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On a Sine Series of Ramanujan

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Abstract

We present sharp bounds for the partial sums of a special sine series studied by Ramanujan. Ramanujan's method is used to deduce limits and inequalities for related sums.

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1. Introduction and Statement of the Main Results

Our work has been inspired by the following remarkable series formula, which is due to Ramanujan; see Berndt [1], p. 397:

$$S(x) = \sum_{k=0}^{\infty} \frac{\sin^3(3^k x)}{3^k} = \frac{3}{4}\sin(x), \quad x \in \mathbb{R}. \tag{1}$$

From (1) we conclude that S is positive on $(0, \pi)$.

The aim of this paper is twofold. First, we study the partial sums

$$S_n(x) = \sum_{k=0}^{n} \frac{\sin^3(3^k x)}{3^k}.$$

More precisely, we provide sharp upper and lower bounds for $S_n(x)$. It turns out that, not only S, but even all partial sums are positive on $(0, \pi)$.

Theorem 1. *For all integers $n \geq 0$ and real numbers $x \in (0, \pi)$, we have*

$$0 < S_n(x) \leq \sin(x). \tag{2}$$

Equality holds in (2) if and only if $n = 0$ and $x = \pi/2$.

As an immediate consequence of (2), we obtain sharp constant bounds for $S_n(x)$.

Corollary 1. *For all integers $n \geq 0$ and real numbers $x \in (0, \pi)$, we have*

$$0 < S_n(x) \leq 1. \tag{3}$$

Both bounds are sharp. Equality holds in (3) if and only if $n = 0$ and $x = \pi/2$.

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Our second aim is to use Ramanujan's method of proof to deduce inequalities for the related sine sums

$$T_n(x) = \sum_{k=0}^{n} \frac{\sin^4(2^k x)}{4^k} \quad \text{and} \quad U_n(x) = \sum_{k=0}^{n} \frac{\sin(5^k x)}{5^k}\Big(\sin^2(5^k x) + \sin^2(2 \cdot 5^k x)\Big).$$

The limits of these sums as $n \to \infty$ are also established.

Here are the results.

Theorem 2. *For all integers $n \geq 0$ and real numbers $x \in (0, \pi)$, we have*

$$T_n(x) \leq \sin^2(x). \tag{4}$$

Equality holds in (4) if and only if $x = m\pi/2^{n+1}$, with $m \in \{1, 2, ..., 2^{n+1} - 1\}$.

An application of (4) leads to the best possible constant upper bound for $T_n(x)$.

Corollary 2. *For all integers $n \geq 0$ and real numbers $x \in (0, \pi)$, we have*

$$T_n(x) \leq 1. \tag{5}$$

The bound is sharp. Equality holds in (5) if and only if $x = \pi/2$.

The two upper bounds for $U_n(x)$, which are given in the next theorem, cannot be compared.

Theorem 3. *For all integers $n \geq 0$ and real numbers $x \in (0, \pi)$, we have*

$$U_n(x) \leq \frac{25}{16}\sin(x) \tag{6}$$

and

$$0 < U_n(x) \leq \frac{3}{4}\sqrt{3}. \tag{7}$$

The sign of equality is valid in (6) if and only if $n = 0$ and $x \in \{\arccos(\sqrt{3/8}), \pi - \arccos(\sqrt{3/8})\}$. The constant bounds given in (7) are sharp. Equality holds in (7) if and only if $n = 0$ and $x \in \{\pi/3, 2\pi/3\}$.

The following formulas play key roles in our proofs of the inequalities (4), (6) and (7):

$$T_n(x) = \sin^2(x) - \frac{\sin^2(2^{n+1}x)}{4^{n+1}}, \tag{8}$$

$$U_n(x) = \frac{5}{4}\sin(x) - \frac{\sin(5^{n+1}x)}{4 \cdot 5^n}. \tag{9}$$

If we let $n \to \infty$, then we obtain from (8) and (9):

$$\sum_{k=0}^{\infty} \frac{\sin^4(2^k x)}{4^k} = \sin^2(x), \quad x \in \mathbb{R}, \tag{10}$$

and

$$\sum_{k=0}^{\infty} \frac{\sin(5^k x)}{5^k}\Big(\sin^2(5^k x) + \sin^2(2 \cdot 5^k x)\Big) = \frac{5}{4}\sin(x), \quad x \in \mathbb{R}. \tag{11}$$

Ramanujan's method can be applied to prove similar identities. We show that the following companions to (10) and (11) are valid:

$$\sum_{k=0}^{\infty} \frac{\cos(2^k x) + 2\sin^4(2^k x)}{2^k} = 3 - \cos(x) - 4\sin^4(x/2), \quad x \in \mathbb{R}, \tag{12}$$

and

$$\sum_{k=0}^{\infty} \frac{\sin^2(2^k x)\cos(2^{k+1}x)}{2^k} = -\sin^2(x), \quad x \in \mathbb{R}. \tag{13}$$

Here are two special cases. Using

$$\sin(5^k \pi/4) = \frac{(-1)^k}{\sqrt{2}}, \quad 0 \le k \in \mathbb{Z},$$

we conclude from (11) with $x = \pi/8$ that

$$\sum_{k=0}^{\infty} \frac{\sin(5^k\pi/8) + 2\sin^3(5^k\pi/8)}{5^k} = \frac{5}{4}\sqrt{2-\sqrt{2}},$$

and from (12) with $x = \pi/5$ we obtain

$$\sum_{k=0}^{\infty} \frac{\cos(2^k\pi/5) + 2\sin^4(2^k\pi/5)}{2^k} = \frac{1}{8}(15+\sqrt{5}).$$

Inequalities for the partial sums of sine and cosine series have attracted the attention of mathematicians for many years. They play an important role in various fields, such as Fourier analysis, theory of functions, and number theory. For instance, they can be used to find bounds for the zeros of polynomials and to show that certain analytic functions are univalent. For detailed information on this subject we refer to Askey [2], Askey and Gasper [3], Milovanović et al. [4], Ch. 4 and to the recently published papers Alzer and Kwong [5–7] and the references therein. Among the numerous published results we state an elegant double inequality, which is related to (3):

$$-\frac{5}{64}(\sqrt{5}-1)\sqrt{10-2\sqrt{5}} \le \sum_{k=1}^{n} \frac{\sin^3(kx)}{k} \le 1, \quad n \ge 1,\ 0 < x < \pi.$$

The constant bounds are sharp. These inequalities were proved by Kim et al. [8] in 2021.

One of the referees mentioned the papers of Kazımoğlu and his co-authors, which "deal with different classes of functions, they may be useful in providing a broader background on inequalities involving partial sums"; see [9–11].

In the next section, we collect some helpful lemmas and in Section 3, we present the proofs of our results. We conclude the paper with an open problem, which is given in Section 4. The numerical calculations have been carried out using the computer software Maple 13.

2. Lemmas

The following elementary inequalities for the sine function play important roles in the proofs of the theorems. The two parts of Lemma 1 are known. For completeness, we offer short proofs.

Lemma 1. *(i) For all integers $m \ge 2$ and real numbers $x \in (0, \pi)$, we have*

$$\sin(mx) < m\sin(x).$$

(ii) For all real numbers $\lambda > 1$ and $x \in (0, \pi/2]$, we have

$$\sin(\lambda x) < \lambda \sin(x). \tag{14}$$

Proof. (i) Let $m \ge 2$ and $0 < x < \pi$. Then

$$\frac{\sin(mx)}{\sin(x)} = \sum_{k=1}^{m-1} \cos((m-k)x)\cos^{k-1}(x) + \cos^{m-1}(x) < m - 1 + 1 = m.$$

(ii) Let

$$f_\lambda(x) = \lambda \sin(x) - \sin(\lambda x).$$

We distinguish two cases.

Case 1. $0 < x \le \pi/(2\lambda)$. Since the cosine function is strictly decreasing on $[0, \pi/2]$, we obtain

$$f'_\lambda(x) = \lambda\big(\cos(x) - \cos(\lambda x)\big) > 0 \quad \text{and} \quad f_\lambda(x) > f_\lambda(0) = 0.$$

In particular,

$$f_\lambda(\pi/(2\lambda)) = \lambda \sin(\pi/(2\lambda)) - 1 > 0. \tag{15}$$

Case 2. $\pi/(2\lambda) < x \le \pi/2$. We use (15) and that the sine function is strictly increasing on $[0, \pi/2]$. This gives

$$\sin(\lambda x) \le 1 < \lambda \sin(\pi/(2\lambda)) < \lambda \sin(x).$$

□

We note that closely related inequalities can be found in Jagers [12]. A referee asked for the largest real number η such that (14) holds for all $\lambda > 1$ and $x \in (0, \eta\pi]$.

Lemma 2. *For all real numbers $\lambda \ge 1$ and $x \in (0, \pi/2]$, we have*

$$0 \le \lambda \sin(x) + \sin(3\lambda x).$$

Equality holds if and only if $\lambda = 1$ and $x = \pi/2$.

Proof. Let

$$g_\lambda(x) = \lambda \sin(x) + \sin(3\lambda x).$$

We consider two cases.

Case 1. $\lambda = 1$. Then

$$g_1(x) = 2 \sin(2x) \cos(x) \ge 0$$

with equality if and only if $x = \pi/2$.

Case 2. $\lambda > 1$. We study two subcases.

Case 2.1. $0 < x \le \pi/(2\lambda)$. From Lemma 1 (ii) we obtain

$$g_\lambda(x) > \sin(\lambda x) + \sin(3\lambda x) = 2 \sin(2\lambda x) \cos(\lambda x) \ge 0.$$

Case 2.2. $\pi/(2\lambda) < x \le \pi/2$. Using the monotonicity of the sine function and (15) leads to

$$g_\lambda(x) > \lambda \sin(\pi/(2\lambda)) - 1 > 0.$$

□

The final two lemmas provide bounds for the function

$$h_m(x) = \sin(x) - \frac{\sin(5^m x)}{5^m}, \tag{16}$$

which we need to prove Theorem 3.

Lemma 3. *For all integers $m \ge 1$ and real numbers $x \in (0, \pi/2]$, we have*

$$h_m(x) \le \frac{5}{4} \sin(x). \tag{17}$$

Equality holds if and only if $m = 1$ and $x = \arccos(\sqrt{3/8})$.

Proof. Inequality (17) is equivalent to

$$-\frac{\sin(5^m x)}{5^m} \le \frac{\sin(x)}{4}, \quad 0 < x \le \pi/2.$$

Case 1. $m = 1$. Then

$$\frac{1}{4}\sin(x) + \frac{1}{5}\sin(5x) = \frac{1}{20}\sin(x)\big(8\cos^2(x) - 3\big)^2 \ge 0$$

with equality if and only if $x = \arccos(\sqrt{3/8})$.

Case 2. $m \ge 2$. Let $\lambda = 5^{m-1}$.

Case 2.1. $0 < x \le \pi/(2\lambda)$. Applying Case 1 and Lemma 1 (ii) gives

$$-\frac{\sin(5^m x)}{5^m} = \frac{\sin(5\lambda x)}{5\lambda} \le \frac{\sin(\lambda x)}{4\lambda} < \frac{\sin(x)}{4}.$$

Case 2.2. $\pi/(2\lambda) < x \le \pi/2$. We use the inequality

$$\frac{4}{5}t < \sin(\pi t/2), \quad 0 < t \le 1/5,$$

with $t = 1/\lambda$ and the monotonicity of the sine function. This yields

$$-\frac{\sin(5^m x)}{5^m} \le \frac{1}{5\lambda} < \frac{1}{4}\sin(\pi/(2\lambda)) \le \frac{\sin(x)}{4}.$$

The proof of Lemma 3 is complete. □

Lemma 4. *For all integers $m \ge 1$ and real numbers $x \in (0, \pi/2]$, we have*

$$0 < h_m(x) \le \frac{3}{5}\sqrt{3}. \tag{18}$$

Equality holds if and only if $m = 1$ and $x = \pi/3$.

Proof. The left-hand side of (18) follows from Lemma 1 (ii). To prove the right-hand side, we distinguish three cases.

Case 1. $m = 1$. We have

$$h_1'(x) = \cos(x) - \cos(5x).$$

Let $x \in [0, \pi/2]$. Since $h_1'(x) = 0$ if and only if $5x = x$ or $5x = 2\pi - x$ or $5x = 2\pi + x$, we obtain $x = 0$ or $x = \pi/3$ or $x = \pi/2$. Moreover, $h_1''(\pi/3) = -3\sqrt{3}$. It follows that h_1 has precisely one local maximum on $(0, \pi/2)$, which is attained at $x = \pi/3$. We have $h_1(0) = 0$ and $h_1(\pi/2) = 4/5$. Thus

$$h_1(x) \le h_1(\pi/3) = \frac{3}{5}\sqrt{3}$$

with equality if and only if $x = \pi/3$.

Case 2. $m = 2$. Let

$$u(x) = -\frac{\sin(25x)}{25} \quad \text{and} \quad v(x) = \frac{3}{5}\sqrt{3} - \sin(x).$$

The function u is decreasing on $[1.5, \pi/2]$ and v is decreasing on $(0, \pi/2]$.

Case 2.1. $0 < x \le 1.5$. Then

$$u(x) \le \frac{1}{25} < 0.041... = v(1.5) \le v(x).$$

Case 2.2. $1.5 < x \le \pi/2$. We have

$$u(x) \le u(1.5) = 0.007... \quad \text{and} \quad v(x) \ge v(\pi/2) = 0.039....$$

Thus

$$h_2(x) = u(x) - v(x) + \frac{3}{5}\sqrt{3} < \frac{3}{5}\sqrt{3}.$$

Case 3. $m \geq 3$. Using

$$-\frac{\sin(5^m x)}{5^m} \leq \frac{1}{5^m} \leq \frac{1}{5^3} = 0.008 \quad \text{and} \quad \frac{3}{5}\sqrt{3} - \sin(x) \geq \frac{3}{5}\sqrt{3} - \sin(\pi/2) = 0.039...$$

leads to (18) with "$<$" instead of "$\leq$". □

3. Proofs

Proof of Theorem 1. The proof of (1) given in [1], pp. 397–398 is based upon the formula

$$S_n(x) = \frac{1}{4}\sum_{k=0}^{n}\Big(\frac{\sin(3^k x)}{3^{k-1}} - \frac{\sin(3^{k+1}x)}{3^k}\Big) = \frac{3}{4}\sin(x) - \frac{\sin(3^{n+1}x)}{4\cdot 3^n}. \tag{19}$$

This follows from

$$\sin^3(t) = \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t)$$

with $t = 3^k x$, dividing by 3^k and summing on k.

Let $n \geq 0$. Applying (19) gives

$$S_n(x) = \frac{3}{4m}\big(m\sin(x) - \sin(mx)\big), \quad m = 3^{n+1},$$

so that Lemma 1 (i) implies that $S_n(x) > 0$.

Since $S_n(x)$ and $\sin(x)$ are symmetrical with respect to $x = \pi/2$, it suffices to prove the right-hand side of (2) for $x \in (0, \pi/2]$. From (19) we obtain

$$4\cdot 3^n\big(\sin(x) - S_n(x)\big) = 3^n\sin(x) + \sin(3^{n+1}x). \tag{20}$$

Let $n \geq 0$. We use (20) and Lemma 2 with $\lambda = 3^n$. It follows that

$$\sin(x) - S_n(x) \geq 0$$

with equality if and only if $n = 0$ and $x = \pi/2$. □

Proof of Theorem 2. Using the identities

$$\cos(2t) = 2\cos^2(t) - 1 \quad \text{and} \quad \cos(4t) = 8\cos^4(t) - 8\cos^2(t) + 1$$

leads to

$$\cos(4t) - 4\cos(2t) = 8\cos^4(t) - 16\cos^2(t) + 5 = 8\sin^4(t) - 3. \tag{21}$$

It follows that

$$\sin^4(t) + \frac{1}{4}\sin^4(2t) = \frac{1}{32}\cos(8t) - \frac{1}{2}\cos(2t) + \frac{15}{32}. \tag{22}$$

Next, we set $t = 2^k x$, divide by 4^k and sum both sides of (22) from $k = 0$ to $k = n-1$. Then

$$\sum_{k=0}^{n-1}\Big(\frac{\sin^4(2^k x)}{4^k} + \frac{\sin^4(2^{k+1}x)}{4^{k+1}}\Big) = 2\sum_{k=0}^{n}\frac{\sin^4(2^k x)}{4^k} - \sin^4(x) - \frac{\sin^4(2^n x)}{4^n} \tag{23}$$

and

$$\sum_{k=0}^{n-1}\Big(\frac{1}{32}\frac{\cos(2^{k+3}x)}{4^k} - \frac{1}{2}\frac{\cos(2^{k+1}x)}{4^k} + \frac{15}{32\cdot 4^k}\Big) = -\frac{1}{2}\cos(2x) - \frac{1}{8}\cos(4x) + \frac{\cos(2^{n+1}x)}{2\cdot 4^n}$$
$$+\frac{\cos(2^{n+2}x)}{2\cdot 4^{n+1}} + \frac{5}{8}\Big(1 - \frac{1}{4^n}\Big). \tag{24}$$

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We have

$$\sin^4(x) - \frac{1}{2}\cos(2x) - \frac{1}{8}\cos(4x) + \frac{5}{8} = 2\sin^2(x)$$

and

$$\sin^4(t) + \frac{1}{2}\cos(2t) + \frac{1}{8}\cos(4t) - \frac{5}{8} = -\frac{1}{2}\sin^2(2t). \tag{25}$$

We set $t = 2^n x$ and divide both sides of (25) by 4^n. Applying (23) and (24) gives

$$2T_n(x) = 2\sum_{k=0}^{n} \frac{\sin^4(2^k x)}{4^k} = 2\sin^2(x) - \frac{2}{4^{n+1}}\sin^2(2^{n+1}x).$$

This implies that (4) is valid for $x \in (0, \pi)$ with equality if and only if $\sin(2^{n+1}x) = 0$, that is, $x = m\pi/2^{n+1}$ with $m \in \{1, 2, ..., 2^{n+1} - 1\}$. □

Proof of Theorem 3. Because of symmetry, it suffices to prove (6) and (7) for $x \in (0, \pi/2]$. We apply

$$\sin(5t) - 5\sin(t) = -4\sin^3(t) - 4\sin(t)\sin^2(2t), \quad t \in \mathbb{R},$$

with $t = 5^k x$. Then we divide both sides by 5^{k+1} and sum from $k = 0$ to $k = n$. This leads to the representation

$$U_n(x) = \frac{5}{4}h_{n+1}(x),$$

where h_m is defined in (16). Using Lemma 3 gives (6) with equality if and only if $n = 0$ and $x = \arccos(\sqrt{3/8})$.

From Lemma 4 we obtain (7). Equality holds if and only if $n = 0$ and $x = \pi/3$. □

Proofs of (12) and (13). We use (21) with $t = 2^k x$, divide by 2^{k+2} and sum from $k = 0$ to $k = n$. This yields

$$\sum_{k=0}^{n} \frac{\cos(2^k x) + 2\sin^4(2^k x)}{2^k} = 3 - \cos(x) - 4\sin^4(x/2) - \frac{3}{2^{n+1}} + \frac{\cos(2^{n+1}x)}{2^{n+1}} + \frac{\sin^4(2^n x)}{2^{n-1}}.$$

We let $n \to \infty$ and obtain (12).

For $t \in \mathbb{R}$, we have

$$\sin^2(2t) - 2\sin^2(t) = 2\sin^2(t)\cos(2t).$$

We set $t = 2^k x$, divide by 2^{k+1} and sum. Then

$$\sum_{k=0}^{n} \frac{\sin^2(2^k x)\cos(2^{k+1}x)}{2^k} = \frac{\sin^2(2^{n+1}x)}{2^{n+1}} - \sin^2(x),$$

so that $n \to \infty$ gives (13). □

4. Open Problem

We define the alternating counterpart of $S_n(x)$ by

$$V_n(x) = \sum_{k=0}^{n} (-1)^k \frac{\sin^3(3^k x)}{3^k}.$$

With regard to (3) it is natural to ask: do there exist constant upper and lower bounds for $V_n(x)$? We have

$$|V_n(x)| \le \sum_{k=0}^{n} \frac{|\sin^3(3^k x)|}{3^k} \le \sum_{k=0}^{n} \frac{1}{3^k} = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) < \frac{3}{2} \tag{26}$$

and from

$$V_n(\pi/2) = \sum_{k=0}^{n} \frac{1}{3^k} = \frac{3}{2} - \frac{1}{2 \cdot 3^n}$$

we conclude that

$$\lim_{n\to\infty} V_n(\pi/2) = \frac{3}{2}.$$

Thus

$$V_n(x) < \frac{3}{2}, \quad n \geq 0, \quad x \in \mathbb{R},$$

where 3/2 is the best possible constant upper bound.

From (26) we conclude that $V_n(x) > -3/2$. We guess that this lower bound can be improved. Numerous computer calculations and plots led us to the conjecture that the following refinement is valid.

Conjecture. *For all integers $n \geq 0$ and real numbers $x \in [0, \pi]$, we have*

$$-\frac{3}{8} < V_n(x). \tag{27}$$

The lower bound is sharp.

Remark 1. *Since*

$$\sin^3(\pi/6) = \frac{1}{8} \quad \textit{and} \quad \sin^3(3^k\pi/6) = (-1)^{k+1}, \quad k \in \mathbb{N},$$

we obtain

$$V_n(\pi/6) = \frac{1}{8} - \sum_{k=1}^{n} \frac{1}{3^k} = -\frac{3}{8} + \frac{1}{2 \cdot 3^n}.$$

Thus

$$\lim_{n\to\infty} V_n(\pi/6) = -\frac{3}{8}.$$

It follows that if (27) is valid, then the given bound is best possible.

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