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Operator Interpolation Method for Variable-Order Fractional Laplacians on the Real Line

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Abstract

We develop and analyze interpolation in the fractional-order parameter for the left-quantized variable-order fractional Laplacian on the real line. Chebyshev–Lobatto interpolation expresses the variable-order action as a finite weighted sum of constant-order fractional Laplacians. Under a Fourier-side analytic-majorant condition, we prove geometric convergence of this continuum operator interpolation in the uniform norm and provide a practical weighted-integrability criterion for the hypothesis. We also establish Lipschitz stability with respect to perturbations of the spatially varying order profile. The number of constant-order evaluations required to achieve a prescribed continuum interpolation tolerance grows at most logarithmically with the inverse tolerance. For a mapped Chebyshev realization, we derive a conditional consistency estimate separating order interpolation error, strong-norm spatial projection error, and the defect between the continuum fractional Laplacian and fractional powers of a discrete mapped Laplacian. A matrix-free nonlinear formulation, an Allen–Cahn lifting, and exact Gaussian and algebraically decaying examples illustrate the scope of the construction. The analysis proves geometric convergence only of the continuum order interpolation; convergence of the mapped L^2 projection in the required strong norm, decay of the discrete fractional power realization defect, convergence of the fully discrete nonlinear solver, and a quasi-linear implementation cost are not established.

Keywords: variable-order fractional Laplacian; mapped spectral method; Fourier-like mapped Chebyshev functions; operator interpolation; fractional partial differential equations; unbounded domains; Allen–Cahn equation; nonlocal diffusion

MSC: 35R11; 65N35; 65N12; 65T50; 35J60

Check for updates

Academic Editor: Tao Liu

Received: 14 September 2026
Revised: 4 November 2026
Accepted: 10 November 2026
Published: 18 November 2026

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1. Introduction

Fractional partial differential equations provide a natural framework for the description of transport, diffusion, and phase-transition processes in which long-range interactions or anomalous spatial dynamics cannot be represented adequately by classical local operators. In particular, the fractional Laplacian has become a widely used nonlocal diffusion operator in models of anomalous transport, interface dynamics, phase separation, and heterogeneous media. When the spatial domain is unbounded, however, its numerical approximation remains challenging because of the simultaneous presence of nonlocal interactions and slowly decaying solution profiles.

A further level of complexity arises when the order of the fractional operator is allowed to vary in space. The variable-order fractional Laplacian $\mathcal{L}_{s(\cdot)}$, which we denote formally by

$$\mathcal{L}_{s(\cdot)}u(x) = (-\Delta)^{s(x)}u(x), \qquad 0 < s_{\min} \le s(x) \le s_{\max} < 1, \qquad x \in \mathbb{R}, \tag{1}$$

can be used to describe media in which the degree of nonlocality changes spatially. Such models are particularly relevant in heterogeneous systems, where classical and anomalous diffusion mechanisms may coexist or where the range of spatial interactions changes from one region to another. From the numerical point of view, variable-order operators are considerably more difficult than their constant-order counterparts. For a constant value s, the fractional Laplacian admits the Fourier representation

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s}\hat{u}(\xi), \tag{2}$$

so that the operator acts as a fixed Fourier multiplier. In the variable-order case, the corresponding symbol depends simultaneously on the physical and frequency variables $|\xi|^{2s(x)}$, and the resulting operator is no longer diagonal in Fourier space. This loss of separability is one of the main computational difficulties associated with variable-order fractional models.

A number of accurate methods have been developed for constant-order fractional Laplacians [1] on unbounded domains. Rational spectral methods based on mapped Gegenbauer or Chebyshev functions [2] were introduced to approximate algebraically decaying solutions directly on unbounded domains [3]. In particular, Sheng et al. [4] proposed Fourier-like biorthogonal mapped Chebyshev functions [2] and showed that the constant-order integral fractional Laplacian can be fully diagonalized in the corresponding spectral representation. Their formulation leads to quasi-optimal computational complexity and avoids the dense matrices that generally arise from direct Galerkin discretizations of the nonlocal operator. These developments provide an efficient framework for constant-order fractional PDEs on $\mathbb{R}^d$ and constitute an important starting point for the method developed in the present work.

The numerical approximation of variable-order fractional Laplacians has received increasing attention only more recently. Wu and Zhang [5] developed meshfree methods based on globally supported radial basis functions and explicit formulas for the action of the variable-order operator. Their approach avoids direct numerical evaluation of the hypersingular integral and was applied to heterogeneous diffusion, wave-propagation problems, and variable-order Allen–Cahn models [6]. In a different direction, fast Fourier-based approximations have been proposed for variable-order space-fractional phase-field problems on periodic domains, where the dependence of the Fourier multiplier on the spatially varying order is approximated through suitable expansions [7]. More recently, Hao et al. [8] introduced a second-order finite-difference approximation of the integral variable-order fractional Laplacian together with a quasi-linear fast implementation. These approaches demonstrate that accurate and efficient computation of variable-order fractional operators is possible, but they are based respectively on meshfree discretizations, periodic Fourier representations, or finite differences on bounded computational domains. For a comprehensive overview of modern computational methods for fractional differential equations, including spectral, finite difference, and meshfree approaches, we refer to the recent monograph by Jafari, Tajadodi, and Gasimov [9]. The constant-order diagonalization described above does not directly extend to a spatially varying fractional order, because the symbol $|\xi|^{2s(x)}$ depends on both x and ξ. To the best of our knowledge, the possibility of recovering the efficiency of constant-order mapped spectral methods [2,10,11] by interpolating the fractional operator with respect to its order has not been systematically developed for variable-order fractional PDEs on the whole real line.

The objective of this work is to isolate and analyze the approximation of the *order dependence* in the left-quantized variable-order fractional Laplacian on $\mathbb{R}$. For $s(x) \in [a,b] \subset (0,1)$, we approximate the continuum operator by

$$\mathcal{L}_{s(\cdot)}u(x) \approx \sum_{m=0}^{M} \ell_m(s(x))(-\Delta)^{s_m}u(x), \tag{3}$$

where s_m are Chebyshev–Lobatto nodes in the order interval. Unlike interpolation of the order field itself, this construction interpolates the family of constant-order operators. The principal contributions are fourfold. First, we establish a uniform geometric order interpolation bound under a Fourier-side analytic-majorant hypothesis and give a practical sufficient integrability criterion. Second, we prove a Lipschitz bound quantifying sensitivity to perturbations of the spatial order profile. Third, we formulate a mapped Chebyshev realization and derive a conditional consistency estimate that explicitly distinguishes continuum interpolation, spatial projection, and the realization defect of discrete fractional matrix powers. Fourth, we give a matrix-free nonlinear residual and a lifting formulation for Allen–Cahn profiles with nonzero limits at infinity, together with exact analytical illustrations for Gaussian and algebraically decaying functions. The scope is deliberately analytical: no convergence rate for the mapped realization defect, fully discrete PDE convergence theorem, numerical benchmark, or quasi-linear runtime bound is claimed. In particular, diagonalization of the ordinary mapped stiffness matrix is not identified with exact diagonalization of the projected continuum fractional Laplacian.

The remainder of the paper is organized as follows. Section 2 defines the variable-order operator. Section 3 recalls the mapped spectral representation, and Section 4 develops interpolation in the fractional-order parameter and a possible discrete realization. Section 5 establishes the interpolation and conditional consistency estimates. Sections 6 and 7 describe the nonlinear formulation and Allen–Cahn lifting. Section 8 gives an analytical illustration, and Section 9 concludes.

2. Variable-Order Fractional Equations on $\mathbb{R}$

We consider nonlinear equations on the whole real line involving a spatially varying fractional diffusion operator. The variable fractional order is denoted by $s : \mathbb{R} \longrightarrow (0,1)$, and is assumed throughout the paper to satisfy

$$0 < s_{\min} \le s(x) \le s_{\max} < 1, \qquad x \in \mathbb{R}\,. \tag{4}$$

Here $s_{\min}$ and $s_{\max}$ are fixed bounds satisfying $0 < s_{\min} \le s(x) \le s_{\max} < 1$ for every $x \in \mathbb{R}$; the bounds need not be attained. When desired, they may be chosen as the essential infimum and supremum. Unless otherwise stated, we further assume that s is sufficiently smooth, for instance $s \in C^{0,\eta}(\mathbb{R})$, $\quad 0 < \eta \le 1$, and, for the spectral interpolation developed later, we shall consider profiles with additional regularity whenever required. We denote with σ the constant fractional order $s(x) = \text{const.} = \sigma$.

The principal class of problems studied in this work is

$$\mathcal{L}_{s(\cdot)}u(x) + V(x)u(x) + F(x,u(x)) = g(x), \qquad x \in \mathbb{R}, \tag{5}$$

where $\mathcal{L}_{s(\cdot)}$ denotes the variable-order fractional Laplacian (1), V is a given potential or reaction coefficient, F is a nonlinear term, and g is a prescribed source. The stationary variable-order fractional Allen–Cahn equation will be introduced as a particular instance of (5).

For constant $s(x) = \sigma \in (0, 1)$, we use the Fourier definition (2). A natural heterogeneous extension is obtained by allowing the exponent to depend on the observation point x. In the present work, we adopt the pseudodifferential definition

$$\mathcal{L}_{s(\cdot)}u(x) := (-\Delta)^{s(x)}u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^{2s(x)}\hat{u}(\xi)e^{ix\xi}\,d\xi. \tag{6}$$

Here the Fourier transform convention is $\hat{u}(\xi) = \int_{\mathbb{R}} u(x)e^{-ix\xi}\,dx$.

Several non-equivalent definitions of variable-order fractional Laplacians have been proposed in the literature; see, for example, [5,12]. In the present work, we adopt the left-quantized pseudodifferential definition (6), with symbol $p(x, \xi) = |\xi|^{2s(x)}$. All interpolation results below refer to this specific realization and should not automatically be transferred to other variable-order integral kernels.

Because the symbol $p(x, \xi) = |\xi|^{2s(x)}$ depends on both the physical and frequency variables, $\mathcal{L}_{s(\cdot)}$ is a genuine pseudodifferential operator rather than a Fourier multiplier. Hence, the diagonal spectral representation available for a constant fractional order cannot be used directly when s varies in space. This loss of diagonal structure is the computational difficulty addressed by the operator interpolation method developed below. Another consequence of the spatial dependence of s is that $\mathcal{L}_{s(\cdot)}$ is not, in general, symmetric with respect to the standard $L^2(\mathbb{R})$ inner product. In particular,

$$\left(\mathcal{L}_{s(\cdot)}u, v\right)_{L^2(\mathbb{R})} \neq \left(u, \mathcal{L}_{s(\cdot)}v\right)_{L^2(\mathbb{R})}$$

for general nonconstant $s(x)$. Hence, variational and energy arguments valid for the constant-order fractional Laplacian cannot be transferred without modification to the variable-order setting.

2.1. Functional Framework

The main emphasis of the present paper is the operator interpolation rather than a complete well-posedness theory for (5). We therefore introduce a basic functional framework sufficient for the construction and analysis of the spectral method.

Let

$$H^{\sigma}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : (1 + |\xi|^2)^{\sigma/2}\hat{u}(\xi) \in L^2(\mathbb{R})\right\},$$

equipped with the norm

$$\|u\|^2_{H^{\sigma}(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^{\sigma}|\hat{u}(\xi)|^2\,d\xi.$$

For $u \in \mathcal{S}(\mathbb{R})$, definition (6) is pointwise meaningful. More generally, if $u \in H^r(\mathbb{R})$ with $r > 2s_{\max} + 1/2$, Fourier inversion and Cauchy–Schwarz imply $\mathcal{L}u \in L^{\infty}(\mathbb{R})$. We work initially on such a common domain; the existence and regularity of solutions to (5) are separate questions.

This choice is also convenient because solutions of fractional equations on unbounded domains need not decay exponentially. Fractional diffusion frequently generates algebraic far-field behavior, and therefore the approximation space should not impose a Gaussian decay structure a priori. This consideration motivates the mapped spectral representation introduced in Section 3.

The approximation requires regularity both of the solution and of the fractional-order profile. Throughout the analysis, we therefore assume that u has the spatial regularity and far-field decay required by the mapped spectral approximation, while $s(x)$ is sufficiently smooth and satisfies (4). Precise assumptions will be introduced when the interpolation and spatial errors are analyzed.

2.2. Nonlinear Variable-Order Model Problems

We consider the semilinear equation (5). The assumptions on V, F, and g will be specified for each model problem. We assume that the corresponding solution is sufficiently regular and decays sufficiently rapidly for the mapped spectral representation to be effective.

The simplest linear benchmark is the variable-order fractional reaction–diffusion problem (5) where we assume for simplicity $F(x, u(x)) = 0$ and $V(x) = \mu = \text{const}$.

As a representative nonlinear application, we shall later consider a stationary variable-order fractional Allen–Cahn equation. Its detailed formulation, treatment of non-decaying far-field states, and heterogeneous order profiles are deferred to Section 7.

3. Fourier-like Mapped Chebyshev Representation

We use an $L^2(\mathbb{R})$-orthonormal mapped Chebyshev basis and diagonalize the finite-dimensional stiffness matrix of the ordinary Laplacian. Fractional powers of this matrix yield a convenient family of discrete operators. In general, however, these fractional matrix powers do not coincide with the Galerkin projection of the continuum integral fractional Laplacian; this distinction is made explicit in Section 3.3 and in the realization defect term of Section 5.

3.1. Mapped Basis Functions

Let $T_n(y) = \cos(n \arccos y)$, $(-1 < y < 1)$, denote the Chebyshev polynomial of the first kind of degree n. We recall the orthogonality relation

$$\int_{-1}^{1} T_n(y) T_m(y) \omega(y)\, dy = \frac{\pi c_n}{2} \delta_{nm}, \qquad \omega(y) = (1 - y^2)^{-1/2}, \tag{7}$$

where $c_0 = 2$, $c_n = 1$, $n \geq 1$, and δ_{nm} denotes the Kronecker symbol.

To extend this system to the real line, we introduce the scaled algebraic mapping

$$x = \frac{\nu y}{\sqrt{1 - y^2}}, \qquad y = \frac{x}{\sqrt{\nu^2 + x^2}}, \qquad \nu > 0. \tag{8}$$

The parameter ν sets the spatial scale of the mapped basis and influences approximation errors. Small values of ν concentrate the mapped grid around the origin, whereas larger values distribute the approximation points over a wider physical region.

The Jacobian of (8) is

$$\frac{dx}{dy} = \frac{\nu}{(1 - y^2)^{3/2}}, \qquad \frac{dy}{dx} = \frac{\nu^2}{(\nu^2 + x^2)^{3/2}}.$$

We define the scaled mapped Chebyshev functions by

$$\overset{\frown}{T}_n^{\nu}(x) = \left(\frac{2}{\pi c_n} \right)^{1/2} \frac{\nu^{1/2}}{\sqrt{\nu^2 + x^2}} T_n\left(\frac{x}{\sqrt{\nu^2 + x^2}} \right), \qquad n \geq 0. \tag{9}$$

Equivalently,

$$\overset{\frown}{T}_n^{\nu}(x) = \nu^{-1/2} \overset{\frown}{T}_n^{1}(x/\nu). \tag{10}$$

The factor multiplying the Chebyshev polynomial in (9) is essential: unlike a direct polynomial composition with the mapping, it ensures decay as $|x| \to \infty$ and produces an $L^2(\mathbb{R})$-orthonormal system.

Indeed, using (8) and (7), one obtains

$$\left(\overset{\frown}{T}_n^{\nu}, \overset{\frown}{T}_m^{\nu} \right)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \overset{\frown}{T}_n^{\nu}(x) \overset{\frown}{T}_m^{\nu}(x)\, dx = \delta_{nm}.$$

For a prescribed spectral degree N, we therefore define

$$V_N^\nu = \operatorname{span}\{\mathcal{T}_0^\nu, \mathcal{T}_1^\nu, \dots, \mathcal{T}_N^\nu\} \subset L^2(\mathbb{R}). \tag{11}$$

The scaling parameter will be kept explicit throughout the present work. This is useful for variable-order problems because the characteristic spatial scale may depend strongly on the range and spatial variation of the fractional order. The mapping scale ν is a numerical parameter; no optimality claim is made here.

For quadrature and nodal evaluations, we employ the Chebyshev–Gauss points

$$y_j = -\cos\left(\frac{(2j-1)\pi}{2N_q}\right), \qquad j = 1, \dots, N_q, \tag{12}$$

and map them onto the real line according to

$$x_j = \frac{\nu y_j}{\sqrt{1-y_j^2}}. \tag{13}$$

In the collocation implementation, the number of distinct physical nodes must match the dimension $N+1$ of V_N^ν when the nodal transform is taken to be square and invertible. Any use of a larger quadrature grid $N_q > N+1$ for error measurement or dealiasing is a separate operation. The L^2 projection P_N used in Section 5 is not the same operator as interpolation from nodal values, and the corresponding error estimates must not be interchanged.

Thus, no finite physical boundary is introduced. Increasing the number of nodes enlarges the represented portion of the real axis automatically, while the scale ν controls the relative concentration of nodes near the origin and in the far field.

3.2. Biorthogonality and Spectral Transforms

Although the mapped functions $\{\mathcal{T}_n^\nu\}_{n=0}^N$ are orthonormal in $L^2(\mathbb{R})$, they do not diagonalize the usual H^1 seminorm. We therefore introduce the stiffness matrix

$$S_{mn}^\nu = \int_{\mathbb{R}} \partial_x \mathcal{T}_n^\nu(x) \partial_x \mathcal{T}_m^\nu(x)\, dx, \qquad 0 \le m, n \le N.$$

For the unscaled system, we write simply $S = S^1$. The mapped stiffness matrix is symmetric positive definite [4]. We denote its eigendecomposition by $SE = E\Lambda$, $E^T E = I$, where

$$\Lambda = \operatorname{diag}(\lambda_0, \lambda_1, \dots, \lambda_N), \qquad \lambda_p > 0, \tag{14}$$

and the columns of $E = (e_{np})_{n,p=0,\dots,N}$ are the orthonormal eigenvectors of S.

Thus, the p-th Fourier-like mapped basis function Φ_p^ν is obtained as a linear combination of the original mapped Chebyshev functions $\mathcal{T}_n^\nu$, with coefficients given by the components e_{np} of the p-th eigenvector. The corresponding Fourier-like mapped Chebyshev functions are defined as

$$\Phi_p^\nu(x) = \sum_{n=0}^{N} e_{np} \mathcal{T}_n^\nu(x), \qquad p = 0, \dots, N. \tag{15}$$

Since E is orthogonal, the functions $\{\Phi_p^\nu\}_{p=0}^N$ span the same approximation space V_N^ν of (11). The scaling relation (10) gives $S^\nu = \nu^{-2} S$, and consequently the eigenvectors are independent of ν, while the eigenvalues scale as $\lambda_p^\nu = \nu^{-2}\lambda_p$. The transformed functions (15) therefore satisfy the two orthogonality relations

$$\left(\Phi_p^\nu, \Phi_q^\nu\right)_{L^2(\mathbb{R})} = \delta_{pq}, \quad \text{and} \quad \left(\partial_x \Phi_p^\nu, \partial_x \Phi_q^\nu\right)_{L^2(\mathbb{R})} = \lambda_p^\nu \delta_{pq}.$$

Hence, the transformed basis diagonalizes both the mass and discrete Laplacian forms.

If

$$u_N(x) = \sum_{n=0}^{N} c_n \mathcal{T}_n^\nu(x) = \sum_{p=0}^{N} \hat{u}_p \Phi_p^\nu(x), \tag{16}$$

then the two sets of coefficients are related by $\mathbf{c} = E\hat{\mathbf{u}}$, $\hat{\mathbf{u}} = E^T\mathbf{c}$. The mapped Chebyshev representation retains the standard cosine-transform structure in the reference variable y. Therefore, nodal values at (12) can be transformed to mapped Chebyshev coefficients, and conversely, using FFT-based discrete cosine transforms. This feature will be used later to evaluate nonlinear terms in physical space and return them to coefficient space without constructing dense multilinear Galerkin tensors.

Nodal values are converted to mapped Chebyshev coefficients by a fast cosine transform, while multiplication by E^T gives the Fourier-like coefficients. These transformations are precomputed or applied repeatedly in the operator evaluation described below.

3.3. Constant-Order Fractional Operator

Let A_N denote the positive mapped discrete Laplacian associated with the mass and stiffness forms, so that $A_N\Phi_p^\nu = \lambda_p^\nu\Phi_p^\nu$. Its spectral fractional power is defined by

$$A_N^s v_N = \sum_{p=0}^{N} (\lambda_p^\nu)^s (v_N, \Phi_p^\nu)_{L^2(\mathbb{R})}\Phi_p^\nu, \qquad 0 < s < 1.$$

Let $A = -\Delta$ be the nonnegative self-adjoint continuum Laplacian on $L^2(\mathbb{R})$, and let P_N denote the L^2-orthogonal projection onto V_N^ν. The discrete stiffness operator is $A_N = P_N A|_{V_N^\nu}$ whenever the expressions are defined (equivalently, it is the operator associated with the restricted energy form). The discrete fractional operator used here is A_N^s, *not* the projected continuum operator $P_N A^s|_{V_N^\nu}$. In general,

$$(P_N A P_N)^s\big|_{V_N^\nu} \neq P_N A^s\big|_{V_N^\nu}, \qquad 0 < s < 1,$$

because compression and fractional functional calculus do not commute. For example, for $A = \mathrm{diag}(1, 4)$ and P the orthogonal projection onto the span of $(1, 1)^\mathsf{T}$, the compressed power is $(5/2)^s$, whereas the compression of the power is $(1+4^s)/2$; these differ for $0 < s < 1$. Accordingly, diagonalizing A_N supplies an implementable surrogate, but its approximation to the continuum fractional Laplacian is *conditional* on a separate consistency estimate. For $v_N \in V_N^\nu$, we measure the defect in the continuum function space by

$$\delta_N(s; v_N) = A^s v_N - A_N^s v_N,$$

where the second term is embedded in $L^2(\mathbb{R})$ (and both terms must admit the pointwise bounds required by Theorem 2). No bound of the form $\|\delta_N(s; P_N u)\|_\infty \to 0$ follows solely from diagonalization. Section 5 therefore retains this defect explicitly instead of identifying the two operators. Hence, for any $s \in [s_{\min}, s_{\max}]$,

$$D_N(s) = \mathrm{diag}\big((\lambda_0^\nu)^s, \dots, (\lambda_N^\nu)^s\big). \tag{17}$$

All constant-order operators therefore share the same eigenvectors and differ only through their diagonal multipliers. This property is exploited in the operator interpolation method of the next section.

4. Operator Interpolation in the Fractional Order

We first interpolate the continuum family $\mathcal{L}_\sigma = (-\Delta)^\sigma$ in the order parameter $\sigma \in I_s = [s_{\min}, s_{\max}]$. Its discrete realization is introduced separately in Section 4.1.

Let the $M + 1$ constants $\{s_m\}_{m=0,\dots,M}$ be defined as

$$s_m = \frac{s_{\max} + s_{\min}}{2} + \frac{s_{\max} - s_{\min}}{2}\cos\left(\frac{m\pi}{M}\right), \qquad m = 0, \dots, M, \tag{18}$$

and ℓ_m denote the associated Lagrange cardinal functions:

$$\ell_m(s) = \prod_{\substack{k=0 \\ k\neq m}}^{M} \frac{s - s_k}{s_m - s_k}, \qquad m = 0, \dots, M.$$

We define the operator interpolant by

$$\mathcal{L}^{(M)}_{s(x)} = \sum_{m=0}^{M} \ell_m(s(x))\, \mathcal{L}_{s_m}. \tag{19}$$

or equivalently, according to (6), by

$$\mathcal{L}^{(M)}_{s(\cdot)} u(x) = \sum_{m=0}^{M} \ell_m(s(x))(-\Delta)^{s_m} u(x). \tag{20}$$

The cardinal functions are evaluated in barycentric form for numerical stability.

Thus, at every physical point x, the dependence of the operator on $s(x)$ is approximated by interpolation of the constant-order family $\sigma \mapsto (-\Delta)^{\sigma}$ over $I_s = [s_{\min}, s_{\max}]$.

The coefficients $\ell_m(s(x))$ contain all the spatial dependence of the fractional order, whereas the operators $\mathcal{L}_{s_m}$ are independent of x and can therefore be precomputed and applied using constant-order spectral solvers.

It is important to distinguish (19) from a spatial interpolation of $s(x)$. The function $s(x)$ itself is not approximated here. Rather, the nonlinear dependence of the fractional operator on its order is interpolated over the interval I_s. Consequently, the order profile $s(x)$ may be evaluated at the physical nodes with its full available resolution.

For the Fourier definition of the constant-order fractional Laplacian (2), the dependence on the fractional order is generated by $|\xi|^{2\sigma} = \exp\big(2\sigma \log|\xi|\big)$. Differentiation with respect to σ introduces powers of $\log|\xi|$,

$$\partial^q_\sigma |\xi|^{2\sigma} = (2\log|\xi|)^q |\xi|^{2\sigma}.$$

Although $\log|\xi|$ is singular at $\xi = 0$, the product $|\xi|^{2\sigma}|\log|\xi||^q$ tends to zero for every fixed $\sigma > 0$. Hence, the zero-frequency mode does not create a divergence, but it requires some care when establishing uniform regularity estimates with respect to the order variable. Although $\sigma \mapsto |\xi|^{2\sigma}$ is entire for each fixed $\xi \neq 0$, analyticity of the integrated operator family requires an integrable majorant uniform on a complex neighborhood of the order interval. Such a hypothesis is imposed explicitly in Proposition 1. The finite-dimensional discrete matrix powers are analytic in σ because their eigenvalues are positive.

For the following evaluation, we use the normalized coordinate

$$\eta(\sigma) = \frac{2\sigma - s_{\max} - s_{\min}}{s_{\max} - s_{\min}}, \qquad \eta(\sigma) \in [-1, 1].$$

and evaluate the Lagrange functions associated with the nodes (18) in barycentric form. Thus $s_m \in [s_{\min}, s_{\max}]$, with the end points of the admissible interval included.

The interpolation formula (20) may be evaluated using the barycentric form. Let

$$\omega_m = (-1)^m d_m, \qquad d_m = \begin{cases} \frac{1}{2}, & m = 0 \text{ or } m = M, \\ 1, & 1 \le m \le M-1. \end{cases} \tag{21}$$

Then, for $\eta \neq \eta_m$,

$$\ell_m(\eta) = \frac{\dfrac{\omega_m}{\eta - \eta_m}}{\displaystyle\sum_{k=0}^{M} \frac{\omega_k}{\eta - \eta_k}}.$$

At an interpolation node, the usual limiting definition is used, $\ell_m(\eta_k) = \delta_{mk}$.

For the spatially varying order $s(x)$, we therefore define

$$b_m(x) := \ell_m(\eta(s(x))), \qquad m = 0, \dots, M. \tag{22}$$

The cardinal property immediately gives $\sum_{m=0}^{M} b_m(x) = 1,\ x \in \mathbb{R}$.

The proposed approximation of the variable-order fractional Laplacian (19) takes the compact form

$$\mathcal{L}^{(M)}_{s(\cdot)} u = \sum_{m=0}^{M} b_m(\cdot)\, \mathcal{L}_{s_m} u. \tag{23}$$

This formula possesses an important consistency property. If the fractional order is constant and coincides with one of the interpolation orders, $s(x) \equiv s_j$, then

$$\mathcal{L}^{(M)}_{s(\cdot)} u = \mathcal{L}_{s_j} u, \tag{24}$$

since $b_m(x) = \delta_{mj}$. More generally, for a constant order $s(x) \equiv s^*$ not coinciding with a node, the only approximation introduced by (23) is the Chebyshev interpolation error with respect to the parameter σ.

The choice of Chebyshev–Lobatto nodes (18) is particularly convenient because it avoids the large oscillations associated with equispaced high-degree interpolation and permits the weights (21) to be generated explicitly. Furthermore, if the map $\sigma \longmapsto \mathcal{L}_\sigma u$ admits an analytic continuation to a complex neighborhood of I_s, the interpolation error decreases geometrically with M. A precise statement of this property, together with its interaction with the spatial spectral error, will be given in Section 5.

4.1. Discrete Variable-Order Operator

We now combine the order interpolation with the mapped spectral discretization introduced in Section 3.

Let V_N^ν, as defined in (11), denote the Fourier-like mapped Chebyshev approximation space, and $\hat{\boldsymbol{u}} = (\hat{u}_0, \dots, \hat{u}_N)^T$ be the spectral coefficient vector, according to (16).

For each fixed interpolation order s_m, the discrete constant-order fractional operator $\mathsf{D}_N^{(s_m)} : V_N^\nu \longrightarrow V_N^\nu$ is defined through its action on the Fourier-like mapped Chebyshev basis $\{\Phi_p^\nu\}_{p=0}^N$ by

$$\mathsf{D}_N^{(s_m)} \Phi_p^\nu = (\lambda_p^\nu)^{s_m} \Phi_p^\nu, \qquad p = 0, \dots, N,$$

where λ_p^ν are the eigenvalues of the discrete mapped Laplacian.

In its diagonalizing spectral representation, $\mathsf{D}_N^{(s_m)} = \mathsf{T}_N^{-1} \boldsymbol{\Lambda}^{(s_m)} \mathsf{T}_N$, where T_N denotes the relevant mapped spectral transform and $\boldsymbol{\Lambda}^{(s_m)}$ is diagonal. The precise form of $\boldsymbol{\Lambda}^{(s_m)}$ depends on the constant-order realization described in Section 3; importantly, no dense fractional differentiation matrix needs to be assembled for each application of the operator.

Let $\{x_j\}_{j=0}^N$ be the mapped Chebyshev collocation points (13) and define $s_j = s(x_j)$. To avoid confusion between the physical-node index and the order interpolation index, we write

$$B_{jm} = b_m(x_j)\,, \qquad 0 \le j \le N, \quad 0 \le m \le M.$$

with $b_m(x)$ defined by (22). For each order node s_m, compute $v^{(m)} = A_N^{s_m} u_N$ and evaluate this discrete fractional power action at the physical nodes. Its difference from the continuum action $(-\Delta)^{s_m} u_N$ is the realization defect defined in Section 5.

The discrete variable-order fractional Laplacian is then assembled pointwise as

$$[\mathsf{D}_{N,M}^{s(\cdot)} \boldsymbol{u}]_j = \sum_{m=0}^{M} B_{jm}\, v_j^{(m)}, \qquad j = 0, \dots, N.$$

Equivalently, introducing

$$\boldsymbol{B}_m = \mathrm{diag}(B_{0m}, \dots, B_{Nm}),$$

we obtain the matrix-free representation

$$\mathsf{D}_{N,M}^{s(\cdot)} = \sum_{m=0}^{M} \boldsymbol{B}_m \mathsf{D}_N^{(s_m)}. \tag{25}$$

Equation (25) expresses the discrete variable-order action as a sum of $M+1$ constant-order discrete actions followed by diagonal pointwise weighting. The representation can be applied without assembling the full variable-order matrix. Its arithmetic cost, however, depends on the transforms used to apply the constant-order operators; a dense change of eigenbasis entails quadratic cost in N. The nonlinear extension is described in Section 6.

4.2. Matrix-Free Evaluation and Operation Count

A direct discretization of a variable-order fractional Laplacian generally produces a non-translation-invariant dense operator. Explicit assembly of such a matrix requires $O(N^2)$ storage and a comparable cost for each matrix–vector product in one dimension. Moreover, if the hypersingular integral defining the variable-order operator is evaluated separately at all physical nodes, the dependence of the kernel on $s(x_j)$ prevents the direct use of a single global convolution. The interpolated representation (25) avoids this construction. The required order nodes, diagonal constant-order multipliers, and barycentric weights are precomputed once. Each online application then consists of $M+1$ constant-order spectral evaluations followed by their pointwise weighted accumulation, as summarized in Algorithm 1.

Algorithm 1 Matrix-free evaluation of the interpolated variable-order discrete operator

Require: nodal values $\boldsymbol{u}$; order profile $s(x_j)$; order nodes $\{s_m\}_{m=0}^{M}$; interpolation matrix $\boldsymbol{B}$; constant-order diagonal data $\{\boldsymbol{\Lambda}^{(s_m)}\}_{m=0}^{M}$.

Ensure: $\boldsymbol{q} \approx \{(-\Delta)^{s(x_j)} u_N(x_j)\}_{j=0}^{N}$.

1: Set $\boldsymbol{q} = \boldsymbol{0}$.

2: **for** $m = 0, \dots, M$ **do**

3: Apply the chosen constant-order operator

$$\boldsymbol{v}^{(m)} = \mathsf{D}_N^{(s_m)} \boldsymbol{u}.$$

4: Accumulate

$$q_j \leftarrow q_j + B_{jm} v_j^{(m)}, \qquad j = 0, \dots, N.$$

5: **end for**

6: **return** $\boldsymbol{q}$.

Let C_N denote the cost of one constant-order application, including *all* transformations between physical nodal values, mapped Chebyshev coefficients, and the eigenvector basis. Algorithm 1 requires $M+1$ such applications plus $O(MN)$ scalar accumulation operations. A cosine transform between the nodal and mapped Chebyshev representations may cost $O(N \log N)$; however, multiplication by the generally dense eigenvector matrices E and E^{T} costs $O(N^2)$ per vector without additional structure. Thus, $C_N = O(N^2)$ for this generic dense implementation, and the total online work is $O((M+1)N^2)$, apart from any separate nonlinear or Krylov iterations. Explicit storage of E costs $O(N^2)$; storing the interpolation weights costs $O(MN)$. A quasi-linear $O((M+1)N \log N)$ complexity would require an independently demonstrated fast eigenbasis transform and is not claimed here. These bounds refer to application after the eigenpair precomputation, whose cost is not included in online work.

Storing all nodal interpolation weights B_{jm} requires $O(MN)$ memory, in addition to the eigenvector matrix E, which requires $O(N^2)$ storage when kept explicitly. The temporary working vectors can be reused in a streaming implementation, requiring only $O(N)$ additional workspace. Thus, $O(N)$ workspace must not be confused with the total storage of the dense eigenvector realization.

The parameters M and N independently control the order interpolation and spatial discretization, respectively. Their error contributions are quantified in Section 5.

Finally, the proposed construction does not depend on the particular nonlinearity appearing in the underlying PDE. Once the family of constant-order operators $\{\mathsf{D}_N^{(s_m)}\}_{m=0}^M$ has been constructed, the same variable-order operator can be reused for linear elliptic equations, reaction–diffusion systems, and nonlinear phase-field models. In Section 7, this framework will be specialized to the variable-order fractional Allen–Cahn equation.

5. Approximation and Error Analysis

Let P_N denote the mapped Chebyshev projection onto the approximation space V_N (16), i.e.,

$$P_N u = \sum_{p=0}^{N} \left(u, \Phi_p^{\nu}\right)_{L^2(\mathbb{R})} \Phi_p^{\nu}.$$

and $\mathcal{L}_{s(\cdot)}^{(M)}$ the operator interpolant (20).

The total approximation error decomposes as

$$\begin{aligned} \mathcal{L}u - \sum_{m=0}^{M} \ell_m(s(\cdot)) A_N^{s_m} P_N u = {} & \underbrace{[\mathcal{L}u - \mathcal{J}_M \mathcal{L}u]}_{\text{order interpolation error}} + \underbrace{\mathcal{J}_M \mathcal{L}(u - P_N u)}_{\text{spatial spectral error}} \\ & + \underbrace{\sum_{m=0}^{M} \ell_m(s(\cdot))[(-\Delta)^{s_m} - A_N^{s_m}] P_N u}_{\text{discrete operator realization defect}}, \end{aligned} \tag{26}$$

where $\mathcal{J}_M \mathcal{L} v := \sum_{m=0}^{M} \ell_m(s(\cdot))(-\Delta)^{s_m} v$. The three terms are the order interpolation error, the spatial projection error, and the discrete operator realization defect, respectively. This decomposition is useful because the errors originate from completely different mechanisms and can therefore be controlled independently.

5.1. Order Interpolation Error

The first source of error is caused by replacing the continuously varying operator $\mathcal{L}_{s(x)}$ by a finite combination of constant-order operators. For fixed $\xi \neq 0$, consider the function

$$\varphi_\xi(\sigma) = |\xi|^{2\sigma}, \qquad \sigma \in I_s.$$

Its repeated integer order q derivatives with respect to the fractional order σ are given explicitly

$$\frac{\partial^q}{\partial \sigma^q} \varphi_\xi(\sigma) = (2 \log |\xi|)^q |\xi|^{2\sigma}, \qquad q \geq 1. \tag{27}$$

Thus, the symbol is analytic with respect to σ for every fixed $\xi \neq 0$. However, differentiation with respect to the fractional order introduces powers of $\log|\xi|$. This observation is important: regularity with respect to the order variable is therefore connected to logarithmically weighted regularity of the underlying function in Fourier space.

The Fourier-side integrability assumption in Proposition 1 supplies the uniform analytic-majorant required for order interpolation.

Proposition 1 (Uniform order interpolation error)**.** *Let* $0 < a < b < 1$, *let* $s : \mathbb{R} \to [a, b]$, *and let* s_m, $m = 0, \ldots, M$, *with* $M \geq 1$, *be the Chebyshev–Lobatto nodes in* $[a, b]$. Set $\sigma(z) = \frac{a+b}{2} + \frac{b-a}{2} z$.

Assume that there exist a number $\beta > b$ and a Bernstein ellipse E_ρ (including its interior), $\rho > 1$, such that

$$0 < \alpha_\rho := \inf_{z \in E_\rho} \operatorname{Re} \sigma(z) \le \sup_{z \in E_\rho} \operatorname{Re} \sigma(z) =: \gamma_\rho < \beta,$$

and that

$$A_\rho(u) := \frac{1}{2\pi} \int_{\mathbb{R}} \sup_{z \in E_\rho} |\xi|^{2 \operatorname{Re} \sigma(z)} |\hat{u}(\xi)| \, d\xi < \infty.$$

Here the integrand at $\xi = 0$ is defined to be zero. Then the continuum interpolation estimate stated in Proposition 1 holds.

For the pseudodifferential definition $\mathcal{L}_{s(\cdot)} u(x) = (2\pi)^{-1} \int |\xi|^{2s(x)} \hat{u}(\xi) e^{ix\xi} \, d\xi$, and the continuum operator interpolant $\mathcal{I}_M \mathcal{L} u = \sum_{m=0}^{M} \ell_m(s(x)) \mathcal{L}_{s_m} u$, one has

$$\| \mathcal{L}_{s(\cdot)} u - \mathcal{I}_M \mathcal{L} u \|_{L^\infty(\mathbb{R})} \le \frac{4 A_\rho(u)}{\rho - 1} \rho^{-M}.$$

Proof. For each fixed $\xi \neq 0, f_\xi(z) = |\xi|^{2\sigma(z)}$ is entire in z. The ellipse condition is imposed on the entire complex ellipse, not merely on its real diameter. With $c_\rho = (\rho + \rho^{-1})/2$, the extrema are explicit:

$$\alpha_\rho = \frac{a+b}{2} - \frac{b-a}{2} c_\rho, \qquad \gamma_\rho = \frac{a+b}{2} + \frac{b-a}{2} c_\rho.$$

Since $\alpha_\rho \to a > 0$ and $\gamma_\rho \to b < \beta$ as $\rho \downarrow 1$, an admissible ellipse exists for every fixed $\beta > b$. The integrability of the uniform majorant is an additional assumption of this proposition, verified by the criterion in Proposition 2. The standard Chebyshev–Lobatto interpolation inequality for functions analytic in $\mathcal{E}_\rho$ gives $\sup_{t \in [-1,1]} |f_\xi(t) - I_M f_\xi(t)| \le 4\rho^{-M}(\rho - 1)^{-1} \sup_{z \in \mathcal{E}_\rho} |f_\xi(z)|$. At $\xi = 0$, the symbol is set to zero (all orders have a positive real part). Substitute $t = \frac{2s(x) - a - b}{b - a}$, multiply by $|\hat{u}(\xi)|$, integrate in ξ, and use the triangle inequality. The majorant $A_\rho(u)$ is independent of x, so taking the supremum in x proves the result. No composition theorem for negative Sobolev norms is invoked. □

Proposition 2 (A practical criterion for geometric order convergence)**.** *Let $0 < a < b < 1$, and suppose $u \in L^1(\mathbb{R})$ has a Fourier transform satisfying*

$$\int_{|\xi| \le 1} |\hat{u}(\xi)| \, d\xi + \int_{|\xi| > 1} |\xi|^{2\beta} |\hat{u}(\xi)| \, d\xi < \infty \tag{28}$$

for some $\beta > b$. Then there exists $\rho > 1$ such that the ellipse hypothesis and $A_\rho(u) < \infty$ of Proposition 1 hold. Consequently, the order interpolation error decays geometrically in M.

Proof. Fix $\beta > b$ from condition (28) and let $c_\rho = (\rho + \rho^{-1})/2$. Because $c_\rho \downarrow 1$ as $\rho \downarrow 1$, one may choose $\rho > 1$ so close to 1 that

$$0 < \frac{a+b}{2} - \frac{b-a}{2} c_\rho \le \frac{a+b}{2} + \frac{b-a}{2} c_\rho < \beta.$$

Thus, the above ellipse condition holds throughout the closed ellipse. For $0 < |\xi| \le 1$, positivity of α_ρ implies $\sup_{z \in E_\rho} |\xi|^{2 \operatorname{Re} \sigma(z)} \le 1$; for $|\xi| > 1$, the upper bound $\gamma_\rho < \beta$ yields $\sup_{z \in E_\rho} |\xi|^{2 \operatorname{Re} \sigma(z)} \le |\xi|^{2\beta}$. Consequently,

$$A_\rho(u) \le \frac{1}{2\pi} \left(\int_{|\xi| \le 1} |\hat{u}(\xi)| \, d\xi + \int_{|\xi| > 1} |\xi|^{2\beta} |\hat{u}(\xi)| \, d\xi \right) < \infty.$$

Proposition 1 then gives geometric convergence in the order interpolation degree M. □

https://doi.org/10.3390/axioms15110860

Theorem 1 (Lipschitz stability with respect to the fractional-order profile)**.** *Let* $0 < a \le s(x), t(x) \le b < 1$ *and suppose*

$$B_{a,b}(u) := \frac{1}{\pi}\int_{\mathbb{R}} |\hat{u}(\xi)|\,|\log|\xi||\begin{cases}|\xi|^{2a}, & 0<|\xi|\le 1,\\ |\xi|^{2b}, & |\xi|>1\end{cases} d\xi < \infty. \tag{29}$$

Then

$$\left\|\mathcal{L}_{s(\cdot)}u - \mathcal{L}_{t(\cdot)}u\right\|_{L^\infty(\mathbb{R})} \le B_{a,b}(u)\|s-t\|_{L^\infty(\mathbb{R})}. \tag{30}$$

Proof. For $\xi \neq 0$, the mean-value theorem applied to $\sigma \mapsto |\xi|^{2\sigma}$ gives

$$\left||\xi|^{2s(x)} - |\xi|^{2t(x)}\right| \le 2|s(x)-t(x)|\,|\log|\xi|| \max_{\sigma\in[a,b]} |\xi|^{2\sigma}.$$

The maximum is $|\xi|^{2a}$ for $|\xi| \le 1$ and $|\xi|^{2b}$ for $|\xi| > 1$. Multiply by $|\hat{u}(\xi)|/(2\pi)$, integrate, and take the supremum in x. □

Corollary 1 (Perturbation of a sampled order profile)**.** *Under the hypotheses of the theorem, if* $\tilde{s}$ *is an approximation to* s *taking values in* $[a,b]$*, then the modeling error caused by using* $\tilde{s}$ *is at most* $B_{a,b}(u)\|s-\tilde{s}\|_\infty$*. In particular, the order interpolation in the manuscript does* not *require interpolation of* s *itself; if a sampled or measured order field is used, its error can be controlled separately.*

Corollary 2 (Interpolation with an approximate order profile)**.** *Under the hypotheses of Proposition 1 and Theorem 1, suppose* $s, \tilde{s} : \mathbb{R} \to [a,b]$*. Let* $\mathcal{J}_M\mathcal{L}_{\tilde{s}(\cdot)}u$ *be the Chebyshev order interpolant evaluated at* $\tilde{s}(x)$*. Then*

$$\left\|\mathcal{L}_{s(\cdot)}u - \mathcal{J}_M\mathcal{L}_{\tilde{s}(\cdot)}u\right\|_{L^\infty(\mathbb{R})} \le B_{a,b}(u)\|s-\tilde{s}\|_{L^\infty(\mathbb{R})} + \frac{4A_\rho(u)}{\rho-1}\rho^{-M}.$$

Proof. Insert and subtract $\mathcal{L}_{\tilde{s}(\cdot)}u$ and apply Theorem 1 and Proposition 1. □

Corollary 3 (Number of interpolation operators)**.** *Under the hypotheses of Proposition 1, suppose* $A_\rho(u) > 0$*. For* $\varepsilon_{\mathrm{ord}} > 0$*, it is sufficient to choose an integer* $M \ge 1$*, satisfying*

$$M \ge \max\left\{1, \left\lceil \frac{\log\left(4A_\rho(u)/((\rho-1)\varepsilon_{\mathrm{ord}})\right)}{\log\rho}\right\rceil\right\}.$$

$$M + 1 = O\left(1 + \log\frac{1}{\varepsilon_{\mathrm{ord}}}\right) \qquad (\varepsilon_{\mathrm{ord}} \downarrow 0).$$

5.2. Mapped Spectral Approximation

We now analyze the error introduced by the Fourier-like mapped Chebyshev approximation in the spatial variable. Let V_N denote the mapped Chebyshev approximation space introduced in Section 3, and let $P_N : L^2(\mathbb{R}) \to V_N$ be the associated L^2-orthogonal projection.

The estimate below is conditional on the strong-Sobolev projection error $u - P_N u \in H^r(\mathbb{R})$. An explicit convergence rate in N requires an independently proved mapped-projection estimate in this norm.

Proposition 3 (Spatial approximation in a strong norm)**.** *Let* $a \le s(x) \le b$*,* $0 < a < b < 1$*, and* $r > 2b + \frac{1}{2}$*. Let* $e_N = u - P_N u \in H^r(\mathbb{R})$*. Then*

$$\|\mathcal{J}_M\mathcal{L}e_N\|_{L^\infty(\mathbb{R})} \le C_{r,b}\Lambda_M\|e_N\|_{H^r(\mathbb{R})}, \qquad \Lambda_M = \sup_{\sigma\in[a,b]}\sum_{m=0}^{M}|\ell_m(\sigma)| = O(\log(M+1)).$$

Proof. For each x, the triangle inequality yields $|\mathcal{J}_M \mathcal{L} e_N(x)| \le \sum_m |\ell_m(s(x))| \, \|\mathcal{L}_{s_m} e_N\|_\infty$. By Fourier inversion and Cauchy–Schwarz, uniformly for $s_m \in [a, b]$,

$$\begin{aligned} \|\mathcal{L}_{s_m} e_N\|_\infty &\le \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{2s_m} |\widehat{e}_N(\xi)| \, d\xi \\ &\le C_{r,b} \|e_N\|_{H^r}, \\ C_{r,b} &= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} \frac{\max(1, |\xi|^{4b})}{(1+|\xi|^2)^r} \, d\xi \right)^{1/2} < \infty. \end{aligned}$$

The Lebesgue constant controls the scalar sum at the *same point* x, and taking $\sup_x$ completes the proof. □

Proposition 3 is a continuity estimate, not an approximation theorem [13] for the L^2 projection P_N. In particular, membership of $u - P_N u$ in $H^r(\mathbb{R})$ for each N does not imply $\|u - P_N u\|_{H^r} \to 0$. If a separate approximation result supplies numbers $\varepsilon_N(u) \to 0$ with

$$\|u - P_N u\|_{H^r(\mathbb{R})} \le \varepsilon_N(u), \qquad r > 2b + \tfrac{1}{2},$$

then the spatial term in Proposition 3 is at most $C_{r,b} \Lambda_M \varepsilon_N(u)$. One possible route to establishing this Equation would be to prove both density of $\bigcup_N V_N^\nu$ in $H^r(\mathbb{R})$ and uniform H^r-boundedness of the particular L^2 projectors P_N; neither property is proved here. Therefore, convergence of this mapped L^2 projection in the required strong norm, and any rate in N, remain open for the present analysis. An estimate solely in $H^{s_{\max}}$ would not suffice.

Theorem 2 (Conditional consistency estimate)**.** *Assume the hypotheses of Propositions 1 and 3. Suppose also that $P_N u$ belongs to the domain of the relevant continuum operators and that all realization defects*

$$\delta_N(\sigma; P_N u) := (-\Delta)^\sigma (P_N u) - A_N^\sigma (P_N u) \qquad \sigma \in \{s_0, \dots, s_M\}$$

are bounded pointwise for $\sigma = s_m$. Then

$$\begin{aligned} \left\| \mathcal{L} u - \sum_{m=0}^{M} \ell_m(s(\cdot)) A_N^{s_m} P_N u \right\|_{L^\infty} \le{}& \frac{4A_\rho(u)}{\rho - 1} \rho^{-M} + C_{r,b} \Lambda_M \|u - P_N u\|_{H^r} \\ &+ \Lambda_M \max_{0 \le m \le M} \|\delta_N(s_m; P_N u)\|_{L^\infty}. \end{aligned}$$

Proof. Insert and subtract $\mathcal{J}_M \mathcal{L} u$ and $\mathcal{J}_M \mathcal{L}(P_N u)$; apply Propositions 1 and 3 and the Lebesgue constant bound. □

Theorem 2 controls the consistency error of the operator applied to a projected test function; it does not prove convergence of solutions of the fully discrete nonlinear PDE. More precisely, the operator consistency bound contains (i) the order interpolation term $4A_\rho(u)\rho^{-M}/(\rho - 1)$, (ii) the unproved strong projection term $C_{r,b}\Lambda_M \|u - P_N u\|_{H^r}$, and (iii) the discrete realization term $\Lambda_M \max_{0 \le m \le M} \|\delta_N(s_m; P_N u)\|_\infty$. Only term (i) is shown here to vanish as $M \to \infty$ under the stated analytic-majorant assumptions. For fixed M, convergence of terms (ii) and (iii) as $N \to \infty$ would require independent estimates. A joint limit $M = M(N) \to \infty$ would additionally require their rates to dominate the growth of the Lebesgue factor $\Lambda_M = O(\log(M + 1))$. Finally, even vanishing operator consistency error does not imply convergence of the solution of the nonlinear collocation equations without an appropriate stability/invertibility theory and control of nonlinear residual and solver errors. None of those additional conditions is proved in this paper; no unconditional complete-method convergence theorem or quasi-linear complexity statement is asserted.

6. Nonlinear Solver

We consider the nonlinear problems

$$\mathcal{L}_{s(\cdot)}u + F(x,u) = g$$

and discretize the reaction term pseudospectrally at the mapped nodes. The variable-order contribution is applied matrix-free using the operator constructed in Section 4.

6.1. Pseudospectral Discretization

Let $V_N = \mathrm{span}\{\Phi_0, \Phi_1, \dots, \Phi_N\}$ denote the mapped Chebyshev approximation space introduced in Section 3, and let $u_N(x) = c\sum_{n=0}^{N} \hat{u}_n \Phi_n(x)$. We denote by $\mathcal{T}_N$ and $\mathcal{T}_N^{-1}$ the forward and inverse mapped spectral transforms, respectively.

Let $\{x_q\}_{q=0}^{N}$ be the mapped collocation nodes (13) associated with the Fourier-like mapped Chebyshev representation, and define

$$U_q = u_N(x_q), \qquad \mathbf{U} = (U_0, \dots, U_N)^T.$$

The nodal and spectral representations are related by

$$\hat{\mathbf{u}} = \mathcal{T}_N \mathbf{U}, \qquad \mathbf{U} = \mathcal{T}_N^{-1}\hat{\mathbf{u}}.$$

The discrete differential operator acting on the nodal vector U can be determined as follows. For every fixed s_m, the mapped Chebyshev discretization provides a diagonal spectral representation

$$\mathcal{L}_{s_m,N} = \mathcal{T}_N^{-1}\Lambda^{(m)}\mathcal{T}_N,$$

where $\Lambda^{(m)} = \mathrm{diag}((\lambda_0^{\nu})^{s_m}, \dots, (\lambda_N^{\nu})^{s_m})$ is the fractional power of the *discrete* mapped Laplacian eigenvalue matrix (14).

Introducing the diagonal matrices

$$D_m = \mathrm{diag}(\ell_m(s(x_0)), \dots, \ell_m(s(x_N))),$$

according to the operator interpolation (19), the discrete variable-order operator can therefore be written as

$$\mathcal{L}_{M,N} = \sum_{m=0}^{M} D_m \mathcal{T}_N^{-1}\Lambda^{(m)}\mathcal{T}_N. \tag{31}$$

Expression (31) is not diagonal because the spatially varying matrices D_m do not commute with the spectral transforms. Nevertheless, its action on a vector can be evaluated without forming $\mathcal{L}_{M,N}$ explicitly. This observation permits matrix-free application of the discrete operator, although the cost of a generic dense eigenbasis transformation remains quadratic in N.

The nonlinear term $F(x,u)$ is evaluated directly at the physical nodes:

$$\mathbf{F}(\mathbf{U}) = (F(x_0, U_0), \dots, F(x_N, U_N))^T.$$

Hence, no third- or fourth-order Galerkin tensors are required. For a cubic nonlinearity, for example, U^3 is evaluated pointwise and transformed only when a spectral representation is required.

This pseudospectral treatment is particularly advantageous for Allen–Cahn-type nonlinearities [6]. For

$$F(u) = \varepsilon^{-2}(u^3 - u), \tag{32}$$

we simply evaluate

$$F(U_q) = \varepsilon^{-2}\left(U_q^3 - U_q\right), \qquad q = 0, \dots, N$$

and

$$F'(U_q) = \varepsilon^{-2}\big(3U_q^2 - 1\big), \qquad q = 0, \dots, N$$

whereas for the more general asymmetric reaction

$$F(u) = \alpha u(1-u) + \beta(u^3 - u),$$

it is

$$F'(U_q) = \alpha(1 - 2U_q) + \beta(3U_q^2 - 1).$$

6.2. Residual and Jacobian

Let $\mathbf{G} = (g(x_0), \dots, g(x_N))^T$. The fully discrete nonlinear system is

$$\mathbf{R}(\mathbf{U}) = \mathcal{L}_{M,N}\mathbf{U} + \mathbf{F}(\mathbf{U}) - \mathbf{G} = \mathbf{0}. \tag{33}$$

At a Newton iterate $\mathbf{U}^{(k)}$, the Jacobian associated with (33) is

$$J(\mathbf{U}^{(k)}) = \mathcal{L}_{M,N} + D_F(\mathbf{U}^{(k)}), \tag{34}$$

where

$$D_F(\mathbf{U}) = \operatorname{diag}(F_u(x_0, U_0), \dots, F_u(x_N, U_N)).$$

In particular, for the symmetric Allen–Cahn nonlinearity (32),

$$D_F(\mathbf{U}) = \varepsilon^{-2} \operatorname{diag}\big(3U_0^2 - 1, \dots, 3U_N^2 - 1\big) .$$

A key feature of (34) is that neither $\mathcal{L}_{M,N}$ nor the full Jacobian needs to be assembled. For any vector $\mathbf{v}$, the Jacobian–vector product is

$$J(\mathbf{U})\mathbf{v} = \sum_{m=0}^{M} D_m \widehat{\mathcal{I}}_N^{-1} \Lambda^{(m)} \widehat{\mathcal{I}}_N \mathbf{v} + D_F(\mathbf{U})\mathbf{v}. \tag{35}$$

Thus, a Jacobian–vector product requires only diagonal multiplications, forward and inverse mapped transforms, and pointwise operations. Because F_u is available analytically for the nonlinearities considered here, we use the exact matrix-free Jacobian–vector product (35).

6.3. Newton–Krylov Iteration and Preconditioning

Starting from an initial approximation $\mathbf{U}^{(0)}$, Newton's method generates a sequence $\{\mathbf{U}^{(k)}\}$ by solving

$$J(\mathbf{U}^{(k)})\delta\mathbf{U}^{(k)} = -\mathbf{R}(\mathbf{U}^{(k)}) \tag{36}$$

and setting

$$\mathbf{U}^{(k+1)} = \mathbf{U}^{(k)} + \omega_k \delta\mathbf{U}^{(k)}, \qquad 0 < \omega_k \le 1.$$

When the Newton direction is well defined and is a descent direction for $\frac{1}{2}\|R(U)\|_2^2$, a backtracking line search may be used to seek a reduction in the residual norm. Global convergence is not established.

For large N, the Newton system may be solved using a Krylov subspace method such as GMRES. Its matrix–vector products can be formed using the Jacobian vector formula (35), without explicitly storing the dense Jacobian; no iteration count or convergence guarantee is asserted here.

The efficiency of the Newton–Krylov method depends strongly on an appropriate preconditioner. The operator interpolation formulation suggests a natural choice. Let $s_* = \dfrac{1}{2}(s_{\min} + s_{\max})$ or, alternatively, let s_* denote a suitable spatial average of $s(x)$. We

define the frozen-order fractional operator $A_* = \mathcal{J}_N^{-1}\Lambda^{(s_*)}\mathcal{J}_N$. The nonlinear diagonal part may similarly be represented by a scalar shift

$$d_k = \frac{\sum_{q=0}^{N} w_q F_u(x_q, U_q^{(k)})}{\sum_{q=0}^{N} w_q},$$

where w_q are the mapped quadrature weights. A simple preconditioner is then

$$P_k = A_* + (d_k + \sigma_k)I, \tag{37}$$

where $\sigma_k \geq 0$ is a stabilizing shift chosen, if necessary, to keep the inverse of P_k uniformly well defined.

The crucial advantage of (37) is that its inverse is diagonal in spectral space:

$$P_k^{-1}\mathbf{r} = \mathcal{J}_N^{-1}\big(\Lambda^{(s_*)} + (d_k + \sigma_k)I\big)^{-1}\mathcal{J}_N\mathbf{r}.$$

Consequently, the application of P_k^{-1} requires one forward transform, one inverse transform, and a diagonal division. No matrix factorization is required. No mesh-independent preconditioning bound or iteration-complexity estimate is established here.

7. Variable-Order Fractional Allen–Cahn Problem

The stationary variable-order fractional Allen–Cahn equation is used here to illustrate the formulation of a nonlinear residual and the treatment of non-decaying far-field states. No numerical validation or existence theorem for heterogeneous layers is asserted in this section.

Throughout this section, $\mathcal{L}_{s(\cdot)}$ denotes the variable-order operator (6). The definition (2) is particularly convenient in the present context because the dependence on the spatial variable enters through the symbol $|\xi|^{2s(x)}$. Consequently, interpolation in the order parameter reduces the continuum operator to a weighted sum of constant-order actions, whose discrete evaluation cost depends on the chosen spectral transforms.

We consider the stationary variable-order fractional Allen–Cahn equation

$$\mathcal{L}_{s(\cdot)}u(x) + \frac{1}{\varepsilon^2}W'(u(x)) = 0, \qquad x \in \mathbb{R}, \tag{38}$$

where $\varepsilon > 0$ controls the characteristic width of the transition region and W is a double-well potential. Unless stated otherwise, we use the classical quartic potential

$$W(u) = \frac{1}{4}(1-u^2)^2, \qquad W'(u) = u^3 - u.$$

Equation (38) therefore becomes

$$\mathcal{L}_{s(\cdot)}u(x) + \frac{1}{\varepsilon^2}\big(u^3(x) - u(x)\big) = 0, \qquad x \in \mathbb{R}. \tag{39}$$

Of particular interest are interface-type solutions connecting the two stable phases,

$$\lim_{x\to-\infty} u(x) = -1, \qquad \lim_{x\to+\infty} u(x) = 1.$$

For a constant order s, these conditions define the classical fractional Allen–Cahn layer problem [6]. In the variable-order setting, the same far-field states are retained while the mechanism controlling the transition between them varies spatially through $s(x)$.

Write $u = h + v$, where $h(x) = \tanh((x - x_0)/\ell)$ is bounded and v is the decaying unknown. The lifting is not in $L^2(\mathbb{R})$, so it must not be inserted into an L^2-only Fourier–Galerkin routine without an extension of its domain. For $0 < \sigma < 1$, the continuum

fractional Laplacian of the smooth bounded lifting can instead be defined by the convergent principal-value integral

$$(-\Delta)^{\sigma} h(x) = c_{1,\sigma}\,\mathrm{PV}\int_{\mathbb{R}} \frac{h(x)-h(y)}{|x-y|^{1+2\sigma}}\,dy, \qquad c_{1,\sigma} = \frac{4^{\sigma}\Gamma(\frac{1}{2}+\sigma)}{\sqrt{\pi}|\Gamma(-\sigma)|}.$$

For the bounded smooth lifting h, we define $\mathcal{L}_{s(\cdot)}h(x)$ pointwise by the constant-order principal-value formula at $\sigma = s(x)$. Compatibility of this extension with the Fourier pseudodifferential definition on Schwartz functions follows from the corresponding constant-order identity, but a full mapping theory on nondecaying functions is not developed here. Define $f_h(x) = \mathcal{L}_{s(\cdot)}h(x)$ independently of the mapped L^2 basis. The lifted residual is

$$R(v) = \mathcal{L}_{s(\cdot)}v + f_h + \varepsilon^{-2}[(h+v)^3 - (h+v)].$$

At the discrete level, the unknown is the decaying correction $v_N \in V_N^{\nu}$, not the non-decaying total profile $u = h + v_N$. The nodal nonlinear residual must therefore be formed as

$$R_{N,M}(V) = L_{N,M}V + F_{h,N} + \varepsilon^{-2}[(H+V)^{\circ 3} - (H+V)],$$

where $H_j = h(x_j)$ and $F_{h,N}$ approximates $\mathcal{L}h(x_j)$ independently of the L^2 mapped basis. The approximation of $F_{h,N}$ is a separate step; its quadrature and interpolation errors are not controlled by Theorem 1 or by the conditional consistency estimate (Theorem 2). The lifting contribution $f_h = \mathcal{L}h$ is well defined in the continuum through the constant-order principal-value formula evaluated at $\sigma = s(x)$. Any discretization of f_h by order interpolation, quadrature, or truncation introduces an additional error that is not estimated in the present work.

8. An Analytical Illustration of Order Interpolation

Consider $u(x) = e^{-x^2}$ and the variable-order profile

$$s(x) = 0.55 + 0.25\tanh(x/2), \qquad 0.30 < s(x) < 0.80.$$

For each constant order $\sigma \in (0,1)$, the fractional Laplacian is given by

$$(-\Delta)^{\sigma} e^{-x^2} = 4^{\sigma}\frac{\Gamma(\sigma+\frac{1}{2})}{\sqrt{\pi}}\,{}_1F_1\Big(\sigma+\tfrac{1}{2};\tfrac{1}{2};-x^2\Big).$$

Consequently, the variable-order operator defined in Section 2 is obtained by substituting $\sigma = s(x)$ in this expression. Its order-interpolated approximation is

$$\mathcal{I}_M\mathcal{L}u(x) = \sum_{m=0}^{M} \ell_m(s(x))\,4^{s_m}\frac{\Gamma(s_m+\frac{1}{2})}{\sqrt{\pi}}\,{}_1F_1\Big(s_m+\tfrac{1}{2};\tfrac{1}{2};-x^2\Big).$$

This identity illustrates that the order interpolation can be formed from a finite set of constant-order expressions. Its error is governed by Proposition 1 under the stated hypotheses. No numerical convergence or runtime claim is inferred from this example.

An Algebraically Decaying Analytical Example

Let $u_{\alpha}(x) = (1+x^2)^{-\alpha}$, $\alpha > 1/4$, so that $u_{\alpha} \in L^2(\mathbb{R})$ and the mapped L^2 framework is applicable. The closed-form continuum expression itself also makes sense under broader conditions. For the Fourier-multiplier fractional Laplacian, the exact constant-order expression is

$$(-\Delta)^{\sigma} u_{\alpha}(x) = \frac{4^{\sigma}\Gamma(\sigma+\frac{1}{2})\Gamma(\alpha+\sigma)}{\sqrt{\pi}\,\Gamma(\alpha)}\,{}_2F_1\Big(\sigma+\tfrac{1}{2},\alpha+\sigma;\tfrac{1}{2};-x^2\Big).$$

Consequently, for the left-quantized variable-order definition, the exact expression is obtained by evaluating the right-hand side at $\sigma = s(x)$. The interpolated expression is obtained by evaluating the same closed form at the Chebyshev–Lobatto orders s_m and taking the weighted sum $\sum_{m=0}^{M} \ell_m(s(x))(-\Delta)^{s_m} u_\alpha(x)$. This illustrates the continuum interpolation independently of any mapped discrete fractional power. When the Fourier weighted-integrability criterion of Proposition 2 holds, geometric interpolation convergence follows. No spatial discretization error or runtime is inferred.

9. Conclusions

We have analyzed Chebyshev interpolation with respect to the fractional-order parameter for the left-quantized variable-order fractional Laplacian on the real line. The construction represents the spatially dependent operator through a finite family of constant-order actions and scalar interpolation weights. Under a Fourier-side analytic-majorant assumption, we proved a geometric uniform norm interpolation estimate; a weighted Fourier-integrability criterion provides a direct sufficient condition for this result. We also established Lipschitz stability with respect to perturbations of the variable-order profile, which separates uncertainty in the order field from interpolation error. Together, these estimates clarify the accuracy of the continuum operator approximation and show that a prescribed order interpolation tolerance requires only logarithmically many constant-order contributions. For a mapped Chebyshev discretization, we identified three distinct error mechanisms: continuum order interpolation, spatial projection, and the discrepancy between continuum fractional Laplacians and fractional powers of a discrete mapped Laplacian. The resulting consistency estimate is conditional on strong-norm spatial approximation and control of this realization defect. The nonlinear residual and the lifting of Allen–Cahn far-field states indicate how the construction can be incorporated into model equations; exact Gaussian and algebraically decaying examples illustrate its continuum action without relying on unverified numerical rates. The principal remaining analytical tasks are to prove quantitative mapped-projection and realization defect estimates and to establish stability and convergence of the fully discrete nonlinear problem. Reproducible numerical comparisons and an analysis of the cost of the required eigenbasis transformations are also needed before stronger claims about solver accuracy or computational complexity can be made.

Author Contributions: Conceptualization, C.C.; methodology, C.C.; formal analysis, C.C.; writing—original draft preparation, C.C.; writing—review and editing, C.C.; supervision, C.C.; resources, Y.G.; methodology, Y.G.; writing—original draft preparation, Y.G.; writing—review and editing, Y.G.; funding acquisition, Y.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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