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Physical-Symbol Duality and Delsarte Theory for Semisimple Mixed-Alphabet Codes

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Abstract

Let q be an odd prime power, $P = \mathbb{F}_q[v]/\langle v^2 - \tau_2 v\rangle$ and $Q = \mathbb{F}_q[u,v]/\langle u^3 - \tau_1^2 u, v^2 - \tau_2 v\rangle$ with $\tau_1\tau_2 \neq 0$. We study Q-submodules of $\mathbb{F}_q^n \times P^m \times Q^k$ under the symbol metric, that is, the Hamming metric of the mixed alphabet, in which a nonzero P- or Q-coordinate is one symbol error, rather than under the Hamming metric of the Gray image. The metric space is the product Hamming scheme $H(n,q) \otimes H(m,q^2) \otimes H(k,q^6)$. We write the Q-valued module dual in the original coordinates and prove that, for submodules, it equals the character annihilator of this scheme; examples show that the submodule hypothesis and the idempotent weights are both necessary. Specializing Delsarte's theory, we give a three-block MacWilliams identity with explicit eigenmatrices, prove that the one-variable symbol enumerator is not closed under duality, and treat linear-programming, Singleton, Hamming and Plotkin bounds. Our main structural result reduces the minimum distance, the dual distance, the orthogonal-array strength profile and the maximal size of a Q-submodule to six classical linear codes over $\mathbb{F}_q$; hence, Singleton-optimal submodules are exactly those with MDS constituents, and submodules can be strictly smaller than $\mathbb{F}_q$-linear codes. A channel model, mixed split orthogonal arrays, exact examples and numerical comparisons with known bounds are included.

Keywords: mixed-alphabet code; symbol metric; product Hamming scheme; MacWilliams identity; Krawtchouk polynomial; Delsarte bound; constituent code; MDS code; mixed orthogonal array; split orthogonal array

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1. Introduction

1.1. Mixed-Alphabet and Ring-Linear Codes

A mixed-alphabet (or polyalphabetic) code is a set of words whose coordinates take values in alphabets of different sizes, and its distance is the number of coordinates in which two words differ. Such codes go back to the study of perfect mixed codes [1], tables and bounds for mixed binary/ternary codes were obtained by Brouwer, Hämäläinen, Östergård and Sloane [2], a Singleton-type bound for polyalphabetic codes by Sidorenko et al. [3]

(see also [4]), and Gilbert–Varshamov, sphere-packing, Elias–Bassalygo and first linear-programming bounds for bounded mixed alphabets by Yehezkeally et al. [5].

A second line of research studies mixed codes that are modules over a ring. The prototype is the family of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, whose duality uses an inner product in which the binary part is weighted by the element $2 \in \mathbb{Z}_4$ [6]; it was extended to $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$- and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes [7,8] and to mixtures of semisimple rings such as $\mathbb{F}_q RS$ [9] and $\mathbb{F}_q PQ$ [10]; see also [11]. The present paper works with the rings

$$P = \frac{\mathbb{F}_q[v]}{\langle v^2 - \tau_2 v\rangle}, \qquad Q = \frac{\mathbb{F}_q[u,v]}{\langle u^3 - \tau_1^2 u, v^2 - \tau_2 v\rangle}, \qquad \mathcal{M} = \mathbb{F}_q^n \times P^m \times Q^k,$$

which were introduced, together with the six orthogonal idempotents and the constituent decomposition recalled in Section 2, by Almuhaimeed, Ansari and Shujat [12]; that work studies generalized cyclic codes in $\mathcal{M}$, their Gray images and their Euclidean duals. In this literature, the main tools are Gray maps and constituent codes, and the applications include LCD and quantum codes obtained from Gray images [9–12], entanglement-assisted quantum codes [13] and DNA codes [14]. We return to quantum codes in Section 12.

1.2. Symbol Metric Versus Gray Metric

Since $P \cong \mathbb{F}_q^2$ and $Q \cong \mathbb{F}_q^6$, a Gray map sends $\mathcal{M}$ onto $\mathbb{F}_q^{n+2m+6k}$, and the Hamming weight of the Gray image (the Lee weight of [12]) counts one nonzero Q-coordinate as one to six errors. That metric is the right one when the $6k$ scalars of the Gray image are transmitted and corrupted independently. In many situations, however, a whole symbol is lost or corrupted at once, for instance, when each coordinate is stored on one node of a heterogeneous storage system or sent in one packet (Section 3). The appropriate distance is then the number of coordinates of (x, p, z) that differ, i.e., the ordinary Hamming metric of the mixed alphabet $\mathbb{F}_q^n \times P^m \times Q^k$ used in [2,3,5]. We call it the physical or symbol metric. It forgets the ring structure, and the resulting metric space is the product Hamming scheme $H(n, q) \otimes H(m, q^2) \otimes H(k, q^6)$.

1.3. Delsarte Theory

For the Hamming metric over one alphabet, the weight distributions of a linear code and of its dual determine each other through the MacWilliams identity [15,16], whose kernel consists of Krawtchouk polynomials. Delsarte's thesis [17] placed this identity in the framework of association schemes, a notion that originated in the design of experiments [18,19] and is treated in [20,21]: the second eigenmatrix of a scheme maps the distance distribution of any code to a nonnegative vector, which yields linear-programming (LP) bounds for codes and designs [17,22,23]. In a companion paper [24], Delsarte introduced the dual distance and the external distance of an unrestricted code; in particular, the dual distance minus one is the strength of the code viewed as an orthogonal array. For additive codes in translation schemes, the dual distribution is the distance distribution of the character-theoretic dual; over finite Frobenius rings, this underlies the MacWilliams identities of Wood [25], and Gluesing-Luerssen [26] studied the partitions of a finite abelian group that support such identities (Fourier-reflexive partitions). Product schemes appear in LP bounds for mixed binary/ternary codes [2], in mixed-level orthogonal arrays [27–29], and in Levenshtein's split orthogonal arrays [30], which are designs in $H(n_1, q) \otimes H(n_2, q)$ [31]. Multivariate P- and Q-polynomial schemes [32] and codes and designs in them [33] provide a general framework containing such products; new dual solutions of Delsarte's LP were found recently in [34].

1.4. What Is Known and What Is New

Most harmonic-analytic statements in Sections 4–7 are specializations of known theory to the scheme above: the MacWilliams identity and Delsarte's inequalities for additive codes in translation schemes [17,23], the product-scheme versions [2,28,29], the mixed Singleton bound [3,4], and the sphere-packing and Plotkin bounds for mixed alphabets. These results hold for every additive subgroup of $\mathcal{M}$ once the dual is taken with respect to the scalar form B of (6). We give complete proofs with explicit normalizations, because the three local alphabet sizes q, q^2, q^6 must be tracked in all later computations, but we do not claim them as new. The ring-theoretic preliminaries of Section 2 are those of [12]. The contributions that concern Q-submodules and the symbol metric specifically are the following.

(i) An explicit formula for the Q-valued module dual in the original coordinates (x, p, z), its description through the trace form of Q, and the identification of the module dual with the character annihilator of the product scheme for submodules (Theorem 2 and Proposition 2). Examples show that neither the submodule hypothesis nor the idempotent weights in the form can be dropped (Example 1 and Remark 1).

(ii) A proof that the one-variable symbol enumerator is not closed under duality (Proposition 5), and a channel model in which this matters: on a memoryless mixed symmetric channel, the undetected-error probability is determined by the three-block enumerator but not by the one-variable enumerator (Section 3, Corollary 2).

(iii) A constituent-reduction theorem (Theorem 11): the minimum symbol distance, the dual distance, the orthogonal-array strength profile and the largest possible size of a Q-submodule are determined by six classical linear codes over $\mathbb{F}_q$. Consequently, the submodules meeting the symbol Singleton bound are exactly those whose constituents are MDS codes (Corollary 5), and Q-submodules can be strictly smaller than $\mathbb{F}_q$-linear codes with the same parameters (Example 3). This is the restriction that the linear program does not see.

(iv) Mixed split orthogonal arrays with an anisotropic profile (t_X, t_P, t_Q), a three-block, mixed-alphabet version of Levenshtein's split orthogonal arrays, together with their dual-support characterization and factorial-moment identities (Section 7) and examples in which the profile carries information beyond the total strength (Sections 9.2 and 9.3).

(v) Exact examples and a numerical comparison with the Gilbert–Varshamov, Singleton, Hamming and Plotkin bounds, including parameters for which the LP bound certifies that a Q-submodule is optimal among all $\mathbb{F}_q$-linear codes (Section 9), together with reproducible verification code (Section 10).

1.5. Organization

Section 2 recalls the constituent decomposition and establishes the character duality. Section 3 introduces the symbol metric, the eigenmatrices of the product scheme and a channel model. The MacWilliams identity and the Delsarte bounds are treated in Sections 4 and 5, the elementary bounds in Section 6, and mixed orthogonal arrays in Section 7. Section 8 contains the constituent-reduction theorem. Examples and the comparison of bounds are in Section 9, the algorithms and the verification in Section 10, and the conclusion and future directions in Sections 11 and 12.

2. Semisimple Constituents and Character Duality

Throughout, $q = p^s$ is an odd prime power and $\tau_1, \tau_2 \in \mathbb{F}_q^\times$. The rings P, Q, the idempotents below (up to their ordering) and the constituent decomposition of Theorem 1 are those of [12]; we recall them in the notation needed here. Put

$$\eta_1 = 1 - \frac{v}{\tau_2}, \qquad \eta_2 = \frac{v}{\tau_2}$$

in P. For the u-factor of Q, put

$$e_0 = 1 - \frac{u^2}{\tau_1^2}, \qquad e_+ = \frac{u(u+\tau_1)}{2\tau_1^2}, \qquad e_- = \frac{u(u-\tau_1)}{2\tau_1^2}.$$

We order the primitive idempotents of Q as

$$\varepsilon_1 = e_0\eta_1, \quad \varepsilon_2 = e_0\eta_2, \quad \varepsilon_3 = e_+\eta_1, \quad \varepsilon_4 = e_+\eta_2, \quad \varepsilon_5 = e_-\eta_1, \quad \varepsilon_6 = e_-\eta_2. \tag{1}$$

Thus, $\varepsilon_1, \dots, \varepsilon_6$ correspond, respectively, to the common roots

$$(u,v) = (0,0),\ (0,\tau_2),\ (\tau_1,0),\ (\tau_1,\tau_2),\ (-\tau_1,0),\ (-\tau_1,\tau_2).$$

Proposition 1. *The elements η_1, η_2 are the primitive orthogonal idempotents of P, and the six elements in (1) are the primitive orthogonal idempotents of Q. Evaluation at the roots gives*

$$P \cong \mathbb{F}_q^2, \qquad Q \cong \mathbb{F}_q^6.$$

Thus, every $p \in P$ and $z \in Q$ have unique forms

$$p = \eta_1 p_1 + \eta_2 p_2, \qquad z = \sum_{r=1}^{6} \varepsilon_r z_r, \qquad p_i, z_r \in \mathbb{F}_q.$$

Proof. Since q is odd, 2 is invertible in $\mathbb{F}_q$ and $\tau_1 \neq -\tau_1$; since $\tau_1\tau_2 \neq 0$, the roots $0, \tau_1, -\tau_1$ of $u^3 - \tau_1^2 u$ are pairwise distinct, and so are the roots $0, \tau_2$ of $v^2 - \tau_2 v$. Hence,

$$v^2 - \tau_2 v = v(v - \tau_2), \qquad u^3 - \tau_1^2 u = u(u-\tau_1)(u+\tau_1)$$

have distinct linear factors. The displayed elements are their Lagrange interpolation idempotents: each takes the value 1 at its own root and 0 at the others. Their pairwise products are zero, their sums are one, and the Chinese remainder theorem gives the asserted decompositions. □

For $a = \sum_{r=1}^{6} \varepsilon_r a_r \in Q$, define

$$\rho(a) = a_1, \qquad \sigma(a) = \eta_1 a_1 + \eta_2 a_2. \tag{2}$$

Equivalently, $\rho(a) = a \bmod \langle u, v\rangle$ and $\sigma(a) = a \bmod \langle u\rangle$ are the natural ring epimorphisms $Q \to Q/\langle u,v\rangle \cong \mathbb{F}_q$ and $Q \to Q/\langle u\rangle \cong P$. The ambient alphabet $\mathcal{M} = \mathbb{F}_q^n \times P^m \times Q^k$ is a Q-module under

$$a \cdot (x,p,z) = \big(\rho(a)x, \sigma(a)p, az\big). \tag{3}$$

A mixed-alphabet code is a Q-submodule $C \leq_Q \mathcal{M}$. Write the CRT components coordinatewise as

$$p = \eta_1 p^{(1)} + \eta_2 p^{(2)}, \qquad z = \sum_{r=1}^{6} \varepsilon_r z^{(r)}.$$

Define $L_1 = n + m + k$, $L_2 = m + k$, $L_3 = \cdots = L_6 = k$,

$$V_r = \mathbb{F}_q^{L_r} \quad (1 \le r \le 6),$$
$$\Psi(x,p,z) = \big((x, p^{(1)}, z^{(1)}), (p^{(2)}, z^{(2)}), z^{(3)}, z^{(4)}, z^{(5)}, z^{(6)}\big).$$

Theorem 1. *For every $C \leq_Q \mathcal{M}$, there are unique $\mathbb{F}_q$-linear codes $C_r \leq V_r$ such that*

$$\Psi(C) = C_1 \oplus C_2 \oplus \cdots \oplus C_6. \tag{4}$$

Conversely, every six-tuple of such field-linear codes defines a unique Q-submodule. If $k_r = \dim_{\mathbb{F}_q} C_r$, then

$$|C| = q^K, \qquad K = \sum_{r=1}^{6} k_r, \qquad |\mathcal{M}| = q^N, \quad N = n + 2m + 6k.$$

Proof. Under Ψ, multiplication by $a = \sum_r \varepsilon_r a_r$ acts diagonally as $(w_1, \ldots, w_6) \longmapsto (a_1 w_1, \ldots, a_6 w_6)$. In particular, multiplication by ε_r is the projection onto component r. A submodule therefore contains the six component projections of each of its words and is their direct sum. Conversely, a direct sum of field subspaces is invariant under every diagonal scalar. The cardinalities follow from the constituent dimensions. □

For $w, w' \in \mathcal{M}$ with $\Psi(w) = (w_1, \ldots, w_6)$ and $\Psi(w') = (w'_1, \ldots, w'_6)$, define the Q-valued form

$$[w, w']_Q = \sum_{r=1}^{6} \varepsilon_r \langle w_r, w'_r \rangle \tag{5}$$

and its scalar contraction

$$B(w, w') = \sum_{r=1}^{6} \langle w_r, w'_r \rangle, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the standard dot product of V_r. Explicit expressions in the original coordinates (x, p, z) are given in Proposition 2 below. Both forms are nondegenerate; the former is Q-bilinear and the latter is $\mathbb{F}_q$-bilinear. We use the module dual

$$C^\perp = \{w \in \mathcal{M} : [w, c]_Q = 0 \text{ for every } c \in C\}.$$

Fix the canonical nontrivial additive character

$$\psi(a) = \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)\right)$$

and identify $w \in \mathcal{M}$ with the character $\chi_w(c) = \psi(B(w, c))$. Since B is nondegenerate, $w \mapsto \chi_w$ is an isomorphism from $\mathcal{M}$ onto its character group. Let

$$C^\circ = \{w \in \mathcal{M} : \chi_w(c) = 1 \text{ for every } c \in C\}$$

be the character annihilator. For every additive subgroup $C \subseteq \mathcal{M}$, one has $|C|\,|C^\circ| = |\mathcal{M}|$ by the duality theory of finite abelian groups.

Theorem 2. *For every mixed-alphabet code $C \leq_Q \mathcal{M}$,*

$$C^\circ = C^\perp, \qquad \Psi(C^\perp) = C_1^\perp \oplus \cdots \oplus C_6^\perp. \tag{7}$$

Consequently,

$$|C|\,|C^\perp| = q^N \qquad \textit{and} \qquad C^{\perp\perp} = C.$$

Proof. We use three facts: the elements $\varepsilon_1, \ldots, \varepsilon_6$ form an $\mathbb{F}_q$-basis of Q (Proposition 1); C is closed under the action of Q; and the form $(\lambda, \mu) \mapsto \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\lambda\mu)$ on $\mathbb{F}_q$ is nondegenerate, so that, for $\alpha \in \mathbb{F}_q$, the character $\lambda \mapsto \psi(\lambda\alpha)$ of $\mathbb{F}_q$ is trivial only when $\alpha = 0$.

$C^\perp \subseteq C^\circ$. If $[w, c]_Q = 0$, then all six coefficients $\langle w_r, c_r \rangle$ vanish because the ε_r are linearly independent; hence, $B(w, c) = 0$ and $\chi_w(c) = 1$.

$C^\circ \subseteq C^\perp$. Let $w \in C^\circ$, $c \in C$, $1 \le r \le 6$ and $\lambda \in \mathbb{F}_q$. Because C is a Q-submodule, the word $(\lambda\varepsilon_r)\cdot c$ belongs to C, and $\Psi((\lambda\varepsilon_r)\cdot c)$ has λc_r in position r and zeros elsewhere. Therefore,

$$1 = \chi_w\big((\lambda\varepsilon_r)\cdot c\big) = \psi(\lambda\langle w_r, c_r\rangle) \quad \text{for every } \lambda \in \mathbb{F}_q,$$

and the nondegeneracy of the trace form gives $\langle w_r, c_r\rangle = 0$. As this holds for every r, $[w,c]_Q = 0$.

Since $\Psi(C) = \bigoplus_r C_r$, the condition $[w,c]_Q = 0$ for all $c \in C$ means that $w_r \in C_r^\perp$ for every r, which proves the second equality in (7). Finally, $|C_r||C_r^\perp| = q^{L_r}$ and $C_r^{\perp\perp} = C_r$ in V_r, and $\sum_r L_r = N$. □

Proposition 2. *Embed $\mathbb{F}_q \subset P \subset Q$ by constants and by $v \mapsto v$ (well defined, since $v^2 - \tau_2 v = 0$ in Q). For $w = (x,p,z)$ and $w' = (x',p',z')$ in $\mathcal{M}$,*

$$[w,w']_Q = \varepsilon_1 \sum_{i=1}^{n} x_i x_i' + e_0 \sum_{j=1}^{m} p_j p_j' + \sum_{\ell=1}^{k} z_\ell z_\ell', \tag{8}$$

$$B(w,w') = \mathrm{Tr}_{Q/\mathbb{F}_q}\big([w,w']_Q\big) = \sum_i x_i x_i' + \sum_j \big(p_j^{(1)} p_j'^{(1)} + p_j^{(2)} p_j'^{(2)}\big) + \sum_\ell \sum_{r=1}^{6} z_\ell^{(r)} z_\ell'^{(r)}, \tag{9}$$

where $\mathrm{Tr}_{Q/\mathbb{F}_q}(a)$ is the trace of the $\mathbb{F}_q$-linear map $y \mapsto ay$ on Q. Moreover, $[a\cdot w, w']_Q = a\,[w,w']_Q$ for all $a \in Q$.

Proof. Because $e_0 + e_+ + e_- = 1$, inside Q we have $\eta_1 = \varepsilon_1 + \varepsilon_3 + \varepsilon_5$ and $\eta_2 = \varepsilon_2 + \varepsilon_4 + \varepsilon_6$. Hence, for $p_j, p_j' \in P \subset Q$,

$$p_j p_j' = (\varepsilon_1 + \varepsilon_3 + \varepsilon_5) p_j^{(1)} p_j'^{(1)} + (\varepsilon_2 + \varepsilon_4 + \varepsilon_6) p_j^{(2)} p_j'^{(2)}, \qquad e_0 p_j p_j' = \varepsilon_1 p_j^{(1)} p_j'^{(1)} + \varepsilon_2 p_j^{(2)} p_j'^{(2)},$$

since $e_0\varepsilon_1 = \varepsilon_1$, $e_0\varepsilon_2 = \varepsilon_2$ and $e_0\varepsilon_r = 0$ for $r \ge 3$. Also, $z_\ell z_\ell' = \sum_r \varepsilon_r z_\ell^{(r)} z_\ell'^{(r)}$. Collecting the coefficient of each ε_r and comparing with the definition of Ψ gives (8). Multiplication by $a = \sum_r \varepsilon_r a_r$ is diagonal in the basis $\varepsilon_1, \dots, \varepsilon_6$, so $\mathrm{Tr}_{Q/\mathbb{F}_q}(a) = a_1 + \cdots + a_6$, and (9) follows. For the last statement, $\varepsilon_1\rho(a) = \varepsilon_1 a_1 = a\varepsilon_1$; moreover, $\sigma(a) = \eta_1 a_1 + \eta_2 a_2 = (\varepsilon_1 + \varepsilon_3 + \varepsilon_5)a_1 + (\varepsilon_2 + \varepsilon_4 + \varepsilon_6)a_2$ in Q, so $e_0\sigma(a) = \varepsilon_1 a_1 + \varepsilon_2 a_2 = e_0 a$. The Q-block is Q-bilinear trivially. □

Example 1. *Let $n = m = k = 1$ and let $C = \{(0,a,0) : a \in \mathbb{F}_q\}$ consist of the constant P-symbols. It is an $\mathbb{F}_q$-subspace but not a Q-submodule, since $\varepsilon_1 \cdot (0,1,0) = (0,\eta_1,0) \notin C$. A constant $a \in P$ has $a^{(1)} = a^{(2)} = a$, so, by Proposition 2,*

$$C^\perp = \{(x',p',z') : p'^{(1)} = p'^{(2)} = 0\}, \qquad C^\circ = \{(x',p',z') : p'^{(1)} + p'^{(2)} = 0\}.$$

Thus, $|C^\perp| = q^7$ while $|C^\circ| = q^8 = q^9/|C|$: for this $\mathbb{F}_q$-linear code, the Q-valued annihilator is strictly smaller than the character annihilator, and the MacWilliams identity of Section 4 fails for it. The Q-module structure is used exactly in the second step of the proof of Theorem 2.

Remark 1. *Several papers on $\mathbb{F}_q PQ$-type codes define the dual through the unweighted product $x\cdot x' + p\cdot p' + z\cdot z' \in Q$; see, for instance, [10], where a different pair of semisimple rings is used. For the action (3), this product is $\mathbb{F}_q$-bilinear but not Q-bilinear, since $\rho(a)xx' \ne a\,xx'$ in general. The weights ε_1 and e_0 in (8) realize the Q-module isomorphisms $\mathbb{F}_q \cong \varepsilon_1 Q$ and $P \cong e_0 Q$, in the same way that the factor 2 embeds $\mathbb{Z}_2$ into $\mathbb{Z}_4$ in the inner product of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes [6]. They are needed for the duality theory. For the code $C_{\mathrm{rep}} = \{(a, \eta_1 a, \varepsilon_1 a) : a \in \mathbb{F}_q\}$ of Section 9.1, a direct computation in CRT coordinates shows that the annihilator under the unweighted product is $\{x' = p'^{(1)} = z'^{(1)} = 0\}$, which has q^6 elements, whereas $|C_{\mathrm{rep}}^\perp| = q^8 = q^9/|C_{\mathrm{rep}}|$. Both this count and those of Example 1 were also confirmed by exhaustive search over all 3^9 words for $q = 3$ (Section 10).*

Remark 2. *The map Ψ is an algebraic isomorphism, but it is not an isometry for the physical metric introduced below. A P-coordinate is nonzero when at least one of its two CRT components is nonzero; similarly, a Q-coordinate is nonzero when at least one of its six components is nonzero. Thus, the physical support is obtained by a coordinatewise logical "or," not by adding the six constituent supports.*

3. The Symbol Metric and the Product Hamming Scheme

For $c = (x, p, z) \in \mathcal{M}$, define the three block weights

$$\mathrm{wt}_X(c) = |\{i : x_i \neq 0\}|, \qquad \mathrm{wt}_P(c) = |\{j : p_j \neq 0\}|, \qquad \mathrm{wt}_Q(c) = |\{\ell : z_\ell \neq 0\}|,$$

and put

$$\mathrm{wt}_{\mathrm{ph}}(c) = \mathrm{wt}_X(c) + \mathrm{wt}_P(c) + \mathrm{wt}_Q(c), \qquad d_{\mathrm{ph}}(c, c') = \mathrm{wt}_{\mathrm{ph}}(c - c'). \tag{10}$$

The number of physical positions is $L = n + m + k$, whereas the logarithmic q-ary capacity is $N = n + 2m + 6k$.

$\mathrm{wt}_{\mathrm{ph}}(c)$ is the number of nonzero coordinates of c regarded as a word of length L over the mixed alphabet $\mathbb{F}_q, \ldots, \mathbb{F}_q, P, \ldots, P, Q, \ldots, Q$. It is therefore the usual Hamming weight of mixed-alphabet codes [2,3,5], and it does not depend on the ring structure. We call it the physical (or symbol) weight only to distinguish it from the Lee weight $W_L(x, p, z) = W_H(x) + W_L(p) + W_L(z)$ of [12], which is the Hamming weight of the Gray image in $\mathbb{F}_q^{n+2m+6k}$ and counts a nonzero P-symbol once or twice and a nonzero Q-symbol between one and six times. The ring structure enters this paper only through the class of codes (submodules) and the duality.

Let $H(s, Q_0)$ denote the Hamming association scheme on an alphabet of size Q_0 and length s. The relations of

$$\mathfrak{X} = H(n, q) \otimes H(m, q^2) \otimes H(k, q^6) \tag{11}$$

are indexed by triples (i, j, ℓ), where two words are related when their three block distances are i, j, ℓ, respectively. For $0 \leq r, i \leq s$, write

$$K_r^{(s,Q_0)}(i) = \sum_{h=0}^{r} (-1)^h (Q_0 - 1)^{r-h} \binom{i}{h} \binom{s-i}{r-h}. \tag{12}$$

They have the generating function

$$\sum_{r=0}^{s} K_r^{(s,Q_0)}(i)\, T^r = (1 + (Q_0 - 1)T)^{s-i} (1 - T)^i. \tag{13}$$

Proposition 3. *Let $A_{(i,j,\ell)}$ be the adjacency matrices and $E_{(r,s,t)}$ the primitive idempotents of $\mathfrak{X}$, with $0 \leq i, r \leq n$, $0 \leq j, s \leq m$, $0 \leq \ell, t \leq k$.*

(a) The valency of relation (i, j, ℓ) is

$$v_{i,j,\ell} = \binom{n}{i} (q-1)^i \binom{m}{j} (q^2-1)^j \binom{k}{\ell} (q^6-1)^\ell. \tag{14}$$

(b) The first eigenmatrix P and the second eigenmatrix Q, defined by $A_{(i,j,\ell)} E_{(r,s,t)} = \mathsf{P}_{(r,s,t),(i,j,\ell)} E_{(r,s,t)}$ and $E_{(r,s,t)} = q^{-N} \sum_{(i,j,\ell)} \mathsf{Q}_{(i,j,\ell),(r,s,t)} A_{(i,j,\ell)}$, are

$$\mathsf{P}_{(r,s,t),(i,j,\ell)} = K_i^{(n,q)}(r)\, K_j^{(m,q^2)}(s)\, K_\ell^{(k,q^6)}(t), \tag{15}$$

$$\mathsf{Q}_{(i,j,\ell),(r,s,t)} = K_r^{(n,q)}(i)\, K_s^{(m,q^2)}(j)\, K_t^{(k,q^6)}(\ell). \tag{16}$$

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Thus, the eigenvalue of $A_{(i,j,\ell)}$ on the eigenspace (r,s,t) is $K_i(r)K_j(s)K_\ell(t)$, whereas the transform of distance distributions in Sections 4 and 5 uses Q. The two are related by $v_{(i,j,\ell)}\mathsf{Q}_{(i,j,\ell),(r,s,t)} = v_{(r,s,t)}\mathsf{P}_{(r,s,t),(i,j,\ell)}$, but they differ in general; for example, in $H(3,3)$, one has $K_1(0) = 6$ and $K_0(1) = 1$.

(c) *Through the pairing (6), $E_{(r,s,t)}$ is the orthogonal projection onto the span of the characters χ_w with block weight (r,s,t). Identifying the relation (i,j,ℓ) with the idempotent (i,j,ℓ), one has $\mathsf{P} = \mathsf{Q}$ entrywise, so $\mathfrak{X}$ is a self-dual translation scheme.*

Proof. (a) For a fixed word, choose the differing positions independently in the three blocks and then a nonzero difference at each position.

(b), (c) Consider first one block A^s, where A has order Q_0 and carries one of the nondegenerate local pairings of (9). For $b \in A^s$ of weight r and $x \in A^s$ of weight i,

$$\sum_{\mathrm{wt}(a)=i} \chi_b(a) = K_i^{(s,Q_0)}(r), \qquad \sum_{\mathrm{wt}(b')=r} \chi_{b'}(x) = K_r^{(s,Q_0)}(i),$$

because both sums factor over the coordinates, and $\sum_{a\in A\setminus\{0\}} \chi_b(a)$ equals $Q_0 - 1$ or -1 as $b = 0$ or $b \neq 0$, respectively; comparing with (13) gives the values. The first identity says that $A_i\chi_b = K_i(r)\chi_b$, so the characters of weight r span the eigenspace E_r with $\mathsf{P}_{r,i} = K_i(r)$. The second identity says that the matrix $Q_0^{-s}\sum_{\mathrm{wt}(b')=r} \chi_{b'}\chi_{b'}^*$, which is the projection E_r, has (x,y)-entry $Q_0^{-s}K_r(\mathrm{wt}(x-y))$, so $\mathsf{Q}_{i,r} = K_r(i)$ [16,17]. For the product scheme, adjacency matrices, idempotents and characters are tensor products of those of the three factors, which gives (15) and (16). The relation between P and Q is the reciprocity $\binom{s}{i}(Q_0-1)^iK_r(i) = \binom{s}{r}(Q_0-1)^rK_i(r)$, applied in each block. Finally, (15) and (16) are given by the same function of the two triples, which is the self-duality. □

Remark 3. *The scalar value $i + j + \ell$ is the physical distance, but retaining the triple is essential for harmonic analysis: the three local Krawtchouk systems have different alphabet parameters. In particular, merging all triples with the same sum before dualizing generally loses information (Proposition 5).*

A Channel Model

We begin with the following definition:

Definition 1. *Let $0 \le \pi_X, \pi_P, \pi_Q < 1$ and write $A_X = \mathbb{F}_q$, $A_P = P$, $A_Q = Q$. The mixed symmetric channel $\mathrm{MSC}(\pi_X, \pi_P, \pi_Q)$ maps an input $c \in \mathcal{M}$ to $y = c + e$, where the coordinates of the error e are independent and, at a position of type $T \in \{X, P, Q\}$, e equals 0 with probability $1 - \pi_T$ and is uniformly distributed on $A_T \setminus \{0\}$ with probability π_T.*

The symbol metric is the natural metric of the following two error models.

(a) *Adversarial symbol errors.* Up to e transmitted symbols may be replaced by arbitrary symbols of the same alphabet. A code corrects all such patterns if and only if $d_{\mathrm{ph}} \ge 2e + 1$, and detects up to $d_{\mathrm{ph}} - 1$ symbol errors.

(b) *A memoryless mixed symmetric channel* (Definition 1).

Such models arise when a codeword is split into L units whose payloads have different sizes and the basic failure event destroys a whole unit, whatever its size. Examples are a distributed storage system whose nodes store one, two or six q-ary symbols, and a packet network in which packets of three lengths are lost or corrupted as wholes. The Gray-image metric would count one failed node of the third kind as up to six errors, whereas the symbol metric counts it once. The codes and bounds of this paper are combinatorial, and the following definition only makes precise the probabilistic model in which the three-block enumerator of Section 4 is the relevant object.

Proposition 4. *On* $\mathrm{MSC}(\pi_X, \pi_P, \pi_Q)$*, an error pattern* e *of block weight* (i, j, ℓ) *occurs with probability*

$$(1-\pi_X)^{n-i}\Big(\frac{\pi_X}{q-1}\Big)^{i}(1-\pi_P)^{m-j}\Big(\frac{\pi_P}{q^2-1}\Big)^{j}(1-\pi_Q)^{k-\ell}\Big(\frac{\pi_Q}{q^6-1}\Big)^{\ell}. \tag{17}$$

Consequently, if $\pi_T < (|A_T|-1)/|A_T|$ *for each* T*, maximum-likelihood decoding minimizes* $\lambda_X \operatorname{wt}_X(y-c) + \lambda_P \operatorname{wt}_P(y-c) + \lambda_Q \operatorname{wt}_Q(y-c)$ *over* c *in the code, where* $\lambda_T = \log\big((|A_T|-1)(1-\pi_T)/\pi_T\big) > 0$*. It is minimum symbol-distance decoding when* $\lambda_X = \lambda_P = \lambda_Q$*.*

Proof. Formula (17) follows from independence and the definition. Hence, $\log \Pr(y \mid c)$ equals a constant minus $\sum_T \lambda_T \operatorname{wt}_T(y-c)$. □

Since (17) depends on the triple (i, j, ℓ) and not only on $i+j+\ell$, even when $\pi_X = \pi_P = \pi_Q$, the performance of a code on this channel is governed by its three-block enumerator; see Corollary 2 and Example 2.

4. Physical MacWilliams and Krawtchouk Transforms

Theorem 3 is the MacWilliams identity of the translation scheme $\mathfrak{X}$, i.e., the specialization of Delsarte's transform for additive codes [17,23]. Its proof uses only the characters of $(\mathcal{M}, +)$ and the nondegeneracy of B, so it holds verbatim for every additive subgroup $C \subseteq \mathcal{M}$ if $C^\perp$ is replaced by C°. For Q-submodules, the two duals coincide by Theorem 2; this is the only place in this section where the module structure is used.

For $\boldsymbol{X} = (X_0, X_1)$, $\boldsymbol{Y} = (Y_0, Y_1)$, and $\boldsymbol{Z} = (Z_0, Z_1)$, define the split physical weight enumerator

$$W_C(\boldsymbol{X}; \boldsymbol{Y}; \boldsymbol{Z}) = \sum_{c \in C} X_0^{n-\operatorname{wt}_X(c)} X_1^{\operatorname{wt}_X(c)} Y_0^{m-\operatorname{wt}_P(c)} Y_1^{\operatorname{wt}_P(c)} Z_0^{k-\operatorname{wt}_Q(c)} Z_1^{\operatorname{wt}_Q(c)}. \tag{18}$$

Theorem 3. *For every mixed-alphabet code* $C \leq_Q \mathcal{M}$*,*

$$\begin{aligned} W_{C^\perp}(\boldsymbol{X}; \boldsymbol{Y}; \boldsymbol{Z}) = \frac{1}{|C|} W_C\big(&X_0 + (q-1)X_1, X_0 - X_1; \\ &Y_0 + (q^2-1)Y_1, Y_0 - Y_1; \\ &Z_0 + (q^6-1)Z_1, Z_0 - Z_1\big). \end{aligned} \tag{19}$$

Proof. *Step 1 (orthogonality).* For $w \in \mathcal{M}$, the restriction of χ_w to the additive group C is a character of C; it is trivial exactly when $w \in C^\circ$. Since the sum of a nontrivial character over a finite group vanishes,

$$\sum_{c \in C} \chi_w(c) = |C|\, \mathbf{1}[w \in C^\circ]. \tag{20}$$

Step 2 (Poisson summation). For a function F on $\mathcal{M}$ with values in a commutative ring containing $\mathbb{C}$, put $\hat{F}(c) = \sum_{w \in \mathcal{M}} F(w)\chi_w(c)$. Summing over $c \in C$ and using (20) and Theorem 2,

$$\sum_{w \in C^\perp} F(w) = \frac{1}{|C|} \sum_{c \in C} \hat{F}(c). \tag{21}$$

Step 3 (factorization). Take $F(w) = \prod_i X_{\mathbf{1}[x_i \neq 0]} \prod_j Y_{\mathbf{1}[p_j \neq 0]} \prod_\ell Z_{\mathbf{1}[z_\ell \neq 0]}$ for $w = (x, p, z)$, so that $\sum_{w \in C^\perp} F(w) = W_{C^\perp}$. By (9), B is a sum of local nondegenerate pairings, one for each physical position: xx' on $\mathbb{F}_q$, $p^{(1)}p'^{(1)} + p^{(2)}p'^{(2)}$ on P, and $\sum_r z^{(r)}z'^{(r)}$ on Q. Hence, $\chi_w(c)$ and $\hat{F}(c)$ factor over the L positions.

Step 4 (local sums). On an additive alphabet A of order Q_0 with a nondegenerate pairing,

$$\sum_{a \in A} T_{\mathbf{1}[a \neq 0]} \psi(\langle a, b \rangle) = \begin{cases} T_0 + (Q_0 - 1)T_1, & b = 0, \\ T_0 - T_1, & b \neq 0, \end{cases}$$

because $\sum_{a\in A}\psi(\langle a,b\rangle)$ equals Q_0 for $b=0$ and 0 for $b\neq 0$. With $Q_0=q,q^2,q^6$ on the three blocks, $\hat{F}(c)$ is the monomial of c in (18) evaluated at the substituted variables of (19). Summing over $c\in C$ and using (21) gives (19). □

Let $A_{i,j,\ell}(C)$ denote the number of words of C having physical block weight (i,j,ℓ).

Corollary 1. *For $0\le r\le n$, $0\le s\le m$, and $0\le t\le k$,*

$$A_{r,s,t}(C^\perp)=\frac{1}{|C|}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{\ell=0}^{k}A_{i,j,\ell}(C)K_r^{(n,q)}(i)K_s^{(m,q^2)}(j)K_t^{(k,q^6)}(\ell). \tag{22}$$

In particular, the right side is a nonnegative integer for every additive code, and hence for every Q-submodule.

Proof. Expand the three substitutions in (19): by (13) with $T=X_1/X_0$, $(X_0+(q-1)X_1)^{n-i}(X_0-X_1)^i=\sum_r K_r^{(n,q)}(i)X_0^{n-r}X_1^r$, and similarly for the other two blocks. The coefficient of $X_0^{n-r}X_1^rY_0^{m-s}Y_1^sZ_0^{k-t}Z_1^t$ is therefore (22). □

The ordinary physical enumerator is the specialization

$$W_C^{\mathrm{ph}}(S,T)=W_C(S,T;S,T;S,T). \tag{23}$$

Unlike the usual one-alphabet situation, it does not determine the ordinary enumerator of the dual.

Proposition 5. *Assume that $n=m=k=1$. There are mixed-alphabet codes C_X and C_P such that*

$$W_{C_X}^{\mathrm{ph}}(S,T)=W_{C_P}^{\mathrm{ph}}(S,T)\quad \textit{but}\quad W_{C_X^\perp}^{\mathrm{ph}}(S,T)\neq W_{C_P^\perp}^{\mathrm{ph}}(S,T).$$

Consequently, no transformation depending only on $W_C^{\mathrm{ph}}(S,T)$ can recover $W_{C^\perp}^{\mathrm{ph}}(S,T)$ for all codes.

Proof. In $V_1=\mathbb{F}_q^3$, let C_X have first constituent $\langle(1,0,0)\rangle$ and let C_P have first constituent $\langle(0,1,0)\rangle$; set all other constituents equal to zero. In original coordinates,

$$C_X=\{(a,0,0):a\in\mathbb{F}_q\},\qquad C_P=\{(0,\eta_1 a,0):a\in\mathbb{F}_q\}.$$

Both codes have one zero word and $q-1$ words of physical weight one, so

$$W_{C_X}^{\mathrm{ph}}=W_{C_P}^{\mathrm{ph}}=S^3+(q-1)S^2T.$$

For the first dual, the field coordinate is forced to zero and the other two physical coordinates are arbitrary. Hence,

$$W_{C_X^\perp}^{\mathrm{ph}}=S\big(S+(q^2-1)T\big)\big(S+(q^6-1)T\big).$$

For the second dual, only the first CRT component of the P-coordinate is forced to zero. Therefore,

$$W_{C_P^\perp}^{\mathrm{ph}}=\big(S+(q-1)T\big)^2\big(S+(q^6-1)T\big),$$

which is different. □

Remark 4. *The obstruction in Proposition 5 is not ring pathology; it is the loss of the coordinate type. In the terminology of Gluesing-Luerssen [26], the partition of $\mathcal{M}$ into block-weight classes is a product of the reflexive Hamming partitions of the three blocks; hence, it supports a MacWilliams identity [26] (Theorem 4.3), which is (19). Proposition 5 shows that its coarsening by total symbol weight is not self-dual, although the two partitions coincide when all coordinates have the same alphabet. The three-block enumerator (18) is the natural product-scheme object closed under duality.*

Corollary 2. *Let $C \le_Q \mathcal{M}$ be used for error detection on $\mathrm{MSC}(\pi_X, \pi_P, \pi_Q)$ and put $\Pi = (1-\pi_X)^n(1-\pi_P)^m(1-\pi_Q)^k$. The probability of an undetected error is*

$$\begin{aligned} P_{\mathrm{ud}}(C) &= W_C\Big(1-\pi_X, \tfrac{\pi_X}{q-1}; 1-\pi_P, \tfrac{\pi_P}{q^2-1}; 1-\pi_Q, \tfrac{\pi_Q}{q^6-1}\Big) - \Pi \\ &= \frac{1}{|C^\perp|} W_{C^\perp}\Big(1, 1-\tfrac{q\pi_X}{q-1}; 1, 1-\tfrac{q^2\pi_P}{q^2-1}; 1, 1-\tfrac{q^6\pi_Q}{q^6-1}\Big) - \Pi. \end{aligned}$$

Proof. An error is undetected exactly when $e \in C \setminus \{0\}$, so the first line follows from (17). For the second, apply Theorem 3 to $C^\perp$, whose dual is C by Theorem 2; the substitution gives $X_0 + (q-1)X_1 = 1$ and $X_0 - X_1 = 1 - q\pi_X/(q-1)$, and similarly for the other two blocks. □

Example 2. *For the codes of Proposition 5, C_X has $q-1$ nonzero words of block weight $(1,0,0)$ and C_P has $q-1$ nonzero words of block weight $(0,1,0)$. Hence,*

$$P_{\mathrm{ud}}(C_X) = \pi_X(1-\pi_P)(1-\pi_Q), \qquad P_{\mathrm{ud}}(C_P) = \frac{(1-\pi_X)\pi_P(1-\pi_Q)}{q+1}.$$

For $\pi_X = \pi_P = \pi_Q = \pi$, the two values differ by the factor $q+1$, although $W^{\mathrm{ph}}_{C_X} = W^{\mathrm{ph}}_{C_P}$. For $q = 5$ and $\pi = 10^{-2}$, they equal $9.801 \cdot 10^{-3}$ and $1.6335 \cdot 10^{-3}$, respectively.

5. Delsarte Inequalities and Linear-Programming Bounds

The results of this section are Delsarte's inequalities and LP bound [17,22] for the scheme $\mathfrak{X}$; compare [2] for mixed binary/ternary codes and [28,29] for designs in product schemes. The association-scheme inequalities do not require linearity. Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty code of size M. Its inner distribution is

$$A_{i,j,\ell}(\mathcal{D}) = \frac{1}{M}\big|\{(a,b) \in \mathcal{D}^2 : (d_X(a,b), d_P(a,b), d_Q(a,b)) = (i,j,\ell)\}\big|. \tag{24}$$

It is the average number of codewords at block distance (i,j,ℓ) from a codeword. Then, $A_{0,0,0} = 1$, $A_{i,j,\ell} \ge 0$ and $\sum_{i,j,\ell} A_{i,j,\ell} = M$. When $\mathcal{D} = C$ is an additive subgroup, translation by the first word shows that (24) is precisely its split weight distribution.

Define the product Krawtchouk transform of the inner distribution by

$$A^\star_{r,s,t} = \frac{1}{M}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{\ell=0}^{k} A_{i,j,\ell} K_r^{(n,q)}(i) K_s^{(m,q^2)}(j) K_t^{(k,q^6)}(\ell) = \frac{1}{M}\big(A\,\mathsf{Q}\big)_{(r,s,t)}. \tag{25}$$

By (13) at $T = 1$, $\sum_r K_r^{(s,Q_0)}(i) = Q_0^s \mathbf{1}[i=0]$. Hence, $A^\star_{0,0,0} = 1$ and $\sum_{r,s,t} A^\star_{r,s,t} = q^N/M$.

Theorem 4. *For every nonempty $\mathcal{D} \subseteq \mathcal{M}$,*

$$A^\star_{r,s,t} \ge 0 \quad (0 \le r \le n,\ 0 \le s \le m,\ 0 \le t \le k).$$

If $\mathcal{D} = C$ is a mixed-alphabet code, then

$$A^\star_{r,s,t} = A_{r,s,t}(C^\perp). \tag{26}$$

Proof. Let $\widehat{\mathcal{M}}_{r,s,t}$ be the characters represented through B by words of block weight (r,s,t). By the proof of Proposition 3, for $x \in \mathcal{M}$ of block weight (i,j,ℓ), one has $\sum_{\chi \in \widehat{\mathcal{M}}_{r,s,t}} \chi(x) = \mathsf{Q}_{(i,j,\ell),(r,s,t)}$. Grouping the pairs $(a,b) \in \mathcal{D}^2$ by their block distance and using $\chi(a-b) = \chi(a)\overline{\chi(b)}$,

$$\begin{aligned} A^{\star}{}_{r,s,t} &= \frac{1}{M^2}\sum_{a,b\in\mathcal{D}}\sum_{\chi\in\widehat{\mathcal{M}}_{r,s,t}} \chi(a-b) \\ &= \frac{1}{M^2}\sum_{\chi\in\widehat{\mathcal{M}}_{r,s,t}}\Big|\sum_{a\in\mathcal{D}}\chi(a)\Big|^2 \ge 0. \end{aligned}$$

If $\mathcal{D} = C$ is a subgroup, the inner character sum is M precisely for characters in C° and is zero otherwise by (20), so $A^\star_{r,s,t} = |C^\circ \cap \widehat{M}_{r,s,t}|$. Apply Theorem 2 to obtain (26). □

Let $\mathrm{LP}_q(n,m,k;d)$ be the optimum of the following linear program in the real variables $A_{i,j,\ell}$, $0 \le i \le n$, $0 \le j \le m$, $0 \le \ell \le k$ (there are $(n+1)(m+1)(k+1)$ of them):

$$\text{maximize} \quad \sum_{i,j,\ell} A_{i,j,\ell}, \tag{27}$$

$$\text{subject to} \quad A_{0,0,0} = 1, \quad A_{i,j,\ell} \ge 0, \tag{28}$$

$$A_{i,j,\ell} = 0 \quad \text{if } 0 < i+j+\ell < d, \tag{29}$$

$$\sum_{i,j,\ell} A_{i,j,\ell} K_r^{(n,q)}(i) K_s^{(m,q^2)}(j) K_t^{(k,q^6)}(\ell) \ge 0 \qquad \text{for all } (r,s,t) \ne (0,0,0). \tag{30}$$

The constraint (30) for $(r,s,t) = (0,0,0)$ would read $\sum A_{i,j,\ell} \ge 0$ and is redundant. Dividing the constraint (r,s,t) by its value $v_{r,s,t} > 0$ at $(0,0,0)$ replaces the Krawtchouk values by normalized values in $[-1,1]$, which is numerically convenient. The program is bounded: by $\sum_r K_r(i) = Q_0^s \mathbf{1}[i=0]$, the sum of the left sides of (30) over all (r,s,t), including $(0,0,0)$, equals $q^N A_{0,0,0} = q^N$; the term for $(0,0,0)$ is $\sum A_{i,j,\ell}$ and the others are nonnegative, so $\sum A_{i,j,\ell} \le q^N$.

Theorem 5. *If $\mathcal{D} \subseteq \mathcal{M}$ has minimum physical distance of at least d, then*

$$|\mathcal{D}| \le \mathrm{LP}_q(n,m,k;d). \tag{31}$$

If $\mathcal{D} = C$ is an additive subgroup of $\mathcal{M}$, for instance, an $\mathbb{F}_q$-linear code or a Q-submodule, then, in addition, $|C|$ is a power of p (of q if C is $\mathbb{F}_q$-linear), and

$$\frac{1}{|C|} \sum_{i,j,\ell} A_{i,j,\ell} K_r^{(n,q)}(i) K_s^{(m,q^2)}(j) K_t^{(k,q^6)}(\ell) = A_{r,s,t}(C^\circ) \in \mathbb{Z}_{\ge 0}.$$

Proof. The inner distribution of $\mathcal{D}$ satisfies (28); minimum distance gives (29); and Theorem 4 gives (30). Its objective value is $|\mathcal{D}|$, proving (31). For an additive subgroup, the integrality statement is (20) together with the second part of the proof of Theorem 4. □

The usual dual description of the LP gives a compact certificate that can be checked without solving the primal program.

Theorem 6. *Suppose that*

$$f(i,j,\ell) = \sum_{r=0}^{n} \sum_{s=0}^{m} \sum_{t=0}^{k} f_{r,s,t} K_r^{(n,q)}(i) K_s^{(m,q^2)}(j) K_t^{(k,q^6)}(\ell) \tag{32}$$

satisfies

(a) $f_{0,0,0} > 0$ *and* $f_{r,s,t} \ge 0$ *for every* (r,s,t)*;*
(b) $f(i,j,\ell) \le 0$ *whenever* $i+j+\ell \ge d$ *and* $(i,j,\ell) \ne (0,0,0)$.

Then, every code $\mathcal{D} \subseteq \mathcal{M}$ of minimum physical distance of at least d satisfies

$$|\mathcal{D}| \le \frac{f(0,0,0)}{f_{0,0,0}}. \tag{33}$$

By linear-programming duality, the smallest value of the right side over all such f equals $\mathrm{LP}_q(n,m,k;d)$.

Proof. Let A be the inner distribution of $\mathcal{D}$ and $M = |\mathcal{D}|$. For $(i,j,\ell) \neq (0,0,0)$, either $0 < i+j+\ell < d$ and then $A_{i,j,\ell} = 0$, or $i+j+\ell \geq d$ and then $A_{i,j,\ell} \geq 0$ and $f(i,j,\ell) \leq 0$ by (b). Since $A_{0,0,0} = 1$,

$$\sum_{i,j,\ell} A_{i,j,\ell} f(i,j,\ell) \leq f(0,0,0).$$

On the other hand, inserting (32) and exchanging the sums,

$$\sum_{i,j,\ell} A_{i,j,\ell} f(i,j,\ell) = M \sum_{r,s,t} f_{r,s,t} A^{\star}_{r,s,t} \;\geq\; M f_{0,0,0} A^{\star}_{0,0,0} = M f_{0,0,0},$$

because $f_{r,s,t} \geq 0$ by (a), $A^{\star}_{r,s,t} \geq 0$ by Theorem 4, and $A^{\star}_{0,0,0} = 1$. Combining the two displays gives (33). The last statement is the duality theorem of linear programming applied to (27)–(30), normalized by $f_{0,0,0} = 1$. □

Remark 5. *The ordinary distance only forbids triples by the half-space $i+j+\ell < d$, whereas the spectral constraints remain genuinely three-dimensional. Collapsing the triples to their sums before forming the constraints would lose information, as in Proposition 5. Certificates that genuinely combine two blocks occur already for small parameters (Example 5).*

Remark 6. *The integrality conditions in Theorem 5 are satisfied by every $\mathbb{F}_q$-linear code in $\mathcal{M} \cong \mathbb{F}_q^n \times \mathbb{F}_q^{2m} \times \mathbb{F}_q^{6k}$ with the block metric, when the dual is taken as the B-annihilator. They do not distinguish Q-submodules from other $\mathbb{F}_q$-linear codes. The restrictions that are specific to Q-submodules are those of the constituent-reduction theorem (Theorem 11); they can make the largest submodule strictly smaller than the LP bound and smaller than the largest $\mathbb{F}_q$-linear code (Example 3, Table 3).*

6. Capacity-Aware Metric Bounds

The bounds in this section are the standard Singleton, sphere-packing and Plotkin bounds for mixed alphabets written in our parameters. Theorem 7 is the Singleton-type bound for polyalphabetic codes of Sidorenko et al. [3] (see also [4] (Cor. 2.15) and [5] (Thm. 10)), and the arguments for Theorem 8 are the classical ones [16] applied position by position. We include the short proofs to make the dependence on the three alphabet sizes explicit.

For $0 \leq t \leq L = n+m+k$, define the maximum q-ary capacity of any t physical positions by

$$\mathsf{W}_{\max}(t) = 6\min\{t,k\} + 2\min\{(t-k)_+, m\} + \min\{(t-k-m)_+, n\}, \tag{34}$$

where $(a)_+ = \max\{a,0\}$. This simply selects Q-positions first, then P-positions, then field positions. The greedy choice is optimal: if a set Ω of t positions omits a position of capacity c' while containing one of capacity $c < c'$, exchanging the two increases the capacity of Ω.

Theorem 7. *Let $C \leq_Q \mathcal{M}$ have $|C| = q^K$ and minimum physical distance d. Then,*

$$K \leq N - \mathsf{W}_{\max}(d-1). \tag{35}$$

More generally, (35) holds with $K = \log_q |\mathcal{D}|$ for every not necessarily linear $\mathcal{D} \subseteq \mathcal{M}$.

Proof. Puncture any set Ω of $d-1$ physical positions. Two codewords with the same punctured image differ only inside Ω, so their distance is at most $d-1$; hence, the puncturing map is injective on the code. If Ω contains a field, b P-, and c Q-positions, the target alphabet of the punctured code has size $q^{N-a-2b-6c}$. Choose Ω to maximize $a+2b+6c$ subject to $a+b+c = d-1$. The maximum is (34). □

Corollary 3. *If equality holds in* (35)*, then puncturing a maximum-capacity set of* $d-1$ *positions maps the code bijectively onto the entire complementary alphabet. Thus, the complement of every maximum-capacity set of* $d-1$ *positions is an information set of a physical Singleton-optimal code.*

Proof. The injection in the proof of Theorem 7 is between finite sets of the same cardinality and is therefore a bijection. □

The radius-t physical ball has volume

$$\mathsf{V}(t) = \sum_{i+j+\ell \le t} \binom{n}{i}(q-1)^i \binom{m}{j}(q^2-1)^j \binom{k}{\ell}(q^6-1)^\ell. \tag{36}$$

Theorem 8. *Let* $\mathcal{D} \subseteq \mathcal{M}$ *have size* M *and minimum physical distance* d*. Put*

$$e = \left\lfloor \frac{d-1}{2} \right\rfloor, \qquad \mathsf{S} = n(1-q^{-1}) + m(1-q^{-2}) + k(1-q^{-6}).$$

Then,

$$M\,\mathsf{V}(e) \le q^N. \tag{37}$$

If $d > \mathsf{S}$*, then*

$$M \le \left\lfloor \frac{d}{d-\mathsf{S}} \right\rfloor. \tag{38}$$

Proof. If a word y were at a distance of at most e from two codewords, the triangle inequality would put these codewords at a distance of at most $2e < d$. Hence, the balls of radius e about the codewords are disjoint. Each contains $\mathsf{V}(e)$ words, because the words at block distance (i,j,ℓ) from a fixed word are counted by (14). This proves (37).

Let $\Sigma = \sum_{(a,b)} d_{\mathrm{ph}}(a,b)$, the sum over ordered pairs of distinct codewords. Each of the $M(M-1)$ terms is at least d, so $\Sigma \ge M(M-1)d$. On the other hand, Σ is the sum over the L positions of the number of ordered pairs of codewords that differ there. At a position with alphabet of size Q_0, if the symbol α occurs M_α times in that position, this number is $M^2 - \sum_\alpha M_\alpha^2 \le M^2 - M^2/Q_0$ by the Cauchy–Schwarz inequality. Summing over the positions of the three blocks gives $\Sigma \le M^2\mathsf{S}$. Hence, $(M-1)d \le M\mathsf{S}$, i.e., $M(d-\mathsf{S}) \le d$, and for $d > \mathsf{S}$ we get $M \le d/(d-\mathsf{S})$; integrality gives (38). □

Remark 7. *Replacing* (n,m,k) *by the Gray length* N *in the classical* q*-ary bounds would bound codes for the Gray-image metric, in which one corrupted* Q*-symbol may count as six errors. In the bounds above, the error radius is measured in symbols, i.e., in the* L *physical positions, while* N *enters only through the alphabet sizes. This is the separation appropriate for the error models of Section 3.* Conversely, a greedy argument gives the Gilbert–Varshamov lower bound: there is a code of minimum symbol distance of at least d with at least $\lceil q^N/\mathsf{V}(d-1)\rceil$ words [5].

Lemma 1. *For* $0 \le t \le L$*,*

$$N - \mathsf{W}_{\max}(t) = \sum_{r=1}^{6} (L_r - t)_+,$$

where $L_1 = n+m+k$*,* $L_2 = m+k$ *and* $L_3 = \cdots = L_6 = k$ *are the lengths of the constituent spaces* V_r*.*

Proof. The capacity of a physical position is the number of constituent spaces V_r in which it occurs: a field position occurs only in V_1, a P-position in V_1, V_2, and a Q-position in all six. The position sets of $V_1 \supseteq V_2 \supseteq V_3 = \cdots = V_6$ are initial segments of the greedy order (Q-positions, then P-positions, then field positions). Hence, a maximum-capacity set Ω of t positions meets the position set of V_r in exactly $\min\{t, L_r\}$ positions, and $N - \mathsf{W}_{\max}(t) = \sum_r \big(L_r - \min\{t, L_r\}\big) = \sum_r (L_r - t)_+$. □

7. Mixed Orthogonal Arrays and Dual Support Profiles

Throughout this section, $C \leq_Q \mathcal{M}$; all statements remain true for an arbitrary additive subgroup of $\mathcal{M}$ if $C^\perp$ is replaced by C°, and Remark 9 treats unrestricted codes. Mixed orthogonal arrays are Delsarte designs in $\mathfrak{X}$ [17,28,29]. The profile of Definition 3 is a three-block, mixed-alphabet version of Levenshtein's split orthogonal arrays [30], which are designs in $H(n_1, q) \otimes H(n_2, q)$ whose set of vanishing dual coefficients is a rectangle with the origin removed [31] (Example 8.4).

For coordinate sets $A \subseteq \{1, \dots, n\}$, $B \subseteq \{1, \dots, m\}$, and $D \subseteq \{1, \dots, k\}$, let

$$\pi_{A,B,D} : C \longrightarrow \mathbb{F}_q^A \times P^B \times Q^D$$

be the physical coordinate projection.

Definition 2. *A mixed-alphabet code C is a mixed orthogonal array of strength t if $\pi_{A,B,D}$ is surjective whenever $|A| + |B| + |D| \leq t$. Since the projection is a group homomorphism, its fibers are cosets of its kernel, so every target tuple then occurs exactly*

$$\lambda_{A,B,D} = \frac{|C|}{q^{|A|+2|B|+6|D|}} \tag{39}$$

times. Thus, for additive codes, surjectivity and uniformity of a projection are equivalent.

The product scheme also supports an anisotropic notion that records how much strength is required in each physical block.

Definition 3. *For $\boldsymbol{\tau} = (t_X, t_P, t_Q)$ with $0 \leq t_X \leq n$, $0 \leq t_P \leq m$, and $0 \leq t_Q \leq k$, the code C is a mixed orthogonal array of profile $\boldsymbol{\tau}$ if every $\pi_{A,B,D}$ is surjective whenever $|A| \leq t_X$, $|B| \leq t_P$, and $|D| \leq t_Q$.*

Theorem 9. *A mixed-alphabet code C has profile $\boldsymbol{\tau} = (t_X, t_P, t_Q)$ if and only if*

$$A_{i,j,\ell}(C^\perp) = 0 \tag{40}$$

for every nonzero triple satisfying $0 \leq i \leq t_X$, $0 \leq j \leq t_P$, and $0 \leq \ell \leq t_Q$. In particular,

$$C \text{ has total strength } t \iff d_{\mathrm{ph}}(C^\perp) \geq t + 1. \tag{41}$$

Proof. Fix (A, B, D), let $S = A \cup B \cup D$ be the corresponding set of physical positions and $\Omega_S = \mathbb{F}_q^A \times P^B \times Q^D$. The image $\pi_S(C)$ is a subgroup of Ω_S, and a proper subgroup of a finite abelian group is annihilated by some nontrivial character. By (9), the characters of Ω_S are $\omega \mapsto \psi\big(\sum_{\pi \in S} \langle b_\pi, \omega_\pi \rangle\big)$ with $b \in \Omega_S$; extending b by zero outside S gives a word $w_b \in \mathcal{M}$ with $\mathrm{supp}(w_b) \subseteq S$, and the character evaluated at $\pi_S(c)$ equals $\chi_{w_b}(c)$. Hence, π_S fails to be surjective if and only if $C^\circ = C^\perp$ contains a nonzero word supported in S.

A nonzero word whose block weight is at most (t_X, t_P, t_Q) is supported in its own support, a set S with block sizes at most (t_X, t_P, t_Q). Conversely, a word supported in such a set S has block weight of at most (t_X, t_P, t_Q). Therefore, C has profile $\boldsymbol{\tau}$ if and only if $C^\perp$ has no nonzero word of block weight of at most $\boldsymbol{\tau}$, which is (40). The same argument with the condition $|S| \leq t$ gives (41). □

Corollary 4. *If C has profile (t_X, t_P, t_Q) and $|C| = q^K$, then*

$$K \geq t_X + 2t_P + 6t_Q. \tag{42}$$

If C has total strength t, then

$$K \geq \mathsf{W}_{\max}(t). \tag{43}$$

Proof. Project onto a set containing exactly t_X, t_P, t_Q coordinates of the three types. Surjectivity requires at least $q^{t_X+2t_P+6t_Q}$ codewords. For total strength, choose t positions of maximum capacity and apply the same argument. □

The next identities are mixed analogs of the factorial moments that arise in the Pless power-moment framework [35].

Theorem 10. *Suppose that C has profile (t_X, t_P, t_Q). For $0 \le a \le t_X$, $0 \le b \le t_P$, and $0 \le c \le t_Q$,*

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{\ell=0}^{k}\binom{i}{a}\binom{j}{b}\binom{\ell}{c}A_{i,j,\ell}(C) = |C|\binom{n}{a}\binom{m}{b}\binom{k}{c}(1-q^{-1})^a(1-q^{-2})^b(1-q^{-6})^c. \tag{44}$$

If C has total strength t, the identity holds whenever $a + b + c \le t$.

Proof. Count the tuples (w, A', B', D') with $w \in C$ and A', B', D' sets of a, b, c positions of the three blocks on which w is nonzero. Counting first by w gives the left side of (44). Now, fix (A', B', D'); there are $\binom{n}{a}\binom{m}{b}\binom{k}{c}$ choices. Because $(a, b, c) \le (t_X, t_P, t_Q)$, the profile hypothesis says that the joint projection $\pi_{A',B',D'}$ onto the product $\mathbb{F}_q^{A'} \times P^{B'} \times Q^{D'}$ is surjective. By (39), every tuple of this product is attained by exactly $|C|/q^{a+2b+6c}$ codewords. This joint uniformity, and not merely the uniformity of every single coordinate, is what is needed here: the number of codewords that are nonzero on all selected positions is the number of target tuples with all entries nonzero, $(q-1)^a(q^2-1)^b(q^6-1)^c$, times $|C|/q^{a+2b+6c}$, that is, $|C|(1-q^{-1})^a(1-q^{-2})^b(1-q^{-6})^c$. Summing over the choices of (A', B', D') gives the right side. For total strength t, the joint projection onto $a + b + c \le t$ positions is surjective by definition, and the same count applies. If the joint projection is not uniform, (44) can fail; see Section 9.2. □

Remark 8. *The profile criterion is strictly more informative than a single strength number. For example, demanding one-factor uniformity in the Q block costs six q-ary dimensions, whereas one-factor uniformity in the field block costs only one. Equations (42) and (44) retain this asymmetry. Examples in which the profile gives more than the total strength are in Sections 9.2 and 9.3.*

Remark 9. *For an arbitrary code $\mathcal{D} \subseteq \mathcal{M}$, call $\mathcal{D}$ a mixed orthogonal array of profile $\boldsymbol{\tau}$ if each projection $\pi_{A,B,D}$ with $(|A|, |B|, |D|) \le \boldsymbol{\tau}$ takes every value equally often. The projection of the uniform distribution on $\mathcal{D}$ onto Ω_S is uniform if and only if all its nontrivial Fourier coefficients vanish, and these coefficients are $M^{-1}\sum_{a\in\mathcal{D}}\chi_w(a)$ with $w \ne 0$ supported in S. By the formula for $A^\star$ in the proof of Theorem 4, $\mathcal{D}$ has profile $\boldsymbol{\tau}$ if and only if $A^\star_{r,s,t}(\mathcal{D}) = 0$ for every nonzero $(r, s, t) \le \boldsymbol{\tau}$. This is Delsarte's notion of a T-design [17,28] for the rectangle $T = \{(r, s, t) \ne 0 : (r, s, t) \le \boldsymbol{\tau}\}$, and for Q-submodules, it reduces to Theorem 9 by (26).*

8. Constituent Reduction: What the Module Structure Adds

The results of Sections 4–7 hold for all additive codes. We now use the decomposition $\Psi(C) = \bigoplus_r C_r$ of a Q-submodule in an essential way. Recall from Remark 2 that the coordinates of V_1 correspond to all L physical positions $(x_i, p_j^{(1)}, z_\ell^{(1)})$, those of V_2 to the P- and Q-positions $(p_j^{(2)}, z_\ell^{(2)})$, and those of V_r, $r \ge 3$, to the Q-positions $(z_\ell^{(r)})$, injectively in each case. For $y \in V_r$, let $\beta(y) = (\beta_X, \beta_P, \beta_Q)$ be the numbers of field, P- and Q-positions in the support of y; then, $\beta_X + \beta_P + \beta_Q = \mathrm{wt}_H(y)$, the Hamming weight of y in V_r. Let d_H denote minimum Hamming distance, and let $k_q(\ell, d)$ denote the largest dimension of a linear $[\ell, \kappa, \ge d]_q$ code, with $k_q(\ell, d) = 0$ for $\ell < d$.

Theorem 11. *Let $C \leq_Q \mathcal{M}$ have constituents $C_1, \ldots, C_6$.*

(a) For $c \in C$, the physical support of c is the union of the physical supports of its components $\varepsilon_r \cdot c \in C$, and $\varepsilon_r \cdot c$ has block weight $\beta(c_r)$. Consequently,

$$d_{\mathrm{ph}}(C) = \min\{d_H(C_r) : C_r \neq 0\}.$$

(b) $d_{\mathrm{ph}}(C^\perp) = \min\{d_H(C_r^\perp) : C_r^\perp \neq 0\}$.

(c) C has profile $\boldsymbol{\tau}$ if and only if, for every r, no nonzero $y \in C_r^\perp$ satisfies $\beta(y) \leq \boldsymbol{\tau}$. In particular, C has total strength t if and only if $d_H(C_r^\perp) \geq t+1$ for every r with $C_r^\perp \neq 0$, i.e., if and only if every constituent is a classical orthogonal array of strength $\min\{t, L_r\}$ over $\mathbb{F}_q$.

(d) For $1 \leq d \leq L$,

$$\max\{\log_q |C| : C \leq_Q \mathcal{M},\ d_{\mathrm{ph}}(C) \geq d\} = \sum_{r=1}^{6} k_q(L_r, d) = k_q(L, d) + k_q(m+k, d) + 4\,k_q(k, d).$$

Proof. (a) By Theorem 1, $\varepsilon_r \cdot c \in C$ and $\Psi(\varepsilon_r \cdot c)$ has c_r in position r and zeros elsewhere. A physical position of c is nonzero if and only if one of its CRT components is nonzero (Remark 2), which gives the union. Since the coordinates of V_r correspond to distinct physical positions, $\varepsilon_r \cdot c$ has block weight $\beta(c_r)$ and symbol weight $\mathrm{wt}_H(c_r)$. If $c \neq 0$, some $c_r \neq 0$, and $\mathrm{wt}_{\mathrm{ph}}(c) \geq \mathrm{wt}_H(c_r) \geq d_H(C_r)$. Conversely, every nonzero $y \in C_r$ is Ψ of a word of C of symbol weight $\mathrm{wt}_H(y)$.

(b) Apply (a) to $C^\perp$, which is a submodule with constituents $C_r^\perp$ by Theorem 2.

(c) By Theorem 9, C has profile $\boldsymbol{\tau}$ if and only if $C^\perp$ has no nonzero word of block weight of at most $\boldsymbol{\tau}$. By (a) applied to $C^\perp$, the block weight of a nonzero dual word w dominates $\beta(w_r)$ for each r, and each component $\varepsilon_r \cdot w$ is itself a dual word of block weight $\beta(w_r)$. Hence, such a word exists if and only if some $C_r^\perp$ contains a nonzero y with $\beta(y) \leq \boldsymbol{\tau}$. The statement on total strength is the same argument with $\mathrm{wt}_H(y) \leq t$, combined with the classical fact that a linear code is an orthogonal array of strength t exactly when its dual has minimum distance of at least $t+1$ [24,27].

(d) By (a), $d_{\mathrm{ph}}(C) \geq d$ if and only if every nonzero constituent has minimum distance of at least d. The constituents can be chosen independently (Theorem 1), and $|C| = q^{\sum_r \dim C_r}$. □

For codes over a single finite principal ideal ring decomposed by the Chinese remainder theorem, the analog of (a) is due to Dougherty, Kim and Kulosman [36]. In the mixed setting, the six constituents have different lengths $L, m+k, k, k, k, k$, and this produces the following phenomena.

Corollary 5. *Let $C \leq_Q \mathcal{M}$ with $d_{\mathrm{ph}}(C) = d$, $1 \leq d \leq L$. Equality holds in the Singleton bound (35) if and only if, for every r, the constituent C_r is an MDS code $[L_r, L_r - d + 1, d]_q$ when $L_r \geq d$, and $C_r = 0$ when $L_r < d$. In particular, Singleton-optimal Q-submodules exist for all d whenever $L \leq q+1$, since MDS codes of every dimension and length of at most $q+1$ exist over $\mathbb{F}_q$ (doubly extended Reed–Solomon codes [16] (Ch. 11)).*

Proof. By Lemma 1, $N - \mathrm{W}_{\max}(d-1) = \sum_r (L_r - d + 1)_+$. By Theorem 11(a), each nonzero C_r has minimum distance of at least d, so the classical Singleton bound gives $\dim C_r \leq (L_r - d + 1)_+$, and $C_r = 0$ if $L_r < d$. Hence, $K = \sum_r \dim C_r \leq \sum_r (L_r - d + 1)_+$, with equality if and only if $\dim C_r = L_r - d + 1$ for every r with $L_r \geq d$; an $[L_r, L_r - d + 1]$ code of minimum distance of at least d is MDS with minimum distance of exactly d. Conversely, such constituents give $d_{\mathrm{ph}}(C) = d$ by Theorem 11(a), because $C_1 \neq 0$. □

Example 3. *Let $q = 3$, $(n, m, k) = (1, 1, 5)$ and $d = 4$, so that the constituent lengths are $7, 6, 5, 5, 5, 5$. Exhaustive search (Section 10) gives $k_3(7, 4) = 3$, $k_3(6, 4) = 2$ and $k_3(5, 4) = 1$.*

By Theorem 11(d), every Q-submodule with $d_{\mathrm{ph}} \geq 4$ has at most $3^{3+2+4} = 3^9$ words. Now, let $\theta : \mathbb{F}_{3^6} \to Q$ be an $\mathbb{F}_3$-linear bijection (an identification of vector spaces, not of rings) and let $\mathrm{RS} \subseteq \mathbb{F}_{3^6}^5$ be a $[5,2,4]$ Reed–Solomon code. The $\mathbb{F}_3$-linear code

$$C' = \{(0,0,\theta(c_1),\dots,\theta(c_5)) : (c_1,\dots,c_5) \in \mathrm{RS}\}$$

has 3^{12} words and $d_{\mathrm{ph}}(C') = 4$, since the symbol weight of a word of C' is the Hamming weight of the corresponding Reed–Solomon codeword. Thus, the largest $\mathbb{F}_3$-linear code is at least 3^3 times larger than the largest Q-submodule; the symbol Singleton bound is 3^{15}. The LP bound and the integrality conditions of Theorem 5 apply to both codes and cannot detect this restriction, while Theorem 11 does. Similarly, for $(q;n,m,k;d) = (3;9,1,0;3)$, the parity-check matrix whose nine columns are the points of $\mathrm{PG}(2,3)$ off a line U, with the P-coordinate mapped onto U, defines an $\mathbb{F}_3$-linear perfect mixed code in the sense of [1] with 3^8 words. Every Q-submodule with the same parameters has at most $3^{k_3(10,3)} = 3^7$ words.

9. Exact Examples and Comparison of Bounds

Sections 9.1 and 9.2 use $q = 5$, $\tau_1 = \tau_2 = 1$ and $n = m = k = 1$, so $N = 1+2+6 = 9$ and $|\mathcal{M}| = 5^9$. Sections 9.3–9.5 use $q = 3$ and larger lengths. We retain the idempotent ordering in (1).

9.1. A Singleton-Optimal Repetition Code

Consider the one-dimensional code

$$C_{\mathrm{rep}} = \{(a, \eta_1 a, \varepsilon_1 a) : a \in \mathbb{F}_5\}. \tag{45}$$

Its first constituent is $\langle(1,1,1)\rangle \leq \mathbb{F}_5^3$ and its other constituents are zero. Every nonzero word occupies all three physical positions, so

$$W_{C_{\mathrm{rep}}} = X_0Y_0Z_0 + 4X_1Y_1Z_1, \qquad |C_{\mathrm{rep}}| = 5, \qquad d_{\mathrm{ph}}(C_{\mathrm{rep}}) = 3. \tag{46}$$

The MacWilliams identity gives the compact exact formula

$$\begin{aligned} W_{C_{\mathrm{rep}}^{\perp}} = \frac{1}{5}[&(X_0 + 4X_1)(Y_0 + 24Y_1)(Z_0 + 15624Z_1) \\ &+ 4(X_0 - X_1)(Y_0 - Y_1)(Z_0 - Z_1)]. \end{aligned} \tag{47}$$

The coefficient distribution is displayed in Table 1; its sum is $390{,}625 = 5^8$, as required by duality.

Table 1. Physical split distribution of $C_{\mathrm{rep}}^{\perp}$ ($q = 5$).

(i,j,ℓ)	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
$A_{i,j,\ell}$	1	3124	4	74,996	0	12,500	20	299,980

Since $\mathrm{W}_{\max}(2) = 6+2 = 8$, Theorem 7 gives $K \leq 9-8 = 1$, so C_{rep} is Singleton-optimal. This is the case $d = L$ of Corollary 5: only C_1 can be nonzero, and it must be the repetition code $[3,1,3]_5$. For $d = L$, the bound $|\mathcal{D}| \leq q$ is elementary, because the projection onto the field coordinate is injective. We therefore use this example only to illustrate Theorem 3 and Corollary 5; nontrivial applications of the linear program are given in Sections 9.4 and 9.5.

9.2. A Mixed Orthogonal Array from Dual Distance

Let

$$D = \{(a, \eta_1 a + \eta_2 b, \varepsilon_1 a + \varepsilon_2 b) : a, b \in \mathbb{F}_5\}. \tag{48}$$

Its first two constituents are

$$D_1 = \langle (1,1,1) \rangle \leq \mathbb{F}_5^3, \qquad D_2 = \langle (1,1) \rangle \leq \mathbb{F}_5^2,$$

and $D_3 = \cdots = D_6 = 0$. Directly from (48),

$$W_D = X_0Y_0Z_0 + 4X_0Y_1Z_1 + 20X_1Y_1Z_1, \qquad d_{\mathrm{ph}}(D) = 2. \tag{49}$$

Put $C = D^{\perp}$. Equivalently, in CRT coordinates, C is defined by the two independent checks

$$x + p^{(1)} + z^{(1)} = 0, \qquad p^{(2)} + z^{(2)} = 0. \tag{50}$$

It follows that $|C| = 5^7 = 78125$. Since $C^{\perp} = D$ has minimum physical distance two, Theorem 9 proves that C is a mixed orthogonal array of strength exactly one. Its split distribution, obtained from (49) by Theorem 3, is given in Table 2.

Table 2. Physical split distribution of the strength-one array $C = D^{\perp}$ ($q = 5$).

(i,j,ℓ)	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
$A_{i,j,\ell}$	1	624	0	15,000	0	2500	4	59,996

Uniformity in each single physical factor has three different fiber sizes:

$$\begin{aligned} \frac{|C|}{5} &= 15625 \quad (\mathbb{F}_5 \text{ coordinate}), \\ \frac{|C|}{25} &= 3125 \quad (P \text{ coordinate}), \\ \frac{|C|}{15625} &= 5 \quad (Q \text{ coordinate}). \end{aligned}$$

The first factorial moments provide independent numerical checks:

$$\begin{aligned} \sum_{i,j,\ell} iA_{i,j,\ell} &= 62500 = 78125\left(1 - \frac{1}{5}\right), \\ \sum_{i,j,\ell} jA_{i,j,\ell} &= 75000 = 78125\left(1 - \frac{1}{25}\right), \\ \sum_{i,j,\ell} \ell A_{i,j,\ell} &= 78120 = 78125\left(1 - \frac{1}{15625}\right). \end{aligned}$$

These values verify (44) for each of the three one-factor profiles. The code is not of total strength two because its dual contains the weight-two words appearing in (49).

The profile gives more than the total strength. The dual constituents D_1 and D_2 consist of words of block type $(1,1,1)$ and $(0,1,1)$, respectively. By Theorem 11(c), C has profile $\boldsymbol{\tau}$ if and only if neither $(1,1,1) \leq \boldsymbol{\tau}$ nor $(0,1,1) \leq \boldsymbol{\tau}$, i.e., if and only if $t_P = 0$ or $t_Q = 0$. The maximal profiles are therefore $(1,1,0)$ and $(1,0,1)$, although the total strength is only one. Concretely, the projection of C onto the field and P positions is uniform with index $|C|/5^3 = 625$; the projection onto the field and Q positions is a bijection onto $\mathbb{F}_5 \times Q$, so C is systematic on these two symbols, and the projection onto the P and Q positions has image of size $5^7 < 5^8$. The second factorial moments behave accordingly:

$$\begin{aligned} \sum ij\, A_{i,j,\ell} &= 60000 = 78125(1 - \tfrac{1}{5})(1 - \tfrac{1}{25}), \\ \sum i\ell\, A_{i,j,\ell} &= 62496 = 78125(1 - \tfrac{1}{5})(1 - \tfrac{1}{15625}), \\ \sum j\ell\, A_{i,j,\ell} &= 74996 \neq 78125(1 - \tfrac{1}{25})(1 - \tfrac{1}{15625}) = 74995.2. \end{aligned}$$

The first two identities are instances of Theorem 10; the third fails, as it must, because the joint projection onto the P and Q positions is not uniform.

9.3. An Anisotropic Profile over $\mathbb{F}_3$

Let $q = 3$ and $(n, m, k) = (4, 1, 1)$, so $L = 6$ and $N = 12$. Let D be the submodule with constituents

$$D_1 = \langle (1, 1, 1, 1, 1, 0) \rangle \leq \mathbb{F}_3^6, \qquad D_2 = \langle (1, 1) \rangle \leq \mathbb{F}_3^2, \qquad D_3 = \cdots = D_6 = 0,$$

where the coordinates of V_1 are $(x_1, \ldots, x_4, p^{(1)}, z^{(1)})$ and those of V_2 are $(p^{(2)}, z^{(2)})$. Then,

$$W_D = X_0^4 Y_0 Z_0 + 2X_0^4 Y_1 Z_1 + 2X_1^4 Y_1 Z_0 + 4X_1^4 Y_1 Z_1,$$

and $C = D^\perp$ is given by the checks $x_1 + x_2 + x_3 + x_4 + p^{(1)} = 0$ and $p^{(2)} + z^{(2)} = 0$, so $|C| = 3^{10}$. The dual words have block types $(4, 1, 0)$, $(0, 1, 1)$ and $(4, 1, 1)$, so $d_{\mathrm{ph}}(C^\perp) = 2$ and the total strength of C is only one. By Theorem 9, however, the maximal profiles of C are $(4, 0, 1)$ and $(3, 1, 0)$. Thus, any four field coordinates together with the Q-coordinate are jointly uniform; since $|C| = 3^{4+6}$, the projection onto $\mathbb{F}_3^4 \times Q$ is a bijection, i.e., these five positions form an information set. Likewise any three field coordinates together with the P-coordinate are uniform with index $3^{10}/3^5 = 243$, while the total strength alone guarantees uniformity of single coordinates only. The moment identities of Theorem 10 hold for the profile $(4, 0, 1)$; for instance,

$$\sum \binom{i}{4} \binom{\ell}{1} A_{i,j,\ell}(C) = 11648 = 3^{10} \left(\frac{2}{3}\right)^4 \frac{728}{729}.$$

The corresponding identity for $(a, b, c) = (1, 1, 1)$ fails, since $(1, 1, 1)$ is dominated by neither maximal profile. All statements in this subsection were also checked by full enumeration of C.

9.4. The Linear-Programming Bound in Action

Let $q = 3$. We give two examples in which the LP bound is strictly better than the Singleton, Hamming and Plotkin bounds.

Example 4. *Let $(n, m, k) = (6, 1, 1)$ and $d = 5$, so $L = 8$ and $N = 14$. The Singleton bound gives $3^4 = 81$ and the Hamming bound gives 309, while the Plotkin bound does not apply because $\mathsf{S} = 4 + \frac{8}{9} + \frac{728}{729} > 5$. Let $K_r = K_r^{(6,3)}$ and*

$$f(i, j, \ell) = 1 + \tfrac{3}{5} K_1(i) + \tfrac{3}{10} K_2(i) + \tfrac{1}{10} K_3(i) + \tfrac{1}{10} K_6(i).$$

The coefficients are nonnegative with $f_{0,0,0} = 1$, and $f(i, j, \ell)$ depends only on i, with values $\frac{243}{5}, \frac{81}{5}, \frac{81}{10}, 0, 0, 0, 0$ for $i = 0, \ldots, 6$. Since $j, \ell \leq 1$, the condition $i + j + \ell \geq 5$ forces $i \geq 3$, where $f = 0$. By Theorem 6, every code with $d_{\mathrm{ph}} \geq 5$ has at most $\lfloor 243/5 \rfloor = 48$ words, and $243/5$ is also the LP optimum. Hence, every $\mathbb{F}_3$-linear code, and in particular every Q-submodule, has at most 27 words. This bound is attained. By Theorem 11(d), the constituents $C_2, \ldots, C_6$ of length 2 or 1 must vanish, and C_1 can be an optimal $[8, 3, 5]_3$ code, e.g., with generator matrix

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 \end{pmatrix}$$

on the coordinates $(x_1, \ldots, x_6, p^{(1)}, z^{(1)})$. The resulting submodule has split enumerator

$$W_C = X_0^6 Y_0 Z_0 + 6X_0^3 X_1^3 Y_1 Z_1 + 4X_0^2 X_1^4 (Y_0 Z_1 + Y_1 Z_0 + Y_1 Z_1) + 2X_0 X_1^5 (Y_0 Z_0 + Y_0 Z_1 + Y_1 Z_0) + 2X_1^6 Y_1 Z_1,$$

so $d_{\mathrm{ph}}(C) = 5$ and $|C| = 27$. Thus, the LP bound certifies that this Q-submodule is optimal among all $\mathbb{F}_3$-linear codes in $\mathcal{M}$, which neither the Singleton nor the Hamming bound can do.

Example 5. *Let* $(n, m, k) = (7, 1, 1)$ *and* $d = 7$. *Since* $K_1^{(7,3)}(i) = 14 - 3i$ *and* $K_1^{(1,9)}(j) = 8 - 9j$, *the function*

$$f(i, j, \ell) = 1 + \tfrac{3}{4}K_1^{(7,3)}(i) + \tfrac{1}{4}K_1^{(1,9)}(j) = \tfrac{9}{4}(6 - i - j)$$

has nonnegative coefficients and $f \le 0$ *whenever* $i + j + \ell \ge 7$, *because then* $i + j \ge 6$. *Theorem 6 gives* $|\mathcal{D}| \le \lfloor 27/2 \rfloor = 13$, *which is the LP optimum, whereas the Plotkin, Singleton and Hamming bounds give* 15, 27 *and* 89. *The certificate uses the Krawtchouk polynomials of the field block and of the P-block simultaneously. For linear codes, the LP bound gives at most* 9 *words, while the best Q-submodule has* $3^{k_3(9,7)} = 3$ *words; whether an* $\mathbb{F}_3$*-linear code with* 9 *words exists for these parameters is left open.*

9.5. Comparison with Known Bounds

Table 3 compares, for $q = 3$, the Gilbert–Varshamov lower bound with the upper bounds of Singleton [3], Hamming and Plotkin (Theorems 7 and 8) and with the Delsarte LP bound (Theorem 5). The last column gives the exact maximum size of a Q-submodule, obtained from Theorem 11(d) and an exhaustive computation of the numbers $k_3(L_r, d)$. Every LP value is certified by an exact rational dual solution (Section 10).

Table 3. Bounds on the size of codes in $\mathbb{F}_3^n \times P^m \times Q^k$ with minimum symbol distance d. GV is a lower bound for unrestricted codes; Singleton, Hamming, Plotkin and LP are upper bounds for unrestricted codes; "Sub" is the exact maximum over Q-submodules. A dagger marks the rows in which the best submodule attains the largest power of 3 not exceeding the LP bound, so it is optimal among all $\mathbb{F}_3$-linear codes.

$(n, m, k; d)$	GV	Singleton	Hamming	Plotkin	LP	Sub
$(2, 1, 1; 3)$	7	9	79	–	9	$3^{2\,\dagger}$
$(4, 1, 1; 3)$	43	81	713	–	81	3^3
$(4, 1, 1; 4)$	7	27	713	–	27	3^2
$(6, 1, 1; 4)$	37	243	6385	–	243	3^4
$(6, 1, 1; 5)$	9	81	309	–	48	$3^{3\,\dagger}$
$(6, 1, 1; 6)$	3	27	309	53	18	$3^{2\,\dagger}$
$(6, 1, 1; 7)$	2	9	36	6	5	$3^{1\,\dagger}$
$(8, 1, 1; 5)$	37	729	2331	–	340	$3^{5\,\dagger}$
$(8, 1, 1; 6)$	10	243	2331	–	139	$3^{4\,\dagger}$
$(8, 1, 1; 7)$	4	81	221	–	40	3^2
$(6, 2, 1; 6)$	7	81	2006	–	48	$3^{3\,\dagger}$
$(6, 2, 1; 7)$	3	27	169	31	18	3^1
$(5, 1, 2; 6)$	5	27	2083	–	18	$3^{2\,\dagger}$
$(5, 1, 2; 7)$	2	9	113	8	6	$3^{1\,\dagger}$

Three observations can be read off Table 3. For small d, the LP bound coincides with the Singleton bound. For $(2, 1, 1; 3)$, where $L = 4 = q + 1$, the Singleton bound is attained by a submodule, as predicted by Corollary 5. In the rows $(4, 1, 1; 3)$, $(4, 1, 1; 4)$ and $(6, 1, 1; 4)$, the best submodule is smaller. By Corollary 5, equality would require ternary MDS constituents $[6, 4, 3]$, $[6, 3, 4]$ and $[8, 5, 4]$, respectively, and these do not exist (as confirmed by the search of Section 10). For larger d, the LP bound is strictly smaller than all three elementary bounds; this happens in ten of the fourteen rows. In nine rows, the best Q-submodule attains the largest power of 3 not exceeding the LP bound, which proves its optimality among all $\mathbb{F}_3$-linear codes of $\mathcal{M}$. The remaining rows leave gaps that the present methods do not close.

10. Separable Computation and Verification

The product form of the transform avoids a dense multiplication by a matrix of order $(n+1)(m+1)(k+1)$. Let

$$\mathbf{K}_X = (K_r^{(n,q)}(i))_{0\le r,i\le n},$$
$$\mathbf{K}_P = (K_s^{(m,q^2)}(j))_{0\le s,j\le m},$$
$$\mathbf{K}_Q = (K_t^{(k,q^6)}(\ell))_{0\le t,\ell\le k}.$$

Proposition 6. *If* $\mathbf{A} = (A_{i,j,\ell}(C))$ *is stored as a three-dimensional array, then*

$$\mathbf{A}^{\perp} = \frac{1}{|C|}\big(\mathbf{A} \times_1 \mathbf{K}_X \times_2 \mathbf{K}_P \times_3 \mathbf{K}_Q\big), \tag{51}$$

where $\times_r$ *denotes a mode-r tensor–matrix product. The three contractions require*

$$O\big((n+1)^2(m+1)(k+1) + (n+1)(m+1)^2(k+1) + (n+1)(m+1)(k+1)^2\big)$$

arithmetic operations and may be performed over the integers.

Proof. Contract first in i, then in j, and finally in ℓ. The resulting entry (r,s,t) is the triple sum in (22). Counting the entries multiplied at each of the three stages gives the stated complexity. Formula (12) has integer values, so exact integer arithmetic followed by exact division by $|C|$ suffices. □

10.1. Algorithms

All values in Tables 1–3 and in the examples were obtained with the following procedures, implemented in Python 3 with exact integer and rational arithmetic. The program uses NumPy for array operations and the HiGHS solver through SciPy for the linear programs, and it is provided as Supplementary Materials (`mixed_codes_verify.py`); a full run takes well under a minute on a laptop. For simplicity, the program assumes that q is an odd prime; all examples use $q \in \{3,5\}$.

Algorithm 1 (split distribution of a submodule)**.** *Input:* q, (n,m,k) and generator matrices $G_1,\dots,G_6$ of the constituents over $\mathbb{F}_q$.

1. Row-reduce each G_r modulo q to a basis of C_r; let k_r be its size.
2. Enumerate each C_r as the q^{k_r} combinations of its basis.
3. For every tuple $(c_1,\dots,c_6) \in C_1 \times \cdots \times C_6$, compute the block weight: wt_X counts the nonzero x_i in c_1; wt_P counts the j with $p_j^{(1)} \neq 0$ or $p_j^{(2)} \neq 0$; wt_Q counts the ℓ with $z_\ell^{(r)} \neq 0$ for some r. This is the logical "or" of Remark 2.
4. Accumulate the counts $A_{i,j,\ell}$.

Algorithm 2 (dual distribution, two independent ways)**.** *(i) Compute bases of the null spaces* $C_r^{\perp}$ *by Gaussian elimination and apply Algorithm 1. (ii) Apply* (51) *with exact integer Krawtchouk matrices and exact division by* $|C|$*. The two results are compared entry by entry, and their sum is compared with* $q^N/|C|$*.*

Algorithm 3 (LP bound with an exact certificate)**.** *(i) Set up* (27)–(30) *with the constraints normalized by* $v_{r,s,t}$*. (ii) Solve it numerically. (iii) Convert the dual multipliers into rational numbers* $f_{r,s,t} \ge 0$ *with denominators at most* 10^6*. (iv) Verify in exact rational arithmetic that* $f(i,j,\ell) \le 0$ *on all admissible nonzero triples, rescaling slightly if rounding requires it, and report*

$f(0,0,0)$. *By Theorem 6, the reported value is a rigorous upper bound; in all cases, it agrees with the numerical optimum.*

Algorithm 4 (best linear constituents). *$k_q(\ell, d)$ is computed by exhaustive backtracking over systematic generator matrices $[I \mid A]$. The rows of A are taken in lexicographic order, which is no loss of generality up to equivalence, each row of A must have weight of at least $d-1$, and all codeword weights are checked. Together with Theorem 11(d), this gives the column "Sub" of Table 3.*

10.2. Verification

The following independent checks were carried out.

(1) Ring level: the idempotent relations of Proposition 1, the module action, and the identities (8) and (9) were checked on random words using genuine arithmetic in Q. The counts in Example 1 and Remark 1 were confirmed by exhaustive search over all 3^9 words of $\mathbb{F}_3 \times P \times Q$.

(2) Random constituent generator matrices were drawn in every admissible dimension: 40 instances for $q = 3$ and $(n, m, k) = (1, 1, 1)$, 15 for $q = 5$ and $(1, 1, 1)$, and 15 for $q = 3$ and $(2, 1, 1)$. For each instance, the enumerated dual distribution agreed with (51), and $d_{\mathrm{ph}}(C)$ agreed with Theorem 11(a).

(3) Tables 1 and 2, the factorial moments of Section 9.2, and the uniformity statements of Sections 9.2 and 9.3 were recomputed by full enumeration of the codes and of the projections.

(4) The certificates of Examples 4 and 5 and all LP values of Table 3 were verified in exact arithmetic. The minimum distances of the codes in Examples 3 and 4 were verified by enumeration; the Reed–Solomon code of Example 3 was built in an explicit model of $\mathbb{F}_{3^6}$.

(5) The undetected-error probabilities of Example 2, and both expressions of Corollary 2 for the code $C = D^{\perp}$ of Section 9.2, were evaluated exactly.

The computations are audits, not substitutes for the proofs in Sections 4, 5 and 8.

11. Conclusions

With the symbol (mixed Hamming) metric, codes in $\mathbb{F}_q^n \times P^m \times Q^k$ live in the product scheme $H(n, q) \otimes H(m, q^2) \otimes H(k, q^6)$, while the algebraic dual of a Q-submodule is controlled by six semisimple field constituents. Theorem 2 and Proposition 2 reconcile the two descriptions by identifying the module dual, written explicitly in the original coordinates, with the character annihilator. As a consequence, Delsarte's theory of additive codes applies without change: the three-block MacWilliams identity, the Delsarte inequalities, the LP bound and the dual-distance characterization of mixed orthogonal arrays. These results are specializations of known theory, and their integrality refinements are shared by all $\mathbb{F}_q$-linear codes. What the module structure adds is the constituent reduction of Theorem 11. It expresses the minimum distance, dual distance, strength profile, and largest size of a submodule through six classical linear codes, characterizes Singleton-optimal submodules by MDS constituents, and shows that submodules can be strictly smaller than $\mathbb{F}_q$-linear codes. Together with the LP bound, it certifies optimality among all $\mathbb{F}_q$-linear codes in several of the cases of Table 3. We also showed that the one-variable symbol enumerator is not closed under duality, and that this matters for the undetected-error probability on a mixed symmetric channel.

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12. Future Research Directions

We propose the following directions for further research.

(1) *Semidefinite bounds.* The Terwilliger algebra of the product scheme can be used to strengthen the LP bound by semidefinite programming, as Schrijver did for binary codes [37]. For Q-submodules, such bounds should be combined with the constituent reduction of Theorem 11.

(2) *Quantum codes in the symbol metric.* Quantum and entanglement-assisted codes from rings of this type are usually obtained from Gray images [10–13]. The present results provide two ingredients for a symbol-metric theory. First, by Theorem 2, a Q-submodule satisfies $C^{\perp} \subseteq C$ if and only if $C_r^{\perp} \subseteq C_r$ for every r, so dual containment can be checked constituent by constituent. Second, if $C^{\perp} \subseteq C$, then the symbol weight of $C \setminus C^{\perp}$ is $\min\{i + j + \ell : A_{i,j,\ell}(C) > A_{i,j,\ell}(C^{\perp})\}$, and both split distributions are available from Theorem 3. Developing quantum error correction for mixed alphabets in which one corrupted Q-symbol counts as one error is a natural next step; DNA codes over such rings [14] could be treated similarly.

(3) Complete support-profile enumerators may be used to study unequal error costs or coordinate-dependent channels, for example, the weighted decoding rule of Proposition 4.

(4) The relation between the symbol metric and the generalized quasi-cyclic Gray images with block profiles studied in [12] deserves a systematic comparison.

(5) When q is even, or one of τ_1, τ_2 vanishes, the rings acquire repeated-root components; extending the physical theory then requires radical-aware duality in place of the six-field decomposition.

Supplementary Materials: The following supporting information can be downloaded at https://www.mdpi.com/article/10.3390/axioms15110856/s1, Table: Program Table.

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