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$L^2 \times L^2 \to L^1$ Boundedness of the Bilinear Marcinkiewicz Integral Operator with Rough Kernels

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Abstract

We study the bilinear Marcinkiewicz integral operator associated with a rough, integrable kernel Ω on $\mathbb{S}^{2n-1}$. For every $n \geq 1$, we prove that the operator maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ boundedly into $L^1(\mathbb{R}^n)$ whenever $\Omega \in L^q(\mathbb{S}^{2n-1})$, $2 \leq q \leq \infty$, satisfies the mean-zero condition. The proof uses a parameter-dependent product wavelet decomposition adapted to the square-function structure. To the best of our knowledge, this is the first boundedness result for the bilinear Marcinkiewicz integral operator with rough angular kernels, without any angular smoothness assumption.

Keywords: bilinear Marcinkiewicz integral; rough kernels; L^p boundedness; square functions; wavelet decomposition

MSC: 42B20; 42B25

1. Introduction

Singular integrals and Littlewood–Paley square functions are fundamental objects in harmonic analysis. Their boundedness depends on the interplay between cancellation, kernel singularity, and angular regularity. Since the foundational work of Calderón and Zygmund [1,2], the study of integral operators with rough kernels has attracted considerable attention.

A classical example is the Marcinkiewicz integral. Let $n \geq 2$, and let $\omega \in L^1(\mathbb{S}^{n-1})$ satisfy

$$\int_{\mathbb{S}^{n-1}} \omega(\theta)\, d\sigma(\theta) = 0. \tag{1}$$

The Marcinkiewicz integral associated with ω is defined by

$$\mu_\omega^{\mathrm{lin}} f(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y| \leq 2^t} \frac{\omega(y')}{|y|^{n-1}} f(x-y)\, dy \right|^2 \frac{dt}{2^{2t}} \right)^{1/2}, \qquad y' = y/|y|.$$

Stein [3] proved that this operator is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$ and is of weak type $(1,1)$ when $\omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{n-1})$ with $0 < \alpha \leq 1$. Benedek, Calderón, and Panzone [4] subsequently established boundedness for all $1 < p < \infty$ under the assumption $\omega \in C^1(\mathbb{S}^{n-1})$. Later developments substantially weakened the angular regularity requirements. Ding, Fan, and Pan [5] proved $L^p(\mathbb{R}^n)$ boundedness for all $1 < p < \infty$ when $\omega \in H^1(\mathbb{S}^{n-1})$, and Al-Salman, Al-Qassem, Cheng, and Pan [6] obtained the same conclusion under the condition $\omega \in L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$. Here the notation $h \in L(\log^+ L)^\alpha(\mathbb{S}^{n-1})$, for $\alpha > 0$, means that

$$\int_{\mathbb{S}^{n-1}} |h(\theta)| \big(1 + \log^+ |h(\theta)|\big)^\alpha\, d\sigma(\theta) < \infty,$$

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where $\log^+ t = \max\{\log t, 0\}$. The case $\alpha = 1$ is denoted by $L\log^+ L(\mathbb{S}^{n-1})$.

The Marcinkiewicz integral is closely related to the rough Calderón–Zygmund singular integral

$$T_\omega^{\mathrm{lin}} f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\omega(y')}{|y|^n} f(x-y)\, dy,$$

where p. v. denotes the Cauchy principal value. The operators $\mu_\omega^{\mathrm{lin}}$ and T_ω^{lin} involve the same angular cancellation (1), but their radial structures differ. The kernel inside the truncated integral defining $\mu_\omega^{\mathrm{lin}}$ has order $|y|^{1-n}$, whereas the singular integral kernel has order $|y|^{-n}$. Thus the former has a weaker singularity inside each truncation, while the outer square-function norm introduces an additional scale-dependent structure.

For T_ω^{lin}, Calderón and Zygmund [2] established $L^p(\mathbb{R}^n)$ boundedness for $1 < p < \infty$ when $\omega \in L\log^+ L(\mathbb{S}^{n-1})$. The space $L\log^+ L(\mathbb{S}^{n-1})$ is strictly contained in $L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$, the kernel class in the result for $\mu_\omega^{\mathrm{lin}}$ cited above. Connett [7] and Ricci and Weiss [8] obtained the same conclusion under the condition $\omega \in H^1(\mathbb{S}^{n-1})$. Thus the H^1 condition yields L^p boundedness for both T_ω^{lin} and $\mu_\omega^{\mathrm{lin}}$ throughout the range $1 < p < \infty$. Grafakos and Stefanov [9] established L^p bounds for T_ω^{lin} under a logarithmic integrability condition on the kernel. Chen, Fan, and Pan [10] obtained related L^p bounds for the Marcinkiewicz integral $\mu_\omega^{\mathrm{lin}}$ for kernels in the Grafakos–Stefanov class. These similar results further illustrate the connection between the two operators and motivate the investigation of their bilinear counterparts.

Let $n \geq 1$, and let $\Omega \in L^1(\mathbb{S}^{2n-1})$ satisfy

$$\int_{\mathbb{S}^{2n-1}} \Omega((y,z)')\, d\sigma((y,z)') = 0. \tag{2}$$

The bilinear rough singular integral associated with Ω is defined by

$$T_\Omega(f,g)(x) = \mathrm{p.v.} \int_{\mathbb{R}^{2n}} \frac{\Omega((y,z)')}{|(y,z)|^{2n}} f(x-y) g(x-z)\, dy\, dz,$$

where p. v. denotes the Cauchy principal value and $(y,z)' = (y,z)/|(y,z)|$. Coifman and Meyer [11] introduced this operator and established boundedness results under regularity assumptions. In dimension $n = 1$, they obtained $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^p(\mathbb{R})$ boundedness for $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$, when Ω has bounded variation on $\mathbb{S}^1$. For Lipschitz kernels in higher dimensions, the multilinear Calderón–Zygmund theory of Grafakos and Torres [12] yields the boundedness of T_Ω from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ throughout the range

$$1 < p_1, p_2 < \infty, \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{2} < p < \infty. \tag{3}$$

The rough-kernel case poses additional challenges. Grafakos, He, and Honzík [13] proved that, for mean-zero $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $2 \leq q \leq \infty$, T_Ω is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, and also from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for exponent triples $(1/p_1, 1/p_2, 1/p)$ in a neighborhood of $(1/2, 1/2, 1)$. When $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, they further established the boundedness of T_Ω throughout the same full exponent range stated in (3).

Their proof begins with a dyadic decomposition of T_Ω into operators denoted by T_j in their paper. For $j < 0$, the Coifman–Meyer multiplier theorem provides summable bounds for T_j. For large positive j, the decaying $L^2 \times L^2 \to L^1$ estimate is obtained by decomposing the multiplier of T_j into compactly supported product wavelets and exploiting combinatorial grouping and frequency orthogonality. To reach the full exponent range for bounded kernels, they also verified that, for $j \geq 0$, the kernel of T_j satisfies the bilinear Calderón–Zygmund size and Hölder continuity estimates with kernel constant

$$A_j \leq C_{n,\varepsilon} \|\Omega\|_{L^\infty(\mathbb{S}^{2n-1})} 2^{j\varepsilon}, \qquad 0 < \varepsilon < 1.$$

Combining these kernel estimates with the $L^2 \times L^2 \to L^1$ bound, they applied the multilinear Calderón–Zygmund theorem of Grafakos and Torres [12] to obtain

$$\|T_j\|_{L^{p_1}\times L^{p_2}\to L^p} \le C\big(A_j + \|T_j\|_{L^2\times L^2\to L^1}\big), \qquad 1 < p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Since ε can be chosen arbitrarily small, interpolation with the decaying $L^2 \times L^2 \to L^1$ estimate yields sufficient decay for large positive j. Together with the bounds for $j < 0$, this allows the operators T_j to be summed throughout the full exponent range.

Subsequent work weakened the angular integrability assumptions in the bilinear setting and extended the theory to multilinear operators. In the bilinear case, He and Park [14] proved $L^{p_1}\times L^{p_2} \to L^p$ boundedness of T_Ω for $1 < p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2$, assuming that $\Omega \in L^q(\mathbb{S}^{2n-1})$ has mean zero and $q > \max\{4/3, p/(2p-1)\}$. For rough m-linear singular integrals with mean-zero angular kernels in $L^q(\mathbb{S}^{mn-1})$, Grafakos, He, Honzík, and Park [15] established boundedness in the optimal open range of input exponents for $2 \le q \le \infty$, and Dosidis and Slavíková [16] extended the boundedness result to $1 < q < 2$. When $m = 2$, these results imply that T_Ω maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ boundedly into $L^1(\mathbb{R}^n)$ whenever Ω has mean zero and belongs to $L^q(\mathbb{S}^{2n-1})$ for some $q > 1$.

Motivated by these results and the relationship between singular integrals and Marcinkiewicz integrals, we study the corresponding bilinear square function. For $\Omega \in L^1(\mathbb{S}^{2n-1})$ satisfying (2), define

$$\mu_\Omega(f,g)(x) = \left(\int_{-\infty}^{\infty} |F_{\Omega,t}(f,g)(x)|^2 \frac{dt}{2^{2t}}\right)^{1/2},$$

where

$$F_{\Omega,t}(f,g)(x) = \int_{|(y,z)|\le 2^t} \frac{\Omega((y,z)')}{|(y,z)|^{2n-1}} f(x-y)g(x-z)\,dy\,dz.$$

The square function μ_Ω is the pointwise $L^2(\mathbb{R}, dt)$ norm of the bilinear map $(f,g) \mapsto (2^{-t}F_{\Omega,t}(f,g))_{t\in\mathbb{R}}$. This formulation relates μ_Ω to Hart's vector-valued theory [17], which covers output exponents below one for discrete bilinear square functions. Hart's framework imposes size and local Hölder-type difference estimates on the vector-valued kernel ([17], Definition 5.1). The weighted theory of Bu, Fu, and Zhang [18] imposes similar kernel size and difference conditions for non-convolution bilinear Marcinkiewicz integrals.

In what follows, we study μ_Ω for mean-zero kernels $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $2 \le q \le \infty$. For finite q, this assumption allows essentially unbounded kernels and therefore does not guarantee the pointwise kernel size bounds required by Hart ([17], Equation (5.2)) and by Bu, Fu, and Zhang ([18], Definition 4). Even for $q = \infty$, angular jumps can violate the kernel difference estimates required in both frameworks. Thus neither framework applies directly to the full class of kernels considered here.

From the Fourier multiplier perspective, μ_Ω has the dilation form of the bilinear square functions studied by Si, Xue, and Yabuta [19]. Let $K(u,v) = \Omega((u,v)')|(u,v)|^{1-2n}\chi_{\{|(u,v)|\le 1\}}$ and set $m = \widehat{K}$. For $R = 2^t$, the kernel of $2^{-t}F_{\Omega,t}$ is $R^{-2n}K(u/R, v/R)$, whose multiplier is $m_R(\xi,\eta) = m(R\xi, R\eta)$. Since $dt = dR/(R\log 2)$, we obtain

$$\mu_\Omega(f,g)(x) = (\log 2)^{-1/2}\left(\int_0^\infty \left|T_{m_R}(f,g)(x)\right|^2 \frac{dR}{R}\right)^{1/2}.$$

The weighted theorem of Si, Xue, and Yabuta ([19], Theorem 1.1) assumes a family of a priori unweighted weak-type estimates for the square function. In particular, applying their theorem at $(p_1, p_2, p) = (2, 2, 1)$ requires an unweighted $L^2 \times L^2 \to L^{1,\infty}$ bound.

In this paper, we prove the following $L^2 \times L^2 \to L^1$ estimate for μ_Ω directly from the angular L^q condition and spherical cancellation, without imposing any of the additional hypotheses discussed above.

Theorem 1. *Let $n \geq 1$ and $2 \leq q \leq \infty$. Suppose that $\Omega \in L^q(\mathbb{S}^{2n-1})$ satisfies (2). Then*

$$\|\mu_\Omega(f,g)\|_{L^1(\mathbb{R}^n)} \leq C_{n,q}\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)},$$

where $C_{n,q}$ is a positive constant depending only on n and q.

The restriction $q \geq 2$ arises in the Plancherel step, where we use the embedding $L^q(\mathbb{S}^{2n-1}) \hookrightarrow L^2(\mathbb{S}^{2n-1})$ to control the L^2 norm of the unit-scale kernel piece τ. As for the mapping exponents, the scalar-valued bilinear Calderón–Zygmund argument for T_Ω does not extend the range of boundedness for μ_Ω, which is sublinear in each input.

The proof adapts the product wavelet method to the mixed norm $L^1_x(\mathbb{R}^n; L^2_t(\mathbb{R}))$. We first introduce a continuous frequency decomposition and express the truncated kernel as a sum of rescaled annular kernels. The resulting multiplier is separated into a part where both frequency variables are bounded away from zero and two parts where one frequency variable is small. Writing $t = h + u$, with $h \in \mathbb{Z}$ and $u \in [0,1)$, separates the local parameter dependence from the dyadic scale.

A key ingredient in our proof is the local parameter-dependent multiplier estimate in Lemma 4. This estimate is similar in form to the fixed-symbol criterion of Grafakos, He, and Slavíková ([20], Theorem 1.3 with $q = 2$). Both estimates use bounds L and A for the L^2 norm of the symbol and the L^∞ norms of its derivatives, respectively. With these bounds uniform in $u \in [0,1]$, applying their criterion to each a_u gives an $L^2 \times L^2 \to L^2_u(L^1_x)$ bound of order $L^{1/2}A^{1/2}$. By Minkowski's inequality, this controls the weaker mixed norm. Our lemma yields the $L^2 \times L^2 \to L^1_x(L^2_u)$ bound needed for the square-function argument, with operator norm at most $CL^{2/3}A^{1/3}$. We expand the symbols in a common product wavelet basis, view the coefficient functions as elements of $L^2([0,1])$, and group the wavelet indices into rows, columns, and matching sets to exploit frequency orthogonality. When one frequency variable is small, maximal-function bounds, orthogonality, and bounded overlap of frequency annuli provide the decay needed for summation.

The paper is organized as follows. Section 2 presents the frequency decomposition and reduces Theorem 1 to two estimates for the localized operators. Section 3 proves Proposition 1. Section 4 establishes Proposition 2 and proves the local parameter-dependent multiplier estimate (Lemma 4).

Throughout the paper, C denotes a positive constant that is independent of the essential variables and may vary from line to line. Its dependence on relevant parameters will be specified when necessary.

2. Preliminaries for the Proof of Theorem 1

Set

$$\sigma_t(y,z) = 2^{-t}\frac{\Omega((y,z)')}{|(y,z)|^{2n-1}}\chi_{\{|(y,z)|\leq 2^t\}}.$$

Then

$$\mu_\Omega(f,g)(x) = \left(\int_{-\infty}^{\infty} |\sigma_t * (f\otimes g)(x,x)|^2\,dt\right)^{1/2},$$

where $(f\otimes g)(y,z) = f(y)g(z)$, and the convolution is taken on $\mathbb{R}^{2n}$.

Choose a nonnegative function $\Phi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\Phi \subset (1/2,2)$ and

$$\int_0^\infty \Phi(r)\frac{dr}{r} = \log 2.$$

For $t \in \mathbb{R}$, define the Fourier multiplier operator Δ_t on $\mathbb{R}^{2n}$ by

$$\widehat{\Delta_t f}(\xi,\eta) = \Phi(2^t|(\xi,\eta)|)\hat{f}(\xi,\eta).$$

For $s, t \in \mathbb{R}$, set

$$M_{s,t}(y,z) = \Delta_{t-s}\sigma_t(y,z).$$

Thus,

$$m_{s,t}(\xi,\eta) := \widehat{M_{s,t}}(\xi,\eta) = \Phi(2^{t-s}|(\xi,\eta)|)\widehat{\sigma_t}(\xi,\eta).$$

By the change of variables $r = 2^{t-s}|(\xi,\eta)|$, we have

$$\int_{-\infty}^{\infty} \Phi(2^{t-s}|(\xi,\eta)|)\, ds = \frac{1}{\log 2}\int_0^{\infty} \Phi(r)\frac{dr}{r} = 1.$$

Consequently,

$$\int_{-\infty}^{\infty} \widehat{M_{s,t}}(\xi,\eta)\, ds = \widehat{\sigma_t}(\xi,\eta).$$

This gives the decomposition

$$\sigma_t = \int_{-\infty}^{\infty} M_{s,t}\, ds.$$

Define the bilinear multiplier operator $T_{m_{s,t}}$ by

$$\begin{aligned} T_{m_{s,t}}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{s,t}(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} M_{s,t}(x-y,x-z)f(y)g(z)\, dy\, dz. \end{aligned}$$

Using the preceding decomposition, we obtain

$$\begin{aligned} \mu_\Omega(f,g)(x) &= \left(\int_{-\infty}^{\infty}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma_t(x-y,x-z)f(y)g(z)\, dy\, dz\right|^2 dt\right)^{1/2} \\ &= \left(\int_{-\infty}^{\infty}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\int_{-\infty}^{\infty} M_{s,t}(x-y,x-z)\, ds\right)f(y)g(z)\, dy\, dz\right|^2 dt\right)^{1/2} \\ &= \left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} M_{s,t}(x-y,x-z)f(y)g(z)\, dy\, dz\, ds\right|^2 dt\right)^{1/2} \\ &= \left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} T_{m_{s,t}}(f,g)(x)\, ds\right|^2 dt\right)^{1/2}. \end{aligned}$$

By Minkowski's integral inequality,

$$\mu_\Omega(f,g)(x) \le \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |T_{m_{s,t}}(f,g)(x)|^2\, dt\right)^{1/2} ds.$$

Applying Minkowski's integral inequality once more, we obtain, for $1 \le p < \infty$,

$$\begin{aligned} \|\mu_\Omega(f,g)\|_{L^p(\mathbb{R}^n)} &\le \int_{-\infty}^{\infty}\left\|\left(\int_{-\infty}^{\infty} |T_{m_{s,t}}(f,g)|^2\, dt\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} ds \\ &= \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n}\left(\int_{-\infty}^{\infty} |T_{m_{s,t}}(f,g)(x)|^2\, dt\right)^{p/2} dx\right)^{1/p} ds. \end{aligned} \tag{4}$$

For $0 < p < \infty$ and a family of measurable functions $\{h_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}^n$, define

$$\|h_t\|_{L^p_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} = \left(\int_{\mathbb{R}^n}\left(\int_{-\infty}^{\infty} |h_t(x)|^2\, dt\right)^{p/2} dx\right)^{1/p}.$$

This is a norm when $p \ge 1$ and a quasi-norm when $0 < p < 1$. To prove Theorem 1, we will establish the following two propositions.

Proposition 1. *Let $1/2 < p < \infty$ and $1 < p_1, p_2 < \infty$ satisfy $1/p = 1/p_1 + 1/p_2$. Suppose that $\Omega \in L^q(\mathbb{S}^{2n-1})$ satisfies (2), where $1 < q \le \infty$, and let $0 < \delta < 1/q'$. Then, for $s < 0$,*

$$\|T_{m_{s,t}}(f,g)\|_{L^p_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \le C2^s\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

For $s \ge 0$,

$$\|T_{m_{s,t}}(f,g)\|_{L^p_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \le C2^{s(2n-\delta)}\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

Proposition 2. *Suppose that* $\Omega \in L^q(\mathbb{S}^{2n-1})$ *satisfies* (2), $2 \le q \le \infty$, *and let* $0 < \delta < 1/q'$. *Then, for* $s > 0$,

$$\|T_{m_{s,t}}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \le C(s+1)2^{-\delta s/3}\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{5}$$

The conclusion of Theorem 1 then follows from (4) and the preceding two propositions. These propositions will be proved in the next two sections.

3. Proof of Proposition 1

We first state a lemma whose proof follows the same argument as that of the Coifman–Meyer multiplier theorem (see Theorem 7.5.3 of [21]).

Lemma 1. *Let* $\{H_t\}_{t\in\mathbb{R}}$ *be a family of functions on* $\mathbb{R}^{2n} \setminus \{(0,0)\}$ *satisfying*

$$\left(\int_{\mathbb{R}} |\partial_\xi^{\alpha_1}\partial_\eta^{\alpha_2} H_t(\xi,\eta)|^2\,dt\right)^{1/2} \le C_{\alpha_1,\alpha_2}(|\xi|+|\eta|)^{-|\alpha_1|-|\alpha_2|}$$

for every $(\xi,\eta) \ne (0,0)$ *and all multi-indices* $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$ *with* $|\alpha_1|+|\alpha_2| \le 2n$. *Let* $1/2 < p < \infty$ *and* $1 < p_1, p_2 < \infty$ *satisfy* $1/p = 1/p_1 + 1/p_2$. *Define*

$$T_{H_t}(f,g)(x) = \int_{\mathbb{R}^{2n}} H_t(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta.$$

Then there exists a constant $C > 0$ *such that*

$$\left\|\left(\int_{\mathbb{R}} |T_{H_t}(f,g)|^2\,dt\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

To prove Proposition 1, it suffices to verify the derivative estimates required by Lemma 1, with the appropriate dependence on s. We first record the following estimates, which follow by adapting the Fourier transform argument in ([22], p. 550).

Lemma 2. *Let* $\Omega \in L^q(\mathbb{S}^{2n-1})$, *where* $1 < q \le \infty$, *satisfy the mean-zero condition* (2), *and let* $0 < \delta < 1/q'$. *Then*

$$|\widehat{\sigma_0}(\xi,\eta)| \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\min\{|(\xi,\eta)|, |(\xi,\eta)|^{-\delta}\}. \tag{6}$$

Moreover, for every nonzero multi-index $\alpha \in \mathbb{N}_0^{2n}$,

$$|\partial^\alpha\widehat{\sigma_0}(\xi,\eta)| \le C_\alpha\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\min\{1, |(\xi,\eta)|^{-\delta}\}. \tag{7}$$

By (6), the scaling identity $\widehat{\sigma_t}(\xi,\eta) = \widehat{\sigma_0}(2^t(\xi,\eta))$, and the support condition $\operatorname{supp}\Phi \subset (1/2,2)$, we obtain

$$\begin{aligned}\left(\int_{-\infty}^{\infty} |m_{s,t}(\xi,\eta)|^2\,dt\right)^{1/2} &= \left(\int_{-\infty}^{\infty} |\Phi(2^{t-s}|(\xi,\eta)|)|^2|\widehat{\sigma_0}(2^t(\xi,\eta))|^2\,dt\right)^{1/2} \\ &\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\left(\int_F \min\{|2^t(\xi,\eta)|, |2^t(\xi,\eta)|^{-\delta}\}^2\,dt\right)^{1/2} \\ &\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\min\{2^s, 2^{-s\delta}\},\end{aligned} \tag{8}$$

where

$$F = \{t \in \mathbb{R} : 1/2 < 2^{t-s}|(\xi,\eta)| < 2\}.$$

For each fixed $(\xi,\eta)\neq(0,0)$, the set F is an interval of length 2, and $|2^t(\xi,\eta)|$ is comparable to 2^s on F.

To estimate the derivatives of $m_{s,t}$, set

$$\Gamma(\xi,\eta)=\Phi(|(\xi,\eta)|).$$

For $\alpha\in\mathbb{N}_0^{2n}$ with $1\le|\alpha|\le 2n$, Leibniz's rule gives

$$\partial^\alpha m_{s,t}(\xi,\eta)=\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}2^{t|\alpha_1|}(\partial^{\alpha_1}\widehat{\sigma_0})(2^t(\xi,\eta))2^{(t-s)|\alpha_2|}(\partial^{\alpha_2}\Gamma)(2^{t-s}(\xi,\eta)).$$

Since the derivatives of Γ are bounded and supported in the annulus $\{(\xi,\eta)\in\mathbb{R}^{2n}: 1/2<|(\xi,\eta)|<2\}$, it follows that

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}|\partial^\alpha m_{s,t}(\xi,\eta)|^2\,dt\right)^{1/2}\\&\quad\le C_\alpha\sum_{\alpha_1+\alpha_2=\alpha}\left(\int_F\left|2^{t|\alpha_1|}(\partial^{\alpha_1}\widehat{\sigma_0})(2^t(\xi,\eta))2^{(t-s)|\alpha_2|}\right|^2dt\right)^{1/2}.\end{aligned}\tag{9}$$

If $|\alpha_1|\neq 0$, then (7) yields

$$\begin{aligned}&\left(\int_F\left|2^{t|\alpha_1|}(\partial^{\alpha_1}\widehat{\sigma_0})(2^t(\xi,\eta))2^{(t-s)|\alpha_2|}\right|^2dt\right)^{1/2}\\&\le C_\alpha\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\left(\int_F\left|2^{t|\alpha_1|}\min\{1,|2^t(\xi,\eta)|^{-\delta}\}2^{(t-s)|\alpha_2|}\right|^2dt\right)^{1/2}\\&\le C_\alpha\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}I(s)|(\xi,\eta)|^{-|\alpha|},\end{aligned}\tag{10}$$

where

$$I(s)=\begin{cases}2^s, & s<0,\\ 2^{s(2n-\delta)}, & s\ge 0.\end{cases}$$

Indeed, on F,

$$2^{t|\alpha_1|}2^{(t-s)|\alpha_2|}\le C_\alpha 2^{s|\alpha_1|}|(\xi,\eta)|^{-|\alpha|}.$$

Since $1\le|\alpha_1|\le 2n$, we have

$$2^{s|\alpha_1|}\min\{1,2^{-s\delta}\}\le I(s).$$

If $|\alpha_1|=0$, then (6) gives

$$\begin{aligned}&\left(\int_F\left|2^{t|\alpha_1|}(\partial^{\alpha_1}\widehat{\sigma_0})(2^t(\xi,\eta))2^{(t-s)|\alpha_2|}\right|^2dt\right)^{1/2}\\&=\left(\int_F\left|\widehat{\sigma_0}(2^t(\xi,\eta))2^{(t-s)|\alpha|}\right|^2dt\right)^{1/2}\\&\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\left(\int_F\left|\min\{|2^t(\xi,\eta)|,|2^t(\xi,\eta)|^{-\delta}\}2^{(t-s)|\alpha|}\right|^2dt\right)^{1/2}\\&\le C_\alpha\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\min\{2^s,2^{-s\delta}\}|(\xi,\eta)|^{-|\alpha|}.\end{aligned}\tag{11}$$

Combining (8), (9), (10) and (11), we obtain, for every $\alpha\in\mathbb{N}_0^{2n}$ with $|\alpha|\le 2n$,

$$\left(\int_{-\infty}^{\infty}|\partial^\alpha m_{s,t}(\xi,\eta)|^2\,dt\right)^{1/2}\le C_\alpha\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}I(s)|(\xi,\eta)|^{-|\alpha|}.$$

This estimate, together with Lemma 1, completes the proof of Proposition 1.

4. Proof of Proposition 2

Define

$$\tau(y,z) = \frac{\Omega((y,z)')}{|(y,z)|^{2n-1}}\chi_{\{(y,z)\in\mathbb{R}^{2n}:1/2<|(y,z)|\le 1\}},$$

and, for $\ell \in \mathbb{N}_0$, set

$$\tau_\ell(y,z) = 2^{2n\ell}\tau(2^\ell(y,z)).$$

The dyadic decomposition of the unit ball gives

$$\sigma_0 = \sum_{\ell\ge 0} 2^{-\ell}\tau_\ell.$$

Taking Fourier transforms, we obtain

$$\widehat{\sigma_0}(\xi,\eta) = \sum_{\ell\ge 0} 2^{-\ell}\widehat{\tau}(2^{-\ell}(\xi,\eta)). \tag{12}$$

Using polar coordinates, we have

$$\begin{aligned}\|\tau\|^2_{L^2(\mathbb{R}^{2n})} &= \int_{\mathbb{S}^{2n-1}}\int_{1/2}^1 |\Omega(\theta)|^2 r^{-2(2n-1)} r^{2n-1}\,dr\,d\sigma(\theta)\\ &= \|\Omega\|^2_{L^2(\mathbb{S}^{2n-1})}\int_{1/2}^1 r^{1-2n}\,dr\\ &\le C\|\Omega\|^2_{L^2(\mathbb{S}^{2n-1})}.\end{aligned}$$

Consequently, for $2 \le q \le \infty$,

$$\|\tau\|_{L^2(\mathbb{R}^{2n})} \le C\|\Omega\|_{L^2(\mathbb{S}^{2n-1})} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}. \tag{13}$$

Moreover, the same argument as that used to establish (6) and (7) gives, for every multi-index $\alpha \in \mathbb{N}_0^{2n}$, including $\alpha = 0$,

$$|\partial^\alpha\widehat{\tau}(\xi,\eta)| \le C_\alpha\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\min\{1,|(\xi,\eta)|^{-\delta}\}. \tag{14}$$

For $j \in \mathbb{R}$, let $d_j = \min\{j,0\}$ and define

$$q_j(\xi,\eta) = \Phi(2^{d_j-j}|(\xi,\eta)|)\widehat{\tau}(2^{d_j}(\xi,\eta)).$$

Equivalently,

$$q_j(\xi,\eta) = \begin{cases}\Phi(2^{-j}|(\xi,\eta)|)\widehat{\tau}(\xi,\eta), & j \ge 0,\\ \Phi(|(\xi,\eta)|)\widehat{\tau}(2^j(\xi,\eta)), & j < 0.\end{cases}$$

We have the following estimates.

Lemma 3. *For every $j \in \mathbb{R}$,*

$$\|q_j\|_{L^2(\mathbb{R}^{2n})} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}. \tag{15}$$

Moreover, for every integer $K > 2n$,

$$\max_{|\alpha|\le K}\|\partial^\alpha q_j\|_{L^\infty(\mathbb{R}^{2n})} \le C_K\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\min\{2^{-\delta j},1\}. \tag{16}$$

Proof. We first prove (15). For $j \ge 0$, Plancherel's theorem and (13) yield

$$\|q_j\|_{L^2(\mathbb{R}^{2n})} \le \|\Phi\|_{L^\infty(\mathbb{R})}\|\widehat{\tau}\|_{L^2(\mathbb{R}^{2n})} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}.$$

For $j < 0$, we use

$$|\widehat{\tau}(2^j(\xi,\eta))| \le \|\tau\|_{L^1(\mathbb{R}^{2n})} \le C\|\Omega\|_{L^1(\mathbb{S}^{2n-1})}.$$

Since $\Phi(|(\xi,\eta)|)$ is supported in a fixed annulus, it follows that

$$\|q_j\|_{L^2(\mathbb{R}^{2n})} \le C\|\Omega\|_{L^1(\mathbb{S}^{2n-1})} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}.$$

To prove (16), recall that $\Gamma(\xi,\eta) = \Phi(|(\xi,\eta)|)$. For $j \ge 0$, Leibniz's rule gives

$$\partial^\alpha q_j(\xi,\eta) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} 2^{-j|\beta|} (\partial^\beta \Gamma)(2^{-j}(\xi,\eta))(\partial^{\alpha-\beta}\widehat{\tau})(\xi,\eta).$$

On the support of $(\partial^\beta\Gamma)(2^{-j}(\xi,\eta))$, the quantity $|(\xi,\eta)|$ is comparable to 2^j. Thus, by (14),

$$|\partial^\alpha q_j(\xi,\eta)| \le C_\alpha 2^{-\delta j}\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}.$$

For $j < 0$, we have

$$\partial^\alpha q_j(\xi,\eta) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\partial^\beta\Gamma)(\xi,\eta) 2^{j|\alpha-\beta|} (\partial^{\alpha-\beta}\widehat{\tau})(2^j(\xi,\eta)).$$

Since $2^{j|\alpha-\beta|} \le 1$, the uniform bound in (14) gives

$$|\partial^\alpha q_j(\xi,\eta)| \le C_\alpha \|\Omega\|_{L^q(\mathbb{S}^{2n-1})}.$$

This proves (16). □

Let γ be a smooth radial function on $\mathbb{R}^n$ such that $0 \le \gamma \le 1$, $\gamma(\xi) = 0$ for $|\xi| \le 1/16$, and $\gamma(\xi) = 1$ for $|\xi| \ge 1/8$. Define

$$a_j(\xi,\eta) = \gamma(\xi)\gamma(\eta)q_j(\xi,\eta),$$

and

$$b_j(\xi,\eta) = (1-\gamma(\xi))q_j(\xi,\eta), \qquad c_j(\xi,\eta) = \gamma(\xi)(1-\gamma(\eta))q_j(\xi,\eta).$$

Let $J = \max\{j, 0\}$. Then

$$q_j = a_j + b_j + c_j.$$

The support conditions imply

$$\operatorname{supp} a_j \subset \{(\xi,\eta) \in \mathbb{R}^{2n} : 1/16 \le |\xi| \le 2^{J+1},\ 1/16 \le |\eta| \le 2^{J+1}\}, \tag{17}$$

$$\operatorname{supp} b_j \subset \{(\xi,\eta) \in \mathbb{R}^{2n} : |\xi| \le 1/8,\ 2^J/3 \le |\eta| \le 2^{J+1}\}, \tag{18}$$

$$\operatorname{supp} c_j \subset \{(\xi,\eta) \in \mathbb{R}^{2n} : |\eta| \le 1/8,\ 2^J/3 \le |\xi| \le 2^{J+1}\}. \tag{19}$$

For example, on the support of b_j,

$$|\eta|^2 = |(\xi,\eta)|^2 - |\xi|^2 \ge 2^{2J-2} - (1/8)^2 \ge 2^{2J}/9.$$

We will establish

$$\|T_{a_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \le C(J+1)\|\Omega\|_{L^q(\mathbb{S}^{2n-1})} 2^{-\delta J/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)} \tag{20}$$

and

$$\|T_{b_j(2^t\cdot)}(f,g)\|_{L^1_x(L^2_t)} + \|T_{c_j(2^t\cdot)}(f,g)\|_{L^1_x(L^2_t)} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})} 2^{-\delta J}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{21}$$

These two estimates imply Proposition 2.

Indeed, assuming (20) and (21), we obtain

$$\|T_{q_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \le C(J+1)2^{-\delta J/3}\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{22}$$

By (12) and the definition of $m_{s,t}$,

$$m_{s,t}(\xi,\eta) = \sum_{\ell\geq 0} 2^{-\ell}\Phi(2^{t-s}|(\xi,\eta)|)\widehat{\tau}(2^{t-\ell}(\xi,\eta)).$$

Set $j = s-\ell$. By the definition of q_j,

$$\begin{aligned} q_j(2^{t-\ell-d_j}(\xi,\eta)) &= \Phi(2^{d_j-j}2^{t-\ell-d_j}|(\xi,\eta)|)\widehat{\tau}(2^{d_j}2^{t-\ell-d_j}(\xi,\eta)) \\ &= \Phi(2^{t-s}|(\xi,\eta)|)\widehat{\tau}(2^{t-\ell}(\xi,\eta)). \end{aligned}$$

Therefore,

$$m_{s,t}(\xi,\eta) = \sum_{\ell\geq 0} 2^{-\ell} q_{s-\ell}(2^{t-\ell-d_{s-\ell}}(\xi,\eta)).$$

Using (22) and the translation invariance of integration in t, we obtain

$$\begin{aligned} &\|T_{m_{s,t}}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \\ &\quad\leq C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}\sum_{\ell\geq 0} 2^{-\ell}(1+\max\{s-\ell,0\})2^{-\delta\max\{s-\ell,0\}/3}. \end{aligned}$$

For $s>0$,

$$\begin{aligned} \sum_{0\leq\ell\leq s} 2^{-\ell}(s-\ell+1)2^{-\delta(s-\ell)/3} &= 2^{-\delta s/3}\sum_{0\leq\ell\leq s}(s-\ell+1)2^{-(1-\delta/3)\ell} \\ &\leq (s+1)2^{-\delta s/3}\sum_{\ell\geq 0} 2^{-(1-\delta/3)\ell} \\ &\leq C(s+1)2^{-\delta s/3}. \end{aligned}$$

Moreover,

$$\sum_{\ell>s} 2^{-\ell} \leq C2^{-s} \leq C(s+1)2^{-\delta s/3}.$$

Consequently,

$$\|T_{m_{s,t}}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \leq C(s+1)2^{-\delta s/3}\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

To prove (20) and (21), we first state the following lemma, whose proof will be given at the end of this section.

Lemma 4. *Let $n\geq 1$, and let $T>2n$ be an integer. Suppose that $a_u(\xi,\eta)$, $u\in[0,1]$, is a jointly measurable family of symbols such that $a_u\in C^T_c(\mathbb{R}^{2n})$ for almost every u. Assume that*

$$\operatorname*{ess\,sup}_{u\in[0,1]}\|a_u\|_{L^2(\mathbb{R}^{2n})}\leq L$$

and

$$\operatorname*{ess\,sup}_{u\in[0,1]}\max_{|\alpha|\leq T}\|\partial^\alpha a_u\|_{L^\infty(\mathbb{R}^{2n})}\leq A.$$

Define

$$T_{a_u}(f,g)(x) = \int_{\mathbb{R}^{2n}} a_u(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta.$$

Then

$$\int_{\mathbb{R}^n}\left(\int_0^1 |T_{a_u}(f,g)(x)|^2\,du\right)^{1/2} dx \leq C_{n,T}L^{2/3}A^{1/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{23}$$

The constant is independent of the locations and sizes of the supports of the symbols. Consequently, the operator extends to a bounded bilinear map from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n;L^2([0,1]))$.

4.1. Estimate for $\|T_{a_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))}$

By (15),

$$\|a_j\|_{L^2(\mathbb{R}^{2n})} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}. \tag{24}$$

Moreover, (16) and Leibniz's rule imply that, for every integer $K > 2n$,

$$\max_{|\alpha|\le K} \|\partial^\alpha a_j\|_{L^\infty(\mathbb{R}^{2n})} \le C_K\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J}. \tag{25}$$

For $t \in \mathbb{R}$, write

$$t = h + u,$$

where $h \in \mathbb{Z}$ and $0 \le u < 1$. By (24),

$$\|a_j(2^u\cdot)\|_{L^2(\mathbb{R}^{2n})} = 2^{-nu}\|a_j\|_{L^2(\mathbb{R}^{2n})} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})},$$

with a constant independent of u. Furthermore, since

$$\partial^\alpha[a_j(2^u(\xi,\eta))] = 2^{u|\alpha|}(\partial^\alpha a_j)(2^u(\xi,\eta)),$$

(25) gives

$$\max_{|\alpha|\le K} \|\partial^\alpha[a_j(2^u\cdot)]\|_{L^\infty(\mathbb{R}^{2n})} \le C_K\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J}.$$

Applying Lemma 4, we obtain

$$\|T_{a_j(2^u\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_u([0,1]))} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{26}$$

A change of variables gives

$$T_{a_j(2^{h+u}\cdot)}(f,g)(x) = 2^{-2hn}T_{a_j(2^u\cdot)}(f^{(h)},g^{(h)})(2^{-h}x),$$

where

$$\widehat{f^{(h)}}(\xi) = \hat{f}(2^{-h}\xi), \qquad \widehat{g^{(h)}}(\eta) = \hat{g}(2^{-h}\eta).$$

By Plancherel's theorem,

$$\|f^{(h)}\|_{L^2(\mathbb{R}^n)} = 2^{hn/2}\|f\|_{L^2(\mathbb{R}^n)}, \qquad \|g^{(h)}\|_{L^2(\mathbb{R}^n)} = 2^{hn/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

Thus, by (26),

$$\begin{aligned} &\|T_{a_j(2^{h+u}\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_u([0,1]))} \\ &= 2^{-hn}\|T_{a_j(2^u\cdot)}(f^{(h)},g^{(h)})\|_{L^1_x(\mathbb{R}^n;L^2_u([0,1]))} \\ &\le C2^{-hn}\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J/3}\|f^{(h)}\|_{L^2(\mathbb{R}^n)}\|g^{(h)}\|_{L^2(\mathbb{R}^n)} \\ &\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{27}$$

Let

$$E_{h,J} = \{\xi \in \mathbb{R}^n : (1/32)2^{-h} \le |\xi| \le 2^{J+1}2^{-h}\},$$

and define

$$\widehat{f_{h,J}} = \hat{f}\,\chi_{E_{h,J}}, \qquad \widehat{g_{h,J}} = \hat{g}\,\chi_{E_{h,J}}.$$

By (17), for $0 \le u < 1$,

$$T_{a_j(2^{h+u}\cdot)}(f,g) = T_{a_j(2^{h+u}\cdot)}(f_{h,J},g_{h,J}).$$

For each $\xi \ne 0$, the condition $\xi \in E_{h,J}$ restricts h to an interval of length $J + 6$. Hence,

$$\sum_{h\in\mathbb{Z}} \chi_{E_{h,J}}(\xi) \le C(J+1).$$

Then, by (27), Plancherel's theorem, and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\|T_{a_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))}\\
&\quad=\int_{\mathbb{R}^n}\left(\sum_{h\in\mathbb{Z}}\int_0^1|T_{a_j(2^{h+u}\cdot)}(f,g)(x)|^2\,du\right)^{1/2}dx\\
&\quad\le\sum_{h\in\mathbb{Z}}\|T_{a_j(2^{h+u}\cdot)}(f_{h,J},g_{h,J})\|_{L^1_x(\mathbb{R}^n;L^2_u([0,1]))}\\
&\quad\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J/3}\sum_{h\in\mathbb{Z}}\|f_{h,J}\|_{L^2(\mathbb{R}^n)}\|g_{h,J}\|_{L^2(\mathbb{R}^n)}\\
&\quad\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J/3}\left(\sum_{h\in\mathbb{Z}}\|\widehat{f_{h,J}}\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}\left(\sum_{h\in\mathbb{Z}}\|\widehat{g_{h,J}}\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}\\
&\quad\le C(J+1)\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

This proves (20).

4.2. Estimates for $\|T_{b_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))}+\|T_{c_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))}$

The support conditions for b_j and c_j are symmetric in ξ and η, and the same argument applies to both terms after interchanging the roles of f and g. We therefore only prove

$$\|T_{b_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))}\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{28}$$

Let

$$B_{j,u}(\xi,\eta)=b_j(2^u\xi,2^u\eta),\qquad 0\le u<1.$$

To further decompose $B_{j,u}$, we use compactly supported product wavelets with sufficiently high regularity. The existence of such wavelets was established by Daubechies [23] and discussed in Meyer's book [24]. Their construction is also presented in Triebel [25].

Lemma 5 ([25], pp. 8–9, (1.55)–(1.60)). *For any fixed $k\in\mathbb{N}$, there exist real compactly supported functions $\psi_F,\psi_M\in C^k(\mathbb{R})$ satisfying $\|\psi_F\|_{L^2(\mathbb{R})}=\|\psi_M\|_{L^2(\mathbb{R})}=1$, and for $0\le\alpha\le k$, we have $\int_{\mathbb{R}}x^\alpha\psi_M(x)dx=0$, and, if for $\vec{x}=(x_1,x_2,\cdots,x_{2n})\in\mathbb{R}^{2n}$, we define*

$$\Psi^G(\vec{x})=\psi_{G_1}(x_1)\psi_{G_2}(x_2)\cdots\psi_{G_{2n}}(x_{2n}),$$

where $G=(G_1,G_2,\cdots,G_{2n})$ belongs to the set

$$I=\{(G_1,G_2,\cdots,G_{2n}):G_i\in\{F,M\}\},$$

then the family of functions

$$\bigcup_{\vec{\mu}\in\mathbb{Z}^{2n}}\left[\{\Psi^{(F,\cdots,F)}(\vec{x}-\vec{\mu})\}\cup\bigcup_{\lambda=0}^{\infty}\{2^{\lambda n}\Psi^G(2^\lambda\vec{x}-\vec{\mu}):G\in I\backslash\{(F,\cdots,F)\}\}\right]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$.

Let $T>2n$. We use the product wavelets in Lemma 5 with regularity at least C^T and vanishing moments through order T. We choose the generating functions ψ_F and ψ_M with support of diameter approximately 1. Wavelets with the same dilation factor 2^λ have a uniformly bounded overlap N independent of λ.

As in [13], the wavelets at each scale can be divided into at most $C_{n,T,N}$ groups such that the wavelets in each group have disjoint supports and share the same index $G\in I$. For $1\le\kappa\le C_{n,T,N}$, let $D_{\lambda,\kappa}$ denote one such group. The supports of its wavelets have diameters about $2^{-\lambda}$. Therefore, the wavelet expansion of $B_{j,u}$ is

$$B_{j,u} = \sum_{\lambda \geq 0} \sum_{1 \leq \kappa \leq C_{n,T,N}} \sum_{\omega \in D_{\lambda,\kappa}} a_\omega(u)\omega, \tag{29}$$

where

$$a_\omega(u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} B_{j,u}(\xi,\eta)\overline{\omega(\xi,\eta)}\, d\xi\, d\eta.$$

By the definition of b_j,

$$B_{j,u}(\xi,\eta) = (1-\gamma(2^u\xi))q_j(2^u(\xi,\eta)).$$

Thus, for every multi-index $\alpha \in \mathbb{N}_0^{2n}$, Leibniz's rule gives

$$\partial^\alpha B_{j,u}(\xi,\eta) = 2^{u|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta [1-\gamma(\xi)])(2^u\xi)(\partial^{\alpha-\beta} q_j)(2^u(\xi,\eta)),$$

where $1-\gamma(\xi)$ is viewed as a function on $\mathbb{R}^{2n}$ that is independent of η. Since $0 \leq u < 1$, the derivatives of the cutoff are uniformly bounded. Hence, by (16),

$$|\partial^\alpha B_{j,u}(\xi,\eta)| \leq C_\alpha \|\Omega\|_{L^q(\mathbb{S}^{2n-1})} 2^{-\delta J}.$$

Moreover, $B_{j,u}$ and its derivatives are supported in the region $|(\xi,\eta)| \leq 2^{J+1}$. It follows that, for any fixed $M_2 > 0$,

$$|\partial^\alpha B_{j,u}(\xi,\eta)| \leq C_{\alpha,M_2} \|\Omega\|_{L^q(\mathbb{S}^{2n-1})} \frac{2^{-\delta J}}{(1+2^{-J}|(\xi,\eta)|)^{M_2}}. \tag{30}$$

In particular,

$$|B_{j,u}(\xi,\eta)| \leq C_{M_2} \|\Omega\|_{L^q(\mathbb{S}^{2n-1})} \frac{2^{-\delta J}}{(1+2^{-J}|(\xi,\eta)|)^{M_2}}.$$

Using the derivative bounds in (30) for $|\alpha| = T+1$ and the vanishing moments of the wavelets, we apply the argument of Lemma 7 in [13] to obtain

$$|a_\omega(u)| \leq C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})} 2^{-\delta J} 2^{-(T+1+n)\lambda}. \tag{31}$$

For $\omega \in D_{\lambda,\kappa}$, write

$$\begin{aligned}\omega(\xi,\eta) &= 2^{\lambda n/2} \prod_{i=1}^n \psi_{G_i}(2^\lambda \xi_i - \mu_i) \times 2^{\lambda n/2} \prod_{i=1}^n \psi_{G_{n+i}}(2^\lambda \eta_i - \mu_{n+i}) \\ &= \omega_{1,k}(\xi)\omega_{2,l}(\eta),\end{aligned}$$

where $k = (\mu_1, \ldots, \mu_n)$, and $l = (\mu_{n+1}, \ldots, \mu_{2n})$. We accordingly write $a_\omega(u) = a_{k,l}(u)$. Define

$$p_{1,k} = 2^{-\lambda n/2}\omega_{1,k}, \qquad p_{2,l} = 2^{-\lambda n/2}\omega_{2,l},$$

and set

$$c_{k,l}(u) = 2^{\lambda n} a_{k,l}(u).$$

By (31),

$$|c_{k,l}(u)| \leq C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})} 2^{-\delta J} 2^{-T\lambda}. \tag{32}$$

The decomposition (29) can therefore be written as

$$B_{j,u} = \sum_{\lambda,\kappa} \sum_{(k,l) \in \mathcal{I}_{\lambda,\kappa}} c_{k,l}(u) p_{1,k} p_{2,l} = \sum_{\lambda,\kappa} \sum_k \sum_{l:(k,l)\in\mathcal{I}_{\lambda,\kappa}} c_{k,l}(u) p_{1,k} p_{2,l}, \tag{33}$$

where

$$\mathcal{I}_{\lambda,\kappa} = \{(k,l) : \omega_{1,k}\omega_{2,l} \in D_{\lambda,\kappa}\}.$$

By (18), for every $u \in [0,1)$,

$$\operatorname{supp} B_{j,u} \subset \{(\xi,\eta) \in \mathbb{R}^{2n} : |\xi| \le 1/8\}.$$

If $c_{k,l}(u) \neq 0$, the support of the corresponding wavelet intersects this region. Since its support has diameter at most $C2^{-\lambda}$, there is a fixed constant C_0, independent of j, u, and λ, such that

$$\operatorname{supp} p_{1,k} = \operatorname{supp} \omega_{1,k} \subset \{\xi \in \mathbb{R}^n : |\xi| \le C_0\}.$$

On the other hand, for a fixed constant $c \approx 1$ depending only on the generating functions, $\operatorname{supp} p_{1,k}$ is contained in the cube

$$\prod_{i=1}^{n} [2^{-\lambda}(\mu_i - c), 2^{-\lambda}(\mu_i + c)].$$

Consequently, there are at most $C2^{\lambda n}$ possible values of k for which some $c_{k,l}(u)$ is nonzero.

Let

$$E_h = \{\eta \in \mathbb{R}^n : (2^J/6)2^{-h} \le |\eta| \le 2^{J+1}2^{-h}\},$$

and define

$$\widehat{g_h}(\eta) = \hat{g}(\eta)\chi_{E_h}(\eta).$$

By (18), for every $u \in [0,1)$,

$$T_{b_j(2^{h+u}\cdot)}(f,g) = T_{b_j(2^{h+u}\cdot)}(f,g_h).$$

Set

$$F_{k,h} = (p_{1,k}(2^h\cdot)\hat{f})^\vee, \qquad G_{l,h} = (p_{2,l}(2^h\cdot)\widehat{g_h})^\vee.$$

By (33),

$$\begin{aligned} T_{b_j(2^{h+u}\cdot)}(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} B_{j,u}(2^h(\xi,\eta))\hat{f}(\xi)\widehat{g_h}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \\ &= \sum_{\lambda,\kappa}\sum_{(k,l)\in\mathcal{I}_{\lambda,\kappa}} c_{k,l}(u)F_{k,h}(x)G_{l,h}(x). \end{aligned}$$

We next establish the pointwise bound

$$|F_{k,h}(x)| \le CMf(x),$$

where M is the Hardy–Littlewood maximal operator on $\mathbb{R}^n$. For fixed λ and κ, write

$$\psi_1(\xi) = \prod_{i=1}^{n} \psi_{G_i}(\xi_i).$$

Then

$$p_{1,k}(\xi) = \psi_1(2^\lambda\xi - k),$$

and hence

$$\check{p}_{1,k}(x) = 2^{-\lambda n}e^{2\pi i 2^{-\lambda}k\cdot x}\check{\psi}_1(2^{-\lambda}x).$$

The compact support and sufficient regularity of ψ_1 imply

$$|\check{\psi}_1(x)| \le C(1+|x|)^{-n-1}.$$

Therefore, the convolution kernel $K_{k,h} = (p_{1,k}(2^h\cdot))^\vee$ satisfies

$$K_{k,h}(x) = 2^{-(h+\lambda)n}e^{2\pi i 2^{-(h+\lambda)}k\cdot x}\check{\psi}_1(2^{-(h+\lambda)}x),$$

and thus

$$|K_{k,h}(x)| \le C2^{-(h+\lambda)n}(1+2^{-(h+\lambda)}|x|)^{-n-1}.$$

Since $F_{k,h} = K_{k,h} * f$, this gives

$$|F_{k,h}(x)| \le C \int_{\mathbb{R}^n} r^{-n}(1 + |y|/r)^{-n-1}|f(x-y)|\,dy, \qquad r = 2^{h+\lambda}.$$

Splitting the integral into $|y| \le r$ and $2^m r < |y| \le 2^{m+1}r$, $m \ge 0$, yields

$$|F_{k,h}(x)| \le C\left(1 + \sum_{m\ge 0} 2^{-m}\right) Mf(x) \le CMf(x).$$

The constant is independent of k, h, and λ.

Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\|T_{b_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} \qquad (34)\\
&= \int_{\mathbb{R}^n}\left(\sum_{h\in\mathbb{Z}}\int_0^1 |T_{b_j(2^{h+u}\cdot)}(f,g)(x)|^2\,du\right)^{1/2} dx\\
&\le C\sum_{\lambda,\kappa}\sum_k \int_{\mathbb{R}^n} Mf(x)\left(\sum_{h\in\mathbb{Z}}\int_0^1\left|\sum_{l:(k,l)\in\mathcal{I}_{\lambda,\kappa}} c_{k,l}(u)G_{l,h}(x)\right|^2 du\right)^{1/2} dx\\
&\le C\sum_{\lambda,\kappa}\sum_k \|Mf\|_{L^2(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}\sum_{h\in\mathbb{Z}}\int_0^1\left|\sum_{l:(k,l)\in\mathcal{I}_{\lambda,\kappa}} c_{k,l}(u)G_{l,h}(x)\right|^2 du\,dx\right)^{1/2}.
\end{aligned}$$

For fixed λ, κ, and k, the supports of the functions $\omega_{2,l}$ appearing in this sum are pairwise disjoint. Consequently, Plancherel's theorem and (32) give

$$\begin{aligned}
&\sum_{h\in\mathbb{Z}}\int_0^1\int_{\mathbb{R}^n}\left|\sum_{l:(k,l)\in\mathcal{I}_{\lambda,\kappa}} c_{k,l}(u)G_{l,h}(x)\right|^2 dx\,du\\
&= \sum_{h\in\mathbb{Z}}\int_0^1\left\|\sum_{l:(k,l)\in\mathcal{I}_{\lambda,\kappa}} c_{k,l}(u)p_{2,l}(2^h\cdot)\widehat{g_h}\right\|^2_{L^2(\mathbb{R}^n)} du\\
&= \sum_{h\in\mathbb{Z}}\sum_{l:(k,l)\in\mathcal{I}_{\lambda,\kappa}} \|c_{k,l}\|^2_{L^2([0,1])}\|p_{2,l}(2^h\cdot)\widehat{g_h}\|^2_{L^2(\mathbb{R}^n)}\\
&\le C\|\Omega\|^2_{L^q(\mathbb{S}^{2n-1})}2^{-2\delta J}2^{-2T\lambda}\sum_{h\in\mathbb{Z}}\|g_h\|^2_{L^2(\mathbb{R}^n)}\\
&\le C\|\Omega\|^2_{L^q(\mathbb{S}^{2n-1})}2^{-2\delta J}2^{-2T\lambda}\|g\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Combining this estimate with (34), the $L^2(\mathbb{R}^n)$ boundedness of M, and the bound $C2^{\lambda n}$ for the number of relevant indices k, we conclude that

$$\begin{aligned}
\|T_{b_j(2^t\cdot)}(f,g)\|_{L^1_x(\mathbb{R}^n;L^2_t(\mathbb{R}))} &\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J}\sum_{\lambda\ge 0}2^{-(T-n)\lambda}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}\\
&\le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}2^{-\delta J}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The analogous estimate for c_j follows by interchanging the two frequency variables and the roles of f and g. Together, these estimates establish (21).

4.3. Proof of Lemma 4

Set $H = L^2([0,1])$. We use the wavelet construction and notation introduced above. Applying Lemma 5 with $k = T$, we obtain fixed generating functions satisfying

$$\psi_F, \psi_M \in C^T(\mathbb{R}), \qquad \int_{\mathbb{R}} t^j \psi_M(t)\,dt = 0, \qquad j = 0,\dots,T.$$

Recall that each group $D_{\lambda,\kappa}$ introduced above has a fixed product type $G = (G_1,\dots,G_{2n})$. For this type, write

$$\psi_1(\xi) = \prod_{i=1}^n \psi_{G_i}(\xi_i), \qquad \psi_2(\eta) = \prod_{i=1}^n \psi_{G_{n+i}}(\eta_i).$$

The normalized factors defined above are then

$$p_{1,k}(\xi) = \psi_1(2^\lambda \xi - k), \qquad p_{2,l}(\eta) = \psi_2(2^\lambda \eta - l),$$

and the wavelet with translation indices (k, l) is

$$\omega_{1,k}(\xi)\omega_{2,l}(\eta) = 2^{\lambda n} p_{1,k}(\xi) p_{2,l}(\eta).$$

Choose $R > 0$ such that $\operatorname{supp} \psi_F \cup \operatorname{supp} \psi_M \subset [-R, R]$, and fix an integer $d > 2R$. For $r, s \in \{0, \dots, d-1\}^n$, let $D_{\lambda,\kappa}^{r,s}$ consist of the wavelets in $D_{\lambda,\kappa}$ whose translation indices (k, l) satisfy

$$k - r \in d\mathbb{Z}^n, \qquad l - s \in d\mathbb{Z}^n.$$

Each pair (k, l) determines a unique pair of residue vectors (r, s), so the nonempty sets $D_{\lambda,\kappa}^{r,s}$ partition $D_{\lambda,\kappa}$ into at most d^{2n} subgroups.

Within a fixed subgroup, if $k \neq k'$, then $k - k' \in d\mathbb{Z}^n \setminus \{0\}$, so $|k_i - k_i'| \geq d$ for some coordinate i. Since

$$\operatorname{supp} p_{1,k} \subset 2^{-\lambda}\big(k + [-R, R]^n\big)$$

and $d > 2R$, the boxes corresponding to k and k' have disjoint projections onto the ith coordinate axis. Hence $p_{1,k}$ and $p_{1,k'}$ have disjoint supports. The same argument shows that $p_{2,l}$ and $p_{2,l'}$ have disjoint supports whenever $l \neq l'$ within the subgroup.

The number of original groups is uniformly bounded in λ, and d depends only on the wavelet system fixed in terms of n and T. Thus the number of refined groups at each scale is bounded by an integer $K_0 \leq C_{n,T}$. After adding empty groups if necessary, we index the refined groups at each scale by $\kappa = 1, \dots, K_0$ and continue to denote them by $D_{\lambda,\kappa}$. We also retain the notation $\mathcal{J}_{\lambda,\kappa}$ for the set of translation-index pairs (k, l) in the refined group $D_{\lambda,\kappa}$.

Using the wavelet basis introduced above, we expand a_u for almost every $u \in [0, 1]$ as

$$a_u(\xi, \eta) = \sum_{\lambda \geq 0} \sum_{\kappa=1}^{K_0} \sum_{\omega \in D_{\lambda,\kappa}} \vartheta_\omega(u)\omega(\xi, \eta), \tag{35}$$

where

$$\vartheta_\omega(u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a_u(\xi, \eta)\overline{\omega(\xi, \eta)}\, d\xi\, d\eta.$$

Since $H = L^2([0, 1])$, Parseval's identity gives

$$\begin{aligned} \sum_\omega \|\vartheta_\omega\|_H^2 &= \int_0^1 \sum_\omega |\vartheta_\omega(u)|^2\, du \\ &= \int_0^1 \|a_u\|_{L^2(\mathbb{R}^{2n})}^2\, du \leq L^2, \end{aligned} \tag{36}$$

where the sum runs over the entire wavelet basis.

For $(k, l) \in \mathcal{J}_{\lambda,\kappa}$, set

$$e_{k,l} = 2^{\lambda n}\vartheta_{\omega_{1,k}\omega_{2,l}}, \qquad P_k = (p_{1,k}\hat{f})^\vee, \qquad Q_l = (p_{2,l}\hat{g})^\vee.$$

The functions $p_{1,k}$ and $p_{2,l}$ are uniformly bounded. Within the fixed group, their supports are pairwise disjoint for distinct first indices and distinct second indices, respectively. Consequently, Plancherel's theorem yields

$$\sum_k \|P_k\|_{L^2(\mathbb{R}^n)}^2 \leq C\|f\|_{L^2(\mathbb{R}^n)}^2, \qquad \sum_l \|Q_l\|_{L^2(\mathbb{R}^n)}^2 \leq C\|g\|_{L^2(\mathbb{R}^n)}^2. \tag{37}$$

where the sums run over the distinct indices occurring in the group.

In addition, (36) gives that

$$\left(\sum_{(k,l)\in\mathcal{J}_{\lambda,\kappa}} \|e_{k,l}\|_H^2\right)^{1/2} = 2^{\lambda n}\left(\sum_{\omega \in D_{\lambda,\kappa}} \|\vartheta_\omega\|_H^2\right)^{1/2} \leq 2^{\lambda n} L. \tag{38}$$

Define

$$\mathcal{S}_{\lambda,\kappa}(x,u) = \sum_{(k,l)\in\mathcal{I}_{\lambda,\kappa}} e_{k,l}(u)P_k(x)Q_l(x).$$

Set

$$\mathfrak{B}_2 = \left(\sum_{(k,l)\in\mathcal{I}_{\lambda,\kappa}} \|e_{k,l}\|_H^2\right)^{1/2}, \qquad \mathfrak{B}_\infty = \sup_{(k,l)\in\mathcal{I}_{\lambda,\kappa}} \|e_{k,l}\|_H.$$

We claim that

$$\|\mathcal{S}_{\lambda,\kappa}\|_{L^1(\mathbb{R}^n;H)} \le C\mathfrak{B}_2^{2/3}\mathfrak{B}_\infty^{1/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{39}$$

The estimate is immediate if $\mathfrak{B}_\infty = 0$. We may therefore assume $\mathfrak{B}_\infty > 0$.

For $r \in \mathbb{N}_0$, define

$$\nu_r = 2^{-r}\mathfrak{B}_\infty, \qquad \mathcal{U}_r = \{(k,l) \in \mathcal{I}_{\lambda,\kappa} : \nu_r/2 < \|e_{k,l}\|_H \le \nu_r\}.$$

These sets partition the pairs with nonzero coefficients. Moreover,

$$\frac{\nu_r^2}{4}|\mathcal{U}_r| \le \sum_{(k,l)\in\mathcal{U}_r} \|e_{k,l}\|_H^2 \le \mathfrak{B}_2^2,$$

so

$$|\mathcal{U}_r| \le \frac{4\mathfrak{B}_2^2}{\nu_r^2}. \tag{40}$$

Fix a positive integer m_r. For each first-coordinate index k, let

$$\mathcal{U}_r(k) = \{l : (k,l) \in \mathcal{U}_r\}, \qquad \mathcal{R}_r = \{k : |\mathcal{U}_r(k)| \ge m_r\}.$$

Thus, $\mathcal{R}_r$ indexes the rows containing at least m_r pairs. Since distinct rows are disjoint,

$$m_r|\mathcal{R}_r| \le \sum_{k\in\mathcal{R}_r} |\mathcal{U}_r(k)| \le |\mathcal{U}_r|,$$

and hence

$$|\mathcal{R}_r| \le \frac{|\mathcal{U}_r|}{m_r}.$$

For $k \in \mathcal{R}_r$, define

$$\mathcal{H}_k(x,u) = \sum_{l\in\mathcal{U}_r(k)} e_{k,l}(u)Q_l(x).$$

The supports of the Fourier transforms $\widehat{Q_l}$ are pairwise disjoint. Plancherel's theorem therefore gives

$$\begin{aligned}
\|\mathcal{H}_k\|_{L^2(\mathbb{R}^n;H)}^2 &= \int_0^1 \left\| \sum_{l\in\mathcal{U}_r(k)} e_{k,l}(u)Q_l \right\|_{L^2(\mathbb{R}^n)}^2 du \\
&= \sum_{l\in\mathcal{U}_r(k)} \|e_{k,l}\|_H^2 \|Q_l\|_{L^2(\mathbb{R}^n)}^2 \\
&\le C\nu_r^2\|g\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

Using the triangle inequality in H and Cauchy–Schwarz, first in x and then in k, we obtain

$$\begin{aligned}
\left\| \sum_{k\in\mathcal{R}_r} P_k\mathcal{H}_k \right\|_{L^1(\mathbb{R}^n;H)} &\le \sum_{k\in\mathcal{R}_r} \|P_k\|_{L^2(\mathbb{R}^n)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}^n;H)} \\
&\le C\nu_r\|g\|_{L^2(\mathbb{R}^n)}|\mathcal{R}_r|^{1/2}\left(\sum_k \|P_k\|_{L^2(\mathbb{R}^n)}^2\right)^{1/2} \\
&\le C\nu_r\left(\frac{|\mathcal{U}_r|}{m_r}\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)} \\
&\le C\mathfrak{B}_2 m_r^{-1/2}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{41}$$

where we used (37) and (40).

Remove all pairs belonging to these rows, and write

$$\mathcal{U}_r^{(1)} = \{(k,l) \in \mathcal{U}_r : k \notin \mathcal{R}_r\}.$$

For each second-coordinate index l, define

$$\mathcal{U}_r^{(1)}(l) = \{k : (k,l) \in \mathcal{U}_r^{(1)}\}, \qquad C_r = \{l : |\mathcal{U}_r^{(1)}(l)| \geq m_r\}.$$

Thus, C_r indexes the remaining columns containing at least m_r pairs. Since distinct columns are disjoint,

$$m_r|C_r| \leq \sum_{l \in C_r} |\mathcal{U}_r^{(1)}(l)| \leq |\mathcal{U}_r|,$$

and therefore

$$|C_r| \leq \frac{|\mathcal{U}_r|}{m_r}.$$

Interchanging the roles of the two coordinates in the row estimate gives

$$\left\| \sum_{l \in C_r} Q_l \sum_{k \in \mathcal{U}_r^{(1)}(l)} e_{k,l} P_k \right\|_{L^1(\mathbb{R}^n;H)} \leq C\mathfrak{B}_2 m_r^{-1/2} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}. \tag{42}$$

After removing these columns, denote the residual set by

$$\mathcal{U}_r^{(2)} = \{(k,l) \in \mathcal{U}_r^{(1)} : l \notin C_r\}.$$

Every row and every column of $\mathcal{U}_r^{(2)}$ contains at most $(m_r - 1)$ pairs. We show that this set can be partitioned into at most $2m_r$ subsets $\mathcal{V}$ such that distinct pairs (k,l) and (k',l') in the same subset satisfy $k \neq k'$, and $l \neq l'$. We call subsets with this property matching sets.

If $m_r = 1$, the residual set is empty. Suppose that $m_r \geq 2$. Since $\mathcal{U}_r^{(2)}$ is finite, enumerate its pairs and place them successively into $2m_r$ initially empty sets. When placing a pair (k,l), exclude every set that already contains a pair with first index k or second index l. Apart from (k,l) itself, its row contains at most $(m_r - 2)$ pairs, and its column contains at most $(m_r - 2)$ pairs. Thus, at most $2(m_r - 2)$ previously placed pairs can prevent a set from being used, and at most that many sets are excluded. Since $2(m_r - 2) < 2m_r$, an available set always exists.

This procedure places every pair into exactly one set and ensures that no set contains two pairs sharing a first or second index. Discarding the empty sets gives the desired partition into at most $2m_r$ matching sets.

For each matching set $\mathcal{V}$, the triangle inequality in H and Cauchy–Schwarz yield

$$\begin{aligned} \left\| \sum_{(k,l) \in \mathcal{V}} e_{k,l} P_k(x) Q_l(x) \right\|_H &\leq \nu_r \sum_{(k,l) \in \mathcal{V}} |P_k(x)||Q_l(x)| \\ &\leq \nu_r \left(\sum_{(k,l) \in \mathcal{V}} |P_k(x)|^2 \right)^{1/2} \left(\sum_{(k,l) \in \mathcal{V}} |Q_l(x)|^2 \right)^{1/2}. \end{aligned}$$

Here each first index and each second index occurs at most once in $\mathcal{V}$. Integrating in x and applying (37), we obtain

$$\left\| \sum_{(k,l) \in \mathcal{V}} e_{k,l} P_k Q_l \right\|_{L^1(\mathbb{R}^n;H)} \leq C\nu_r \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$

Since the residual set is partitioned into at most $2m_r$ matching sets, the triangle inequality gives

$$\left\| \sum_{(k,l) \in \mathcal{U}_r^{(2)}} e_{k,l} P_k Q_l \right\|_{L^1(\mathbb{R}^n;H)} \leq Cm_r\nu_r \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}. \tag{43}$$

Combining the estimates (41), (42), and (43), we obtain

$$\left\| \sum_{(k,l)\in \mathcal{U}_r} e_{k,l} P_k Q_l \right\|_{L^1(\mathbb{R}^n;H)} \le C(\mathfrak{B}_2 m_r^{-1/2} + m_r \nu_r) \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$

Choose

$$m_r = \lceil (\mathfrak{B}_2/\nu_r)^{2/3} \rceil.$$

Since $\mathfrak{B}_2 \ge \mathfrak{B}_\infty \ge \nu_r > 0$, we have

$$(\mathfrak{B}_2/\nu_r)^{2/3} \le m_r \le 2(\mathfrak{B}_2/\nu_r)^{2/3}.$$

Consequently,

$$\mathfrak{B}_2 m_r^{-1/2} + m_r \nu_r \le 3\mathfrak{B}_2^{2/3} \nu_r^{1/3}.$$

Substituting this bound into the preceding estimate and using $\nu_r = 2^{-r}\mathfrak{B}_\infty$, we obtain

$$\left\| \sum_{(k,l)\in \mathcal{U}_r} e_{k,l} P_k Q_l \right\|_{L^1(\mathbb{R}^n;H)} \le C\mathfrak{B}_2^{2/3} \mathfrak{B}_\infty^{1/3} 2^{-r/3} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$

The sets $\mathcal{U}_r$ partition the pairs in $\mathcal{J}_{\lambda,\kappa}$ with nonzero coefficients. Since $\sum_{r\ge 0} 2^{-r/3} < \infty$, the triangle inequality and summation over r prove (39).

To estimate each wavelet coefficient $\vartheta_\omega(u)$, first consider a wavelet of type $G \ne (F, \dots, F)$. Writing $\zeta = (\xi, \eta)$ and $v = (k, l) \in \mathbb{Z}^{2n}$, the corresponding wavelet has the form

$$\omega(\zeta) = 2^{\lambda n} \prod_{i=1}^{2n} \psi_{G_i}(2^\lambda \zeta_i - v_i).$$

There is a coordinate i_0 such that $G_{i_0} = M$. The vanishing moments of ψ_M and a change of variables imply

$$\int_{\mathbb{R}} t^j \overline{\psi_M(2^\lambda t - v_{i_0})}\, dt = 0, \qquad j = 0, \dots, T.$$

For every multi-index α with $|\alpha| \le T$, we have $\alpha_{i_0} \le T$. Factoring the integral into one-dimensional integrals therefore gives

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \zeta^\alpha \overline{\omega(\xi, \eta)}\, d\xi\, d\eta = 0.$$

In particular, every polynomial of total degree at most $(T-1)$ has zero integral against $\overline{\omega}$.

Choose $\zeta_0 \in \operatorname{supp} \omega$. For almost every u, define the Taylor polynomial of degree $(T-1)$ by

$$P_{u,\zeta_0}(\zeta) = \sum_{|\alpha|\le T-1} \frac{\partial^\alpha a_u(\zeta_0)}{\alpha!} (\zeta - \zeta_0)^\alpha.$$

The preceding cancellation gives

$$\vartheta_\omega(u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (a_u(\xi, \eta) - P_{u,\zeta_0}(\xi, \eta)) \overline{\omega(\xi, \eta)}\, d\xi\, d\eta. \tag{44}$$

Taylor's formula with integral remainder gives

$$a_u(\zeta) - P_{u,\zeta_0}(\zeta) = T \sum_{|\alpha|=T} \frac{(\zeta - \zeta_0)^\alpha}{\alpha!} \int_0^1 (1-t)^{T-1} \partial^\alpha a_u(\zeta_0 + t(\zeta - \zeta_0))\, dt.$$

Since $\|\partial^\alpha a_u\|_{L^\infty(\mathbb{R}^{2n})} \le A$ for $|\alpha| = T$, it follows that

$$|a_u(\zeta) - P_{u,\zeta_0}(\zeta)| \le C_{n,T} A |\zeta - \zeta_0|^T.$$

Since the support of ω has diameter at most $C2^{-\lambda}$, then

$$|a_u(\zeta) - P_{u,\zeta_0}(\zeta)| \le CA2^{-T\lambda}, \qquad \zeta \in \operatorname{supp} \omega.$$

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Moreover, the definition of ω and the change of variables give

$$\|\omega\|_{L^1(\mathbb{R}^{2n})} = 2^{-\lambda n} \prod_{i=1}^{2n} \|\psi_{G_i}\|_{L^1(\mathbb{R})} \le C2^{-\lambda n}.$$

Consequently, combining (44) with the preceding two estimates, we obtain

$$|\vartheta_\omega(u)| \le CA2^{-T\lambda}\|\omega\|_{L^1(\mathbb{R}^{2n})} \le CA2^{-(T+n)\lambda}.$$

The type $G = (F, \dots, F)$ occurs only at $\lambda = 0$ and is estimated directly, without using cancellation. For these scaling functions,

$$\|\omega\|_{L^1(\mathbb{R}^{2n})} = \|\psi_F\|_{L^1(\mathbb{R})}^{2n}.$$

Hence, for almost every $u \in [0,1]$,

$$|\vartheta_\omega(u)| \le \|a_u\|_{L^\infty(\mathbb{R}^{2n})}\|\omega\|_{L^1(\mathbb{R}^{2n})} \le CA.$$

Thus, the bound $|\vartheta_\omega(u)| \le CA2^{-(T+n)\lambda}$ also holds for this type.

Since $[0,1]$ has measure one, the coefficient estimates for all types imply

$$\sup_{(k,l)\in\mathcal{J}_{\lambda,\kappa}} \|e_{k,l}\|_H = 2^{\lambda n} \sup_{\omega\in D_{\lambda,\kappa}} \|\vartheta_\omega\|_H \le CA2^{-T\lambda}. \tag{45}$$

Combining (38), (39), and (45) gives

$$\|\mathcal{S}_{\lambda,\kappa}\|_{L^1(\mathbb{R}^n;H)} \le CL^{2/3}A^{1/3}2^{-(T-2n)\lambda/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

By the wavelet expansion (35) and the definitions of $e_{k,l}$, P_k, Q_l and $\mathcal{S}_{\lambda,\kappa}$,

$$T_{a_u}(f,g)(x) = \sum_{\lambda\ge 0}\sum_{\kappa=1}^{K_0} \mathcal{S}_{\lambda,\kappa}(x,u).$$

Therefore,

$$\begin{aligned} \|T_{a_u}(f,g)\|_{L^1(\mathbb{R}^n;H)} &\le \sum_{\lambda\ge 0}\sum_{\kappa=1}^{K_0} \|\mathcal{S}_{\lambda,\kappa}\|_{L^1(\mathbb{R}^n;H)} \\ &\le CK_0L^{2/3}A^{1/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)} \sum_{\lambda\ge 0} 2^{-(T-2n)\lambda/3} \\ &\le C_{n,T}L^{2/3}A^{1/3}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{46}$$

where the last inequality follows from $T > 2n$ and $K_0 \le C_{n,T}$. This completes the proof.

5. Conclusions

In this paper, we establish the $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ boundedness of the bilinear Marcinkiewicz integral μ_Ω for mean-zero kernels $\Omega \in L^q(\mathbb{S}^{2n-1})$, $2 \le q \le \infty$. The proof combines a product wavelet decomposition with a local parameter-dependent bilinear multiplier estimate adapted to the square-function structure.

A natural question is whether, for every mean-zero kernel $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, the bilinear Marcinkiewicz integral satisfies the estimate

$$\|\mu_\Omega(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^\infty(\mathbb{S}^{2n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

throughout the range

$$1 < p_1, p_2 < \infty, \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{2} < p < \infty.$$

For rough bilinear singular integrals T_Ω, Grafakos, He, and Honzík [13] obtained estimates throughout this range for bounded kernels by combining their initial boundedness estimate with bilinear Calderón–Zygmund kernel estimates and interpolation. This scalar-valued argument does not apply directly to μ_Ω, whose definition involves an outer L^2 norm in the truncation parameter. An extension of that approach would require estimates adapted to the corresponding Hilbert space-valued operator. To the best of our knowledge, boundedness of μ_Ω throughout the above exponent range remains unresolved for mean-zero $\Omega \in L^\infty(\mathbb{S}^{2n-1})$.

Beyond the bounded-kernel case, a further problem is to determine the admissible combinations of q, p_1, and p_2 for which

$$\|\mu_\Omega(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^q(\mathbb{S}^{2n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

holds for every mean-zero $\Omega \in L^q(\mathbb{S}^{2n-1})$, with $1/p = 1/p_1 + 1/p_2$. In particular, it remains to determine how the integrability of the angular kernel restricts the range of boundedness.

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