* Correspondence: waleed@cu.edu.eg

Abstract

We investigate the Euler polynomials of the second kind (EPs2) and develop a collection of structural identities for this modified Euler family. The analysis begins with an explicit expansion and a compact inverse relation, followed by connections with the classical Euler basis. We then obtain formulas governing differentiated moments and products, together with linearization relations. Higher-order derivatives are expanded in symmetric and non-symmetric bases, including Bernoulli, Laguerre, Jacobi, and generalized Fibonacci and Lucas polynomials. Converse expansions are established as well. We then evaluate several definite integrals to illustrate how the derived identities can be applied.

Keywords: Euler and Bernoulli numbers; Euler polynomials; generating functions; recurrence relations; definite integrals

MSC: 11B39; 11B83; 33C45

1. Introduction

Special functions and polynomial systems provide effective analytic and computational representations in pure and applied mathematics. They occur naturally in approximation, numerical computation, mathematical physics, engineering models, and many related settings; useful general accounts may be found in [1–3].

Current research on polynomial sequences encompasses both algebraic theory and applications. Recurrences for Sobolev orthogonal polynomials were obtained in [4], whereas identities for degenerate Korobov and poly-Korobov polynomials were reported in [5]. Algebraic features of generalized Fibonacci polynomials were examined in [6]. Further contributions concern Changhee identities [7], unifying polynomial constructions [8], orthogonalized q -Bernoulli systems [9], and particular Sheffer families [10]. Formulas for non-symmetric Jacobi polynomials appear in [11], while a parabolic-mean construction yields an Appell-type family in [12]. Combined Fibonacci–Lucas polynomials were developed in [13]. Other recent directions include q -Hermite-based Apostol–Frobenius–Genocchi



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polynomials [14], generalized Gould–Hopper hybrid polynomials [15], extensions of degenerate polynomial families with computational applications [16], and degenerate Laguerre–Bernoulli polynomials [17]. Identities for fully degenerate Dowling polynomials and numbers were obtained in [18]. Polynomial bases also support numerical schemes for differential and integral models; for example, Lucas coefficient polynomials were applied to two such models in [19]. Among the corresponding numerical developments, Pell coefficient polynomials were incorporated into a tau discretization for linear hyperbolic first-order equations [20]. Modified Chebyshev bases were used for boundary-value problems [21], and shifted Horadam polynomials were employed for KdV-type equations [22]. Delannoy-based collocation procedures were proposed for integro-differential models [23] and the time-fractional cable problem [24]. A further Chebyshev construction was applied to the FitzHugh–Nagumo equation [25], while convolved Fermat polynomials furnished a treatment of a fractional Burgers model [26].

Euler numbers and polynomials form a classical sequence with strong links to combinatorics and number theory, as well as a growing range of computational uses. Determinantal representations involving Euler polynomials were developed in [27]. Probabilistic viewpoints for Bernoulli and Euler families were introduced in [28], while degenerate variants and their identities were considered in [29]. Related probabilistic constructions appear in [30]; connections between degenerate (p, q) -Euler polynomials and other sequences were studied in [31]. Multi-Euler–Genocchi families and their degenerate counterparts were investigated in [32], and ordinary together with degenerate Euler numbers and polynomials were addressed in [33]. On the computational side, Euler wavelets were used for fractional optimal-control systems [34]; an Euler-polynomial integration matrix was applied to integro-differential equations [35]; delayed predator–prey models were treated by collocation [36]; and fractional Euler operational matrices were employed for delay integro-differential problems [37].

The principal contributions are summarized below:

- Two independent derivations of differential and connection identities linking EPs2 with classical Euler polynomials.
- Expansions of EPs2 derivatives in several symmetric and non-symmetric polynomial bases.
- Converse derivative representations of other polynomial families in the EPs2 basis.
- Product identities coupling EPs2 with important families, particularly generalized Fibonacci and Lucas polynomials.
- Definite-integral evaluations illustrating direct consequences of the structural results.

The paper proceeds as follows. Section 2 assembles the required fundamentals about Euler polynomials, EPs2, and the auxiliary polynomial families. Section 3 develops two routes to the differential connections between the first- and second-kind Euler bases. Differentiated moments are treated in Section 4, and Section 5 establishes product expansions. Sections 6 and 7 present, respectively, direct and inverse derivative transformations involving symmetric and non-symmetric bases. Definite-integral consequences are collected in Section 8, after which Section 9 closes the paper.

2. Preliminaries and Some Useful Formulas

First, we collect the Euler- and Bernoulli-polynomial identities needed later. The discussion also records the defining EPs2 relations, establishes their inversion formula, and introduces the other polynomial bases used throughout the analysis.

2.1. An Account of Euler Polynomials

The Euler family is characterized by the exponential generating relation [38]:

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^t + 1}. \quad (1)$$

In this paper, the Euler numbers E_k are defined by

$$E_k = 2^k E_k\left(\frac{1}{2}\right), \quad (2)$$

or, equivalently, by the generating function

$$\operatorname{sech} t = \frac{2}{e^t + e^{-t}} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}. \quad (3)$$

In particular, $E_{2k+1} = 0$ for $k \geq 0$. With this convention, the Euler polynomials admit the centered expansion

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}. \quad (4)$$

Throughout the paper, the symbol E_k is retained for the Euler numbers defined by (2). Conversely, the monomial in (4) can be represented by the following formula [39]:

$$x^n = E_n(x) + \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} E_k(x), \quad n \geq 0. \quad (5)$$

The first-order derivative formula of Euler polynomials can be written as

$$\frac{d}{dx} E_n(x) = n E_{n-1}(x), \quad n \geq 1, \quad (6)$$

and the higher-order derivatives take the form

$$D^\ell E_n(x) = \frac{n!}{(n-\ell)!} E_{n-\ell}(x), \quad n \geq \ell. \quad (7)$$

In addition, it is well known that Euler polynomials satisfy the symmetry property

$$E_n(1-x) = (-1)^n E_n(x), \quad (8)$$

which implies that

$$E_{2m+1}\left(\frac{1}{2}\right) = 0, \quad m \geq 0. \quad (9)$$

2.2. An Account of Bernoulli Polynomials

The standard Bernoulli polynomials $\{B_n(x)\}_{n \geq 0}$ are generated by the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \quad (10)$$

where B_n are the celebrated Bernoulli numbers that are defined as $B_n = B_n(0)$. $B_n(x)$ may be represented as

$$B_n(x) = \sum_{\ell=0}^n \binom{n}{\ell} B_\ell x^{n-\ell}. \quad (11)$$

The inverse formula of (11) is given by

$$x^n = \frac{1}{n+1} \sum_{r=0}^n \binom{n+1}{r} B_r(x). \quad (12)$$

2.3. An Account of the EPs2

The authors of [40] introduced the EPs2 as

$$E_n^*(x) = 2^n E_n\left(\frac{1+x}{2}\right). \quad (13)$$

We define the EPs2 through the exponential generating relation

$$\sum_{n=0}^{\infty} E_n^*(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^{-t} + e^t}. \quad (14)$$

The polynomials $E_r^*(x)$ can be written in series form as

$$E_r^*(x) = \sum_{j=0}^{\lfloor r/2 \rfloor} \binom{r}{2j} E_{2j} x^{r-2j}. \quad (15)$$

From property (7), and the definition in (13), the derivative formula for the EPs2 may be represented as:

$$D^\ell E_k^*(x) = \frac{k!}{(k-\ell)!} E_{k-\ell}^*(x), \quad k \geq \ell. \quad (16)$$

Our next aim is to invert (15) in a compact form. The following auxiliary identity is used in its derivation.

Lemma 1. *The following identity holds for every non-negative integer m :*

$$\sum_{L=0}^m \frac{E_{2L}}{(2L)!(2(m-L))!} = \delta_{m,0} = \begin{cases} 1, & m = 0, \\ 0, & m \geq 1. \end{cases} \quad (17)$$

Proof. Let

$$S_m = \sum_{L=0}^m \frac{E_{2L}}{(2L)!(2(m-L))!}.$$

Consider the generating series

$$V(t) = \sum_{m=0}^{\infty} S_m t^{2m}, \quad (18)$$

which also takes the form

$$V(t) = \sum_{m=0}^{\infty} \sum_{L=0}^m \frac{E_{2L}}{(2L)!(2(m-L))!} t^{2m}.$$

The Cauchy product formula enables one to write $V(t)$ as

$$V(t) = \left(\sum_{L=0}^{\infty} \frac{E_{2L}}{(2L)!} t^{2L} \right) \left(\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \right). \quad (19)$$

By the definition of the Euler numbers in (3), ref. [38]

$$\sum_{n=0}^{\infty} \frac{E_n}{n!} t^n = \frac{2}{e^t + e^{-t}} = \operatorname{sech}(t).$$

Since $E_{2n+1} = 0$, $n \geq 0$, we can write

$$\sum_{L=0}^{\infty} \frac{E_{2L}}{(2L)!} t^{2L} = \operatorname{sech}(t).$$

Moreover, we have

$$\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} = \cosh(t).$$

Therefore (19) implies that

$$V(t) = \operatorname{sech}(t) \cosh(t) = 1. \quad (20)$$

Comparing the coefficients of t^{2m} in (18) and (20), yields

$$S_m = \delta_{m,0}.$$

Therefore,

$$\sum_{L=0}^m \frac{E_{2L}}{(2L)! (2(m-L))!} = \delta_{m,0}.$$

This completes the proof. $\square$

The next result supplies the inverse transformation associated with (15).

Theorem 1. For every non-negative integer j , one has

$$x^j = j! \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{E_{j-2r}^*(x)}{(j-2r)! (2r)!}. \quad (21)$$

Proof. To prove (21), we consider the polynomial

$$P_j(x) = j! \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{E_{j-2r}^*(x)}{(j-2r)! (2r)!}, \quad (22)$$

and we will prove that

$$P_j(x) = x^j.$$

Inserting the series form (15) in the last formula gives

$$P_j(x) = j! \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{1}{(j-2r)! (2r)!} \sum_{L=0}^{\lfloor \frac{j-2r}{2} \rfloor} \binom{j-2r}{2L} E_{2L} x^{j-2r-2L}, \quad (23)$$

which can be turned into the following formula after rearranging the terms:

$$P_j(x) = \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{j!}{(j-2m)!} \left(\sum_{L=0}^m \frac{E_{2L}}{(2L)! (2(m-L))!} \right) x^{j-2m}. \quad (24)$$

The application of Lemma 1 into (24) immediately yields formula (21). This proves the theorem. $\square$

2.4. An Overview of Some Polynomials

Here, we introduce two families of polynomials, namely the symmetric and non-symmetric ones, denoted by $\phi_n(x)$ and $\psi_n(x)$, respectively, that may be represented as

$$\phi_n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} B_{r,n} x^{n-2r}, \quad (25)$$

$$\psi_n(x) = \sum_{r=0}^n G_{r,n} x^{n-r}, \quad (26)$$

where $B_{r,n}$ and $G_{r,n}$ are assumed to be known.

It can be assumed that the inverse relations associated with (25) and (26) are represented as:

$$x^n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \bar{B}_{r,n} \phi_{n-2r}(x), \quad (27)$$

$$x^n = \sum_{r=0}^n \bar{G}_{r,n} \psi_{n-r}(x), \quad (28)$$

and $\bar{B}_{r,n}$ and $\bar{G}_{r,n}$ are also assumed to be known.

Here, we recall some related polynomial families that will be used in the sequel. The normalized Jacobi polynomials admit the hypergeometric representation [41]

$$R_r^{(\delta,\sigma)}(x) = {}_2F_1 \left(\begin{matrix} -r, r + \delta + \sigma + 1 \\ \delta + 1 \end{matrix} \middle| \frac{1-x}{2} \right). \quad (29)$$

The normalized ultraspherical (Gegenbauer) polynomials are defined by

$$C_r^{(\delta)}(x) = R_r^{(\delta-\frac{1}{2}, \delta-\frac{1}{2})}(x). \quad (30)$$

The shifted normalized Jacobi polynomials on $[0, 1]$ are defined by

$$\tilde{R}_r^{(\delta,\sigma)}(x) = R_r^{(\delta,\sigma)}(2x-1). \quad (31)$$

Their power representation is [41]

$$\tilde{R}_m^{(\delta,\sigma)}(x) = \sum_{\ell=0}^m \frac{(-1)^\ell m! \Gamma(\delta+1) (\sigma+1)_m (\delta+\sigma+1)_{2m-\ell}}{\ell! (m-\ell)! \Gamma(m+\delta+1) (\delta+\sigma+1)_m (\sigma+1)_{m-\ell}} x^{m-\ell}. \quad (32)$$

The corresponding inverse formula is

$$x^j = \sum_{r=0}^j \frac{\binom{j}{r} (\delta+1)_{j-r} (j-r+\sigma+1)_r}{(2j-2r+\delta+\sigma+2)_r (j-r+\delta+\sigma+1)_{j-r}} \tilde{R}_{j-r}^{(\delta,\sigma)}(x). \quad (33)$$

The generalized Fibonacci- and Lucas-type families required below are specified by the recurrences

$$W_m^{A,B}(x) = A x W_{m-1}^{A,B}(x) + B W_{m-2}^{A,B}(x), \quad W_0^{A,B}(x) = 1, \quad W_1^{A,B}(x) = A x, \quad m \geq 2, \quad (34)$$

$$w_m^{R,T}(x) = R x w_{m-1}^{R,T}(x) + T w_{m-2}^{R,T}(x), \quad w_0^{R,T}(x) = 2, \quad w_1^{R,T}(x) = R x, \quad m \geq 2, \quad (35)$$

with non-zero constants A, B, R, T .

The two Binet forms of these polynomials are represented as

$$W_k^{A,B}(x) = \frac{\left(Ax + \sqrt{A^2x^2 + 4B}\right)^{k+1} - \left(Ax - \sqrt{A^2x^2 + 4B}\right)^{k+1}}{2^{k+1}\sqrt{A^2x^2 + 4B}}, \quad (36)$$

$$w_k^{R,T}(x) = \frac{\left(Rx + \sqrt{R^2x^2 + 4T}\right)^k + \left(Rx - \sqrt{R^2x^2 + 4T}\right)^k}{2^k}. \quad (37)$$

In addition, these polynomials may be represented as

$$W_m^{A,B}(x) = \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} A^{m-2s} B^s \binom{m-s}{s} x^{m-2s}, \quad (38)$$

$$w_m^{R,T}(x) = \begin{cases} 2, & m = 0, \\ m \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} \frac{R^{m-2s} T^s \binom{m-s}{s}}{m-s} x^{m-2s}, & m \geq 1. \end{cases} \quad (39)$$

The moment formulas of $W_k^{A,B}(x)$ and $w_k^{R,T}(x)$ are represented as

$$x^r W_k^{A,B}(x) = \sum_{m=0}^r \binom{r}{m} A^{-r} (-B)^m W_{k+r-2m}^{A,B}(x), \quad (40)$$

$$x^r w_k^{R,T}(x) = \sum_{m=0}^r \binom{r}{m} R^{-r} (-T)^m w_{k+r-2m}^{R,T}(x). \quad (41)$$

3. New Formulas Between Euler Polynomials of the First and Second Kinds

This section constructs differential transformations between the EPs2 and the classical Euler basis. Both directions are considered, and two distinct arguments are used to establish the resulting identities.

3.1. Derivatives of the EPs2 in Terms of the Euler Polynomials

Theorem 2. Consider N, ℓ to be positive integers with $N \geq \ell$. The following derivative formula holds:

$$\begin{aligned} \frac{d^\ell}{dx^\ell} E_N^*(x) &= \frac{N!}{(N-\ell)!} \left[E_{N-\ell}(x) + \frac{1}{2} \sum_{\rho=1}^{\lfloor \frac{N-\ell}{2} \rfloor} \binom{N-\ell}{2\rho} E_{2\rho} E_{N-\ell-2\rho}(x) \right. \\ &\quad \left. + \frac{1}{2} \sum_{\rho=0}^{\lfloor \frac{N-\ell-1}{2} \rfloor} \binom{N-\ell}{2\rho+1} \frac{2^{2\rho+1}(2^{2\rho+2}-1)B_{2\rho+2}}{\rho+1} E_{N-\ell-(2\rho+1)}(x) \right]. \end{aligned} \quad (42)$$

Proof. First, we establish the following connection formula:

$$\begin{aligned} E_N^*(x) &= E_N(x) + \frac{1}{2} \sum_{\rho=1}^{\lfloor N/2 \rfloor} \binom{N}{2\rho} E_{2\rho} E_{N-2\rho}(x) \\ &\quad + \frac{1}{2} \sum_{\rho=0}^{\lfloor (N-1)/2 \rfloor} \binom{N}{2\rho+1} \frac{2^{2\rho+1}(2^{2\rho+2}-1)B_{2\rho+2}}{\rho+1} E_{N-(2\rho+1)}(x), \end{aligned} \quad (43)$$

where B_ρ and E_ρ are, respectively, the Bernoulli and Euler numbers. We will use the two generating functions of $E_N(x)$ and $E_N^*(x)$. Now, let

$$F(x, t) = \sum_{N=0}^{\infty} E_N(x) \frac{t^N}{N!} = \frac{2e^{xt}}{e^t + 1}, \quad (44)$$

$$G(x, t) = \sum_{N=0}^{\infty} E_N^*(x) \frac{t^N}{N!} = \frac{2e^{xt}}{e^{-t} + e^t}, \quad (45)$$

and define

$$R(x, t) = \frac{G(x, t)}{F(x, t)}. \quad (46)$$

Using (44) and (45), it can be seen that

$$R(x, t) = \frac{e^t + 1}{e^t + e^{-t}}. \quad (47)$$

Recalling the two standard identities

$$\operatorname{sech} t = \frac{2}{e^t + e^{-t}}, \quad (48)$$

$$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad (49)$$

it can be shown that $R(x, t)$ may be represented as

$$R(x, t) = \frac{1}{2} (1 + \operatorname{sech} t + \tanh t). \quad (50)$$

Now, we make use of the following two expansions of $\operatorname{sech} t$, and $\tanh t$:

$$\operatorname{sech} t = \sum_{\rho=0}^{\infty} E_{2\rho} \frac{t^{2\rho}}{(2\rho)!}, \quad (51)$$

$$\tanh t = \sum_{\rho=0}^{\infty} \frac{2^{2\rho+1} (2^{2\rho+2} - 1) B_{2\rho+2}}{\rho + 1} \frac{t^{2\rho+1}}{(2\rho + 1)!}. \quad (52)$$

Inserting (51) and (52) into (50) gives

$$R(x, t) = 1 + \frac{1}{2} \sum_{\rho=1}^{\infty} E_{2\rho} \frac{t^{2\rho}}{(2\rho)!} + \frac{1}{2} \sum_{\rho=0}^{\infty} \frac{2^{2\rho+1} (2^{2\rho+2} - 1) B_{2\rho+2}}{\rho + 1} \frac{t^{2\rho+1}}{(2\rho + 1)!}. \quad (53)$$

From (46), we have

$$G(x, t) = F(x, t) R(x, t), \quad (54)$$

and accordingly, substituting (44) and (53) into (54) yields

$$G(x, t) = \left(\sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!} \right) \left(1 + \frac{1}{2} \sum_{\rho=1}^{\infty} E_{2\rho} \frac{t^{2\rho}}{(2\rho)!} + \frac{1}{2} \sum_{\rho=0}^{\infty} \frac{2^{2\rho+1} (2^{2\rho+2} - 1) B_{2\rho+2}}{\rho + 1} \frac{t^{2\rho+1}}{(2\rho + 1)!} \right). \quad (55)$$

Now, if we consider

$$A(t) = \sum_{N=0}^{\infty} a_N \frac{t^N}{N!}, \quad B(t) = \sum_{N=0}^{\infty} b_N \frac{t^N}{N!}, \quad (56)$$

where a_N and b_N denote the exponential generating coefficients of $A(t)$ and $B(t)$, respectively, that is, $a_N = N! [t^N]A(t)$ and $b_N = N! [t^N]B(t)$, then the Cauchy product formula implies that

$$A(t)B(t) = \sum_{N=0}^{\infty} \left(\sum_{k=0}^N \binom{N}{k} b_k a_{N-k} \right) \frac{t^N}{N!}. \quad (57)$$

This means that the generating function $G(x, t)$ given in (54) can be written as

$$G(x, t) = \sum_{N=0}^{\infty} \left(\sum_{k=0}^N \binom{N}{k} b_k E_{N-k}(x) \right) \frac{t^N}{N!}, \quad (58)$$

where $a_k = E_k(x)$, and the coefficients $b_k = k! [t^k]R(t)$, may be represented explicitly from formula (55) by

$$b_k = \begin{cases} 1, & k = 0, \\ \frac{1}{2} \frac{2^k(2^{k+1} - 1)B_{k+1}}{(k+1)/2}, & k \text{ odd}, \\ \frac{1}{2} E_k, & k \geq 2, \quad k \text{ even}. \end{cases} \quad (59)$$

Substituting (59) into (58) and separating the term $k = 0$, we obtain

$$\begin{aligned} N! [t^N] G(x, t) &= E_N(x) + \frac{1}{2} \sum_{\rho=1}^{\lfloor N/2 \rfloor} \binom{N}{2\rho} E_{2\rho} E_{N-2\rho}(x) \\ &+ \frac{1}{2} \sum_{\rho=0}^{\lfloor (N-1)/2 \rfloor} \binom{N}{2\rho+1} \frac{2^{2\rho+1}(2^{2\rho+2} - 1)B_{2\rho+2}}{\rho+1} E_{N-(2\rho+1)}(x). \end{aligned} \quad (60)$$

In addition, since

$$N! [t^N] G(x, t) = E_N^*(x), \quad (61)$$

then equating (60) and (61) yields the following connection formula:

$$\begin{aligned} E_N^*(x) &= E_N(x) + \frac{1}{2} \sum_{\rho=1}^{\lfloor N/2 \rfloor} \binom{N}{2\rho} E_{2\rho} E_{N-2\rho}(x) \\ &+ \frac{1}{2} \sum_{\rho=0}^{\lfloor (N-1)/2 \rfloor} \binom{N}{2\rho+1} \frac{2^{2\rho+1}(2^{2\rho+2} - 1)B_{2\rho+2}}{\rho+1} E_{N-(2\rho+1)}(x). \end{aligned} \quad (62)$$

Now, using the derivative formula (7), formula (42) can be obtained. $\square$

3.2. Derivatives of Euler Polynomials in Terms of the EPs2

This section is devoted to the derivation of the derivative formula of the Euler polynomials in terms of the second Euler polynomials. To proceed, we require the following two lemmas.

Lemma 2. For each positive integer μ , one has the finite-sum identity

$$\sum_{\rho=0}^{\mu} \binom{2\mu+2}{2\rho+2} B_{2\rho+2} = \mu + B_{2\mu+2}. \quad (63)$$

Proof. We begin with the well-known identity:

$$\sum_{k=0}^N \binom{N+1}{k} B_k = 0, \quad N \geq 1. \quad (64)$$

Replacing N by $(2\mu + 1)$, we get

$$\sum_{k=0}^{2\mu+1} \binom{2\mu+2}{k} B_k = 0, \quad \mu \geq 0. \quad (65)$$

If we note the special values of Bernoulli numbers

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_{2\rho+1} = 0, \quad \rho \geq 1, \quad (66)$$

then identity (65) turns into

$$\sum_{\rho=0}^{\mu} \binom{2\mu+2}{2\rho+2} B_{2\rho+2} = \mu + B_{2\mu+2}. \quad (67)$$

□

Lemma 3. For any integer $\mu \geq 1$, the following identity is satisfied:

$$\sum_{\rho=0}^{\mu} 4^{\rho+1} \binom{2\mu+2}{2\rho+2} B_{2\rho+2} = 2\mu + 1 + (2 - 2^{2\mu+2}) B_{2\mu+2}. \quad (68)$$

Proof. If we set $x = \frac{1}{2}$, in the series form of Bernoulli polynomials in (11), we get

$$2^N B_N \left(\frac{1}{2} \right) = \sum_{k=0}^N \binom{N}{k} 2^k B_k. \quad (69)$$

In addition, the following identity holds [38].

$$B_N \left(\frac{1}{2} \right) = (2^{1-N} - 1) B_N. \quad (70)$$

Thus, combining the two identities (69) and (70) yields

$$\sum_{k=0}^N \binom{N}{k} 2^k B_k = (2 - 2^N) B_N. \quad (71)$$

Replacing N by $(2\mu + 2)$, we get

$$\sum_{k=0}^{2\mu+2} \binom{2\mu+2}{k} 2^k B_k = (2 - 2^{2\mu+2}) B_{2\mu+2}. \quad (72)$$

Noting the special values in (66), one obtains the following formula:

$$\sum_{\rho=0}^{\mu} 4^{\rho+1} \binom{2\mu+2}{2\rho+2} B_{2\rho+2} = 2\mu + 1 + (2 - 2^{2\mu+2}) B_{2\mu+2}. \quad (73)$$

□

Theorem 3. Let N, ℓ be positive integers. The ℓ -th derivative of $E_N(x)$ with $N \geq \ell$ may be represented as

$$D^\ell E_N(x) = 2N! \left[\sum_{\mu=0}^{\lfloor \frac{N-\ell}{2} \rfloor} \frac{1}{2(N-2\mu-\ell)!(2\mu)!} E_{N-\ell-2\mu}^*(x) + \sum_{\mu=0}^{\lfloor \frac{N-\ell-1}{2} \rfloor} \frac{(2^{2\mu+2}-1)B_{2\mu+2}-\mu-1}{(2\mu+2)!(N-2\mu-\ell-1)!} E_{N-\ell-2\mu-1}^*(x) \right]. \quad (74)$$

Proof. We differentiate the power representation of $E_N(x)$ to obtain the following derivative expression:

$$D^\ell E_N(x) = 2N! \sum_{\rho=0}^{N-\ell} \frac{(1-2^{\rho+1})B_{\rho+1}}{(\rho+1)!(N-\ell-\rho)!} x^{N-\ell-\rho}. \quad (75)$$

If we apply the inverse formula of $E_N^*(x)$ in (21), then we obtain

$$D^\ell E_N(x) = 2N! \sum_{\rho=0}^{N-\ell} \frac{(1-2^{\rho+1})B_{\rho+1}}{(\rho+1)!} \sum_{\tau=0}^{\lfloor \frac{N-\ell-\rho}{2} \rfloor} \frac{1}{(N-\ell-\rho-2\tau)!(2\tau)!} E_{N-\ell-\rho-2\tau}^*(x), \quad (76)$$

which, after rearrangement, takes the following equivalent form:

$$D^\ell E_N(x) = 2N! \left[\sum_{\mu=0}^{\lfloor \frac{N-\ell}{2} \rfloor} \frac{1}{(N-2\mu-\ell)!} \sum_{\rho=0}^{\mu} \frac{(1-2^{2\rho+1})B_{2\rho+1}}{(2\mu-2\rho)!(2\rho+1)!} E_{N-\ell-2\mu}^*(x) + \sum_{\mu=0}^{\lfloor \frac{N-\ell-1}{2} \rfloor} \frac{1}{(N-2\mu-\ell-1)!} \sum_{\rho=0}^{\mu} \frac{(1-4^{\rho+1})B_{2\rho+2}}{(2\mu-2\rho)!(2\rho+2)!} E_{N-\ell-2\mu-1}^*(x) \right]. \quad (77)$$

Now, in order to turn (77) into a simplified formula, write it as

$$D^\ell E_N(x) = 2N! \left[\sum_{\mu=0}^{\lfloor \frac{N-\ell}{2} \rfloor} \frac{R_\mu}{(N-2\mu-\ell)!} E_{N-\ell-2\mu}^*(x) + \sum_{\mu=0}^{\lfloor \frac{N-\ell-1}{2} \rfloor} \frac{\bar{R}_\mu}{(N-2\mu-\ell-1)!} E_{N-\ell-2\mu-1}^*(x) \right], \quad (78)$$

where

$$R_\mu = \sum_{\rho=0}^{\mu} \frac{(1-2^{2\rho+1})B_{2\rho+1}}{(2\mu-2\rho)!(2\rho+1)!}, \quad (79)$$

$$\bar{R}_\mu = \sum_{\rho=0}^{\mu} \frac{(1-4^{\rho+1})B_{2\rho+2}}{(2\mu-2\rho)!(2\rho+2)!}. \quad (80)$$

The remaining task is to evaluate R_μ and $\bar{R}_\mu$ explicitly.

Regarding R_μ , since $B_{2\mu+1} = 0$, for all $\mu \geq 1$, and $B_1 = \frac{-1}{2}$, then R_μ is reduced to the form

$$R_\mu = \frac{1}{2(2\mu)!}. \quad (81)$$

Regarding the sum $\bar{R}_\mu$, it may be expressed as

$$\bar{R}_\mu = \frac{1}{(2\mu+2)!} \left(\sum_{\rho=0}^{\mu} \binom{2\mu+2}{2\rho+2} B_{2\rho+2} - \sum_{\rho=0}^{\mu} 4^{\rho+1} \binom{2\mu+2}{2\rho+2} B_{2\rho+2} \right). \quad (82)$$

Lemmas 2, and 3 serve to find a closed form of $\bar{R}_\mu$ in the last form as

$$\bar{R}_\mu = \frac{-1 - \mu + (2^{2\mu+2} - 1) B_{2\mu+2}}{(2\mu + 2)!}. \quad (83)$$

The two closed forms for the two sums in (81) and (83) convert (78) into the following elegant form:

$$D^\ell E_N(x) = 2N! \left[\sum_{\mu=0}^{\lfloor \frac{N-\ell}{2} \rfloor} \frac{1}{2(N-2\mu-\ell)!(2\mu)!} E_{N-\ell-2\mu}^*(x) + \sum_{\mu=0}^{\lfloor \frac{N-\ell-1}{2} \rfloor} \frac{(2^{2\mu+2} - 1) B_{2\mu+2} - \mu - 1}{(2\mu + 2)!(N-2\mu-\ell-1)!} E_{N-\ell-2\mu-1}^*(x) \right].$$

The asserted identity now follows. $\square$

4. Derivatives of the Moments of the EPs2

In this section, we express differentiated EPs2 moments entirely in the original EPs2 basis.

Theorem 4. For $N, \mu, \ell \in \mathbb{N}_0$ with $\ell \leq N + \mu$, the differentiated moment satisfies

$$D^\ell (x^\mu E_N^*(x)) = \sum_{\sigma=0}^{\lfloor \frac{N+\mu-\ell}{2} \rfloor} \sum_{\rho=0}^{\sigma} \frac{\binom{N}{2\rho} E_{2\rho}(N + \mu - 2\rho)!}{(N + \mu - 2\sigma - \ell)!(2(\sigma - \rho))!} E_{N+\mu-\ell-2\sigma}^*(x). \quad (84)$$

Proof. Termwise differentiation of the expansion (15) gives

$$D^\ell (x^\mu E_N^*(x)) = \sum_{\rho=0}^{\lfloor \frac{N+\mu-\ell}{2} \rfloor} \binom{N}{2\rho} E_{2\rho}(1 + N + \mu - \ell - 2\rho)_\ell x^{N+\mu-2\rho-\ell}. \quad (85)$$

The inverse formula in (21) leads to the following expansion:

$$x^{N+\mu-2\rho-\ell} = \sum_{\tau=0}^{\lfloor \frac{N+\mu-\ell}{2} \rfloor - \rho} \frac{(N + \mu - \ell - 2\rho)!}{(N + \mu - \ell - 2\rho - 2\tau)!(2\tau)!} E_{N+\mu-\ell-2\rho-2\tau}^*(x), \quad (86)$$

and thus, formula (85) may be reformulated as

$$D^\ell (x^\mu E_N^*(x)) = \sum_{\rho=0}^{\lfloor \frac{N+\mu-\ell}{2} \rfloor} \binom{N}{2\rho} E_{2\rho}(1 + N + \mu - \ell - 2\rho)_\ell \times \sum_{\tau=0}^{\lfloor \frac{N+\mu-\ell}{2} \rfloor - \rho} \frac{(N + \mu - \ell - 2\rho)!}{(N + \mu - \ell - 2\rho - 2\tau)!(2\tau)!} E_{N+\mu-\ell-2\rho-2\tau}^*(x). \quad (87)$$

Upon expanding and rearranging the previous expression, the following form is obtained:

$$D^\ell (x^\mu E_N^*(x)) = \sum_{\sigma=0}^{\lfloor \frac{N+\mu-\ell}{2} \rfloor} \sum_{\rho=0}^{\sigma} \frac{\binom{N}{2\rho} E_{2\rho}(N + \mu - 2\rho)!}{(N + \mu - 2\sigma - \ell)!(2(\sigma - \rho))!} E_{N+\mu-\ell-2\sigma}^*(x). \quad (88)$$

This is the asserted expression. $\square$

Two useful specializations of Theorem 4 are recorded next.

Corollary 1. Given $N, \mu \in \mathbb{N}_0$, multiplication of $E_N^*(x)$ by x^μ satisfies

$$x^\mu E_N^*(x) = \sum_{p=0}^{\lfloor \frac{N+\mu}{2} \rfloor} \sum_{\ell=0}^p \frac{\binom{N}{2\ell} E_{2\ell} (N+\mu-2\ell)!}{(N+\mu-2p)!(2(p-\ell))!} E_{N+\mu-2p}^*(x). \quad (89)$$

Proof. Setting $\ell = 0$ in (84) gives the stated multiplication formula. $\square$

Corollary 2. Whenever $N \geq \ell \geq 0$ are integers, the ℓ th derivative reduces to

$$D^\ell E_N^*(x) = \frac{N!}{(N-\ell)!} E_{N-\ell}^*(x). \quad (90)$$

Proof. Taking $\mu = 0$ in (84) is admissible under the non-negative-index form of Theorem 4 and produces the following formula:

$$D^\ell E_N^*(x) = \sum_{\sigma=0}^{\lfloor \frac{N-\ell}{2} \rfloor} \frac{N!}{(N-2\sigma-\ell)!} \left(\sum_{\rho=0}^{\sigma} \frac{E_{2\rho}}{(2\rho)!(2\sigma-2\rho)!} \right) E_{N-\ell-2\sigma}^*(x), \quad (91)$$

which can be written equivalently as

$$D^\ell E_N^*(x) = \sum_{\sigma=0}^{\lfloor \frac{N-\ell}{2} \rfloor} \frac{N!}{(2\sigma)!(N-2\sigma-\ell)!} \left(\sum_{\rho=0}^{\sigma} \binom{2\sigma}{2\rho} E_{2\rho} \right) E_{N-\ell-2\sigma}^*(x). \quad (92)$$

Using the identity (17) yields

$$\sum_{\rho=0}^{\sigma} \binom{2\sigma}{2\rho} E_{2\rho} = 0 \quad (\sigma \geq 1),$$

it follows that

$$\sum_{\rho=0}^{\sigma} \frac{E_{2\rho}}{(2\rho)!(2\sigma-2\rho)!} = \delta_{\sigma,0},$$

and accordingly, formula (92) can be reduced into the following form:

$$D^\ell E_N^*(x) = \frac{N!}{(N-\ell)!} E_{N-\ell}^*(x),$$

which proves the corollary. $\square$

5. Some New Linearization Formulas Involving the EPs2

This section derives linearization formulas involving the EPs2 and the generalized Fibonacci and Lucas polynomials.

Theorem 5. For arbitrary $N, M \in \mathbb{N}_0$, the product with the generalized Fibonacci family linearizes as

$$E_N^*(x) W_M^{A,B}(x) = \sum_{\ell=0}^N \left(\sum_{\nu=0}^{\ell} A^{2\nu-N} (-B)^{\ell-\nu} \binom{N}{2\nu} \binom{N-2\nu}{\ell-\nu} E_{2\nu} \right) W_{N+M-2\ell}^{A,B}(x). \quad (93)$$

Proof. Multiplying the expansion (15) by $W_M^{A,B}(x)$ gives

$$E_N^*(x) W_M^{A,B}(x) = \sum_{\sigma=0}^{\lfloor N/2 \rfloor} \binom{N}{2\sigma} E_{2\sigma} x^{N-2\sigma} W_M^{A,B}(x). \quad (94)$$

Applying the generalized Fibonacci moment relation (40) yields

$$x^{N-2\sigma} W_M^{A,B}(x) = \sum_{\tau=0}^{N-2\sigma} A^{2\sigma-N} (-B)^\tau \binom{N-2\sigma}{\tau} W_{N+M-2\sigma-2\tau}^{A,B}(x). \quad (95)$$

Substituting this expansion into (94) gives

$$E_N^*(x) W_M^{A,B}(x) = \sum_{\sigma=0}^{\lfloor N/2 \rfloor} \binom{N}{2\sigma} E_{2\sigma} A^{2\sigma-N} \left(\sum_{\tau=0}^{N-2\sigma} \binom{N-2\sigma}{\tau} (-B)^\tau W_{N+M-2\sigma-2\tau}^{A,B}(x) \right). \quad (96)$$

With $\rho = \sigma + \tau$, an interchange of the finite summations transforms the preceding expression into

$$E_N^*(x) W_M^{A,B}(x) = \sum_{\rho=0}^N \left(\sum_{\sigma=0}^{\rho} A^{2\sigma-N} (-B)^{\rho-\sigma} \binom{N}{2\sigma} \binom{N-2\sigma}{\rho-\sigma} E_{2\sigma} \right) W_{N+M-2\rho}^{A,B}(x), \quad (97)$$

which completes the proof. $\square$

Theorem 6. For arbitrary $N, M \in \mathbb{N}_0$, the analogous generalized Lucas linearization is

$$E_N^*(x) w_M^{R,T}(x) = \sum_{\ell=0}^N \left(\sum_{\nu=0}^{\ell} R^{2\nu-N} (-T)^{\ell-\nu} \binom{N}{2\nu} \binom{N-2\nu}{\ell-\nu} E_{2\nu} \right) w_{N+M-2\ell}^{R,T}(x). \quad (98)$$

Proof. Repeating the argument used for Theorem 5, with the Lucas-type moment relation in place of its Fibonacci counterpart, proves the result. $\square$

6. Derivative Formulas of the EPs2 in Terms of Different Polynomials

Here, we transform EPs2 derivatives into the symmetric and non-symmetric polynomial bases specified by (25) and (26).

6.1. Derivatives as Combinations of Non-Symmetric Polynomials

We begin with a general non-symmetric-basis expansion based on (26); several concrete families are then recovered by specialization.

Theorem 7. Suppose that $N, \mu \in \mathbb{N}_0$, $\mu \leq N$, and that $\psi_i(x)$ is a non-symmetric polynomial sequence having the representation (26). Then the μ th EPs2 derivative has the expansion

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \binom{N}{2\sigma} E_{2\sigma} (1-\mu+N-2\sigma)_\mu \bar{G}_{2\kappa-2\sigma, N-2\sigma-\mu} \psi_{N-\mu-2\kappa}(x) \\ &+ \sum_{\kappa=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \binom{N}{2\sigma} E_{2\sigma} (1-\mu+N-2\sigma)_\mu \bar{G}_{2\kappa-2\sigma+1, N-2\sigma-\mu} \psi_{N-\mu-2\kappa-1}(x), \end{aligned} \quad (99)$$

where $\bar{G}_{N,n}$ are the coefficients appear in (28).

Proof. Differentiating (15) μ times term by term yields

$$\frac{d^\mu}{dx^\mu} E_N^*(x) = \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \binom{N}{2\sigma} E_{2\sigma} (1-\mu+N-2\sigma)_\mu x^{N-2\sigma-\mu}. \quad (100)$$

Inserting formula (28) into the last formula yields

$$\frac{d^\mu}{dx^\mu} E_N^*(x) = \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \binom{N}{2\sigma} E_{2\sigma} (1-\mu+N-2\sigma)_\mu \sum_{\tau=0}^{N-2\sigma-\mu} \bar{G}_{\tau, N-2\sigma-\mu} \psi_{N-2\sigma-\mu-\tau}(x). \quad (101)$$

Finally, we split the index τ into even and odd parts by writing $\tau = 2\kappa - 2\sigma$ and $\tau = 2\kappa - 2\sigma + 1$, and then rearrange the order of summation. This leads directly to

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \binom{N}{2\sigma} E_{2\sigma} (1 - \mu + N - 2\sigma)_\mu \bar{G}_{2\kappa-2\sigma, N-2\sigma-\mu} \psi_{N-\mu-2\kappa}(x) \\ &+ \sum_{\kappa=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \binom{N}{2\sigma} E_{2\sigma} (1 - \mu + N - 2\sigma)_\mu \bar{G}_{2\kappa-2\sigma+1, N-2\sigma-\mu} \psi_{N-\mu-2\kappa-1}(x). \end{aligned} \quad (102)$$

which completes the proof. $\square$

Remark 1. Theorem 7 serves to obtain explicit formulas of the derivatives of the EPs2 in terms of other non-symmetric polynomials. In this regard, we present some formulas.

Corollary 3. Let N, μ be non-negative integers with $\mu \leq N$. Then,

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= N! \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{2^{2\kappa+2} (2^{2\kappa+2} - 1) B_{2\kappa+2}}{(2\kappa+2)! (N-\mu-2\kappa)!} B_{N-\mu-2\kappa}(x) \\ &- N! \sum_{\kappa=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \frac{E_{2\kappa+2}}{(2\kappa+2)! (N-\mu-2\kappa-1)!} B_{N-\mu-2\kappa-1}(x). \end{aligned} \quad (103)$$

Proof. Combining Theorem 7 with the Bernoulli inversion relation (12) yields

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= N! \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{1}{(N-\mu-2\kappa)!} \left(\sum_{\sigma=0}^{\kappa} \frac{E_{2\sigma}}{(2\kappa-2\sigma+1)! (2\sigma)!} \right) B_{N-\mu-2\kappa}(x) \\ &+ N! \sum_{\kappa=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \frac{1}{(N-\mu-2\kappa-1)!} \left(\sum_{\sigma=0}^{\kappa} \frac{E_{2\sigma}}{(2\kappa-2\sigma+2)! (2\sigma)!} \right) B_{N-\mu-2\kappa-1}(x). \end{aligned} \quad (104)$$

Now, we make use of the following two identities:

$$\sum_{\sigma=0}^{\kappa} \frac{E_{2\sigma}}{(2\kappa-2\sigma+1)! (2\sigma)!} = \frac{2^{2\kappa+2} (2^{2\kappa+2} - 1) B_{2\kappa+2}}{(2\kappa+2)!}, \quad (105)$$

$$\sum_{\sigma=0}^{\kappa} \frac{E_{2\sigma}}{(2\kappa-2\sigma+2)! (2\sigma)!} = -\frac{E_{2\kappa+2}}{(2\kappa+2)!}. \quad (106)$$

Substituting (105) and (106) into formula (104) yields

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= N! \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{2^{2\kappa+2} (2^{2\kappa+2} - 1) B_{2\kappa+2}}{(2\kappa+2)! (N-\mu-2\kappa)!} B_{N-\mu-2\kappa}(x) \\ &- N! \sum_{\kappa=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \frac{E_{2\kappa+2}}{(2\kappa+2)! (N-\mu-2\kappa-1)!} B_{N-\mu-2\kappa-1}(x), \end{aligned} \quad (107)$$

which is exactly the claimed formula. $\square$

Corollary 4. Let N, μ be non-negative integers with $\mu \leq N$. Then the μ -th derivative of the Euler polynomials of the second kind $E_N^*(x)$ admits the following generalized Laguerre expansion:

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= N! (-1)^{N-\mu} \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \frac{E_{2\sigma} \Gamma(1+N-\mu-2\sigma+\alpha)}{(2\kappa-2\sigma)! (2\sigma)! \Gamma(1+N-\mu-2\kappa+\alpha)} \\ &\quad \times L_{N-\mu-2\kappa}^{(\alpha)}(x) \\ &\quad + N! (-1)^{N-\mu+1} \sum_{\kappa=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \frac{E_{2\sigma} \Gamma(1+N-\mu-2\sigma+\alpha)}{(2\kappa-2\sigma+1)! (2\sigma)! \Gamma(N-\mu-2\kappa+\alpha)} \\ &\quad \times L_{N-\mu-2\kappa-1}^{(\alpha)}(x). \end{aligned} \quad (108)$$

Proof. The result is obtained by inserting the relevant inverse coefficients into Theorem 7. $\square$

Corollary 5. Let $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$. Then the μ th derivative of the Euler polynomials of the second kind admits the following expansion in terms of the shifted normalized Jacobi polynomials $\tilde{R}_n^{(\delta, \sigma)}(x)$:

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} C_\kappa^{(0)} \tilde{R}_{N-\mu-2\kappa}^{(\delta, \sigma)}(x) \\ &\quad + \sum_{\kappa=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} C_\kappa^{(1)} \tilde{R}_{N-\mu-2\kappa-1}^{(\delta, \sigma)}(x), \end{aligned} \quad (109)$$

where

$$\begin{aligned} C_\kappa^{(0)} &= \sum_{\rho=0}^{\kappa} \binom{N}{2\rho} \binom{N-\mu-2\rho}{2\kappa-2\rho} E_{2\rho} (1+N-\mu-2\rho)_\mu (1+\delta)_{N-2\kappa-\mu} \\ &\quad \times \frac{(1+N-2\kappa-\mu+\sigma)_{2\kappa-2\rho}}{(2+2N-4\kappa-2\mu+\delta+\sigma)_{2\kappa-2\rho} (1+N-2\kappa-\mu+\delta+\sigma)_{N-2\kappa-\mu}}, \\ C_\kappa^{(1)} &= \sum_{\rho=0}^{\kappa} \binom{N}{2\rho} \binom{N-\mu-2\rho}{2\kappa-2\rho+1} E_{2\rho} (1+N-\mu-2\rho)_\mu (1+\delta)_{N-2\kappa-\mu-1} \\ &\quad \times \frac{(N-2\kappa-\mu+\sigma)_{2\kappa-2\rho+1}}{(2N-4\kappa-2\mu+\delta+\sigma)_{2\kappa-2\rho+1} (N-2\kappa-\mu+\delta+\sigma)_{N-2\kappa-\mu-1}}. \end{aligned}$$

Proof. The result follows by inserting the inverse coefficients in (33) into Theorem 7. The indices κ and ρ are used consistently with the preceding formulas. $\square$

6.2. Derivatives of the EPs2 as Combinations of Symmetric Polynomials

Theorem 8. Assume $N, \mu \in \mathbb{N}_0$, $\mu \leq N$, and let $\phi_i(x)$ be a symmetric polynomial family of the form (25). The μ th EPs2 derivative then has the representation

$$\frac{d^\mu}{dx^\mu} E_N^*(x) = \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \binom{N}{2\sigma} E_{2\sigma} (1-\mu+N-2\sigma)_\mu \bar{B}_{\kappa-\sigma, N-2\sigma-\mu} \phi_{N-\mu-2\kappa}(x). \quad (110)$$

Proof. If we differentiate the series form of (15), then we get

$$\frac{d^\mu}{dx^\mu} E_N^*(x) = \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \binom{N}{2\sigma} E_{2\sigma} (1-\mu+N-2\sigma)_\mu x^{N-2\sigma-\mu}, \quad (111)$$

which can be turned into the following form using the inversion formula (27):

$$\frac{d^\mu}{dx^\mu} E_N^*(x) = \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \binom{N}{2\sigma} E_{2\sigma} (1 - \mu + N - 2\sigma)_\mu \sum_{\tau=0}^{\lfloor \frac{N-\mu}{2} \rfloor - \sigma} \bar{B}_{\tau, N-2\sigma-\mu} \phi_{N-2\sigma-\mu-2\tau}(x). \quad (112)$$

Finally, we set $\kappa = \sigma + \tau$. Since $0 \leq \sigma \leq \kappa \leq \lfloor \frac{N-\mu}{2} \rfloor$, rearranging the order of summation yields

$$\frac{d^\mu}{dx^\mu} E_N^*(x) = \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \sum_{\sigma=0}^{\kappa} \binom{N}{2\sigma} E_{2\sigma} (1 - \mu + N - 2\sigma)_\mu \bar{B}_{\kappa-\sigma, N-2\sigma-\mu} \phi_{N-\mu-2\kappa}(x), \quad (113)$$

which completes the proof. $\square$

Corollary 6. For $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$, the Hermite expansion of the μ th EPs2 derivative is

$$\frac{d^\mu}{dx^\mu} E_N^*(x) = N! 2^{\mu-N} \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{1}{(N - \mu - 2\kappa)!} \left(\sum_{\sigma=0}^{\kappa} \frac{2^{2\sigma} E_{2\sigma}}{(\kappa - \sigma)! (2\sigma)!} \right) H_{N-\mu-2\kappa}(x). \quad (114)$$

Proof. Insert the Hermite inversion coefficients from [42] into Theorem 8 to obtain (114). $\square$

Corollary 7. If $N, \mu \in \mathbb{N}_0$ and $\mu \leq N$, the corresponding ultraspherical-basis expansion reads

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= \frac{N! 2^{1+\mu-N} \Gamma(1+\lambda)}{\Gamma(1+2\lambda)} \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{(N - \mu - 2\kappa + \lambda) \Gamma(N - \mu - 2\kappa + 2\lambda)}{(N - \mu - 2\kappa)!} \\ &\times \left(\sum_{\sigma=0}^{\kappa} \frac{2^{2\sigma} E_{2\sigma}}{(\kappa - \sigma)! (2\sigma)! \Gamma(1 - \mu - \kappa + N - \sigma + \lambda)} \right) U_{N-\mu-2\kappa}^{(\lambda)}(x). \end{aligned} \quad (115)$$

Proof. The formula follows by specializing the symmetric-basis coefficients in Theorem 8. $\square$

Corollary 8. For $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$, expansion in the generalized Fibonacci family $W_n^{A,B}(x)$ defined by (34) gives

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= N! A^{\mu-N} \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} (-1)^{1+\kappa} B^\kappa (\mu + 2\kappa - N - 1) \\ &\times \left(\sum_{\sigma=0}^{\kappa} \frac{(-1)^\sigma A^{2\sigma} E_{2\sigma}}{(\kappa - \sigma)! (N - \mu - \kappa - \sigma + 1)! (2\sigma)! B^\sigma} \right) W_{N-\mu-2\kappa}^{A,B}(x). \end{aligned} \quad (116)$$

Proof. The desired result follows as a direct application of Theorem 8. $\square$

Corollary 9. For $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$, the generalized Lucas basis $w_n^{R,T}(x)$ from (35) yields

$$\begin{aligned} \frac{d^\mu}{dx^\mu} E_N^*(x) &= N! R^{\mu-N} \sum_{\kappa=0}^{\lfloor \frac{N-\mu}{2} \rfloor} (-1)^\kappa c_{N-\mu-2\kappa} T^\kappa \\ &\times \left(\sum_{\sigma=0}^{\kappa} \frac{(-1)^\sigma R^{2\sigma} E_{2\sigma}}{(\kappa - \sigma)! (N - \mu - \kappa - \sigma)! (2\sigma)! T^\sigma} \right) w_{N-\mu-2\kappa}^{R,T}(x), \end{aligned} \quad (117)$$

where

$$c_i = \begin{cases} \frac{1}{2}, & i = 0, \\ 1, & i \neq 0. \end{cases} \quad (118)$$

Proof. The desired result follows as a direct application of Theorem 8. $\square$

7. Derivatives of Different Polynomials as Combinations of the EPs2

We now reverse the preceding transformations by expanding derivatives of other polynomial families in the EPs2 basis. Two general results are followed by explicit specializations.

Theorem 9. Assume $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$. For a non-symmetric family $\psi_N(x)$ of the form introduced earlier, its μ th derivative has the following EPs2 expansion:

$$\begin{aligned} \frac{d^\mu}{dx^\mu} \psi_N(x) &= \sum_{p=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \sum_{\ell=0}^p \frac{G_{2\ell,N} (N-2\ell)!}{(N-2p-\mu)! (2p-2\ell)!} E_{N-\mu-2p}^*(x) \\ &+ \sum_{p=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \sum_{\ell=0}^p \frac{G_{2\ell+1,N} (N-2\ell-1)!}{(N-2p-\mu-1)! (2p-2\ell)!} E_{N-\mu-2p-1}^*(x), \end{aligned} \quad (119)$$

where the coefficients $G_{r,n}$ are those appear in (26).

Proof. Termwise differentiation of the non-symmetric expansion (26) gives

$$\frac{d^\mu}{dx^\mu} \psi_N(x) = \sum_{\rho=0}^{N-\mu} G_{\rho,N} (1+N-\mu-\rho)_\mu x^{N-\rho-\mu}. \quad (120)$$

Substitution of the EPs2 monomial inversion (21) transforms this expression into

$$\frac{d^\mu}{dx^\mu} \psi_N(x) = \sum_{\rho=0}^{N-\mu} G_{\rho,N} (1+N-\mu-\rho)_\mu \sum_{\tau=0}^{\lfloor \frac{N-\rho-\mu}{2} \rfloor} \frac{(N-\rho-\mu)!}{(N-\rho-\mu-2\tau)! (2\tau)!} E_{N-\rho-\mu-2\tau}^*(x). \quad (121)$$

Separating even and odd indices and reordering the finite sums recasts this expression as

$$\begin{aligned} \frac{d^\mu}{dx^\mu} \psi_N(x) &= \sum_{p=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \sum_{\ell=0}^p \frac{G_{2\ell,N} (N-2\ell)!}{(N-2p-\mu)! (2p-2\ell)!} E_{N-\mu-2p}^*(x) \\ &+ \sum_{p=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \sum_{\ell=0}^p \frac{G_{2\ell+1,N} (N-2\ell-1)!}{(N-2p-\mu-1)! (2p-2\ell)!} E_{N-\mu-2p-1}^*(x). \end{aligned} \quad (122)$$

This coincides with the formula asserted in the theorem. $\square$

Corollary 10. For $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$, the μ th derivative of the shifted normalized Jacobi polynomial $\tilde{R}_N^{(\delta, \sigma)}(x)$ admits the following EPs2 representation:

$$\begin{aligned} \frac{d^\mu}{dx^\mu} \tilde{R}_N^{(\delta, \sigma)}(x) &= \frac{N! \Gamma(1 + \delta) \Gamma(1 + 2N + \delta + \sigma)}{\Gamma(1 + N + \delta) \Gamma(1 + N + \delta + \sigma)} \\ &\times \sum_{p=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{E_{N-\mu-2p}^*(x)}{(N-\mu-2p)! (2p)!} \\ &\times {}_4F_3 \left(\begin{matrix} -p, \frac{1}{2} - p, -\frac{N}{2} - \frac{\sigma}{2}, \frac{1}{2} - \frac{N}{2} - \frac{\sigma}{2} \\ \frac{1}{2}, -N - \frac{\delta+\sigma}{2}, \frac{1}{2} - N - \frac{\delta+\sigma}{2} \end{matrix} \middle| 1 \right) \\ &+ \frac{(N + \sigma) N! \Gamma(1 + \delta) \Gamma(2N + \delta + \sigma)}{\Gamma(1 + N + \delta) \Gamma(1 + N + \delta + \sigma)} \\ &\times \sum_{p=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \frac{E_{N-\mu-2p-1}^*(x)}{(N-\mu-2p-1)! (2p+1)!} \\ &\times {}_4F_3 \left(\begin{matrix} -p, \frac{1}{2} - p, \frac{1}{2} - \frac{N}{2} - \frac{\sigma}{2}, 1 - \frac{N}{2} - \frac{\sigma}{2} \\ \frac{3}{2}, \frac{1}{2} - N - \frac{\delta+\sigma}{2}, 1 - N - \frac{\delta+\sigma}{2} \end{matrix} \middle| 1 \right). \end{aligned} \quad (123)$$

Proof. From the series representation (32), the coefficients $G_{\ell, m}$ in Theorem 9 are identified directly. Substituting them into that theorem and separating the even and odd indices yields the two terminating ${}_4F_3(1)$ expressions. $\square$

Corollary 11. Suppose $N, \mu \in \mathbb{N}_0$, $\mu \leq N$, and $\alpha > -1$. The generalized Laguerre derivative can then be expanded in the EPs2 basis as

$$\begin{aligned} \frac{d^\mu}{dx^\mu} L_N^{(\alpha)}(x) &= (-1)^N \Gamma(N + \alpha + 1) \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{E_{N-\mu-2\sigma}^*(x)}{(N-\mu-2\sigma)!} \sum_{\rho=0}^{\sigma} \frac{1}{(2\sigma-2\rho)! (2\rho)! \Gamma(N-2\rho+\alpha+1)} \\ &- (-1)^N \Gamma(N + \alpha + 1) \sum_{\sigma=0}^{\lfloor \frac{N-\mu-1}{2} \rfloor} \frac{E_{N-\mu-2\sigma-1}^*(x)}{(N-\mu-2\sigma-1)!} \sum_{\rho=0}^{\sigma} \frac{1}{(2\sigma-2\rho)! (2\rho+1)! \Gamma(N-2\rho+\alpha)}. \end{aligned} \quad (124)$$

Proof. Applying Theorem 9 to the Laguerre polynomial coefficients directly yields the stated result. $\square$

Theorem 10. Let $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$. If $\phi_N(x)$ belongs to the symmetric class defined above, then its μ th derivative expands in EPs2 according to

$$\frac{d^\mu}{dx^\mu} \phi_N(x) = \sum_{p=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{1}{(N-\mu-2p)!} \sum_{\ell=0}^p \frac{B_{\ell, N} (N-2\ell)!}{(2p-2\ell)!} E_{N-\mu-2p}^*(x). \quad (125)$$

Here, $B_{\ell, N}$ denotes the coefficient defined in (25).

Proof. Termwise differentiation of the symmetric power expansion (25) gives

$$\frac{d^\mu}{dx^\mu} \phi_N(x) = \sum_{\rho=0}^{\lfloor \frac{N-\mu}{2} \rfloor} B_{\rho, N} (1 + N - \mu - 2\rho)_\mu x^{N-2\rho-\mu}. \quad (126)$$

Replacing each monomial with its EPs2 inverse expansion gives

$$\begin{aligned} \frac{d^\mu}{dx^\mu} \phi_N(x) &= \sum_{\rho=0}^{\lfloor \frac{N-\mu}{2} \rfloor} B_{\rho,N} (1+N-\mu-2\rho)_\mu \\ &\times \sum_{\tau=0}^{\lfloor \frac{N-\mu}{2} \rfloor - \rho} \frac{(N-2\rho-\mu)!}{(N-2\rho-\mu-2\tau)!(2\tau)!} E_{N-2\rho-\mu-2\tau}^*(x). \end{aligned} \quad (127)$$

Set $\sigma = \rho + \tau$ and interchange the two finite summations to obtain

$$\frac{d^\mu}{dx^\mu} \phi_N(x) = \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{1}{(N-\mu-2\sigma)!} \sum_{\rho=0}^{\sigma} \frac{B_{\rho,N} (N-2\rho)!}{(2\sigma-2\rho)!} E_{N-\mu-2\sigma}^*(x). \quad (128)$$

This proves the claimed representation. $\square$

Corollary 12. For $N, \mu \in \mathbb{N}_0$ with $\mu \leq N$, the μ th Hermite derivative takes the EPs2 form

$$\frac{d^\mu}{dx^\mu} H_N(x) = 2^N N! \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{U(-\sigma, \frac{1}{2}, 1)}{(2\sigma)!(N-2\sigma-\mu)!} E_{N-\mu-2\sigma}^*(x), \quad (129)$$

where $U(A, B, z)$ denotes the Tricomi (confluent hypergeometric) function [42].

Proof. The result is derived directly from Theorem 10. $\square$

Corollary 13. For $N, \mu \in \mathbb{N}_0$, $\mu \leq N$, and $\lambda > -\frac{1}{2}$, the ultraspherical derivative is represented in the EPs2 basis by

$$\begin{aligned} \frac{d^\mu}{dx^\mu} U_N^{(\lambda)}(x) &= \frac{2^{N+2\lambda-1} N! \Gamma(\frac{1}{2} + \lambda) \Gamma(N + \lambda)}{\sqrt{\pi} \Gamma(N + 2\lambda)} \\ &\times \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{(-1)^\sigma \left(\frac{1}{2} - N + \sigma - \lambda\right)_\sigma}{(N-\mu-2\sigma)!(2\sigma)!(N-\sigma+\lambda)_\sigma} E_{N-\mu-2\sigma}^*(x). \end{aligned} \quad (130)$$

Proof. The application of Theorem 10 yields the following formula:

$$\frac{d^\mu}{dx^\mu} U_N^{(\lambda)}(x) = \frac{2^{N-1+2\lambda} N! \Gamma(\frac{1}{2} + \lambda) \Gamma(N + \lambda)}{\sqrt{\pi} \Gamma(N + 2\lambda)} \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{{}_2F_1\left(-\sigma, \frac{1}{2} - \sigma \middle| 1\right)}{(2\sigma)!(N-2\sigma-\mu)!} E_{N-\mu-2\sigma}^*(x). \quad (131)$$

If we apply the Chu–Vandermonde identity [42], then formula (131) can be simplified to give (130). $\square$

Corollary 14. If $N, \mu \in \mathbb{N}_0$ and $\mu \leq N$, the generalized Fibonacci derivative has the EPs2 expansion

$$\frac{d^\mu}{dx^\mu} W_N^{A,B}(x) = A^N N! \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{{}_2F_1\left(-\sigma, \frac{1}{2} - \sigma \middle| -\frac{4B}{A^2}\right)}{(2\sigma)!(N-2\sigma-\mu)!} E_{N-\mu-2\sigma}^*(x). \quad (132)$$

Proof. Substitution of the generalized Fibonacci series coefficients into Theorem 10 proves the assertion. $\square$

Corollary 15. For integers $N \geq 1$ and $0 \leq \mu \leq N$, the generalized Lucas derivative has the EPs2 representation

$$\frac{d^\mu}{dx^\mu} w_N^{R,T}(x) = R^N N! \sum_{\sigma=0}^{\lfloor \frac{N-\mu}{2} \rfloor} \frac{{}_2F_1\left(-\sigma, \frac{1}{2}-\sigma \middle| 1-N \middle| -\frac{4T}{R^2}\right)}{(2\sigma)!(N-2\sigma-\mu)!} E_{N-\mu-2\sigma}^*(x). \quad (133)$$

Proof. Using the generalized Lucas series coefficients in Theorem 10 establishes the formula. $\square$

8. Some Applications to the Derived Formulas

This section derives several integral formulas from those established in this paper.

Theorem 11. For every integer $i \geq 0$, the integral of the EPs2 over the interval $[c, d]$ is given by

$$\int_c^d E_i^*(x) dx = \frac{2^{i+1}}{i+1} \left[E_{i+1}\left(\frac{d+1}{2}\right) - E_{i+1}\left(\frac{c+1}{2}\right) \right], \quad (134)$$

where $E_n(x)$ denotes the n th Euler polynomial.

Proof. By the definition of the EPs2,

$$E_i^*(x) = 2^i E_i\left(\frac{x+1}{2}\right),$$

and therefore

$$\int_c^d E_i^*(x) dx = 2^i \int_c^d E_i\left(\frac{x+1}{2}\right) dx.$$

First, use the substitution: $x = 2u - 1$. Equivalently,

$$u = \frac{x+1}{2}, \quad dx = 2 du.$$

Moreover, the endpoints $x = c$ and $x = d$ are transformed into

$$u = \frac{c+1}{2} \quad \text{and} \quad u = \frac{d+1}{2},$$

respectively. Hence,

$$\int_c^d E_i^*(x) dx = 2^{i+1} \int_{\frac{c+1}{2}}^{\frac{d+1}{2}} E_i(u) du.$$

Based on the following identity:

$$E_i(u) = \frac{1}{i+1} \frac{d}{du} E_{i+1}(u),$$

which follows from the standard derivative relation

$$\frac{d}{du} E_{i+1}(u) = (i+1) E_i(u),$$

we obtain

$$\int_c^d E_i^*(x) dx = \frac{2^{i+1}}{i+1} \int_{\frac{c+1}{2}}^{\frac{d+1}{2}} \frac{d}{du} E_{i+1}(u) du,$$

and hence

$$\int_c^d E_i^*(x) dx = \frac{2^{i+1}}{i+1} \left[E_{i+1} \left(\frac{d+1}{2} \right) - E_{i+1} \left(\frac{c+1}{2} \right) \right].$$

This proves (134). $\square$

Corollary 16. For any non-negative integer i , the following identity holds:

$$M_i = \int_0^1 E_i^*(x) dx = \frac{2^{i+1} E_{i+1}(1) - E_{i+1}}{i+1}, \quad (135)$$

where E_{i+1} denotes the $(i+1)$ st secant Euler number, satisfying

$$E_{i+1} = 2^{i+1} E_{i+1} \left(\frac{1}{2} \right).$$

Proof. Setting $c = 0$ and $d = 1$ in (134) yields

$$\int_0^1 E_i^*(x) dx = \frac{2^{i+1}}{i+1} \left[E_{i+1}(1) - E_{i+1} \left(\frac{1}{2} \right) \right].$$

Using the identity

$$E_{i+1} \left(\frac{1}{2} \right) = 2^{-(i+1)} E_{i+1},$$

we get

$$\begin{aligned} \int_0^1 E_i^*(x) dx &= \frac{2^{i+1}}{i+1} \left[E_{i+1}(1) - 2^{-(i+1)} E_{i+1} \right] \\ &= \frac{2^{i+1} E_{i+1}(1) - E_{i+1}}{i+1}. \end{aligned}$$

Therefore, (135) follows. $\square$

Remark 2. To provide direct computational verification of some of the formulas derived in this section, we present below several representative low-degree examples. These examples are intended only to verify the validity of some of the derived formulas by evaluating both sides explicitly. The calculations were also checked symbolically in Mathematica, and in each case the difference between the left- and right-hand sides was identically zero.

Example 1 (Verification of Equation (135)). For $i = 3$, we have

$$E_3^*(x) = x^3 - 3x,$$

and hence

$$\int_0^1 E_3^*(x) dx = \int_0^1 (x^3 - 3x) dx = -\frac{5}{4}.$$

On the other hand, Formula (135) gives

$$M_3 = \frac{2^4 E_4(1) - E_4}{4} = -\frac{5}{4}.$$

Similarly, for $i = 4$,

$$E_4^*(x) = x^4 - 6x^2 + 5,$$

so that

$$\int_0^1 E_4^*(x) dx = \int_0^1 (x^4 - 6x^2 + 5) dx = \frac{16}{5},$$

whereas (135) yields

$$M_4 = \frac{2^5 E_5(1) - E_5}{5} = \frac{16}{5}.$$

Thus, in both representative cases, the direct integration and the right-hand side of Equation (135) coincide exactly.

Corollary 17. Let k, m be non-negative integers. Then, the following integral for the moments holds:

$$\int_0^1 x^m E_k^*(x) dx = \sum_{s=0}^{\lfloor \frac{k+m}{2} \rfloor} \sum_{r=0}^s \frac{\binom{k}{2r} E_{2r}(k+m-2r)!}{(k+m-2s)!(2(s-r))!} M_{k+m-2s}, \quad (136)$$

where M_i are given in (135).

Proof. Integrate (89) termwise on $[0, 1]$ and evaluate each EPs2 integral by (135). $\square$

Example 2 (Verification of Equation (136)). Let $k = 5$ and $m = 2$. We verify Equation (136) by evaluating its two sides separately.

For the left-hand side, since

$$E_5^*(x) = x^5 - 10x^3 + 25x,$$

we obtain

$$\begin{aligned} \text{LHS} &= \int_0^1 x^2 E_5^*(x) dx \\ &= \int_0^1 (x^7 - 10x^5 + 25x^3) dx \\ &= \frac{1}{8} - \frac{10}{6} + \frac{25}{4} = \frac{113}{24}. \end{aligned}$$

For the right-hand side, using (89), we have

$$x^2 E_5^*(x) = E_7^*(x) + 11E_5^*(x) - 40E_3^*(x) + 32E_1^*(x).$$

Therefore,

$$\text{RHS} = M_7 + 11M_5 - 40M_3 + 32M_1,$$

where, by (135),

$$M_i = \int_0^1 E_i^*(x) dx.$$

In particular,

$$M_1 = \frac{1}{2}, \quad M_3 = -\frac{5}{4}, \quad M_5 = \frac{61}{6}, \quad M_7 = -\frac{1385}{8}.$$

Consequently, we have

$$\begin{aligned} \text{RHS} &= -\frac{1385}{8} + 11\left(\frac{61}{6}\right) - 40\left(-\frac{5}{4}\right) + 32\left(\frac{1}{2}\right) \\ &= \frac{113}{24}. \end{aligned}$$

Thus,

$$\text{LHS} = \text{RHS} = \frac{113}{24},$$

which verifies Equation (136).

Now, we will develop two results of certain definite integrals of the products of the EPs2 with the two generalized classes defined in (34), and (35). First, the following two integral formulas are needed.

Lemma 4. For every integer k , the following identity holds:

$$\int_0^1 W_k^{A,B}(x) dx = Y_k, \quad (137)$$

where

$$Y_k = \begin{cases} 0, & k = -1, \\ \frac{1}{A(k+1)} (W_{k+1}^{A,B} + B W_{k-1}^{A,B}) + G_k, & k \geq 0, \\ (-1)^k B^{k+1} \left(\frac{1}{-A(k+1)} W_{-k-1}^{A,B} + \frac{B}{-A(k+1)} W_{-k-3}^{A,B} + G_{-k-2} \right), & k < -1, \end{cases} \quad (138)$$

and

$$G_k = \begin{cases} -\frac{2B^{(k+1)/2}}{A(k+1)}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \quad (139)$$

Proof. We start with the structure formula of $W_k^{A,B}(x)$ represented as

$$W_k^{A,B}(x) = \frac{1}{A(k+1)} \left(\frac{d}{dx} W_{k+1}^{A,B}(x) + B \frac{d}{dx} W_{k-1}^{A,B}(x) \right), \quad k \geq 0. \quad (140)$$

Integrating (140) over $[0, 1]$, we get

$$\int_0^1 W_k^{A,B}(x) dx = \frac{1}{A(k+1)} (W_{k+1}^{A,B}(1) - W_{k+1}^{A,B}(0) + B(W_{k-1}^{A,B}(1) - W_{k-1}^{A,B}(0))), \quad k \geq 0. \quad (141)$$

The special values at $x = 0$ and at $x = 1$ can be written in explicit forms. From the following, special values can be obtained:

$$W_m^{A,B} = W_m^{A,B}(1) = \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} A^{m-2s} B^s \binom{m-s}{s}, \quad (142)$$

$$W_m^{A,B}(0) = \begin{cases} B^{\frac{m}{2}}, & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases} \quad (143)$$

Inserting these values into formula (141), we get

$$\int_0^1 W_k^{A,B}(x) dx = \frac{W_{k+1}^{A,B}(1) + B W_{k-1}^{A,B}(1)}{A(k+1)} + G_k, \quad k \geq 0, \quad (144)$$

where

$$G_k = \begin{cases} -\frac{2B^{\frac{k+1}{2}}}{A(k+1)}, & k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases} \quad (145)$$

In addition, it can be shown from the Binet form in (36), that for any positive number k , we have

$$W_{-k}^{A,B}(x) = (-1)^k B^{1-k} W_{k-2}^{A,B}(x). \quad (146)$$

Using the above identity together with (144), we obtain the integral formula in (137). $\square$

Corollary 18. Let r, s be non-negative integers. Then, the integral formula holds:

$$\int_0^1 E_r^*(x) W_s^{(A,B)}(x) dx = \sum_{L=0}^r \left(\sum_{j=0}^L A^{2j-r} (-B)^{L-j} \binom{r}{2j} \binom{r-2j}{L-j} E_{2j} \right) Y_{r+s-2L}, \quad (147)$$

where Y_k is defined in (138).

Proof. Integrating from the product formula (93) over $[0, 1]$ yields

$$\int_0^1 E_r^*(x) W_s^{(A,B)}(x) dx = \sum_{L=0}^r \left(\sum_{j=0}^L A^{2j-r} (-B)^{L-j} \binom{r}{2j} \binom{r-2j}{L-j} E_{2j} \right) \int_0^1 W_{r+s-2L}^{(A,B)}(x) dx. \quad (148)$$

Based on Lemma 4, the required result follows. $\square$

Example 3 (Verification of Equation (147)). Choose $r = 2, s = 3, A = 2$, and $B = 1$. We verify Equation (147) by evaluating its two sides separately.

For the left-hand side, since

$$E_2^*(x) = x^2 - 1, \quad W_3^{2,1}(x) = 8x^3 + 4x,$$

we obtain

$$\begin{aligned} \text{LHS} &= \int_0^1 E_2^*(x) W_3^{2,1}(x) dx \\ &= \int_0^1 (x^2 - 1)(8x^3 + 4x) dx \\ &= -\frac{5}{3}. \end{aligned}$$

For the right-hand side, substituting $r = 2, s = 3, A = 2$, and $B = 1$ into Equation (147) gives

$$\text{RHS} = \frac{1}{4} Y_5 - \frac{3}{2} Y_3 + \frac{1}{4} Y_1,$$

where, by (137),

$$Y_k = \int_0^1 W_k^{2,1}(x) dx.$$

Directly,

$$Y_1 = 1, \quad Y_3 = 4, \quad Y_5 = \frac{49}{3}.$$

Consequently,

$$\begin{aligned} \text{RHS} &= \frac{1}{4} \left(\frac{49}{3} \right) - \frac{3}{2} (4) + \frac{1}{4} (1) \\ &= -\frac{5}{3}. \end{aligned}$$

Thus,

$$\text{LHS} = \text{RHS} = -\frac{5}{3},$$

which verifies Equation (147).

Corollary 19. Let N, ℓ, M be non-negative integers with $0 \leq \ell \leq N$, and let $\alpha > -1$. Then,

$$\int_0^\infty x^\alpha e^{-x} \frac{d^\ell}{dx^\ell} E_N^*(x) L_M^{(\alpha)}(x) dx = V_{N,\ell,M}, \quad (149)$$

where

$$V_{N,\ell,M} = \begin{cases} A_{\frac{N-\ell-M}{2},N,\ell} h_M, & N - \ell - M \geq 0 \text{ and even,} \\ B_{\frac{N-\ell-M-1}{2},N,\ell} h_M, & N - \ell - M \geq 0 \text{ and odd,} \\ 0, & M > N - \ell, \end{cases} \quad (150)$$

and

$$A_{\kappa,N,\ell} = N! \sum_{\rho=0}^{\kappa} \frac{(-1)^{N-\ell} E_{2\rho} \Gamma(N - \ell - 2\rho + \alpha + 1)}{(2\kappa - 2\rho)! (2\rho)! \Gamma(N - 2\kappa - \ell + \alpha + 1)}, \quad (151)$$

$$B_{\kappa,N,\ell} = N! \sum_{\rho=0}^{\kappa} \frac{(-1)^{N-\ell+1} E_{2\rho} \Gamma(N - \ell - 2\rho + \alpha + 1)}{(2\kappa - 2\rho + 1)! (2\rho)! \Gamma(N - 2\kappa - \ell + \alpha)}, \quad (152)$$

with

$$h_M = \frac{\Gamma(M + \alpha + 1)}{M!}. \quad (153)$$

Proof. Using the generalized Laguerre expansion

$$\begin{aligned} \frac{d^\ell}{dx^\ell} E_N^*(x) &= \sum_{\kappa=0}^{\lfloor \frac{N-\ell}{2} \rfloor} A_{\kappa,N,\ell} L_{N-\ell-2\kappa}^{(\alpha)}(x) \\ &\quad + \sum_{\kappa=0}^{\lfloor \frac{N-\ell-1}{2} \rfloor} B_{\kappa,N,\ell} L_{N-\ell-2\kappa-1}^{(\alpha)}(x), \end{aligned} \quad (154)$$

together with the orthogonality relation

$$\int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{m,n} = h_n \delta_{m,n}, \quad (155)$$

where $\delta_{m,n}$ denotes the Kronecker delta, we obtain

$$\begin{aligned} &\int_0^\infty x^\alpha e^{-x} \frac{d^\ell}{dx^\ell} E_N^*(x) L_M^{(\alpha)}(x) dx \\ &= \sum_{\kappa=0}^{\lfloor \frac{N-\ell}{2} \rfloor} A_{\kappa,N,\ell} h_M \delta_{N-\ell-2\kappa,M} + \sum_{\kappa=0}^{\lfloor \frac{N-\ell-1}{2} \rfloor} B_{\kappa,N,\ell} h_M \delta_{N-\ell-2\kappa-1,M}. \end{aligned} \quad (156)$$

Applying the definition of the Kronecker delta to (156) immediately gives (149) and (150). $\square$

Example 4 (Verification of Equation (150)). Take

$$N = 4, \quad \ell = 1, \quad M = 2, \quad \alpha = \frac{1}{2}.$$

Since $N - \ell - M = 1$ is odd, the odd case in (150) applies.

For the left-hand side, we have

$$E_4^*(x) = x^4 - 6x^2 + 5, \quad \frac{d}{dx} E_4^*(x) = 4x^3 - 12x,$$

and

$$L_2^{(1/2)}(x) = \frac{x^2}{2} - \frac{5x}{2} + \frac{15}{8}.$$

Therefore,

$$\begin{aligned}\text{LHS} &= \int_0^\infty x^{1/2} e^{-x} (4x^3 - 12x) \left(\frac{x^2}{2} - \frac{5x}{2} + \frac{15}{8} \right) dx \\ &= \frac{315\sqrt{\pi}}{4}.\end{aligned}$$

For the right-hand side, (150) gives

$$\text{RHS} = B_{0,4,1} h_2.$$

Using (152),

$$B_{0,4,1} = 4! \frac{\Gamma(\frac{9}{2})}{\Gamma(\frac{7}{2})} = 84,$$

while (153) gives

$$h_2 = \frac{\Gamma(\frac{7}{2})}{2!} = \frac{15\sqrt{\pi}}{16}.$$

Hence,

$$\text{RHS} = 84 \left(\frac{15\sqrt{\pi}}{16} \right) = \frac{315\sqrt{\pi}}{4}.$$

Thus,

$$\text{LHS} = \text{RHS} = \frac{315\sqrt{\pi}}{4},$$

which verifies Equation (150) and, in particular, its odd case.

Remark 3. The principal formulas and identities derived in this paper were also checked using Mathematica for several admissible values of the polynomial degrees, derivative orders, and involved parameters. In all tested representative cases, the symbolic differences between the left- and right-hand sides simplified identically to zero, providing supporting computational evidence for the presented results.

9. Concluding Remarks

This work has developed an algebraic framework for EPs2 based on explicit basis transformations and operational identities. The results connect this family with several orthogonal and non-orthogonal systems and provide differentiated-moment, derivative, product, inverse, and integral formulas. Taken together, these identities clarify how EPs2 interact with classical and generalized polynomial bases and supply tools suitable for symbolic manipulation and subsequent computational use. Future investigations will consider additional modified or generalized Euler families and examine EPs2-based spectral approximations for differential models arising in applied science.

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