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A Flexible Heavy-Tailed Probability Model for Skewed Positive Data: Theory, Estimation, and Empirical Assessment

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Abstract

In this study, we investigate a Slash Marshall–Olkin (SMO) family obtained by dividing a positive Marshall–Olkin random variable by an independent beta variable. Its four-parameter exponential member (SMOE) has a regularly varying power-law tail and belongs to the subexponential class. Integral representations are provided for all positive parameters. The original hypergeometric expansions are retained within their convergence domain, and an alternative convergent representation is derived for the remaining Marshall–Olkin parameter values. The density and hazard have finite, strictly positive limits at the origin. The density is decreasing for $\eta \leq 2$ and has at least one positive interior mode for $\eta > 2$. We establish regular variation of the survival function with exponent $-\alpha$ and show that a positive-order moment exists exactly below order α. In addition, an exact generator, an observed-information representation, and precautions for likelihood optimization are provided. The simulations and two economic applications are also examined.

Keywords: power-law tails; subexponentiality; positive data; skewness; Marshall–Olkin transformation; beta mixture; maximum likelihood estimation; economic data

MSC: 62G30; 62E10

Check for updates

Academic Editor: Jean Renaud Pycke

Received: 9 September 2026
Revised: 6 October 2026
Accepted: 10 October 2026
Published: 15 October 2026

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1. Introduction

Slash constructions are widely used to generate heavy-tailed models by dividing a parent random variable by an independent variable supported on $(0, 1)$. The development of slash distributions can be traced to [1]; this work introduced the canonical slash distribution. Ref. [2] extended this idea to the multivariate setting and developed the multivariate skew-slash distribution. Ref. [3] further generalized the family by proposing slash distributions with elliptical contours. After that, ref. [4] proposed a new family of skewed slash distributions generated by the normal kernel. Furthermore, ref. [5] suggested the slashed half-normal distribution as an extension of the half-normal model. Ref. [6] introduced the multivariate slash-t and skew-slash-t distributions, adding further flexibility

for modeling skewness and heavy tails. Ref. [7] provided a unified treatment of continuous uniform slash and α-slash distributions and also introduced discrete counterparts. More recently, ref. [8] studied α-slash distributions in connection with generalized convolutions and extreme-value behavior. In a recent applied extension, ref. [9] introduced the extended slash modified Lindley distribution for modeling asymmetric economic variables with high kurtosis. Ref. [10] proposed a novel discrete slash family for epidemiology informatics data.

Motivated by the literature of slash distribution, we define a new slash extension of the MO family. The proposed model inherits the flexibility of the Marshall–Olkin (MO) transformation and acquires additional tail-heaviness from the beta denominator. Here, we consider an absolutely continuous baseline distribution on $(0, \infty)$. Its density is assumed to be measurable and nonnegative, with integral equal to one. Let $G(x;\xi)$ and $g(x;\xi)$ denote the cumulative distribution function (cdf) and probability density function (pdf) of a baseline distribution on $(0, \infty)$, where ξ is a parameter vector.

The MO [11] transformed cdf is defined by

$$F_X(x;\eta,\xi) = \frac{G(x;\xi)}{G(x;\xi) + \eta\big(1 - G(x;\xi)\big)} = \frac{G(x;\xi)}{\eta + (1-\eta)G(x;\xi)}, \qquad x > 0,\ \eta > 0. \tag{1}$$

The corresponding pdf is

$$f_X(x;\eta,\xi) = \frac{\eta\, g(x;\xi)}{[\eta + (1-\eta)G(x;\xi)]^2}, \qquad x > 0. \tag{2}$$

An important motivation for the present study is related to the tail behavior of the MO transformation. This transformation alone does not change the asymptotic tail class of a proper baseline distribution. More precisely, we have

$$\overline{F}_X(x) = \frac{\eta\{1 - G(x;\xi)\}}{1 - (1-\eta)\{1 - G(x;\xi)\}} \sim \eta\{1 - G(x;\xi)\}, \qquad x \to \infty. \tag{3}$$

For an exponential baseline, the MO parent therefore has exponential decay and all positive moments, whatever the value of η. Thus, changing the MO parameter alone does not produce a polynomial tail. In contrast, by introducing the beta denominator, the survival function of the SMOE model becomes asymptotic to a positive constant times $y^{-\alpha}$; see Theorem 3. This change in tail behavior is the main theoretical motivation for the extension, rather than the addition of parameters itself.

The roles of the parameters are also of interest, although their effects are not completely separate. The parameter λ is an inverse scale, while α determines the asymptotic tail exponent. The parameter β influences the beta mixing mechanism, the body of the distribution, and the tail constant. In addition, η changes the shape of the MO parent and the initial slope of the density. These interpretations do not, by themselves, establish joint identifiability or a better fit. Some special and limiting cases are useful here. As $\beta \downarrow 0$, with the other parameters fixed, we obtain the MO parent in distribution, whereas $\eta = 1$ removes the MO transformation. When $\beta = 1$, we get the usual power-denominator slash construction, since a Beta$(\alpha, 1)$ variable has the same distribution as $V^{1/\alpha}$, where V is uniform. In this regard, the proposed extension gives a tractable alternative when both the body shape and tail behavior are important. It does not suggest that every economic data set requires a heavy-tailed model.

The distinctions discussed by Taleb [12] between tail behavior, moment existence, and finite-sample inference are relevant to this study. A distribution may have a finite mean or variance without having all positive-order moments. Moreover, regularly varying power-law distributions form a subclass of subexponential distributions; these two classes

are therefore not mutually exclusive [13]. For the SMOE distribution, each finite $\alpha > 0$ gives a power-law survival tail, and only the moments of order below α are finite. Hence, the proposed construction is not limited to distributions having all moments finite. We consider it for positive data where polynomial tail decay is appropriate, rather than for every form of fat-tailed behavior. This limitation is further discussed in Section 3.6.

In view of the above discussion, the main objectives of this study are:

(i) to introduce the SMO construction and establish the exact conditions for the existence of its moments;
(ii) to derive representations valid over the full parameter domain and study the boundary behavior and tail asymptotics of the SMOE distribution;
(iii) to present exact simulation and likelihood-based inference, together with the limitations concerning identifiability and optimization; and
(iv) to examine empirical performance using information criteria based on the same data, stability diagnostics, and calibrated goodness-of-fit tests, without making unsupported claims of superiority.

The rest of the paper is organized as follows. In Section 2, we introduce the SMO family. In Section 3, we discuss its exponential member. The likelihood-based inference is presented in Section 4. Simulation is considered in Section 5, and the economic applications are discussed in Section 6. Conclusions are given in Section 7.

2. Slash Marshall–Olkin Family

In this section, we introduce SMO family and its statistical properties. Let

$$U \sim \text{Beta}(\alpha, \beta), \qquad \alpha > 0,\ \beta > 0,$$

with density

$$f_U(u) = \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\text{B}(\alpha,\beta)}, \qquad 0 < u < 1, \tag{4}$$

where

$$\text{B}(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,\text{d}u.$$

Assume that X and U are independent.

Definition 1. *A random variable Y is said to follow the* SMO family, *written*

$$Y \sim \text{SMO}(\eta, \xi, \alpha, \beta),$$

if

$$Y = \frac{X}{U}, \tag{5}$$

where X has pdf (2) and $U \sim \text{Beta}(\alpha,\beta)$ is independent of X.

Since $X > 0$ and $0 < U < 1$ almost surely, it follows that $Y \in (0,\infty)$ a.s.

2.1. Distributional Properties of the SMO Family

The following Propositions 1–3 provide the cdf, sf, and pdf of the SMO family, respectively. Subsequently, Corollary 1 presents the hrf.

Proposition 1. *Let $Y = X/U$ as in (5). Then for $y > 0$,*

$$F_Y(y) = \frac{1}{\text{B}(\alpha,\beta)} \int_0^1 \frac{G(yu;\xi)}{\eta + (1-\eta)G(yu;\xi)} u^{\alpha-1}(1-u)^{\beta-1}\,\text{d}u. \tag{6}$$

Also, $F_Y(y) = 0$ *for* $y \leq 0$.

Proof. By conditioning on U,

$$F_Y(y) = \mathbb{P}\left(\frac{X}{U} \leq y\right) = \mathbb{E}[\mathbb{P}(X \leq yU \mid U)] = \mathbb{E}[F_X(yU)].$$

Substituting (1) and the beta density (4) gives

$$F_Y(y) = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^1 \frac{G(yu;\xi)}{\eta + (1-\eta)G(yu;\xi)} u^{\alpha-1}(1-u)^{\beta-1}\,\mathrm{d}u,$$

which proves (6). □

Proposition 2. *For* $y > 0$, *the survival function (sf) of* Y *is*

$$\overline{F}_Y(y) = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^1 \frac{\eta\left(1 - G(yu;\xi)\right)}{\eta + (1-\eta)G(yu;\xi)} u^{\alpha-1}(1-u)^{\beta-1}\,\mathrm{d}u. \tag{7}$$

Proof. Using sf,

$$\overline{F}_Y(y) = \mathbb{P}(Y > y) = \mathbb{E}[\mathbb{P}(X > yU \mid U)] = \mathbb{E}\big[\overline{F}_X(yU)\big].$$

Substituting the density of U yields (7). □

Proposition 3. *The pdf of the SMO family is*

$$f_Y(y) = \frac{\eta}{\mathrm{B}(\alpha,\beta)} \int_0^1 \frac{g(yu;\xi)\, u^{\alpha}(1-u)^{\beta-1}}{[\eta + (1-\eta)G(yu;\xi)]^2}\,\mathrm{d}u, \qquad y > 0. \tag{8}$$

Proof. By independence and absolute continuity, we have the joint density $f_X(x)f_U(u)$ on the open set $\mathcal{D} = (0,\infty) \times (0,1)$. Now, consider the map $T(x,u) = (x/u, u)$. It is one-to-one, onto, and continuously differentiable from $\mathcal{D}$ to itself. Its inverse, given by $T^{-1}(y,v) = (yv, v)$, is also continuously differentiable. The determinant of the inverse is $v > 0$ throughout this open set. The boundary $u = 0$ is not included and has probability zero. Thus, the change-of-variables theorem gives a uniquely defined transformed joint law, with a density unique up to null sets. Here, no separate invertibility condition on two independent variables U and V is needed, since V is simply U. Use the transformation

$$Y = \frac{X}{U}, \qquad V = U.$$

Then

$$X = YV, \qquad U = V.$$

The Jacobian of the inverse transformation is

$$J = \left|\frac{\partial(x,u)}{\partial(y,v)}\right| = \begin{vmatrix} v & y \\ 0 & 1 \end{vmatrix} = v.$$

Hence, by independence,

$$f_{Y,V}(y,v) = f_X(yv)f_U(v)\,v.$$

Integrating out v over $(0,1)$,

$$f_Y(y) = \int_0^1 f_X(yv)f_U(v)\,v\,\mathrm{d}v.$$

Now substitute (2) and (4):

$$f_Y(y) = \frac{\eta}{\mathrm{B}(\alpha,\beta)} \int_0^1 \frac{g(yv;\xi)\, v^{\alpha}(1-v)^{\beta-1}}{[\eta + (1-\eta)G(yv;\xi)]^2}\, \mathrm{d}v.$$

Renaming v as u gives (8). □

Corollary 1. *The hazard rate function (hrf) of Y is*

$$h_Y(y) = \frac{f_Y(y)}{\overline{F}_Y(y)}, \qquad y > 0, \tag{9}$$

where $f_Y(y)$ and $\overline{F}_Y(y)$ are given by (7) and (8).

2.2. Moments of the SMO Family

The following Proposition 4 gives moments of SMO family.

Proposition 4. *Let $r > 0$. Then the rth raw moment of Y exists if and only if $\mathbb{E}(X^r) < \infty$ and $r < \alpha$. Under these conditions, it is given by*

$$\mathbb{E}(Y^r) = \mathbb{E}(X^r)\,\mathbb{E}(U^{-r}) = \mathbb{E}(X^r)\frac{\mathrm{B}(\alpha - r, \beta)}{\mathrm{B}(\alpha,\beta)}. \tag{10}$$

Proof. Since the integrand is nonnegative, we can apply Tonelli's theorem. Together with independence, this gives the following equality in the extended nonnegative reals, without first assuming that the moments are finite. Since $Y = X/U$ and X is independent of U,

$$\mathbb{E}(Y^r) = \mathbb{E}\left(\frac{X^r}{U^r}\right) = \mathbb{E}(X^r)\mathbb{E}(U^{-r}).$$

Now

$$\mathbb{E}(U^{-r}) = \int_0^1 u^{-r}\frac{u^{\alpha-1}(1-u)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}\,\mathrm{d}u = \frac{1}{\mathrm{B}(\alpha,\beta)}\int_0^1 u^{\alpha-r-1}(1-u)^{\beta-1}\,\mathrm{d}u = \frac{\mathrm{B}(\alpha-r,\beta)}{\mathrm{B}(\alpha,\beta)},$$

When $r < \alpha$, the integral is finite because the exponents at its two endpoints satisfy $\alpha - r - 1 > -1$ and $\beta - 1 > -1$. At $r = \alpha$, the integral diverges logarithmically at zero. For $r > \alpha$, it also diverges. Thus, the beta-function expression is used only when the integral converges, and not through analytic continuation. Since $X > 0$ almost surely, we have $\mathbb{E}(X^r) > 0$. Consequently, if either factor diverges, then $\mathbb{E}(Y^r)$ also diverges. Alternatively, the inequality $Y \geq X$ shows why the parent moment must be finite. □

Remark 1. *For a general parent distribution, α gives an upper restriction on the available moment orders. The parent may, however, impose a stronger restriction. Thus, α need not be the tail index for every SMO baseline. Nevertheless, each SMO member is heavy-tailed in the sense of its positive moment-generating function. To see this, choose $c > 0$ such that $\mathbb{P}(X \geq c) > 0$. Since*

$$F_U(t) \sim \frac{t^{\alpha}}{\alpha \mathrm{B}(\alpha,\beta)} \quad (t \downarrow 0), \qquad \mathbb{P}(Y > y) \geq \mathbb{P}(X \geq c)F_U(c/y) \quad (y > c),$$

we obtain a polynomial lower bound for the survival function. Hence, $\mathbb{E}(e^{tY}) = \infty$ for every $t > 0$. This lower bound does not, by itself, prove regular variation for an arbitrary baseline distribution.

3. Slash Marshall–Olkin Exponential Distribution

In this section, we introduce the SMOE distribution as a special case of the SMO family, due to its versatile and simple nature. Here, we take the exponential baseline

$$G_0(x) = 1 - e^{-\lambda x}, \qquad g_0(x) = \lambda e^{-\lambda x}, \qquad x > 0,\ \lambda > 0.$$

Substituting into (1) and (2), the MOE parent has cdf

$$F_X(x) = \frac{1 - e^{-\lambda x}}{1 - (1-\eta)e^{-\lambda x}}, \qquad x > 0, \tag{11}$$

and pdf

$$f_X(x) = \frac{\eta \lambda e^{-\lambda x}}{\left[1 - (1-\eta)e^{-\lambda x}\right]^2}, \qquad x > 0, \tag{12}$$

Definition 2. *A random variable Y is said to follow the SMOE, written*

$$Y \sim \text{SMOE}(\eta, \lambda, \alpha, \beta),$$

if

$$Y = \frac{X}{U},$$

where X has pdf (12) and $U \sim \text{Beta}(\alpha, \beta)$ is independent of X.

3.1. Integral Forms

Proposition 5. *The cdf, sf, and pdf of the SMOE model are, for $y > 0$,*

$$F_Y(y) = \frac{1}{\text{B}(\alpha, \beta)} \int_0^1 \frac{1 - e^{-\lambda y u}}{1 - (1-\eta)e^{-\lambda y u}} u^{\alpha-1}(1-u)^{\beta-1}\, \mathrm{d}u, \tag{13}$$

$$\overline{F}_Y(y) = \frac{\eta}{\text{B}(\alpha, \beta)} \int_0^1 \frac{u^{\alpha-1}(1-u)^{\beta-1} e^{-\lambda y u}}{1 - (1-\eta)e^{-\lambda y u}}\, \mathrm{d}u, \tag{14}$$

and

$$f_Y(y) = \frac{\eta \lambda}{\text{B}(\alpha, \beta)} \int_0^1 \frac{u^{\alpha}(1-u)^{\beta-1} e^{-\lambda y u}}{\left[1 - (1-\eta)e^{-\lambda y u}\right]^2}\, \mathrm{d}u. \tag{15}$$

For each $\eta > 0$ and $t \geq 0$, consider the denominator

$$D_\eta(t) = 1 - (1-\eta)e^{-t} = (1 - e^{-t}) + \eta e^{-t}$$

which satisfies $\min(1, \eta) \leq D_\eta(t) \leq \max(1, \eta)$. We thus observe that the above integral formulas have no denominator singularity over the stated parameter space. In addition, they give a normalized density, since

$$\int_0^\infty f_Y(y)\, \mathrm{d}y = \int_0^1 f_U(u) \left\{ \int_0^\infty u f_X(yu)\, \mathrm{d}y \right\} \mathrm{d}u = 1,$$

by applying Tonelli's theorem and the substitution $x = yu$. Therefore, the validity of the distribution does not depend on the convergence of any particular series representation.

3.2. Series Representation

Let

$$q = 1 - \eta.$$

When $|q| < 1$, the geometric identities

$$\frac{1}{1-qe^{-t}} = \sum_{k=0}^{\infty} q^k e^{-kt}, \qquad \frac{1}{(1-qe^{-t})^2} = \sum_{k=0}^{\infty} (k+1) q^k e^{-kt} \tag{16}$$

hold.

Also define Kummer's confluent hypergeometric function

$$M(a,b,z) = {}_1F_1(a;b;z).$$

Using the beta–Laplace identity (which is the standard Kummer integral; for details, see [14])

$$\int_0^1 u^{a-1}(1-u)^{b-1}e^{-cu}\,\mathrm{d}u = \mathrm{B}(a,b)\,M(a,a+b,-c), \qquad c \geq 0, \tag{17}$$

we obtain the following result.

Proposition 6. *If $|1-\eta| < 1$, then sf and pdf of the SMOE model admit the series forms*

$$\overline{F}_Y(y) = \eta \sum_{k=0}^{\infty} (1-\eta)^k M(\alpha, \alpha+\beta, -(k+1)\lambda y), \tag{18}$$

and

$$f_Y(y) = \eta\lambda \frac{\alpha}{\alpha+\beta} \sum_{k=0}^{\infty} (k+1)(1-\eta)^k M(\alpha+1, \alpha+\beta+1, -(k+1)\lambda y). \tag{19}$$

Consequently,

$$F_Y(y) = 1 - \overline{F}_Y(y). \tag{20}$$

Proof. From (14), using (16),

$$\overline{F}_Y(y) = \frac{\eta}{\mathrm{B}(\alpha,\beta)} \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} e^{-\lambda y u} \sum_{k=0}^{\infty} (1-\eta)^k e^{-k\lambda y u}\,\mathrm{d}u.$$

For $|1-\eta| < 1$, we use the integrable beta weight multiplied by $\sum_{k\geq 0} |1-\eta|^k$ to dominate the absolute survival series. For the density series, the beta weight multiplied by $\sum_{k\geq 0}(k+1)|1-\eta|^k$ gives the corresponding domination. Hence, we can interchange summation and integration to obtain

$$\overline{F}_Y(y) = \frac{\eta}{\mathrm{B}(\alpha,\beta)} \sum_{k=0}^{\infty} (1-\eta)^k \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} e^{-(k+1)\lambda y u}\,\mathrm{d}u.$$

Applying (17) gives (18). Similarly, from (15),

$$f_Y(y) = \frac{\eta\lambda}{\mathrm{B}(\alpha,\beta)} \sum_{k=0}^{\infty} (k+1)(1-\eta)^k \int_0^1 u^{\alpha}(1-u)^{\beta-1} e^{-(k+1)\lambda y u}\,\mathrm{d}u.$$

Again by (17),

$$\int_0^1 u^{\alpha}(1-u)^{\beta-1}e^{-cu}\,\mathrm{d}u = \mathrm{B}(\alpha+1,\beta) M(\alpha+1, \alpha+\beta+1, -c).$$

Hence

$$f_Y(y) = \eta\lambda \frac{\mathrm{B}(\alpha+1,\beta)}{\mathrm{B}(\alpha,\beta)} \sum_{k=0}^{\infty} (k+1)(1-\eta)^k M(\alpha+1, \alpha+\beta+1, -(k+1)\lambda y).$$

Since

$$\frac{\mathrm{B}(\alpha+1,\beta)}{\mathrm{B}(\alpha,\beta)} = \frac{\alpha}{\alpha+\beta},$$

we obtain (19). □

Theorem 1 (Alternative series covering $\eta \geq 2$)**.** *Let $\eta > 1$ and put $\rho = 1-\eta^{-1} \in (0,1)$. For $a, b > 0$ and $t \geq 0$, define*

$$H_k(a,b;t) = \frac{1}{\mathrm{B}(a,b)} \int_0^1 u^{a-1}(1-u)^{b-1} e^{-tu}(1-e^{-tu})^k \,\mathrm{d}u \tag{21}$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} M(a, a+b, -(j+1)t). \tag{22}$$

Then

$$\overline{F}_Y(y) = \sum_{k=0}^{\infty} \rho^k H_k(\alpha,\beta;\lambda y), \tag{23}$$

$$f_Y(y) = \frac{\lambda}{\eta}\frac{\alpha}{\alpha+\beta} \sum_{k=0}^{\infty} (k+1)\rho^k H_k(\alpha+1,\beta;\lambda y). \tag{24}$$

These representations, together with Proposition 6, are valid for all $\eta > 0$.

Proof. First, write $D_\eta(t) = \eta\{1-\rho(1-e^{-t})\}$. Since $0 \leq \rho(1-e^{-t}) \leq \rho < 1$, we can expand its reciprocal and reciprocal square. The resulting integrals have nonnegative terms. Therefore, Tonelli's theorem gives (23) and (24). By applying the finite binomial expansion in (21), followed by the beta–Laplace identity, we get (22). Moreover, $0 \leq H_k \leq 1$, so both outer series converge uniformly and absolutely. Here, the original series in powers of $1-\eta$ is used only within its convergence domain. □

By truncating the series at $k = K$, we obtain the following absolute error bounds:

$$R_S \leq \frac{\rho^{K+1}}{1-\rho},$$

$$R_f \leq \frac{\lambda\alpha}{\eta(\alpha+\beta)} \frac{\rho^{K+1}\{(K+2)-(K+1)\rho\}}{(1-\rho)^2}.$$

For large k, cancellation may occur in the alternating finite sum (22). Thus, for numerical calculations, the nonnegative integral (21) or the original density integral is preferable. Also, the alternative expansion may converge slowly for large η. These numerical difficulties do not change the mathematical validity of the representation.

Figure 1 illustrates the pdf of the SMOE model for different parameter shapes. Similarly, using (9), we can derive the hrf of the SMOE distribution. Different shapes of the hrf of the SMOE distribution are illustrated in Figure 2.

3.3. Behavior at the Origin and Implications for Modes

Theorem 2 (Boundary values and initial slopes)**.** *Let $m_1 = \mathbb{E}(U) = \alpha/(\alpha+\beta)$ and $m_2 = \mathbb{E}(U^2) = \alpha(\alpha+1)/\{(\alpha+\beta)(\alpha+\beta+1)\}$. Then, for all positive parameter values, we have*

$$f_Y(0+) = h_Y(0+) = \frac{\lambda m_1}{\eta}, \tag{25}$$

$$f'_Y(0+) = \frac{\lambda^2}{\eta^2}(\eta - 2)m_2, \tag{26}$$

$$h'_Y(0+) = \frac{\lambda^2}{\eta^2}\{(\eta - 2)m_2 + m_1^2\}. \tag{27}$$

The density is strictly decreasing on $(0, \infty)$ *when* $0 < \eta \leq 2$. *When* $\eta > 2$, *it is initially increasing and has at least one interior mode.*

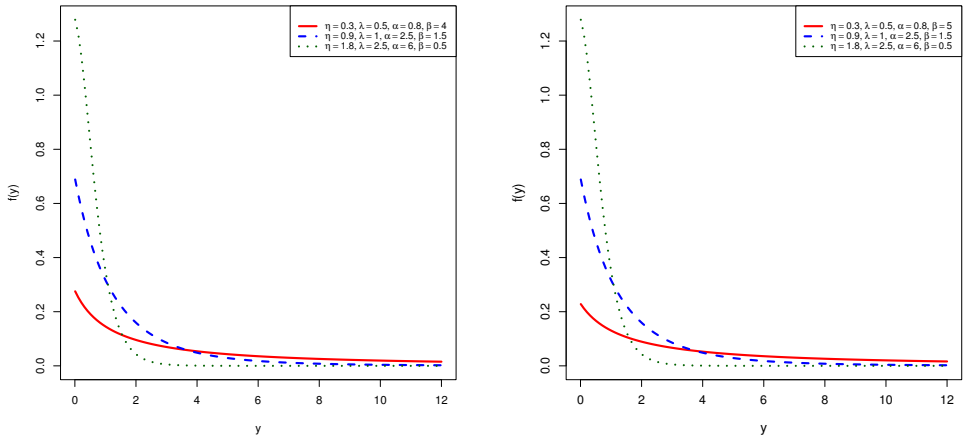


Figure 1. Pdf plots for different parameter values of the SMOE distribution.

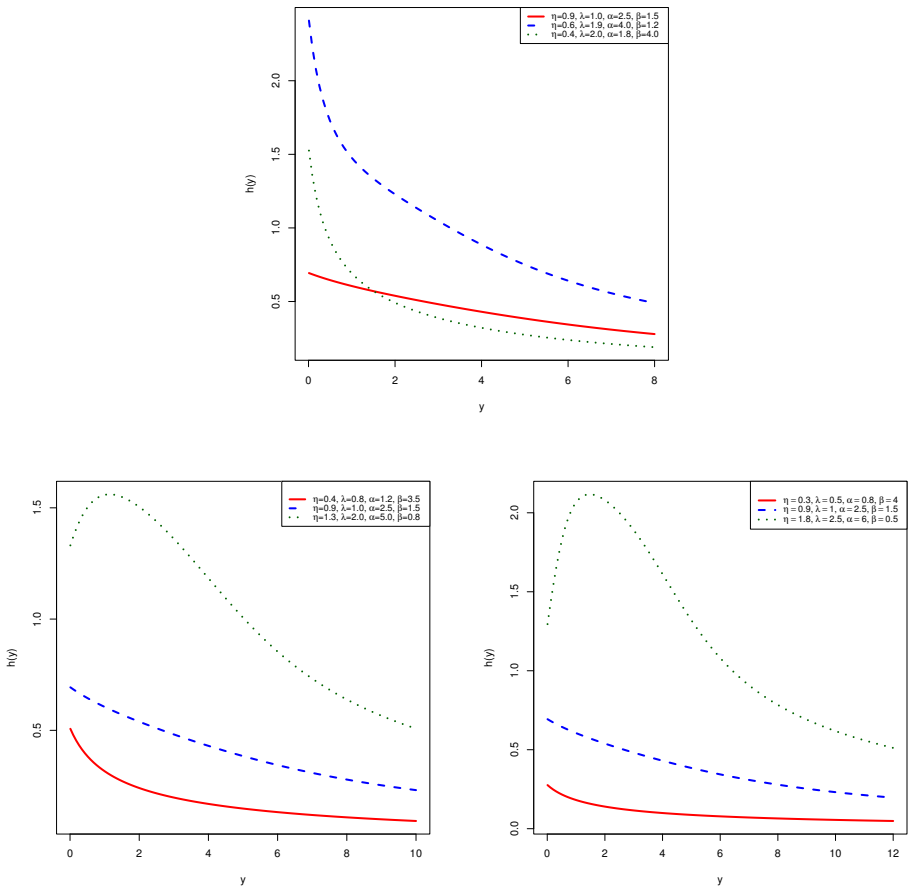


Figure 2. Hrf plots for different parameter values of the SMOE distribution.

Proof. For the MOE parent, we have $f_X(0+) = \lambda/\eta$ and

$$f'_X(x) = -\frac{\eta\lambda^2 e^{-\lambda x}\{1+(1-\eta)e^{-\lambda x}\}}{\{1-(1-\eta)e^{-\lambda x}\}^3}, \qquad f'_X(0+) = \frac{\lambda^2(\eta-2)}{\eta^2}.$$

Since $u \in (0,1)$, the parent density and its first derivative are bounded on the relevant compact intervals near zero. By applying dominated convergence to $f_Y(y) = \mathbb{E}\{Uf_X(yU)\}$ and its derivative, we obtain (25) and (26). Also, $\overline{F}_Y(0+) = 1$, and differentiation of $h_Y = f_Y/\overline{F}_Y$ gives (27). When $\eta \le 2$, we have $1+(1-\eta)e^{-\lambda x} > 0$ for every $x > 0$. Thus, $f'_X(x) < 0$ and $f'_Y(y) < 0$. When $\eta > 2$, the initial derivative is positive. By Theorem 3, the continuous density tends to zero at infinity, and hence it attains a maximum at a positive interior point. This does not establish the uniqueness of the mode. □

For fixed admissible parameters, the density is neither zero nor infinite at the origin. On the closure of the support, the origin is a finite boundary mode if and only if $\eta \le 2$. This boundary mode is not an atom of the distribution. In particular, for $\eta = 2$, the initial derivative is zero, although the density is strictly decreasing for $y > 0$. The hazard is initially increasing when

$$\eta > \eta_H := 2 - \frac{m_1^2}{m_2} = 2 - \frac{\alpha(\alpha+\beta+1)}{(\alpha+\beta)(\alpha+1)} \in (1,2),$$

and initially decreasing when $\eta < \eta_H$. At equality, the first derivative is zero, so higher-order terms must be considered. Thus, the initial slope alone does not describe the shape of the whole hazard curve.

3.4. Moments of the SMOE Model

Proposition 7. *If $|1-\eta| < 1$ and $\alpha > r > 0$, then the rth raw moment of the SMOE model is*

$$\mathbb{E}(Y^r) = \left[\eta\frac{\Gamma(r+1)}{\lambda^r}\sum_{k=0}^{\infty}\frac{(1-\eta)^k}{(k+1)^r}\right]\frac{\mathrm{B}(\alpha-r,\beta)}{\mathrm{B}(\alpha,\beta)}. \tag{28}$$

Proof. For the MO-exponential parent,

$$\overline{F}_X(x) = \frac{\eta e^{-\lambda x}}{1-(1-\eta)e^{-\lambda x}} = \eta\sum_{k=0}^{\infty}(1-\eta)^k e^{-(k+1)\lambda x},$$

provided $|1-\eta| < 1$. Then for $r > 0$,

$$\mathbb{E}(X^r) = r\int_0^{\infty} x^{r-1}\overline{F}_X(x)\,\mathrm{d}x = \eta r\sum_{k=0}^{\infty}(1-\eta)^k\int_0^{\infty} x^{r-1}e^{-(k+1)\lambda x}\,\mathrm{d}x.$$

Using

$$\int_0^{\infty} x^{r-1}e^{-ax}\,\mathrm{d}x = \frac{\Gamma(r)}{a^r}, \qquad a > 0,$$

we get

$$\mathbb{E}(X^r) = \eta\frac{\Gamma(r+1)}{\lambda^r}\sum_{k=0}^{\infty}\frac{(1-\eta)^k}{(k+1)^r}.$$

Now apply (10). □

For all $\eta > 0$ and $0 < r < \alpha$, the moments can also be expressed over the full parameter domain as

$$\mathbb{E}(Y^r) = \frac{\eta r}{\lambda^r}\frac{\mathrm{B}(\alpha-r,\beta)}{\mathrm{B}(\alpha,\beta)}\int_0^{\infty}\frac{t^{r-1}e^{-t}}{1-(1-\eta)e^{-t}}\,\mathrm{d}t. \tag{29}$$

This expression follows by using $\mathbb{E}(X^r) = r\int_0^\infty x^{r-1}\overline{F}_X(x)\,\mathrm{d}x$ and the substitution $t = \lambda x$. Since the denominator is bounded away from zero, the integral is finite for every $r > 0$. From Proposition 4, we then have $\mathbb{E}(Y^r) < \infty$ if and only if $r < \alpha$, without the restriction required for the series. Consequently, the mean is finite exactly when $\alpha > 1$, and the variance exactly when $\alpha > 2$. In particular, Set VI in Section 5 has an infinite population variance for Y. This property alone does not determine the finite-sample MSE of a parameter estimator.

3.5. Regular Variation and Heavy-Tail Classification

Theorem 3 (Tail asymptotics)**.** *For each $\eta, \lambda, \alpha, \beta > 0$, let*

$$C = \frac{\mathbb{E}(X^\alpha)}{\alpha \mathrm{B}(\alpha, \beta)} \in (0, \infty).$$

Then, as $y \to \infty$,

$$\overline{F}_Y(y) \sim Cy^{-\alpha}, \qquad f_Y(y) \sim \alpha C y^{-\alpha-1}, \qquad h_Y(y) \sim \frac{\alpha}{y}. \tag{30}$$

It follows that $\overline{F}_Y(cy)/\overline{F}_Y(y) \to c^{-\alpha}$ for every $c > 0$. Moreover, the positive moment-generating function is infinite at every positive argument.

Proof. For the beta cdf, we have $F_U(t)/t^\alpha \to 1/\{\alpha \mathrm{B}(\alpha, \beta)\}$ as $t \downarrow 0$. Also, $F_U(t) \le Kt^\alpha$ for $0 < t \le 1$ and some finite K. Now, by conditioning on X, we obtain

$$\overline{F}_Y(y) = \mathbb{E}\big[F_U\{\min(X/y, 1)\}\big].$$

For $X < y$, we have $y^\alpha F_U(X/y) \le KX^\alpha$, whereas for $X \ge y$, we have $y^\alpha \le X^\alpha$. The parent has exponential decay, so $\mathbb{E}(X^\alpha) < \infty$. Thus, dominated convergence gives the required survival asymptotic.

For the density, we use the substitution $t = yu$ in the mixture integral and get

$$y^{\alpha+1} f_Y(y) = \frac{1}{\mathrm{B}(\alpha, \beta)} \int_0^y t^\alpha (1 - t/y)^{\beta-1} f_X(t)\,\mathrm{d}t.$$

On $0 < t \le y/2$, the additional factor is bounded by $\max(1, 2^{1-\beta})$, allowing us to apply dominated convergence. On $y/2 < t < y$, we use $f_X(t) \le K_1 e^{-\lambda t}$ to obtain the bound

$$K_1 e^{-\lambda y/2} y^{\alpha+1} \int_{1/2}^1 v^\alpha (1 - v)^{\beta-1}\,\mathrm{d}v \longrightarrow 0.$$

This also covers the integrable endpoint singularity for $0 < \beta < 1$. The density limit is $\mathbb{E}(X^\alpha)/\mathrm{B}(\alpha, \beta) = \alpha C$, from which the hazard limit follows. Finally, a polynomial survival tail cannot have a finite positive exponential moment. □

Corollary 2 (Power-law class and subexponentiality)**.** *For every positive SMOE parameter vector, Y has a regularly varying survival function with exponent $-\alpha$ and belongs to the subexponential class. More precisely, let $Y_1, \ldots, Y_m$ be independent copies of Y. For each fixed integer $m \ge 2$, we have*

$$\mathbb{P}\left(\sum_{j=1}^m Y_j > y\right) \sim \mathbb{P}\left(\max_{1 \le j \le m} Y_j > y\right) \sim m\overline{F}_Y(y), \qquad y \to \infty. \tag{31}$$

Thus, no fixed finite α allows all positive-order moments to exist: $\mathbb{E}(Y^r) < \infty$ if and only if $0 < r < \alpha$, whereas $\mathbb{E}(Y^r) = \infty$ for $r \ge \alpha$.

Proof. The regular-variation property follows from Theorem 3. To show the remaining statement, write $S = \overline{F}_Y$ and choose $b(y) = y^\delta$, where $1/2 < \delta < 1$. By positivity and independence, we have the lower bound

$$\mathbb{P}\left(\sum_{j=1}^{m} Y_j > y\right) \geq 1 - \{1 - S(y)\}^m \sim mS(y).$$

When the sum exceeds y, either its largest term exceeds $y - (m-1)b(y)$ or at least two terms exceed $b(y)$. Therefore,

$$\mathbb{P}\left(\sum_{j=1}^{m} Y_j > y\right) \leq mS\{y - (m-1)b(y)\} + \binom{m}{2} S\{b(y)\}^2.$$

Using $S(y) \sim Cy^{-\alpha}$, the first term divided by $S(y)$ tends to m. In addition, $S\{b(y)\}^2/S(y) \sim Cy^{\alpha(1-2\delta)} \to 0$. Hence, (31) follows. For $m = 2$, this is the defining property of subexponentiality [13]. The moment condition follows from Proposition 4, since all positive moments of the MOE parent are finite. □

3.6. Tail-Class Scope and Practical Implications

Here, the tail classification concerns the asymptotic behavior of the proposed model. It does not mean that every skewed data set has a power-law tail. For the SMOE distribution, $\overline{F}_Y(y) \sim Cy^{-\alpha}$ with a finite positive constant C. Thus, it is a particular polynomial-tail family, rather than the whole regularly varying class with an unrestricted slowly varying factor. In addition, it does not reproduce the asymptotic tails of all-moment-finite subexponential distributions outside the power-law class. The lognormal and Weibull distributions with shape between zero and one are examples; see [15]. Similar fitted curves over a bounded range of observations do not make their tail classes equivalent.

The moment conditions also have different practical implications. For $0 < \alpha \leq 1$, the population mean is infinite. For $1 < \alpha \leq 2$, the mean is finite, but the second moment is infinite. For $\alpha > 2$, both mean and variance are finite, although moments of sufficiently high order still diverge. Therefore, a finite-variance SMOE fit is still a power-law model, not a model having all moments finite. Mean- and variance-based summaries should be used only under the corresponding moment conditions. In particular, the usual finite-variance normal approximation for sample averages is not justified for $\alpha \leq 2$. The fixed-m result in (31) describes asymptotic exceedances caused by one large observation. It does not ensure an accurate approximation at finite thresholds or for finite samples.

The above statements concern the SMOE distribution. For a general SMO parent, the survival asymptotic of Theorem 3 also follows from the same dominated-convergence argument when $\mathbb{E}(X^\alpha) < \infty$. Without this condition, the parent may give a heavier tail. Thus, we do not assign the exponent α to every SMO baseline, nor claim that the general construction covers all fat-tail classes.

4. Maximum Likelihood Estimation

4.1. Score Equations and Regularity

For the general SMO family, the differentiation used below requires a neighborhood in which the first two parameter derivatives of each integrand have an integrable dominating function. Absolute continuity of the baseline alone is not sufficient for this parameter regularity. For the SMOE distribution, the condition holds on compact subsets of $(0, \infty)^4$. Indeed, the denominator has a positive lower bound, while beta weights multiplied by finite powers of $\log u$ and $\log(1-u)$ are integrable. For simplicity, we derive the mle for

the SMO family. By considering the exponential distribution as the baseline, we obtain the mles for the SMOE distribution as a special case. Suppose $y_1, \ldots, y_n$ is a random sample from the SMO family. Define

$$D_i(u) = \eta + (1-\eta)G(y_i u; \xi), \tag{32}$$

and

$$I_i(\eta, \xi, \alpha, \beta) = \int_0^1 \frac{g(y_i u; \xi)\, u^{\alpha}(1-u)^{\beta-1}}{D_i(u)^2}\, \mathrm{d}u. \tag{33}$$

Then from (8),

$$f_Y(y_i) = \frac{\eta}{\mathrm{B}(\alpha, \beta)} I_i.$$

Therefore, the log-likelihood is

$$\ell(\eta, \xi, \alpha, \beta) = n \log \eta - n \log \mathrm{B}(\alpha, \beta) + \sum_{i=1}^{n} \log I_i(\eta, \xi, \alpha, \beta). \tag{34}$$

Score equations are

Derivative with respect to η. Since

$$\frac{\partial D_i(u)}{\partial \eta} = 1 - G(y_i u; \xi),$$

we get

$$\frac{\partial}{\partial \eta}\left[D_i(u)^{-2}\right] = -2\frac{1 - G(y_i u; \xi)}{D_i(u)^3}.$$

Hence

$$\frac{\partial I_i}{\partial \eta} = -2\int_0^1 g(y_i u; \xi)\, u^{\alpha}(1-u)^{\beta-1} \frac{1 - G(y_i u; \xi)}{D_i(u)^3}\, \mathrm{d}u. \tag{35}$$

Therefore,

$$\frac{\partial \ell}{\partial \eta} = \frac{n}{\eta} + \sum_{i=1}^{n} \frac{1}{I_i} \frac{\partial I_i}{\partial \eta}. \tag{36}$$

Derivative with respect to α. Using

$$\frac{\partial}{\partial \alpha} \log \mathrm{B}(\alpha, \beta) = \psi(\alpha) - \psi(\alpha + \beta),$$

we have

$$\frac{\partial I_i}{\partial \alpha} = \int_0^1 \frac{g(y_i u; \xi)\, u^{\alpha}(1-u)^{\beta-1} \log u}{D_i(u)^2}\, \mathrm{d}u. \tag{37}$$

Hence

$$\frac{\partial \ell}{\partial \alpha} = -n\left[\psi(\alpha) - \psi(\alpha + \beta)\right] + \sum_{i=1}^{n} \frac{1}{I_i} \frac{\partial I_i}{\partial \alpha}. \tag{38}$$

Derivative with respect to β. Using

$$\frac{\partial}{\partial \beta} \log \mathrm{B}(\alpha, \beta) = \psi(\beta) - \psi(\alpha + \beta),$$

we obtain

$$\frac{\partial I_i}{\partial \beta} = \int_0^1 \frac{g(y_i u; \xi)\, u^{\alpha}(1-u)^{\beta-1} \log(1-u)}{D_i(u)^2}\, \mathrm{d}u, \tag{39}$$

and

$$\frac{\partial \ell}{\partial \beta} = -n\left[\psi(\beta) - \psi(\alpha + \beta)\right] + \sum_{i=1}^{n} \frac{1}{I_i} \frac{\partial I_i}{\partial \beta}. \tag{40}$$

Derivative with respect to the baseline parameter vector ξ. Assuming differentiation under the integral sign is valid,

$$\frac{\partial I_i}{\partial \xi} = \int_0^1 u^{\alpha}(1-u)^{\beta-1}\left[\frac{\partial_\xi g(y_i u;\xi)}{D_i(u)^2} - 2(1-\eta)\frac{g(y_i u;\xi)\partial_\xi G(y_i u;\xi)}{D_i(u)^3}\right] \mathrm{d}u. \tag{41}$$

Thus

$$\frac{\partial \ell}{\partial \xi} = \sum_{i=1}^{n} \frac{1}{I_i}\frac{\partial I_i}{\partial \xi}. \tag{42}$$

An interior stationary candidate satisfies the following score equations. However, a solution of these equations need not be a maximum, and its uniqueness is not assured:

$$\frac{\partial \ell}{\partial \eta} = 0, \qquad \frac{\partial \ell}{\partial \alpha} = 0, \qquad \frac{\partial \ell}{\partial \beta} = 0, \qquad \frac{\partial \ell}{\partial \xi} = \mathbf{0}.$$

4.2. Observed Information and Identifiability

Let $\theta = (\eta, \lambda, \alpha, \beta)^{\mathsf{T}}$, $a_i = y_i u$, $z_i = e^{-\lambda a_i}$, $q = 1-\eta$, and $D_i = 1 - q z_i$. The conditional density of U, given $Y = y_i$, is then expressed as

$$w_i(u;\theta) = \frac{u f_X(y_i u;\eta,\lambda) f_U(u;\alpha,\beta)}{f_Y(y_i;\theta)}, \quad 0 < u < 1.$$

Apart from constants, the complete-data log-density is given by

$$c_i = \log\eta + \log\lambda - \log \mathrm{B}(\alpha,\beta) + \alpha\log u + (\beta-1)\log(1-u) - \lambda a_i - 2\log D_i.$$

The corresponding score vector is

$$s_i(u) = \begin{pmatrix} \eta^{-1} - 2z_i/D_i \\ \lambda^{-1} - a_i - 2qa_i z_i/D_i \\ \log u - \psi(\alpha) + \psi(\alpha+\beta) \\ \log(1-u) - \psi(\beta) + \psi(\alpha+\beta) \end{pmatrix}. \tag{43}$$

Let ψ_1 denote the trigamma function. Then, the nonzero entries of the symmetric Hessian $H_i = \nabla^2_\theta c_i$ are given by

$$\begin{aligned} H_{i,\eta\eta} &= -\eta^{-2} + 2z_i^2/D_i^2, & H_{i,\eta\lambda} &= 2a_i z_i/D_i^2, \\ H_{i,\lambda\lambda} &= -\lambda^{-2} + 2qa_i^2 z_i/D_i^2, & H_{i,\alpha\beta} &= \psi_1(\alpha+\beta), \\ H_{i,\alpha\alpha} &= -\psi_1(\alpha) + \psi_1(\alpha+\beta), & H_{i,\beta\beta} &= -\psi_1(\beta) + \psi_1(\alpha+\beta). \end{aligned} \tag{44}$$

By differentiating the integral likelihood twice, we obtain the full observed information matrix. Equivalently, the missing-information identity of Louis [16] gives

$$\mathcal{J}_n(\theta) = -\nabla^2\ell(\theta) = -\sum_{i=1}^{n}[\mathbb{E}_{w_i}\{H_i(U)\} + \mathrm{Cov}_{w_i}\{s_i(U)\}]. \tag{45}$$

All the expectations in this expression are one-dimensional integrals. Thus, it provides an explicit formula for the observed information matrix. It is not a numerical matrix computed for the application data, since that calculation requires the individual observations.

Global identifiability requires that equality of the entire distributions implies equality of all four parameters. A Hessian obtained from a finite sample does not prove this property. Under the usual differentiability and local regularity conditions, nonsingularity of the expected Fisher information gives a suitable criterion for local identifiability. From

the tail limit, we can identify α from the distribution. When α and β are fixed, (25) and (26) also yield

$$\eta = 2 + \frac{m_1^2}{m_2}\frac{f_Y'(0+)}{f_Y(0+)^2}, \qquad \lambda = \frac{\eta f_Y(0+)}{m_1}. \tag{46}$$

We thus observe that η and λ cannot be exactly interchanged when the beta parameters are known. This result does not establish global identifiability when all the parameters are unknown.

Some limiting cases are useful for understanding weak information. As $\beta \downarrow 0$ with α fixed, $U \to 1$ in probability. The distribution then approaches the MOE parent, so much of the information about α is lost. As $\beta \to \infty$ with α fixed, βU converges to a Gamma$(\alpha, 1)$ variable V, as follows by scaling the beta density. If $Z_\eta = \lambda X$, then along paths satisfying $\beta/\lambda \to c \in (0, \infty)$, we have

$$Y = \frac{\beta}{\lambda}\frac{Z_\eta}{\beta U} \xrightarrow{d} c\,\frac{Z_\eta}{V}.$$

This relation shows an approximate β–λ ridge. In addition, the density at the origin depends on λ/η. Therefore, numerical stability must be examined directly. It cannot be concluded from the number of parameters or from a small discrepancy statistic.

At an interior stationary estimate, let $v = \log\theta$ componentwise and $T = \mathrm{diag}(\theta)$. We then have $\mathcal{J}_v = T\mathcal{J}_n T$. Its eigenvalues, condition number, and covariance-based parameter correlations should be reported, together with the two-dimensional profile

$$\ell_p(v_\eta, v_\lambda) = \sup_{v_\alpha, v_\beta} \ell(e^v).$$

A positive-definite observed information matrix establishes a strict local maximum, but not a global maximum. When the matrix is nearly singular or the profiles are broad, profile or bootstrap intervals should be considered instead of directly inverting an unstable Hessian.

4.3. Existence, Non-Uniqueness, and Numerical Safeguards

For positive observations, $f_Y(y_i;\theta)$ is positive and continuous in θ on every compact positive parameter box. Therefore, by the extreme-value theorem, the log-likelihood attains a finite maximum on each such box. However, no general strict-concavity or unique-root property has been established for the unrestricted four-parameter likelihood.

In fact, a finite unrestricted maximizer does not exist without further conditions in every case. To illustrate this point, consider the single observation $y = c > 0$ and the path

$$\lambda_m = m, \quad \eta_m = e^{mc}, \quad \alpha_m = m^2, \quad \beta_m = 1.$$

For $U_m \sim \mathrm{Beta}(m^2, 1)$, we have $m(1 - U_m) \to 0$ in probability because $\mathbb{E}\{m(1-U_m)\} = m/(m^2+1) \to 0$. Putting $A_m = e^{mc(1-U_m)}$, the parent density gives

$$\frac{f_{X_m}(cU_m)}{m} = \frac{A_m}{\{1 + (1 - e^{-mc})A_m\}^2} \longrightarrow \frac{1}{4}$$

in probability. Since this ratio is bounded by $1/\{4(1-e^{-mc})\}$, bounded convergence in probability yields $f_{Y_m}(c)/m = \mathbb{E}\{U_m f_{X_m}(cU_m)/m\} \to 1/4$. Hence, the likelihood for this single observation is unbounded. The same argument applies when every observation equals c. This example does not mean that every nondegenerate data set has an unbounded likelihood. Rather, it shows why existence or uniqueness cannot be assumed without further conditions.

For reproducible numerical estimation, we consider optimization in the log parameters with several dispersed starting values. The largest successfully verified likelihood should

be retained, and interior solutions should be distinguished from solutions on the boundary of the parameter box. The box, stopping rules, gradient norm, quadrature tolerance, and failed starts must be reported. At the retained candidate, the quadrature order should be doubled, followed by re-optimization if the likelihood changes materially. The parameter box should also be enlarged and the profiles examined for flat or escaping solutions. Agreement between several starts gives evidence of numerical stability, but does not prove global optimality. In addition, the scores, profiles, observed information, and uncertainty intervals must use the same data scale and the same retained candidate.

4.4. Parsimony and Practical Implementation

The use of four fitted parameters increases the computational work. It can also make estimation less stable for small samples or in the weak-information regimes discussed in Section 4.2. Giving a separate interpretation to each parameter does not remove these difficulties. Thus, the SMOE distribution is an alternative when both the body shape and a polynomial tail are important in the application. It is not proposed as a default replacement for simpler engineering models, and widespread use among engineers is not claimed.

Some exact submodels with fewer parameters can also be obtained. When $\eta = 1$, the MO deformation is removed, leaving the three free parameters (λ, α, β). When $\beta = 1$, we obtain a three-parameter power-denominator slash MOE model with free parameters (η, λ, α). By fixing both $\eta = 1$ and $\beta = 1$, we get a two-parameter slash exponential model with free parameters (λ, α). Each of these restrictions retains the power-law exponent α. The same integral formulas and exact generator can be used by keeping the specified parameters fixed. In contrast, $\beta \downarrow 0$ gives the MOE model as a limit; $\beta = 0$ is not an admissible parameter value in the beta mixture.

To examine their practical performance, the restricted models and the full model should be fitted to the same observations. Only the free parameters should be counted in AIC and BIC, and numerical stability and uncertainty should be considered along with fit. The full model uses $k = 4$, whereas the above submodels use $k = 3$ or $k = 2$. These penalized criteria alone do not establish practical usefulness, and their usual regularity conditions still apply. In this regard, a simpler model is preferable when stable estimation and adequate evidence do not support the additional flexibility.

5. Simulation

In this section, the simulation design considers the average estimates (AE), signed average bias (AB), mean squared error (MSE), root mean squared error (RMSE), and coverage probability (CP) of the SMOE parameter estimates. We first establish that the generator is exact, independently of the optimization method used afterwards. This exactness does not imply that the maximum likelihood estimates are unbiased.

5.1. Exact Random Generation

For each observation in each replication, independently generate $U_i \sim \text{Beta}(\alpha, \beta)$ and $W_i \sim \text{Unif}(0, 1)$. Then, put

$$X_i = Q_X(W_i) = -\frac{1}{\lambda} \log\left\{ \frac{1 - W_i}{1 - (1 - \eta)W_i} \right\}, \qquad Y_i = X_i / U_i. \tag{47}$$

Here, the parent X_i follows the MOE distribution with parameters (η, λ), rather than the SMOE distribution.

Theorem 4 (Exactness of the stochastic generator)**.** *For all positive parameter values, the construction in (47) gives independent observations having the cdf (13).*

Proof. The parent cdf is continuous and strictly increasing from zero to one. By solving $w = F_X(x)$, we obtain $e^{-\lambda x} = (1-w)/\{1-(1-\eta)w\}$. The denominator is positive and exceeds the numerator by ηw. Thus, the inverse gives a positive observation for $0 < w < 1$, and $\mathbb{P}\{Q_X(W) \le x\} = F_X(x)$. By independence, we have

$$\mathbb{P}\{Q_X(W)/U \le y\} = \int_0^1 F_X(yu) \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}\,\mathrm{d}u,$$

which is exactly (13). Since the pairs (U_i, W_i) are independent, the resulting sample is iid. □

For numerical evaluation, $\lambda X = \log\{1+\eta W/(1-W)\}$ can be computed using stable logarithmic routines. The uniform endpoints zero and one should be excluded. This procedure does not require an acceptance–rejection approximation or truncation of a distributional series.

5.2. Simulation Design and Reporting Requirements

The study considers six parameter sets (Sets I–VI) chosen to cover different parameter regimes. Parameter settings are described in Table 1. The sample sizes used are

$$n \in \{25, 75, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000\}.$$

Table 1. True parameter regimes used for SMOE simulations (Sets I–VI).

Set	η	λ	α	β
I	0.9	1.0	2.5	1.5
II	0.5	1.5	3.0	2.0
III	0.6	1.9	4.0	1.2
IV	0.8	0.7	2.2	3.0
V	1.1	1.2	5.0	1.8
VI	0.4	2.0	1.8	4.0

For a scalar parameter component θ_j and R attempted replications, the performance measures, provided every fit is valid, are given by

$$\mathrm{AE}_j = R^{-1}\sum_{r=1}^{R}\widehat{\theta}_{jr}, \qquad \mathrm{AB}_j = \mathrm{AE}_j - \theta_j,$$

$$\mathrm{MSE}_j = R^{-1}\sum_{r=1}^{R}(\widehat{\theta}_{jr}-\theta_j)^2, \qquad \mathrm{RMSE}_j = \sqrt{\mathrm{MSE}_j},$$

$$\mathrm{CP}_j = R^{-1}\sum_{r=1}^{R}\mathbf{1}\{\theta_j \in \mathrm{CI}_{jr}\}.$$

Here, AB denotes signed bias, not absolute bias. The bias plots display $|\mathrm{AB}_j|$. For a well-conditioned interior estimate, a possible 95% interval on the log-parameter scale is given by

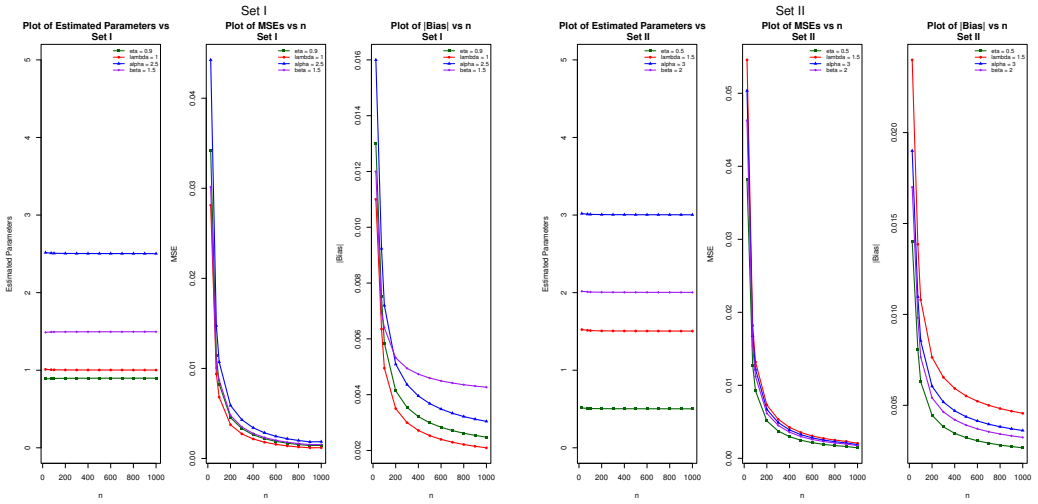
$$\mathrm{CI}_{jr} = \exp\Big[\widehat{v}_{jr} \pm 1.96\sqrt{\{\mathcal{J}_v(\widehat{v}_r)^{-1}\}_{jj}}\Big].$$

This expression gives an interval procedure that can be reproduced. It does not mean that the previously reported CP values were obtained using this procedure.

The previously reported summaries are retained in Tables 2–4 and Figure 3, so that they can be checked against the original results. However, the original analysis does not document the replication count, random seed, optimizer settings, failed-fit counts, or exact

interval construction. Therefore, these entries have not been independently reproduced and are not used as verified evidence of consistency or stability of the estimators. In a new run, the individual estimates and intervals must be saved. Both attempted and valid replications should be reported, together with failure rates and any summaries conditional on successful fits. Failed replications must not be removed from the denominator without explanation. For CP, coverage among valid intervals should be reported along with coverage over all attempts, counting unavailable intervals as noncoverage.

Monte Carlo uncertainty should also be considered. For example, the standard error of an empirical coverage proportion is approximately $\{\widehat{p}(1-\widehat{p})/R\}^{1/2}$ when all R independent intervals are available. Coverage near 0.98 for nominal 0.95 intervals represents overcoverage; it does not automatically show good calibration. Also, bias and MSE need not decrease monotonically in each finite Monte Carlo run. In view of these points, the earlier unconditional statements of convergence and uniformly good performance are not retained.



(a) Set I (b) Set II

(c) Set III (d) Set IV

Figure 3. AE, MSE, and absolute signed-bias curves reported earlier for Sets I–IV. The numerical runs underlying these curves have not been independently reproduced; see Section 5.2.

Table 2. Simulation summaries reported earlier for Sets I and II (not independently reproduced).

n	Parameter	Set I						Set II					
		True	AE	AB	MSE	RMSE	CP	True	AE	AB	MSE	RMSE	CP
25	η	0.9	0.887	−0.013	0.034169	0.184849	0.837965	0.5	0.514	0.014	0.038196	0.195438	0.83664
	λ	1.0	1.011	0.011	0.028121	0.167693	0.838416	1.5	1.524	0.024	0.054576	0.233615	0.837531
	α	2.5	2.516	0.016	0.044256	0.210371	0.838953	3.0	3.019	0.019	0.050361	0.224413	0.838948
	β	1.5	1.488	−0.012	0.030144	0.17362	0.838766	2.0	2.017	0.017	0.046289	0.215149	0.83864
75	η	0.9	0.892494	−0.007506	0.01139	0.106722	0.918825	0.5	0.508083	0.008083	0.012732	0.112836	0.91806
	λ	1.0	1.006351	0.006351	0.009374	0.096818	0.919085	1.5	1.513856	0.013856	0.018192	0.134878	0.918575
	α	2.5	2.509238	0.009238	0.014752	0.121458	0.919395	3.0	3.01097	0.01097	0.016787	0.129565	0.919392
	β	1.5	1.493072	−0.006928	0.010048	0.10024	0.919287	2.0	2.009815	0.009815	0.01543	0.124216	0.919215
100	η	0.9	0.89415	−0.00585	0.008259	0.090879	0.93512	0.5	0.5063	0.0063	0.00924	0.096125	0.93404
	λ	1.0	1.00495	0.00495	0.006806	0.082498	0.935488	1.5	1.5108	0.0108	0.0132	0.114891	0.934768
	α	2.5	2.5072	0.0072	0.010708	0.10348	0.935927	3.0	3.00855	0.00855	0.012188	0.110398	0.935923
	β	1.5	1.4936	−0.0064	0.008704	0.093295	0.93586	2.0	2.00765	0.00765	0.0112	0.10583	0.93573
200	η	0.9	0.895863	−0.004138	0.004535	0.067341	0.951255	0.5	0.504455	0.004455	0.005114	0.071513	0.950012
	λ	1.0	1.003501	0.003501	0.00374	0.061155	0.951678	1.5	1.507637	0.007637	0.007305	0.085469	0.95085
	α	2.5	2.505092	0.005092	0.005887	0.076727	0.952181	3.0	3.006048	0.006048	0.006699	0.081848	0.952176
	β	1.5	1.494682	−0.005318	0.004779	0.06913	0.952104	2.0	2.00541	0.00541	0.00616	0.078485	0.951955
300	η	0.9	0.896457	−0.003543	0.003313	0.057559	0.958914	0.5	0.503815	0.003815	0.003736	0.061123	0.95754
	λ	1.0	1.002998	0.002998	0.002732	0.052268	0.959354	1.5	1.50654	0.00654	0.005341	0.07308	0.958401
	α	2.5	2.50436	0.00436	0.004299	0.065567	0.959877	3.0	3.00518	0.00518	0.004898	0.069986	0.959872
	β	1.5	1.495057	−0.004943	0.003497	0.059135	0.959797	2.0	2.004634	0.004634	0.004506	0.067127	0.959714
400	η	0.9	0.896788	−0.003212	0.002627	0.051254	0.964575	0.5	0.503458	0.003458	0.002964	0.054443	0.963097
	λ	1.0	1.002718	0.002718	0.002167	0.04655	0.965025	1.5	1.505927	0.005927	0.004237	0.065092	0.964002
	α	2.5	2.503953	0.003953	0.00341	0.058391	0.965559	3.0	3.004695	0.004695	0.003886	0.062336	0.965553
	β	1.5	1.495266	−0.004734	0.002774	0.052667	0.965478	2.0	2.004201	0.004201	0.003575	0.05979	0.965394
500	η	0.9	0.897009	−0.002991	0.002193	0.046824	0.96923	0.5	0.503219	0.003219	0.002475	0.049749	0.967665
	λ	1.0	1.002531	0.002531	0.001809	0.042529	0.969686	1.5	1.505518	0.005518	0.003538	0.059481	0.968603
	α	2.5	2.50368	0.00368	0.002848	0.053361	0.970227	3.0	3.004372	0.004372	0.003245	0.056965	0.970221
	β	1.5	1.495406	−0.004594	0.002327	0.048239	0.970145	2.0	2.003912	0.003912	0.002986	0.054644	0.97006

Table 2. *Cont.*

n	Parameter	Set I						Set II					
		True	**AE**	**AB**	**MSE**	**RMSE**	**CP**	**True**	**AE**	**AB**	**MSE**	**RMSE**	**CP**
600	η	0.9	0.897167	−0.002833	0.0019	0.043588	0.973178	0.5	0.503048	0.003048	0.002143	0.046292	0.97154
	λ	1.0	1.002397	0.002397	0.001567	0.039587	0.973638	1.5	1.505224	0.005224	0.003063	0.055342	0.972505
	α	2.5	2.503484	0.003484	0.002463	0.049629	0.974184	3.0	3.004139	0.004139	0.002807	0.052977	0.974178
	β	1.5	1.495506	−0.004494	0.00202	0.044944	0.974101	2.0	2.003704	0.003704	0.002584	0.050836	0.974015
700	η	0.9	0.897288	−0.002712	0.001695	0.041167	0.976039	0.5	0.502917	0.002917	0.00191	0.043703	0.974341
	λ	1.0	1.002295	0.002295	0.001398	0.03739	0.976501	1.5	1.504999	0.004999	0.00273	0.052249	0.975165
	α	2.5	2.503334	0.003334	0.002193	0.046831	0.976743	3.0	3.003961	0.003961	0.0025	0.049996	0.976738
	β	1.5	1.495583	−0.004417	0.001806	0.042497	0.97666	2.0	2.003546	0.003546	0.002302	0.047979	0.976574
800	η	0.9	0.897383	−0.002617	0.001548	0.039346	0.97865	0.5	0.502814	0.002814	0.001744	0.041761	0.976903
	λ	1.0	1.002214	0.002214	0.001277	0.035734	0.979111	1.5	1.504822	0.004822	0.002494	0.04994	0.97769
	α	2.5	2.503216	0.003216	0.001998	0.044694	0.979656	3.0	3.003821	0.003821	0.002278	0.047732	0.979651
	β	1.5	1.495644	−0.004356	0.001653	0.040656	0.979572	2.0	2.003421	0.003421	0.002099	0.045814	0.979486
900	η	0.9	0.89746	−0.00254	0.001438	0.037927	0.981069	0.5	0.502731	0.002731	0.00162	0.040249	0.979231
	λ	1.0	1.002149	0.002149	0.001186	0.034434	0.981529	1.5	1.504679	0.004679	0.002317	0.048136	0.979985
	α	2.5	2.503122	0.003122	0.001845	0.042956	0.982077	3.0	3.003708	0.003708	0.002105	0.045877	0.982073
	β	1.5	1.495692	−0.004308	0.001541	0.039259	0.981993	2.0	2.003319	0.003319	0.00196	0.044271	0.981907
1000	η	0.9	0.897524	−0.002476	0.001455	0.038144	0.98293	0.5	0.502662	0.002662	0.001439	0.037934	0.980821
	λ	1.0	1.002095	0.002095	0.0012	0.034642	0.983389	1.5	1.50456	0.00456	0.002057	0.045353	0.981543
	α	2.5	2.503043	0.003043	0.001866	0.043196	0.983938	3.0	3.003614	0.003614	0.001868	0.043221	0.983934
	β	1.5	1.495733	−0.004267	0.001551	0.03939	0.983853	2.0	2.003235	0.003235	0.001737	0.041677	0.983767

Table 3. Simulation summaries reported earlier for Sets III and IV (not independently reproduced).

n	Parameter	Set III						Set IV					
		True	AE	AB	MSE	RMSE	CP	True	AE	AB	MSE	RMSE	CP
25	η	0.6	0.582	−0.018	0.046324	0.21523	0.836188	0.8	0.785	−0.015	0.040225	0.200562	0.837429
	λ	1.9	1.928	0.028	0.064784	0.254527	0.837656	0.7	0.712	0.012	0.030144	0.17362	0.837726
	α	4.0	3.976	−0.024	0.072576	0.269399	0.838984	2.2	2.217	0.017	0.044289	0.21045	0.838751
	β	1.2	1.184	−0.016	0.036256	0.19041	0.838014	3.0	3.022	0.022	0.058484	0.241835	0.838782
75	η	0.6	0.589608	−0.010392	0.015441	0.124263	0.917799	0.8	0.79134	−0.00866	0.013408	0.115794	0.918515
	λ	1.9	1.916166	0.016166	0.021595	0.146951	0.918647	0.7	0.706928	0.006928	0.010048	0.10024	0.918687
	α	4.0	3.986144	−0.013856	0.024192	0.155538	0.919413	2.2	2.209815	0.009815	0.014763	0.121503	0.919279
	β	1.2	1.190762	−0.009238	0.012085	0.109933	0.918853	3.0	3.012702	0.012702	0.019495	0.139623	0.919297
100	η	0.6	0.5919	−0.0081	0.011396	0.106752	0.933504	0.8	0.79325	−0.00675	0.009266	0.09626	0.93436
	λ	1.9	1.9126	0.0126	0.015876	0.125992	0.934535	0.7	0.7054	0.0054	0.006732	0.082049	0.93461
	α	4.0	3.9892	−0.0108	0.017856	0.133626	0.935466	2.2	2.20765	0.00765	0.009224	0.096042	0.935319
	β	1.2	1.1928	−0.0072	0.008919	0.09444	0.934785	3.0	3.0099	0.0099	0.012531	0.111942	0.935338
200	η	0.6	0.594273	−0.005727	0.00626	0.07912	0.948971	0.8	0.795227	−0.004773	0.005092	0.071356	0.949949
	λ	1.9	1.908908	0.008908	0.008718	0.093371	0.950148	0.7	0.703819	0.003819	0.0037	0.060827	0.950233
	α	4.0	3.992364	−0.007636	0.009808	0.099035	0.95121	2.2	2.20541	0.00541	0.005067	0.071182	0.951042
	β	1.2	1.194909	−0.005091	0.004897	0.069979	0.950433	3.0	3.006999	0.006999	0.006886	0.082981	0.951063
300	η	0.6	0.595096	−0.004904	0.004574	0.067632	0.956307	0.8	0.795913	−0.004087	0.003718	0.060975	0.957369
	λ	1.9	1.907625	0.007625	0.006369	0.079807	0.957589	0.7	0.703275	0.003275	0.002704	0.052001	0.957682
	α	4.0	3.993461	−0.006539	0.007168	0.084663	0.958745	2.2	2.204634	0.004634	0.003703	0.060853	0.958562
	β	1.2	1.195641	−0.004359	0.003578	0.059816	0.957899	3.0	3.005998	0.005998	0.005029	0.070918	0.958585
400	η	0.6	0.595554	−0.004446	0.003626	0.060216	0.96177	0.8	0.796295	−0.003705	0.002948	0.054295	0.962892
	λ	1.9	1.906911	0.006911	0.005048	0.071045	0.963133	0.7	0.702972	0.002972	0.002144	0.046303	0.96323
	α	4.0	3.994072	−0.005928	0.005685	0.075396	0.964363	2.2	2.204201	0.004201	0.002936	0.054186	0.964168
	β	1.2	1.196048	−0.003952	0.002837	0.053267	0.963461	3.0	3.005441	0.005441	0.004113	0.064133	0.964192
500	η	0.6	0.59586	−0.00414	0.003028	0.055025	0.966251	0.8	0.79655	−0.00345	0.002461	0.049612	0.967417
	λ	1.9	1.906434	0.006434	0.004215	0.064923	0.967682	0.7	0.702769	0.002769	0.001748	0.041809	0.967783
	α	4.0	3.99448	−0.00552	0.004747	0.068899	0.968974	2.2	2.203912	0.003912	0.002393	0.04892	0.968769
	β	1.2	1.19632	−0.00368	0.002372	0.048702	0.968025	3.0	3.005069	0.005069	0.003463	0.058846	0.968795

Table 3. *Cont.*

n	Parameter	Set III						Set IV					
		True	**AE**	**AB**	**MSE**	**RMSE**	**CP**	**True**	**AE**	**AB**	**MSE**	**RMSE**	**CP**
600	η	0.6	0.596079	−0.003921	0.002625	0.051236	0.970005	0.8	0.796733	−0.003267	0.002132	0.046174	0.971204
	λ	1.9	1.906093	0.006093	0.003654	0.060449	0.971494	0.7	0.702624	0.002624	0.00148	0.038471	0.971598
	α	4.0	3.994772	−0.005228	0.004115	0.064147	0.972838	2.2	2.203704	0.003704	0.002026	0.045012	0.972625
	β	1.2	1.196514	−0.003486	0.002059	0.045374	0.97185	3.0	3.004807	0.004807	0.003026	0.055009	0.972652
700	η	0.6	0.596247	−0.003753	0.002342	0.048394	0.97291	0.8	0.796873	−0.003127	0.0019	0.043591	0.974135
	λ	1.9	1.905831	0.005831	0.003261	0.057107	0.974445	0.7	0.702513	0.002513	0.001292	0.035938	0.974551
	α	4.0	3.994996	−0.005004	0.003671	0.06059	0.97583	2.2	2.203546	0.003546	0.001769	0.042061	0.97561
	β	1.2	1.196664	−0.003336	0.001839	0.042887	0.974811	3.0	3.004599	0.004599	0.002721	0.05216	0.975638
800	η	0.6	0.596378	−0.003622	0.002134	0.046197	0.975234	0.8	0.796981	−0.003019	0.00173	0.041589	0.97648
	λ	1.9	1.905627	0.005627	0.002971	0.054504	0.976808	0.7	0.702427	0.002427	0.001154	0.033971	0.976915
	α	4.0	3.995171	−0.004829	0.003343	0.057817	0.978228	2.2	2.203421	0.003421	0.001543	0.03928	0.978002
	β	1.2	1.196781	−0.003219	0.001676	0.040943	0.977183	3.0	3.004438	0.004438	0.002494	0.049942	0.978032
900	η	0.6	0.596481	−0.003519	0.001977	0.04446	0.97716	0.8	0.797068	−0.002932	0.001601	0.04001	0.978422
	λ	1.9	1.905466	0.005466	0.002753	0.052468	0.978765	0.7	0.702358	0.002358	0.00105	0.032404	0.978874
	α	4.0	3.995309	−0.004691	0.003096	0.05564	0.980211	2.2	2.203319	0.003319	0.001374	0.037067	0.979981
	β	1.2	1.196873	−0.003127	0.001553	0.039403	0.979145	3.0	3.004311	0.004311	0.002323	0.048202	0.980011
1000	η	0.6	0.596564	−0.003436	0.00185	0.043011	0.978819	0.8	0.797136	−0.002864	0.001497	0.038692	0.980093
	λ	1.9	1.905337	0.005337	0.002576	0.050755	0.980449	0.7	0.702303	0.002303	0.000967	0.031089	0.980559
	α	4.0	3.99542	−0.00458	0.002895	0.053807	0.981917	2.2	2.203235	0.003235	0.001237	0.035165	0.981684
	β	1.2	1.196947	−0.003053	0.001454	0.038132	0.980834	3.0	3.00421	0.00421	0.002185	0.046746	0.981714

Table 4. Simulation summaries reported earlier for Sets V and VI (not independently reproduced).

n	Parameter	Set V						Set VI					
		True	**AE**	**AB**	**MSE**	**RMSE**	**CP**	**True**	**AE**	**AB**	**MSE**	**RMSE**	**CP**
25	η	1.1	1.11	0.01	0.0321	0.179165	0.838667	0.4	0.416	0.016	0.044256	0.210371	0.835569
	λ	1.2	1.186	−0.014	0.038196	0.195438	0.838262	2.0	1.973	−0.027	0.068729	0.262162	0.83784
	α	5.0	5.03	0.03	0.0889	0.298161	0.838971	1.8	1.815	0.015	0.036225	0.190329	0.838683
	β	1.8	1.787	−0.013	0.034169	0.184849	0.838859	4.0	3.974	−0.026	0.064676	0.254315	0.838899
75	η	1.1	1.105774	0.005774	0.0107	0.103441	0.91923	0.4	0.409238	0.009238	0.014752	0.121458	0.917442
	λ	1.2	1.191917	−0.008083	0.012732	0.112836	0.918997	2.0	1.984412	−0.015588	0.02291	0.151359	0.918753
	α	5.0	5.017321	0.017321	0.029633	0.172143	0.919406	1.8	1.80866	0.00866	0.012075	0.109886	0.91924
	β	1.8	1.792494	−0.007506	0.01139	0.106722	0.919341	4.0	3.984989	−0.015011	0.021559	0.146829	0.919364
100	η	1.1	1.1045	0.0045	0.0084	0.091652	0.935264	0.4	0.40675	0.00675	0.009225	0.096047	0.93232
	λ	1.2	1.1937	−0.0063	0.00924	0.096125	0.934779	2.0	1.98885	−0.01115	0.014331	0.119711	0.93427
	α	5.0	5.0135	0.0135	0.02025	0.142302	0.93563	1.8	1.80765	0.00765	0.009261	0.096234	0.935349
	β	1.8	1.7937	−0.0063	0.008259	0.090879	0.935496	4.0	3.989275	−0.010725	0.013611	0.116667	0.935535
200	η	1.1	1.103183	0.003183	0.004461	0.06679	0.951383	0.4	0.40541	0.00541	0.005067	0.07118	0.946022
	λ	1.2	1.195545	−0.004455	0.005114	0.071513	0.950813	2.0	1.991048	−0.008952	0.007422	0.086152	0.950214
	α	5.0	5.009549	0.009549	0.010124	0.100617	0.95181	1.8	1.80541	0.00541	0.005071	0.071209	0.951365
	β	1.8	1.794682	−0.005318	0.004536	0.067351	0.951655	4.0	3.991364	−0.008636	0.007056	0.083999	0.951696
300	η	1.1	1.102728	0.002728	0.003179	0.056383	0.95916	0.4	0.404634	0.004634	0.003703	0.060853	0.951923
	λ	1.2	1.196185	−0.003815	0.003736	0.061123	0.958489	2.0	1.991812	−0.008188	0.005427	0.073667	0.957635
	α	5.0	5.00818	0.00818	0.006652	0.08156	0.959663	1.8	1.804634	0.004634	0.003704	0.060862	0.95918
	β	1.8	1.795057	−0.004943	0.003313	0.057559	0.959288	4.0	3.992574	−0.007426	0.005156	0.071803	0.959143
400	η	1.1	1.102473	0.002473	0.00246	0.049598	0.96491	0.4	0.404201	0.004201	0.002936	0.054186	0.955323
	λ	1.2	1.196542	−0.003458	0.002964	0.054443	0.964131	2.0	1.99224	−0.00776	0.004316	0.065697	0.963257
	α	5.0	5.00742	0.00742	0.005096	0.071383	0.965497	1.8	1.804201	0.004201	0.002936	0.054185	0.964928
	β	1.8	1.795266	−0.004734	0.002627	0.051254	0.964983	4.0	3.993249	−0.006751	0.004101	0.064039	0.965159
500	η	1.1	1.102311	0.002311	0.002005	0.044773	0.969628	0.4	0.403912	0.003912	0.002393	0.04892	0.9582
	λ	1.2	1.196769	−0.003231	0.002475	0.049749	0.96873	2.0	1.992512	−0.007488	0.003599	0.059989	0.968582
	α	5.0	5.006938	0.006938	0.004153	0.06444	0.970305	1.8	1.803912	0.003912	0.002393	0.048918	0.96967
	β	1.8	1.795406	−0.004594	0.002193	0.046824	0.969875	4.0	3.9937	−0.0063	0.0034	0.05831	0.969778

Table 4. *Cont.*

n	Parameter	Set V						Set VI					
		True	**AE**	**AB**	**MSE**	**RMSE**	**CP**	**True**	**AE**	**AB**	**MSE**	**RMSE**	**CP**
600	η	1.1	1.102196	0.002196	0.001698	0.041207	0.973508	0.4	0.403704	0.003704	0.002026	0.045012	0.960718
	λ	1.2	1.196931	−0.003069	0.002143	0.046292	0.972482	2.0	1.992707	−0.007293	0.003113	0.055791	0.972524
	α	5.0	5.006524	0.006524	0.00351	0.059249	0.974283	1.8	1.803704	0.003704	0.002026	0.045011	0.973621
	β	1.8	1.795506	−0.004494	0.0019	0.043588	0.97386	4.0	3.994023	−0.005977	0.002923	0.054065	0.973811
700	η	1.1	1.102109	0.002109	0.001488	0.038574	0.976786	0.4	0.403546	0.003546	0.001769	0.042061	0.962962
	λ	1.2	1.197053	−0.002947	0.00191	0.043703	0.975631	2.0	1.992853	−0.007147	0.002772	0.052648	0.975962
	α	5.0	5.006209	0.006209	0.003072	0.055422	0.977653	1.8	1.803546	0.003546	0.001769	0.042061	0.976521
	β	1.8	1.795583	−0.004417	0.001695	0.041167	0.977088	4.0	3.994268	−0.005732	0.002641	0.051391	0.977084
800	η	1.1	1.102042	0.002042	0.001335	0.036539	0.979603	0.4	0.403421	0.003421	0.001543	0.03928	0.964994
	λ	1.2	1.197147	−0.002853	0.001744	0.041761	0.978321	2.0	1.992966	−0.007034	0.002529	0.050294	0.978635
	α	5.0	5.005965	0.005965	0.002757	0.052511	0.980565	1.8	1.803421	0.003421	0.001543	0.039281	0.979407
	β	1.8	1.795644	−0.004356	0.001548	0.039346	0.980361	4.0	3.994458	−0.005542	0.002427	0.049264	0.980397
900	η	1.1	1.101989	0.001989	0.001217	0.03489	0.98208	0.4	0.403319	0.003319	0.001374	0.037067	0.966854
	λ	1.2	1.197221	−0.002779	0.00162	0.040249	0.980678	2.0	1.993057	−0.006943	0.002346	0.048431	0.980966
	α	5.0	5.005771	0.005771	0.002516	0.050163	0.983121	1.8	1.803319	0.003319	0.001374	0.037068	0.981923
	β	1.8	1.795692	−0.004308	0.001438	0.037927	0.982915	4.0	3.994614	−0.005386	0.002263	0.047574	0.982987
1000	η	1.1	1.101947	0.001947	0.001125	0.033541	0.984297	0.4	0.403235	0.003235	0.001237	0.035165	0.968571
	λ	1.2	1.197279	−0.002721	0.001525	0.039045	0.982785	2.0	1.993132	−0.006868	0.002205	0.046955	0.98305
	α	5.0	5.005615	0.005615	0.002326	0.048228	0.985301	1.8	1.803235	0.003235	0.001237	0.035171	0.984241
	β	1.8	1.795733	−0.004267	0.001355	0.036813	0.985084	4.0	3.994743	−0.005257	0.002141	0.046266	0.985188

6. Applications

In this section, we examine the performance of the SMOE distribution by means of two real data sets. The first concerns charity fundraising expenditure percentages, whereas the second concerns per-capita income for Indian states and union territories. Here, we compare the proposed distribution with the extended slash modified Lindley (ESlaML), beta-slash Lindley (ESlaL), Lindley (Lind) ref. [17], and exponential (Exp) distributions. All numerical results in this section are calculated from the observations given below. In addition, a three-parameter SMOE submodel and the MOE and modified Lindley (ML) parent distributions are considered to examine whether the additional parameters are useful.

6.1. Description of the Data Sets

Data Set 1: Charity fundraising percentage data. The first data set consists of 76 observations [18] describing the percentage of expenditure incurred in charity fundraising. The observations are positive and range from 0.0251 to 9.0960. The complete data used for the analysis are given below:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

No zero observation is present in this sample, and no observation is deleted or replaced. The values are analyzed on their displayed percentage scale, without an additional rescaling.

Data Set 2: Indian state- and union-territory-level per-capita income. The second data set consists of 33 observations. More precisely, these values correspond to per-capita *net state domestic product* (NSDP) for 2020–2021 at constant 2011–2012 prices. They agree with the 2020–2021 column of Annexure II in the Press Information Bureau release of 24 July 2023, after dividing the INR values by 10^4. (Press Information Bureau, Government of India. *State-wise Data on Per Capita Income*, Release ID 1942055, 24 July 2023, Annexure II. https://www.pib.gov.in/PressReleasePage.aspx?PRID=1942055 (accessed on 12 December 2025).

Thus, one unit in the following data represents INR 10,000 at the stated constant prices:

10.5880, 10.6903, 6.1304, 2.8127, 7.3259, 29.8527, 16.0321, 15.5756, 13.3079, 5.1365, 14.9030, 13.2700, 5.6320, 12.7970, 4.6968, 5.6679, 11.8555, 7.2215, 7.0877, 11.1662, 7.3140, 23.9928, 14.3528, 13.9640, 8.0512, 3.9298, 13.7987, 6.3562, 15.2859, 19.9845, 23.4569, 6.5172, 14.4866.

All 33 observations are retained, in the order of the source table. The same displayed values and the same measurement scale are used for every competing distribution. The sample sizes are therefore $n_1 = 76$ and $n_2 = 33$.

Table 5 presents some descriptive measures. Here, the standard deviation uses the divisor $n - 1$, while the empirical skewness and Pearson kurtosis are $m_3/m_2^{3/2}$ and m_4/m_2^2, respectively, with $m_j = n^{-1}\sum_{i=1}^{n}(y_i - \overline{y})^j$. Both samples are positively skewed. In addition, the charity sample has larger empirical skewness and kurtosis than the income sample. These are descriptive sample coefficients. They do not establish a population tail class or the existence of the corresponding population moments.

6.2. Competing Models and Numerical Estimation

For reproducibility, we specify the comparator distributions explicitly. The Lindley and modified Lindley parent densities used here are

$$f_{\mathrm{L}}(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \tag{48}$$

$$f_{\mathrm{ML}}(x;\theta) = \frac{\theta}{1+\theta}e^{-2\theta x}\Big\{(1+\theta)e^{\theta x} + 2\theta x - 1\Big\}, \tag{49}$$

where $x > 0$ and $\theta > 0$. For either parent density f_0, its beta-slash extension is given by

$$f_{\mathrm{BS}}(y;\theta,a,b) = \frac{1}{B(a,b)}\int_0^1 u^a(1-u)^{b-1}f_0(yu;\theta)\,du, \qquad y > 0, \quad a,b > 0. \tag{50}$$

The ESlaML model uses the modified Lindley parent, in agreement with the construction of Gillariose et al. [9]. In this section, ESlaL denotes the explicitly defined three-parameter beta-slash Lindley comparator obtained with $f_0 = f_{\mathrm{L}}$. It should not be confused with the two-parameter power-denominator Lindley slash model: the restriction $b = 1$ gives that construction [19]. The exponential density is $f_{\mathrm{Exp}}(y;\theta) = \theta e^{-\theta y}$. The income observations form a cross-section, with different geographic units observed for the same period [20]. Here, we examine their marginal distribution rather than a regression model.

Table 5. Descriptive measures for the two data sets. Kurtosis is the Pearson coefficient, not excess kurtosis.

Data	*n*	Min.	Max.	Mean	Median	SD	Skewness	Kurtosis
Charity	76	0.0251	9.0960	1.9592	1.7362	1.5740	1.9796	8.1608
Income	33	2.8127	29.8527	11.6133	11.1662	6.2785	0.9665	3.7310

The unknown parameters are estimated by maximizing the log-likelihood in logarithmic coordinates. To make the numerical comparison reproducible, the main searches use

$$10^{-4} \le \eta \le 10^5, \qquad 10^{-5} \le \lambda,\theta \le 10^3, \qquad 10^{-3} \le \alpha,\beta,a,b \le 500,$$

with only the parameters belonging to each model included. The SMOE searches use six initial configurations, and the ESlaML and ESlaL searches use five. The explicit starts, stopping rules, and implementation are supplied with the data. The exponential and Lindley estimates also have analytic expressions, providing independent checks on these two fits.

The mixture integrals are evaluated by Gauss–Jacobi quadrature, initially with 96 nodes. Potential optima are checked by increasing the quadrature order and re-optimizing when necessary. The retained application densities and survival functions are also checked using independent adaptive weighted integration. The resulting differences in the negative log-likelihoods are below 10^{-8}. These calculations verify the numerical evaluations; they do not prove a unique global maximum over an unrestricted parameter space. Accordingly, the reported estimates are bounded numerical likelihood solutions. A dagger identifies an estimate at a search boundary.

For a fitted log-likelihood $\widehat{\ell}$, the information criteria are calculated as

$$\mathrm{AIC} = -2\widehat{\ell} + 2k, \qquad \mathrm{BIC} = -2\widehat{\ell} + k\log n, \tag{51}$$

where k is the number of freely estimated parameters. We use $k = 4$ for SMOE, $k = 3$ for ESlaML and ESlaL, and $k = 1$ for Lindley and exponential. The restriction $\beta = 1$ in SMOE

has $k = 3$, whereas MOE and ML have $k = 2$ and $k = 1$, respectively. All likelihoods are evaluated on the displayed observation scale. For boundary fits, the criteria calculated with these nominal parameter counts are descriptive penalized-likelihood scores; the usual regular asymptotic interpretation is not assumed [21].

6.3. Parameter Estimates and Fitting Results

The estimated parameters are presented in Table 6. Tables 7 and 8 give the negative log-likelihood, AIC, BIC, and the Kolmogorov–Smirnov (D_n), Anderson–Darling (A^2), and Cramér–von Mises (W^2) statistics. The definitions and bootstrap calibration of these statistics are described in Section 6.4. The last three rows in each table provide the additional parsimony comparisons.

Table 6. Bounded numerical likelihood estimates. The additional parsimony comparisons are shown in blue.

Model	Charity ($n_1 = 76$)	Income ($n_2 = 33$)
SMOE	$(10.8104, 2.4875, 3.1558, 1.8916)$	$(438.0374, 60.5604, 5.3155, 500.0000^{\dagger})$
ESlaML	$(0.5858, 500.0000^{\dagger}, 0.0010^{\dagger})$	$(0.1019, 500.0000^{\dagger}, 0.0010^{\dagger})$
ESlaL	$(0.7948, 500.0000^{\dagger}, 0.0010^{\dagger})$	$(0.1603, 500.0000^{\dagger}, 0.0010^{\dagger})$
Lind	0.7948	0.1603
Exp	0.5104	0.0861
SMOE ($\beta = 1$)	$(10.3246, 2.1048, 2.5955)$	$(22.8818, 0.3543, 5.0349)$
MOE	$(3.4546, 0.9111)$	$(15.1253, 0.2551)$
ML	0.5858	0.1019

Parameter orders: SMOE, $(\eta, \lambda, \alpha, \beta)$; ESlaML and ESlaL, (θ, a, b); SMOE with $\beta = 1$, (η, λ, α); MOE, (η, λ); and Lind, Exp, and ML, θ. The dagger marks an active main-search bound. Tables and plots use unrounded estimates, supplied in the computational files.

Table 7. Fitting results for the charity fundraising percentage data ($n_1 = 76$).

Model	k	$-\widehat{\ell}$	AIC	BIC	D_n	A^2	W^2
SMOE	4	**119.9285**	247.8569	257.1798	0.0664	0.2663	0.0437
ESlaML †	3	122.0391	250.0782	257.0704	0.0970	0.9412	0.1702
ESlaL †	3	123.6751	253.3503	260.3425	0.1156	1.4751	0.2650
Lind	1	123.6751	249.3503	251.6810	0.1156	1.4751	0.2650
Exp	1	127.1143	256.2287	258.5594	0.1663	2.9881	0.5708
SMOE ($\beta = 1$)	3	119.9670	**245.9339**	252.9261	**0.0660**	**0.2565**	**0.0414**
MOE	2	122.4211	248.8422	253.5037	0.1043	0.7670	0.1224
ML	1	122.0391	246.0782	**248.4089**	0.0970	0.9412	0.1702

Bold values indicate the minimum across all eight candidates. A dagger denotes a boundary fit; its AIC and BIC use nominal free-parameter counts and are interpreted descriptively. The last three rows address parsimony, rather than being omitted from the comparison.

Table 8. Fitting results for the per-capita income data ($n_2 = 33$).

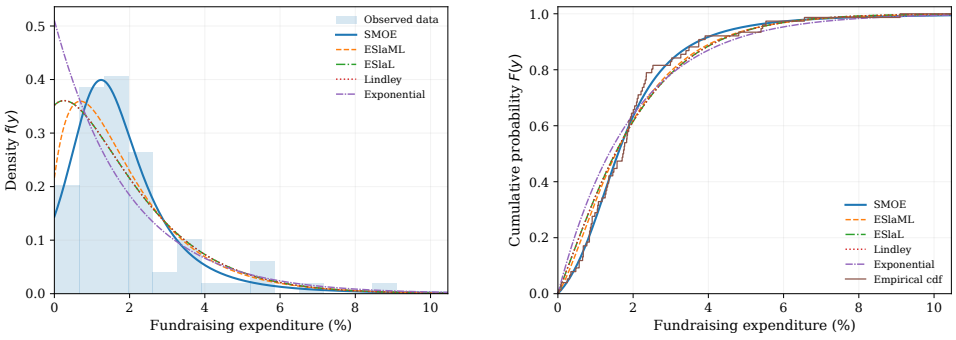
Model	k	$-\widehat{\ell}$	AIC	BIC	D_n	A^2	W^2
SMOE †	4	**103.7869**	215.5737	221.5598	0.1328	0.5449	0.1122
ESlaML †	3	107.3499	220.6998	225.1893	0.1529	1.3704	0.2089
ESlaL †	3	107.4107	220.8215	225.3110	0.1628	1.4360	0.2239
Lind	1	107.4107	216.8215	218.3180	0.1628	1.4360	0.2238
Exp	1	113.9211	229.8422	231.3387	0.2720	3.6550	0.6759
SMOE ($\beta = 1$)	3	104.7303	215.4607	219.9502	0.1356	**0.5212**	0.0869
MOE	2	104.9413	**213.8826**	**216.8756**	**0.1279**	0.5402	**0.0779**
ML	1	107.3499	216.6998	218.1963	0.1529	1.3704	0.2089

Bold values indicate the minimum across all eight candidates. A dagger denotes a boundary fit; its AIC and BIC use nominal free-parameter counts and are interpreted descriptively. The last three rows address parsimony, rather than being omitted from the comparison.

For Data Set 1, the SMOE fit gives $-\widehat{\ell} = 119.9285$, AIC $= 247.8569$, $D_n = 0.0664$, $A^2 = 0.2663$, and $W^2 = 0.0437$. Among the five models comprising SMOE and these four competitors, these are the smallest values of the corresponding measures. For Data Set 2, the retained SMOE solution gives $-\widehat{\ell} = 103.7869$, AIC $= 215.5737$, $D_n = 0.1328$, $A^2 = 0.5449$, and $W^2 = 0.1122$. These values are also smaller than those of ESlaML, ESlaL, Lindley, and exponential for this sample. Thus, the SMOE distribution gives an improvement in likelihood and empirical distributional fit relative to those four competitors.

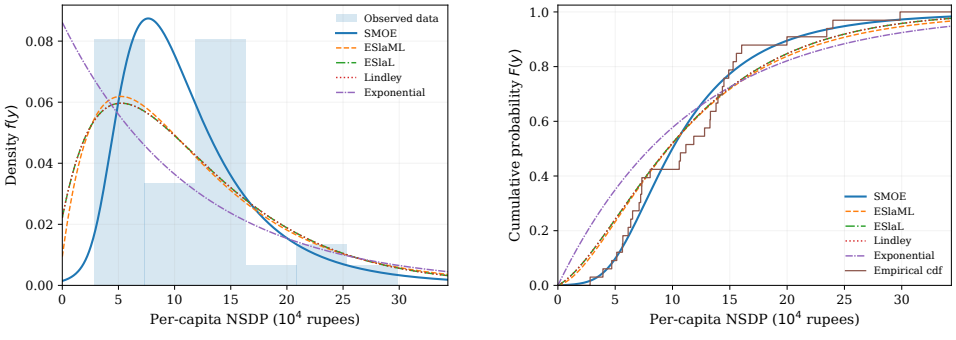
However, the rankings are not the same for every criterion. Among these five models, BIC favors the Lindley distribution for both data sets. The full SMOE model is therefore not described as uniformly better after accounting for complexity. The additional parent and restricted-model comparisons further qualify this conclusion, as discussed in Section 6.5.

Figures 4 and 5 present the newly computed fitted densities and cumulative distribution functions. The curves are calculated from the parameter vectors in Table 6; the same observations and measurement scales are used throughout. Both new SMOE estimates have $\widehat{\eta} > 2$, so their positive interior modes are consistent with Theorem 2. The ESlaL and Lindley curves are almost indistinguishable because the fitted beta denominator in ESlaL is concentrated close to one. An inspection of the plots shows that SMOE follows the concentration of observations more closely than these simpler competitors over much of the observed range.



(**a**) (**b**)

Figure 4. Graphical assessment for the charity fundraising percentage data ($n_1 = 76$). All fitted curves use the newly estimated parameters and the original displayed data scale. (**a**) Fitted densities. (**b**) Empirical and fitted cdfs.



(**a**) (**b**)

Figure 5. Graphical assessment for per-capita NSDP at constant 2011–2012 prices, observed for 2020–2021 ($n_2 = 33$). The horizontal scale is INR 10,000 per person. (**a**) Fitted densities. (**b**) Empirical and fitted cdfs.

6.4. Goodness-of-Fit Tests and Refitted Bootstrap

For the ordered observations $y_{(1)} \le \cdots \le y_{(n)}$, let $p_i = F(y_{(i)}; \widehat{\theta})$. The statistics used in Tables 7 and 8 are

$$D_n = \max_{1 \le i \le n} \left\{ \frac{i}{n} - p_i,\ p_i - \frac{i-1}{n} \right\}, \tag{52}$$

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ p_i - \frac{2i-1}{2n} \right\}^2, \tag{53}$$

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1)\{\log p_i + \log(1 - p_{n+1-i})\}. \tag{54}$$

The composite null hypothesis is that the observations follow some member of the fitted family under a working iid sampling model. Since the parameters are estimated from the same observations, ordinary fixed-parameter K–S probabilities are not used [22].

For each model and each data set, we generate $B = 1999$ parametric-bootstrap samples of size 76 or 33, as appropriate. Every sample is refitted by the bounded likelihood procedure, with the fitted generating vector included as an additional starting value. Each statistic is then recomputed using that sample's refitted cdf. The seed is 02026, with independent reproducible random-number streams indexed by data set, model, and replication. The resulting empirical probability is

$$\widehat{p}_T = \frac{1 + \sum_{b=1}^{B} \mathbf{1}\{T_b^* \ge T_{\text{obs}}\}}{B+1}, \qquad T \in \{D_n, A^2, W^2\}. \tag{55}$$

This is the refitted parametric-bootstrap approach described in SciPy Developers [23]. All 31,984 replications were numerically resolved; 3 required an additional higher-order fitting pass. No replication was discarded. The full SMOE estimate reached a search boundary in 734 charity and 1373 income replications, out of 1999 for each sample. These frequencies are relevant to the stability assessment below. The complete replication records, including numerical checks and boundary solutions, are supplied with the computational files.

Table 9 reports the bootstrap probabilities, with Monte Carlo standard errors at most $1/(2\sqrt{1999}) \simeq 0.0112$. Calibration is approximate, especially for boundary fits and weakly identified parameters; no exact finite-sample test is claimed.

Table 9. Parametric-bootstrap goodness-of-fit probabilities, with $B = 1999$ refitted samples for each model and each data set.

	Charity ($n_1 = 76$)			**Income ($n_2 = 33$)**		
Model	p_D	p_A	p_W	p_D	p_A	p_W
SMOE	0.2470	0.3500	0.2580	0.0390	0.0420	0.0155
ESlaML	0.1065	0.0505	0.0325	0.0730	0.0075	0.0135
ESlaL	0.0370	0.0170	0.0120	0.0590	0.0100	0.0145
Lind	0.0685	0.0260	0.0185	0.0955	0.0300	0.0310
Exp	0.0005	0.0015	0.0005	0.0010	0.0015	0.0015
SMOE ($\beta = 1$)	0.2700	0.4590	0.3315	0.0225	0.0645	0.0380
MOE	0.0115	0.0275	0.0190	0.0790	0.1200	0.1200
ML	0.1580	0.1040	0.0690	0.1265	0.0230	0.0285

p_D, p_A, and p_W correspond to the K–S, Anderson–Darling, and Cramér–von Mises statistics. Boundary solutions are retained and reported in the replication records. These are approximate procedure-specific probabilities; a larger value is not evidence of comparative superiority.

For the charity data, the SMOE probabilities (p_D, p_A, p_W) are $(0.2470, 0.3500, 0.2580)$. For the income data, they are $(0.0390, 0.0420, 0.0155)$. The charity SMOE fit is not rejected by

any of the three tests at the 5% level. In contrast, all three probabilities for the income SMOE fit are below 0.05. Thus, its smaller discrepancies relative to the four beta-slash or classical competitors do not imply an adequate absolute fit under these bootstrap assessments. The income MOE parent is not rejected by these three tests. Probabilities close to 0.05 should also be considered with their Monte Carlo uncertainty. In contrast, the exponential model is rejected by all three tests for both samples. The remaining models are assessed by their separate entries in Table 9, rather than by ranking their probabilities. A large goodness-of-fit probability means that the corresponding discrepancy is not unusually large under the fitted working model and refitting procedure. It does not prove that one non-rejected model is significantly better than another. In addition, the state-level observations form a geographic cross-section, for which independence is not established by the numerical analysis. The bootstrap results are therefore model-based assessments, not design-based inference about an independently sampled population of states.

6.5. Observed Information and Stability of the Estimates

We now examine the numerical information in the SMOE fits. Let $v = (\log\eta, \log\lambda, \log\alpha, \log\beta)^{\mathsf{T}}$ and $J_v = -\partial^2\ell/\partial v\,\partial v^{\mathsf{T}}$. This log-parameter version of the observed information in (45) permits a comparison without domination by the different parameter scales. Numerical differentiation of the analytic log-parameter score is checked with two step sizes and increased quadrature order. At the retained fits, the numerical matrices are

$$\widehat{J}_{v,1} = \begin{pmatrix} 18.6660 & -47.2655 & -19.5402 & 18.6557 \\ -47.2655 & 144.1506 & 68.6061 & -60.1588 \\ -19.5402 & 68.6061 & 46.5811 & -32.8297 \\ 18.6557 & -60.1588 & -32.8297 & 26.5199 \end{pmatrix}, \tag{56}$$

and

$$\widehat{J}_{v,2} = \begin{pmatrix} 3.4562 & -18.6606 & -19.0704 & 18.4697 \\ -18.6606 & 107.3038 & 115.5162 & -106.2725 \\ -19.0704 & 115.5162 & 132.2331 & -114.4828 \\ 18.4697 & -106.2725 & -114.4828 & 105.2611 \end{pmatrix}. \tag{57}$$

For the charity data, the eigenvalues are approximately 0.1149, 2.0372, 13.4209, and 220.3446, with condition number 1916.9. The solution is interior and the curvature is positive, supporting a strict local likelihood maximum. However, the information is not equally strong in every direction. The local standard errors for $(\eta, \lambda, \alpha, \beta)$ are approximately $(7.0900, 2.1928, 2.6290, 5.1345)$, showing substantial uncertainty, particularly for β. The local correlation between the log estimates of η and λ is approximately 0.5652. These local quantities do not prove global identifiability.

For the income data, the condition number is approximately 7.40×10^4, and $\widehat{\beta} = 500$ is the upper search bound. The log-parameter derivative of the negative log-likelihood with respect to $\log\beta$ is approximately -0.00932 at this point. Thus, this is not an unrestricted four-dimensional stationary estimate, and ordinary four-parameter Wald intervals are not reported. The profile calculation makes the issue more explicit: after fixing β at 500, 1000, 5000, and 20,000 and re-estimating the remaining parameters over enlarged bounds, the negative log-likelihoods are approximately 103.7869, 103.7822, 103.7784, and 103.7777, respectively. The likelihood continues to improve slowly as β and λ increase together. This agrees with the large-β scale ridge discussed in Section 4.2.

Figure 6 presents the reoptimized profiles. For the charity sample, the minimum is interior, but a wide range of larger β values has similar likelihood. For the income sample, the selected finite value of β is governed by the main search boundary. Consequently, the four-parameter estimates should not be interpreted as precisely determined structural

quantities. The ESlaML and ESlaL estimates also reach boundary values, with $a = 500$ and $b = 0.001$. Their beta denominators are nearly one, explaining their proximity to their parent distributions.

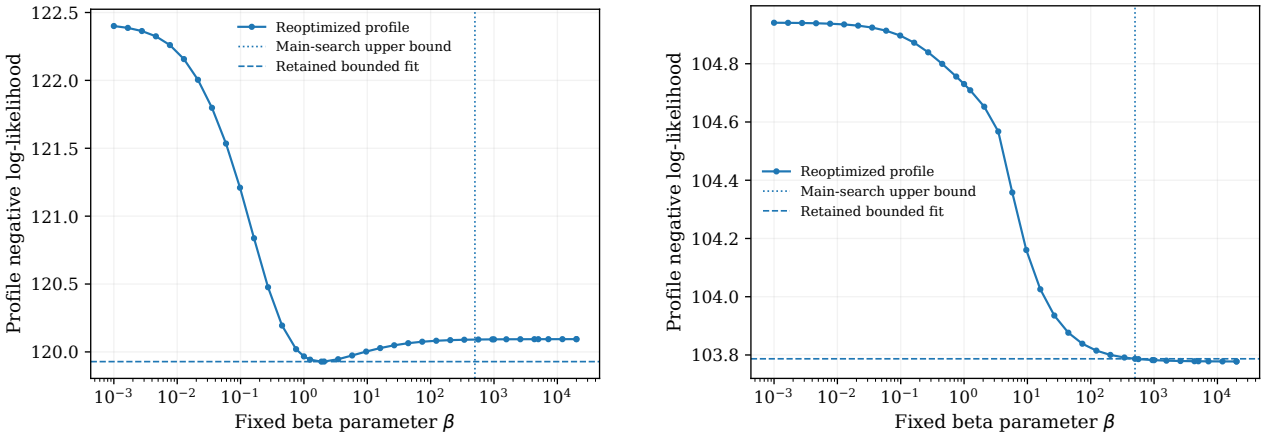


Figure 6. Negative log-likelihood profiles obtained by fixing β and re-estimating the other three SMOE parameters. The vertical line marks the main-search bound $\beta = 500$; profile calculations beyond it use enlarged bounds. (**a**) Charity fundraising percentages. (**b**) Per-capita income.

The additional rows in Tables 6–8 provide a direct examination of model complexity. The three-parameter restriction $\beta = 1$ belongs to the SMOE family and retains its power-law tail. The MOE model is the unslashed parent, obtained as a distributional limit when $\beta \downarrow 0$, rather than by setting an admissible beta parameter equal to zero. The ML model is the one-parameter parent of ESlaML.

For the charity sample, fixing $\beta = 1$ changes the negative log-likelihood from 119.9285 to 119.9670, a loss of only 0.0385. Its AIC decreases from 247.8569 to 245.9339, which is the smallest AIC among all eight candidates. The log-information condition number also decreases from approximately 1916.9 to 81.4. Thus, this sample supports a useful parsimonious member of the new family, rather than providing strong evidence that all four parameters are needed. The AIC advantage of the three-parameter restriction over the ML parent is only 0.1443, so these two AIC values give little separation. The ML parent has the smallest BIC, 248.4089.

For the income sample, the $\beta = 1$ restriction gives an interior solution with AIC 215.4607 and condition number approximately 191.7. Nevertheless, the two-parameter MOE parent has smaller AIC and BIC, namely 213.8826 and 216.8756. Therefore, the additional slash parameters are not preferred by these penalized criteria for this sample. This comparison is particularly relevant in small-sample applications, where simplicity and stability may be more useful than a modest improvement in the unpenalized likelihood.

The interpretation of tail behavior also requires care. When a fundraising percentage is defined as a share of total expenditure, its upper bound is 100, whereas the fitted SMOE model has unbounded support. Under that interpretation, the present fit is an approximation over the observed positive range, not a literal model of arbitrarily large percentages. Its survival probability above 100 is approximately 4.89×10^{-6}, which is small but not zero. The precise expenditure denominator should therefore be considered when interpreting the support. Likewise, fitting the 33 income observations does not by itself establish a power-law population tail.

In view of the obtained results, the charity application supports the flexibility of the SMOE family, especially its three-parameter restriction. The income application also shows

the limitations of the full model: the likelihood improves over ESlaML, ESlaL, Lindley, and exponential, but this alone does not establish adequate absolute fit or justify the additional parameters. The full four-parameter model does not dominate every simpler alternative or every information criterion. The choice between the full model, its restricted members, and the parent distributions should therefore consider goodness of fit, the number of parameters, and the stability of estimation together.

7. Conclusions

In this paper, we introduced a new Slash Marshall–Olkin (SMO) family by dividing a positive Marshall–Olkin random variable by an independent beta random variable. As a special member of this family, the Slash Marshall–Olkin exponential (SMOE) distribution was studied in detail. Several mathematical and statistical properties of the proposed model were derived, including integral and series representations for the cumulative distribution function, survival function, probability density function, and hazard rate function. The behavior of the density and hazard rate function at the origin was also investigated. In addition, the moment properties and the tail behavior of the proposed distribution were discussed. An important property of the SMOE distribution is its polynomially decreasing right tail. More precisely, its survival function is regularly varying with tail exponent $-\alpha$, and hence the distribution belongs to the power-law and subexponential classes. The parameter α plays an important role in controlling the tail behavior, and the rth positive moment exists only when $r < \alpha$. Thus, depending on the value of α, the proposed model can accommodate finite-mean, infinite-variance, and finite-variance situations. However, the SMOE distribution is mainly suitable for positive data showing polynomial-type heavy-tail behavior and is not proposed as a model for all possible types of fat-tailed distributions. The maximum likelihood estimation of the model parameters was also discussed. Moreover, the observed information matrix, some identifiability aspects, and numerical issues related to the likelihood function were investigated. An exact random generation procedure was established for the proposed model. Since the full SMOE distribution contains four parameters, some simpler submodels with two and three parameters were also discussed. These reduced models can be useful in practical situations where a more parsimonious model is preferred.

Finally, the usefulness of the proposed family was examined using two real data sets, namely, the charity fundraising percentage data and the Indian per-capita income data. For the charity fundraising data, the SMOE model provided an adequate fit according to the considered goodness-of-fit procedures. In addition, the three-parameter submodel obtained by taking $\beta = 1$ showed a competitive performance and produced the smallest AIC among the fitted models. For the second data set, the simpler Marshall–Olkin exponential model provided a more parsimonious fit than the full four-parameter SMOE model. These results show that the additional flexibility of the SMOE model can be useful for some data sets, whereas a simpler member of the proposed family may be more appropriate for other data sets. Therefore, the proposed SMO family provides a flexible framework for modeling positive, skewed, and polynomially heavy-tailed data. At the same time, the empirical results indicate the importance of selecting an appropriate member of the family according to the characteristics of the data rather than always preferring the model with the largest number of parameters. In this regard, the proposed model and its simpler submodels can provide useful alternatives for the analysis of positive heavy-tailed data in economics, reliability, finance, and other applied areas. We hope that the present study can provide further motivation for developing other useful members of the SMO family and for studying their theoretical and practical properties in future works.

https://doi.org/10.3390/axioms15100773

Author Contributions: Conceptualization, J.G. and J.J.; methodology, J.G. and J.J.; software, J.J. and M.E.B.; validation, J.G., M.E.B. and M.M.H.; formal analysis, J.G. and J.J.; investigation, J.G. and H.A.; resources, H.A. and N.A.T.; data curation, J.J. and H.A.; writing—original draft preparation, J.G. and J.J.; writing—review and editing, J.G., J.J., H.A., M.E.B., N.A.T. and M.M.H.; visualization, J.J. and M.E.B.; supervision, J.G. and M.M.H.; project administration, J.G. All authors have read and agreed to the published version of the manuscript.

Funding: The study was funded by the Ongoing Research Funding program (ORF-2026-1568), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Acknowledgments: The authors would like to thank the Ongoing Research Funding program (ORF-2026-1568), King Saud University, Riyadh, Saudi Arabia. During the preparation of this manuscript, AI-based tools were used for language refinement.

Conflicts of Interest: The authors declare no conflicts of interest.

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