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# Hybrid Feedback Control for Exponential Stability and Robust $H_\infty$ Control of a Class of Uncertain Neural Network with Mixed Interval and Distributed Time-Varying Delays

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**Abstract:** This paper is concerned the problem of robust  $H_\infty$  control for uncertain neural networks with mixed time-varying delays comprising different interval and distributed time-varying delays via hybrid feedback control. The interval and distributed time-varying delays are not necessary to be differentiable. The main purpose of this research is to estimate robust exponential stability of uncertain neural network with  $H_\infty$  performance attenuation level  $\gamma$ . The key features of the approach include the introduction of a new Lyapunov–Krasovskii functional (LKF) with triple integral terms, the employment of a tighter bounding technique, some slack matrices and newly introduced convex combination condition in the calculation, improved delay-dependent sufficient conditions for the robust  $H_\infty$  control with exponential stability of the system are obtained in terms of linear matrix inequalities (LMIs). The results of this paper complement the previously known ones. Finally, a numerical example is presented to show the effectiveness of the proposed methods.

**Keywords:** neural networks;  $H_\infty$  control; hybrid feedback control; mixed time-varying delay



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## 1. Introduction

During the past decades, the problem of the reliable control has received much attention [1–11]. Neural networks have received considerable due to the effective use of many aspects such as signal processing, automatic control engineering, associative memories, parallel computation, fault diagnosis, combinatorial optimization and pattern recognition and so on [12–14]. It has been shown that the presence of time delay in a dynamical system is often a primary source of instability and performance degradation [15]. Many researchers have paid attentions to the problem of robust stability for uncertain systems with time delays [16–19]. The  $H_\infty$  controller can be used to guarantee closed loop system not only a robust stability but also an adequate level of performance. In practical control systems, actuator faults, sensor faults or some component faults may happen, which often lead to unsatisfactory performance, even loss of stability. Therefore, research on reliable control is necessary.

On the other hand, the  $H_\infty$  control of time-delay systems are practical and theoretical interest since time delay is often encountered in many engineering and industrial processes [20–22].

Most of works have been focused on the problem of designing a robust  $H_\infty$  controller that stabilizes linear uncertain systems with time-varying norm bounded parameter uncertainty in the state and input matrices. The problem of designing a robust reliable  $H_\infty$  controller for neural networks is considered in [23]. Ref [24] have studied the problem of delay dependent robust  $H_\infty$  control for a class of uncertain systems with distributed time-varying delays. The parameter uncertainties are supposed to be time-varying and

norm bounded. The problem of  $H_\infty$  control design usually leads to solving an algebraic Lyapunov equation. It should be noted that some works have been dedicated to the problem of robust reliable control for nonlinear systems with time-varying delay [2,4,7]. However, to the best of the authors’s knowledge, so far the research on robust reliable  $H_\infty$  control is still open problems, which are worth further investigations.

Motivated by above discussion, in this paper we have considered the problem of a robust  $H_\infty$  control for a class of uncertain systems with interval and distributed time-varying delays. The parameter uncertainties are supposed to be time varying and norm bounded. A sufficient condition for the  $H_\infty$  control is presented by LMIs approach using a new LKF with triple integral terms. The robust exponential stability of the system is established which satisfy a formulated  $H_\infty$  performance level for all admissible parameters uncertainties.

The main contributions of this paper are given as follows,

- This research is the first time to study hybrid feedback control for exponential stability and robust  $H_\infty$  control of a class of uncertain neural network with mixed interval and distributed time-varying delays.
- A novel LKF  $V_1(t, x_t) = x^T(t)P_1x(t) + 2x^T(t)P_2 \int_{t-h_2}^t x(s)ds + \left( \int_{t-h_2}^t x(s)ds \right)^T \times P_3 \int_{t-h_2}^t x(s)ds + 2x^T(t)P_4 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds + 2 \left( \int_{t-h_2}^t x(s)ds \right)^T \times P_5 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds + \left( \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds \right)^T P_6 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds$  is first proposed to analyze the problem of robust  $H_\infty$  control for uncertain neural networks with mixed time-varying delays and augmented Lyapunov matrix  $P_i$  ( $i = 1, 2, \dots, 6$ ) do not need to be positive definiteness of the chosen LKF compared with [23].
- The problem of robust  $H_\infty$  control for uncertain neural networks with mixed time-varying delays comprising of interval and distributed time-varying delays, these delays are not necessarily differentiable.
- For the neural networks system (1), the output  $z(t)$  contains the deterministic disturbance input  $w(t)$  and the feedback control  $u(t)$  which is more general and applicable than [23–28].

The rest of this paper is organized as follows. In Section 2, some notations, definitions and some well-known technical lemmas are given. Section 3 presents the  $H_\infty$  control for exponential stability and the robust  $H_\infty$  control for exponential stability. The numerical examples and their computer simulations are provided in Section 4 to indicate the effectiveness of the proposed criteria. Finally, this paper is concluded in Section 5.

## 2. Model Description and Mathematic Preliminaries

The following notation will be used in this paper:  $\mathbb{R}$  and  $\mathbb{R}^+$  denote the set of real numbers and the set of nonnegative real numbers, respectively.  $\mathbb{R}^n$  denotes the  $n$ -dimensional space.  $\mathbb{R}^{n \times r}$  denotes the set of  $n \times r$  real matrices.  $\mathbb{C}[[-\varrho, 0], \mathbb{R}^n]$  denotes the space of all continuous vector functions mapping  $[-\varrho, 0]$  into  $\mathbb{R}^n$  where  $\varrho \in \mathbb{R}^+$ .  $A^T$  and  $A^{-1}$  denote the transpose and the inverse of matrix  $A$ , respectively.  $A$  is symmetric if  $A = A^T$ ,  $\lambda(A)$  denotes all the eigenvalue of  $A$ ,  $\lambda_{\max}(A) = \max\{Re \lambda : \lambda \in \lambda(A)\}$ ,  $\lambda_{\min}(A) = \min\{Re \lambda : \lambda \in \lambda(A)\}$ ,  $A > 0$  or  $A < 0$  denotes that the matrix  $A$  is a symmetric and positive definite or negative definite matrix. If  $A, B$  are symmetric matrices,  $A > B$  means that  $A - B$  is positive definite matrix,  $I$  denotes the identity matrix with appropriate dimensions. The symmetric term in the matrix is denoted by  $*$ . The following norms will be used:  $\|\cdot\|$  refer to the Euclidean vector norm;  $\|\phi\|_c = \sup_{t \in [-\varrho, 0]} \|\phi(t)\|$  stands for the norm of a function  $\phi(\cdot) \in \mathbb{C}[[-\varrho, 0], \mathbb{R}^n]$ .

Consider the following neural network system with mixed time delays

$$\begin{aligned}
 \dot{x}(t) &= -Ax(t) + Bf(x(t)) + Cg(x(t - h(t))) + D \int_{t-d(t)}^t h(x(s))ds + Ew(t) + \mathcal{U}(t), \\
 z(t) &= A_1x(t) + B_4x(t - h(t)) + C_1u(t) + D_1 \int_{t-d(t)}^t x(s)ds + E_1w(t), \\
 x(t) &= \phi(t), \quad t \in [-\varrho, 0],
 \end{aligned}
 \tag{1}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $w(t) \in \mathbb{R}^n$  the deterministic disturbance input,  $z(t) \in \mathbb{R}^n$  the system output,  $f(x(t)), g(x(t)), h(x(t))$  the neuron activation function,  $A = \text{diag}\{a_1, \dots, a_n\} > 0$  is a diagonal matrix,  $B, C, D, E, A_1, B_4, C_1, D_1, E_1$  are the known real constant matrices with appropriate dimensions,  $\phi(t) \in \mathbb{C}[[-\varrho, 0], \mathbb{R}^n]$  the initial function, The state hybrid feedback controller  $\mathcal{U}(t)$  satisfies:

$$\mathcal{U}(t) = B_1u(t) + B_2u(t - \tau(t)) + B_3 \int_{t-d_1(t)}^t u(s)ds,
 \tag{2}$$

where  $u(t) = Kx(t)$  and  $K$  is a constant matrix control gain,  $B_1, B_2, B_3$  are the known real constant matrices with appropriate dimensions. Then, substituting it into (1), it is easy to get the following:

$$\begin{aligned}
 \dot{x}(t) &= [-A + B_1K]x(t) + Bf(x(t)) + Cg(x(t - h(t))) + D \int_{t-d(t)}^t h(x(s))ds \\
 &\quad + Ew(t) + B_2Kx(t - \tau(t)) + B_3K \int_{t-d_1(t)}^t x(s)ds, \\
 z(t) &= [A_1 + C_1K]x(t) + B_4x(t - h(t)) + D_1 \int_{t-d(t)}^t x(s)ds + E_1w(t), \\
 x(t) &= \phi(t), \quad t \in [-\varrho, 0],
 \end{aligned}
 \tag{3}$$

where the time-varying delays function  $h(t), \tau(t), d(t)$  and  $d_1(t)$  satisfy the condition

$$\begin{aligned}
 0 \leq h_1 \leq h(t) \leq h_2, \quad 0 \leq d(t) \leq d, \\
 0 \leq \tau(t) \leq \tau, \quad 0 \leq d_1(t) \leq d_1,
 \end{aligned}
 \tag{4}$$

where  $h_1, h_2, \tau, d, d_1, \varrho = \max\{h_2, \tau, d, d_1\}$  are real constant scalars and we denote  $h_{12} = h_2 - h_1$ .

Throughout this paper, we consider activation functions  $f(\cdot), g(\cdot)$  and  $h(\cdot)$  satisfy Lipschitzian with the Lipschitz constants  $\hat{f}_i, \hat{g}_i$  and  $\hat{h}_i > 0$ :

$$\begin{aligned}
 |f_i(x_1) - f_i(x_2)| &\leq \hat{f}_i|x_1 - x_2|, \\
 |g_i(x_1) - g_i(x_2)| &\leq \hat{g}_i|x_1 - x_2|, \\
 |h_i(x_1) - h_i(x_2)| &\leq \hat{h}_i|x_1 - x_2|,
 \end{aligned}
 \tag{5}$$

where  $i = 1, 2, \dots, n, \forall x_1, x_2 \in \mathbb{R}$  and we denote that

$$\begin{aligned}
 F &= \text{diag}\{\hat{f}_i, i = 1, 2, \dots, n\}, \\
 G &= \text{diag}\{\hat{g}_i, i = 1, 2, \dots, n\}, \\
 H &= \text{diag}\{\hat{h}_i, i = 1, 2, \dots, n\}.
 \end{aligned}
 \tag{6}$$

**Remark 1.** If  $B_2 = 0, B_3 = 0, B_4 = 0, D_1 = 0, E_1 = 0$ , and  $f(\cdot) = g(\cdot) = h(\cdot)$ , the system (3) turns into the neural network with activation functions and time-varying delays proposed by [23]

$$\begin{aligned}
 \dot{x}(t) &= [-A + B_1K]x(t) + Bf(x(t)) + Cf(x(t - h(t))) + D \int_{t-d(t)}^t f(x(s))ds + Ew(t), \\
 z(t) &= [A_1 + C_1K]x(t), \\
 x(t) &= \phi(t), \quad t \in [-\varrho, 0].
 \end{aligned}
 \tag{7}$$

Hence, the system (3) is general neural networks model, with (7) as the special case.

The following definition and lemma are necessary in the proof of the main results:

**Definition 1** ([29]). Given  $\alpha > 0$ . The zero solution of system (1), where  $u(t) = 0, w(t) = 0$ , is  $\alpha$ -stable if there is a positive number  $N > 0$  such that every solution of the system satisfies

$$\|x(t, \phi)\| \leq N\|\phi\|_c e^{-\alpha t}, \quad \forall t \geq 0.$$

**Definition 2** ([29]). Consider  $\alpha > 0$  and  $\gamma > 0$ . The  $H_\infty$  control problem for system (1) has a solution if there exists a memoryless state feedback controller  $u(t) = Kx(t)$  satisfying the following two requirements:

- (i) The zero solution of the closed-loop system, where  $w(t) = 0$ ,  $\dot{x}(t) = -Ax(t) + Bf(x(t)) + Cg(x(t - h(t))) + D \int_{t-d(t)}^t h(x(s))ds + \mathcal{W}(t)$ , is  $\alpha$ -stable.
- (ii) There is a number  $c_0 > 0$  such that

$$\sup \frac{\int_0^\infty \|z(t)\|^2 dt}{c_0 \|\phi\|_c^2 + \int_0^\infty \|w(t)\|^2 dt} \leq \gamma,$$

where the supremum is taken over all  $\phi(t) \in \mathbb{C}[-\varrho, 0], \mathbb{R}^n$  and the non-zero uncertainty  $w(t) \in L_2([0, \infty], \mathbb{R}^n)$ .

**Lemma 1** ([30], Cauchy inequality). For any symmetric positive definite matrix  $N \in M^{n \times n}$  and  $x, y \in \mathbb{R}^n$  we have

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

**Lemma 2** ([30]). For a positive definite matrix  $Z \in \mathbb{R}^{n \times n}$ , and two scalars  $0 \leq r_1 < r_2$  and vector function  $x : [r_1, r_2] \rightarrow \mathbb{R}^n$  such that the following integrals are well defined, one has

$$\left( \int_{r_1}^{r_2} x(s) ds \right)^T Z \left( \int_{r_1}^{r_2} x(s) ds \right) \leq (r_2 - r_1) \int_{r_1}^{r_2} x^T(s) Z x(s) ds.$$

**Lemma 3** ([31]). For any positive definite symmetric constant matrix  $P$  and scalar  $\tau > 0$ , such that the following integrals are well defined, one has

$$- \int_{-\tau}^0 \int_{t+\theta}^t x^T(s) P x(s) ds d\theta \leq - \frac{2}{\tau^2} \left( \int_{-\tau}^0 \int_{t+\theta}^t x(s) ds d\theta \right)^T P \left( \int_{-\tau}^0 \int_{t+\theta}^t x(s) ds d\theta \right).$$

**Lemma 4** ([32]). For given matrices  $H, E$ , and  $F$  with  $F^T F \leq I$  and a scalar  $\epsilon > 0$ , the following inequality holds:

$$HFE + (HFE)^T \leq \epsilon H H^T + \epsilon^{-1} E^T E.$$

### 3. Stability Analysis

In this section, we will present stability criterion for system (3).

Consider a Lyapunov–Krasovskii functional candidate as

$$V(t, x_t) = \sum_{i=1}^{14} V_i(t, x_t), \tag{8}$$

where

$$\begin{aligned} V_1(t, x_t) &= x^T(t)P_1x(t) + 2x^T(t)P_2 \int_{t-h_2}^t x(s)ds \\ &\quad + \left( \int_{t-h_2}^t x(s)ds \right)^T P_3 \int_{t-h_2}^t x(s)ds + 2x^T(t)P_4 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds \\ &\quad + 2 \left( \int_{t-h_2}^t x(s)ds \right)^T P_5 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds \\ &\quad + \left( \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds \right)^T P_6 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds, \\ V_2(t, x_t) &= \int_{t-h_1}^t e^{2\alpha(s-t)} x^T(s)R_1x(s)ds, \\ V_3(t, x_t) &= \int_{t-h_2}^t e^{2\alpha(s-t)} x^T(s)R_2x(s)ds, \\ V_4(t, x_t) &= h_1 \int_{-h_1}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)Q_1\dot{x}(\theta)d\theta ds, \\ V_5(t, x_t) &= h_2 \int_{-h_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)Q_2\dot{x}(\theta)d\theta ds, \\ V_6(t, x_t) &= h_{12} \int_{-h_2}^{-h_1} \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)Z_2\dot{x}(\theta)d\theta ds, \\ V_7(t, x_t) &= \int_{-d}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} h^T(x(\theta))Uh(x(\theta))d\theta ds, \\ V_8(t, x_t) &= \int_{-d_1}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} u^T(\theta)S_2u(\theta)d\theta ds, \\ V_9(t, x_t) &= \tau \int_{-\tau}^0 \int_{t+s}^t e^{-2\alpha(\theta-t)} \dot{u}^T(\theta)S_1\dot{u}(\theta)d\theta ds, \\ V_{10}(t, x_t) &= \int_{-h_2}^{-h_1} \int_s^0 \int_{t+u}^t e^{2\alpha(\theta+u-t)} \dot{x}^T(\theta)Z_1\dot{x}(\theta)d\theta duds, \\ V_{11}(t, x_t) &= \int_{-h_1}^0 \int_\tau^0 \int_{t+s}^t e^{2\alpha(\theta+s-t)} \dot{x}^T(\theta)W_1\dot{x}(\theta)d\theta dsd\tau, \\ V_{12}(t, x_t) &= \int_{-h_2}^0 \int_\tau^0 \int_{t+s}^t e^{2\alpha(\theta+s-t)} \dot{x}^T(\theta)W_2\dot{x}(\theta)d\theta dsd\tau, \\ V_{13}(t, x_t) &= \int_{-h_2}^0 \int_\tau^0 \int_{t+s}^t e^{2\alpha(\theta+s-t)} \dot{x}^T(\theta)W_3\dot{x}(\theta)d\theta dsd\tau, \\ V_{14}(t, x_t) &= \int_{-d}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} x^T(\theta)Q_3x(\theta)d\theta ds. \end{aligned}$$

**Remark 2.** Note that the Lyapunov–Krasovskii functional,  $P_i$  ( $i = 1, 2, \dots, 6$ ) in  $V_1(t, x_t)$  are not necessary to be positive definite.

**Proposition 1.** Given  $\alpha > 0$ , the Lyapunov–Krasovskii functional (8) is positive definite, if there exist matrices  $P_1 = P_1^T, P_3 = P_3^T, P_6 = P_6^T, P_2, P_4, P_5, Q_1 > 0, Q_2 > 0, Q_3 > 0, R_1 > 0,$

$R_2 > 0, S_1 > 0, S_2 > 0, U > 0, W_1 > 0, W_2 > 0, W_3 > 0, Z_1 > 0, Z_2 > 0$  such that the following LMI holds:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & P_5 \\ * & * & \Theta_{33} \end{bmatrix} > 0, \tag{9}$$

where

$$\Theta_{11} = P_1 + h_2 e^{-2\alpha h_2} Q_2 + 0.5 h_2 e^{-4\alpha h_2} W_3, \Theta_{12} = P_2 - e^{-2\alpha h_2} Q_2,$$

$$\Theta_{13} = P_4 - h_2^{-1} e^{-4\alpha h_2} W_3,$$

$$\Theta_{22} = P_3 + h_2^{-1} e^{-2\alpha h_2} (Q_2 + R_2),$$

$$\Theta_{33} = P_6 + h_2^{-3} e^{-4\alpha h_2} (W_3 + W_3^T).$$

**Proof.** Let  $y_1(t) = \int_{t-h_2}^t x(s)ds, y_2(t) = \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds$ , then we can get

$$\begin{aligned} V_1(t, x_t) &= x^T(t)P_1x(t) + 2x^T(t)P_2y_1(t) + y_1^T(t)P_3y_1(t) + 2x^T(t)P_4y_2(t) \\ &\quad + 2y_1^T(t)P_5y_2(t) + y_2^T(t)P_6y_2(t), \end{aligned} \tag{10}$$

$$\begin{aligned} V_3(t, x_t) &\geq e^{-2\alpha h_2} \int_{t-h_2}^t x^T(s)R_2x(s)ds \\ &= h_2^{-1} e^{-2\alpha h_2} y_1^T(t)R_2y_1(t), \end{aligned} \tag{11}$$

$$\begin{aligned} V_5(t, x_t) &\geq h_2 e^{-2\alpha h_2} \int_{-h_2}^0 \int_{t+s}^t \dot{x}^T(\theta)Q_2\dot{x}(\theta)d\theta ds \\ &\geq h_2 e^{-2\alpha h_2} \int_{-h_2}^0 -s^{-1} \left( \int_{t+s}^t \dot{x}(\theta)d\theta \right)^T Q_2 \left( \int_{t+s}^t \dot{x}(\theta)d\theta \right) ds \\ &\geq e^{-2\alpha h_2} \int_{-h_2}^0 [x(t) - x(t+s)]^T Q_2 [x(t) - x(t+s)] ds \\ &= \begin{bmatrix} x(t) \\ y_1(t) \end{bmatrix}^T \begin{bmatrix} h_2 e^{-2\alpha h_2} Q_2 & -e^{-2\alpha h_2} Q_2 \\ * & h_2^{-1} e^{-2\alpha h_2} Q_2 \end{bmatrix} \begin{bmatrix} x(t) \\ y_1(t) \end{bmatrix}, \end{aligned} \tag{12}$$

$$\begin{aligned} V_{13}(t, x_t) &\geq e^{-2\alpha h_2} \int_{-h_2}^0 \int_{\tau}^0 \int_{t+s}^t \dot{x}^T(\theta)W_3\dot{x}(\theta)d\theta ds d\tau \\ &\geq e^{-2\alpha h_2} \int_{-h_2}^0 \int_{\tau}^0 -s^{-1} \left( \int_{t+s}^t \dot{x}(\theta)d\theta \right)^T W_3 \left( \int_{t+s}^t \dot{x}(\theta)d\theta \right) ds d\tau \\ &\geq h_2^{-1} e^{-2\alpha h_2} \int_{-h_2}^0 \int_{\tau}^0 [x(t) - x(t+s)]^T W_3 [x(t) - x(t+s)] ds d\tau \\ &= \begin{bmatrix} x(t) \\ y_2(t) \end{bmatrix}^T \begin{bmatrix} 0.5 h_2 e^{-4\alpha h_2} W_3 & -h_2^{-1} e^{-4\alpha h_2} W_3 \\ * & h_2^{-3} e^{-4\alpha h_2} (W_3 + W_3^T) \end{bmatrix} \begin{bmatrix} x(t) \\ y_2(t) \end{bmatrix}. \end{aligned} \tag{13}$$

Combining with  $V_2(t, x_t), V_4(t, x_t), V_6(t, x_t) - V_{12}(t, x_t), V_{14}(t, x_t)$ , we know that if the LMI (9) holds, the LKF (8) is positive definite. This completes the proof.  $\square$

Let us set

$$\begin{aligned} \lambda_1 &= \lambda_{\min}(P_1), \\ \lambda_2 &= 3h_2^2 \lambda_{\max}(\Theta) + h_1 \lambda_{\max}(R_1) + h_1^2 \lambda_{\max}(Q_1) + h_{12}^2 \lambda_{\max}(Z_2) \\ &\quad + d_1^2 \lambda_{\max}(P_1^{-1} B_1^T S_2 B_1 P_1^{-1}) + \tau^2 \lambda_{\max}(P_1^{-1} B_1^T S_1 B_1 P_1^{-1}) + h_{12} h_2^2 \lambda_{\max}(Z_1) \\ &\quad + h_1^2 \lambda_{\max}(W_1) + h_2^2 \lambda_{\max}(W_2) + d^2 \lambda_{\max}(Q_3) + d^2 \lambda_{\max}(H^T UH). \end{aligned}$$

**Remark 3.** It is noted that the previous works [23–28] consider the Lyapunov matrices  $P_1, P_3$  and  $P_6$  which are positive definite. In our paper, we remove this restriction by applying the method of

constructing complicated Lyapunov  $V_1(t, x_t)$ ,  $V_3(t, x_t)$ ,  $V_5(t, x_t)$  and  $V_{13}(t, x_t)$  as shown in the proof of Proposition 1. Hence,  $P_1$ ,  $P_3$  and  $P_6$  are only real matrices. It can be seen that our paper are more applicable and less conservative than aforementioned works.

**Theorem 1.** Given  $\alpha > 0$ , The  $H_\infty$  control of system (3) has a solution if there exist symmetric positive definite matrices  $Q_1, Q_2, Q_3, R_1, R_2, S_1, S_2, S_3, W_1, W_2, W_3, Z_1, Z_2, Z_3$ , diagonal matrices  $U > 0, U_2 > 0, U_3 > 0$ , and matrices  $P_1 = P_1^T, P_3 = P_3^T, P_6 = P_6^T, P_2, P_4, P_5$  such that the following LMIs hold:

$$\Xi_1 = \begin{bmatrix} \Pi & F^T P_1 & P_1 & 2dP_1 D & 4P_1 B_2 & 2d_1 P_1 B_3 & P_1 E \\ * & -U_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -U_3 & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{1(4,4)} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{1(5,5)} & 0 & 0 \\ * & * & * & * & * & \Xi_{1(6,6)} & 0 \\ * & * & * & * & * & * & -0.5\gamma \end{bmatrix} < 0, \tag{14}$$

$$\Xi_2 = \begin{bmatrix} -0.4R_1 & R_1 B & R_1 C & 2dR_1 D & 4R_1 B_2 & 2d_1 R_1 B_3 & R_1 B_1 & R_1 E \\ * & -U_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -U_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{2(4,4)} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{2(5,5)} & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{2(6,6)} & 0 & 0 \\ * & * & * & * & * & * & -S_3 & 0 \\ * & * & * & * & * & * & * & -0.5\gamma \end{bmatrix} < 0, \tag{15}$$

$$\Xi_3 = [ -0.5e^{-2\alpha h_2} Q_2 + \mathcal{N} ] < 0, \tag{16}$$

$$\Xi_4 = [ -0.1R_1 + \tau^2 B_1^T S_1 B_1 ] < 0, \tag{17}$$

where

$$\Pi = \begin{bmatrix} \Pi_{1,1} & \Pi_{1,2} & 0 & \Pi_{1,4} & \Pi_{1,5} & \Pi_{1,6} & \Pi_{1,7} & 0 & \Pi_{1,9} & -A^T R_1 \\ * & \Pi_{2,2} & \Pi_{2,3} & \Pi_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Pi_{3,3} & \Pi_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{4,4} & 0 & -P_3 & 0 & 0 & -P_5 & 0 \\ * & * & * & * & \Pi_{5,5} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{6,6} & 0 & 0 & \Pi_{6,9} & P_2 \\ * & * & * & * & * & * & \Pi_{7,7} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Pi_{8,8} & 0 & 0 \\ * & * & * & * & * & * & * & * & -2\alpha P_6 & P_4 \\ * & * & * & * & * & * & * & * & * & \Pi_{10,10} \end{bmatrix} < 0,$$

$$\begin{aligned} \Pi_{1,1} &= -AP_1 - P_1 A^T + B_1 B_1 + B_1^T B_1^T + P_2 + P_2^T + h_2 P_4 + h_2 P_4^T - e^{-2\alpha h_1} Q_1 \\ &\quad - 0.5e^{-2\alpha h_2} Q_2 + dQ_3 + dH^T UH - e^{-4\alpha h_1} (W_1 + W_1^T) - e^{-4\alpha h_2} (Z_1 + Z_1^T) \\ &\quad - e^{-4\alpha h_2} (W_2 + W_2^T + W_3 + W_3^T) + F^T U_2 F + B^T U_2 B + R_1 + R_2 - 2\alpha P_1, \end{aligned}$$

$$\Pi_{1,2} = e^{-2\alpha h_1} Q_1, \quad \Pi_{1,4} = -P_2 + e^{-2\alpha h_2} Q_2, \quad \Pi_{1,5} = 2h_1^{-1} e^{-2\alpha h_1} W_1,$$

$$\Pi_{1,6} = P_3 - P_4 + h_2 P_5^T + 2h_2^{-1} e^{-4\alpha h_2} (W_2 + W_3) - 2\alpha P_2, \quad \Pi_{1,7} = 2h_{12}^{-1} e^{-4\alpha h_2} Z_1,$$

$$\Pi_{1,9} = P_5 + h_2 P_6 - 2\alpha P_4, \quad \Pi_{2,2} = -e^{-2\alpha h_1} (R_1 + Q_1) - e^{-2\alpha h_2} Z_2,$$

$$\Pi_{2,3} = e^{-2\alpha h_2} (Z_2 - Z_3), \quad \Pi_{2,4} = e^{-2\alpha h_2} Z_3,$$

$$\Pi_{3,3} = -e^{-2\alpha h_2} (Z_2 + Z_2^T) + e^{-2\alpha h_2} (Z_3 + Z_3^T) + G^T C^T U_3 C G + G^T U_3 G,$$

$$\Pi_{3,4} = e^{-2\alpha h_2} (Z_2 - Z_3), \quad \Pi_{4,4} = -e^{-2\alpha h_2} (Z_2 + R_2 + Q_2),$$

$$\Pi_{5,5} = -h_1^{-2} e^{-4\alpha h_1} (W_1 + W_1^T),$$

$$\Pi_{6,6} = -P_5 - P_5^T - 2\alpha P_3 - h_2^{-2} e^{-4\alpha h_2} (W_2 + W_2^T + W_3 + W_3^T),$$

$$\Pi_{7,7} = -h_{12}^{-2} e^{-4\alpha h_2} (Z_1 + Z_1^T), \quad \Pi_{8,8} = -d^{-1} e^{-2\alpha d} Q_3,$$

$$\Pi_{10,10} = -1.5R_1 + h_1^2 Q_1 + h_2^2 Q_2 + h_{12}^2 Z_2 + h_{12} h_2 Z_1 + h_1 W_1 + h_2 W_2 + h_2 W_3$$

$$\Xi_{1(4,4)} = -2de^{-2\alpha d} U, \quad \Xi_{1(5,5)} = -4e^{-2\alpha \tau} S_1, \quad \Xi_{1(6,6)} = -2d_1 e^{-2\alpha d_1} S_2,$$

$$\begin{aligned} \Xi_{2(4,4)} &= -2de^{-2\alpha d}U, \quad \Xi_{2(5,5)} = -4e^{-2\alpha\tau}S_1, \quad \Xi_{2(6,6)} = -2d_1e^{-2\alpha d_1}S_2, \\ \mathcal{N} &= e^{-2\alpha\tau}B_1^T S_1 B_1 + dB_1^T S_2 B_1 + B_1^T S_3 B_1. \end{aligned}$$

Moreover, stabilizing feedback control is given by

$$u(t) = B_1 P_1^{-1} x(t), \quad t \geq 0,$$

and the solution of the system satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c e^{-\alpha t}, \quad t \geq 0.$$

**Proof.** Choosing the Lyapunov–Krasovskii functional candidate as (8), It is easy to check that

$$\lambda_1 \|x(t)\|^2 \leq V(t, x_t), \forall t \geq 0, \quad \text{and} \quad V(0, x_0) \leq \lambda_2 \|\phi\|_c^2. \tag{18}$$

We take the time-derivative of  $V_i$  along the solutions of system (3)

$$\begin{aligned} \dot{V}_1(t, x_t) &= 2x^T(t)P_1\dot{x}(t) + 2x^T(t)P_2[x(t) - x(t - h_2)] + 2\left(\int_{t-h_2}^t x(s)ds\right)^T P_2\dot{x}(t) \\ &\quad + 2[x(t) - x(t - h_2)]^T P_3 \int_{t-h_2}^t x(s)ds + 2x^T(t)P_4[h_2x(t) - \int_{t-h_2}^t x(s)ds] \\ &\quad + 2\left(\int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds\right)^T P_4\dot{x}(t) \\ &\quad + 2\left(\int_{t-h_2}^t x(s)ds\right)^T P_5[h_2x(t) - \int_{t-h_2}^t x(s)ds] \\ &\quad + 2[x(t) - x(t - h_2)]^T P_5 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds \\ &\quad + 2[h_2x(t) - x(t - h_2)]^T P_6 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds \\ &= -2x^T(t)AP_1x(t) + 2x^T(t)B_1^T B_1x(t) + 2f^T(x(t))B^T P_1x(t) \\ &\quad + 2g^T(x(t - h(t)))C^T P_1x(t) + 2\left(\int_{t-d(t)}^t h(x(s))ds\right)^T D^T P_1x(t) \\ &\quad + 2w^T(t)E^T P_1x(t) + 2u^T(t - \tau(t))B_2^T P_1x(t) \\ &\quad + 2\left(\int_{t-d_1(t)}^t u(s)ds\right)^T B_3^T P_1x(t) + 2x^T(t)P_2[x(t) - x(t - h_2)] \\ &\quad + 2\left(\int_{t-h_2}^t x(s)ds\right)^T P_2\dot{x}(t) + 2[x(t) - x(t - h_2)]^T P_3 \int_{t-h_2}^t x(s)ds \\ &\quad + 2x^T(t)P_4[h_2x(t) - \int_{t-h_2}^t x(s)ds] + 2\left(\int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds\right)^T P_4\dot{x}(t) \\ &\quad + 2\left(\int_{t-h_2}^t x(s)ds\right)^T P_5[h_2x(t) - \int_{t-h_2}^t x(s)ds] \\ &\quad + 2[x(t) - x(t - h_2)]^T P_5 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds \\ &\quad + 2[h_2x(t) - x(t - h_2)]^T P_6 \int_{-h_2}^0 \int_{t+s}^t x(\theta)d\theta ds, \end{aligned}$$



$$\begin{aligned}
 \dot{V}_2(t, x_t) &= x^T(t)R_1x(t) - e^{-2\alpha h_1}x^T(t-h_1)R_1x(t-h_1) - 2\alpha V_2, \\
 \dot{V}_3(t, x_t) &= x^T(t)R_2x(t) - e^{-2\alpha h_2}x^T(t-h_2)R_2x(t-h_2) - 2\alpha V_3, \\
 \dot{V}_4(t, x_t) &\leq h_1^2\dot{x}^T(t)Q_1\dot{x}(t) - h_1e^{-2\alpha h_1}\int_{t-h_1}^t \dot{x}^T(s)Q_1\dot{x}(s)ds - 2\alpha V_4, \\
 \dot{V}_5(t, x_t) &\leq h_2^2\dot{x}^T(t)Q_2\dot{x}(t) - h_2e^{-2\alpha h_2}\int_{t-h_2}^t \dot{x}^T(s)Q_2\dot{x}(s)ds - 2\alpha V_5, \\
 \dot{V}_6(t, x_t) &\leq h_{12}^2\dot{x}^T(t)Z_2\dot{x}(t) - h_{12}e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)ds - 2\alpha V_6, \\
 \dot{V}_7(t, x_t) &\leq dh^T(x(t))Uh(x(t)) - e^{-2\alpha d}\int_{t-d}^t h^T(x(s))Uh(x(s))ds - 2\alpha V_7, \\
 \dot{V}_8(t, x_t) &\leq d_1u^T(t)S_2u(t) - e^{-2\alpha d_1}\int_{t-d_1}^t u^T(s)S_2u(s)ds - 2\alpha V_8, \\
 \dot{V}_9(t, x_t) &\leq \tau^2\dot{u}^T(t)S_1\dot{u}(t) - \tau e^{-2\alpha \tau}\int_{t-\tau}^t \dot{u}^T(s)S_1\dot{u}(s)ds - 2\alpha V_9, \\
 \dot{V}_{10}(t, x_t) &\leq h_{12}h_2\dot{x}^T(t)Z_1\dot{x}(t) - e^{-4\alpha h_2}\int_{-h_2}^{-h_1}\int_{t+\theta}^t \dot{x}^T(u)Z_1\dot{x}(u)dud\theta - 2\alpha V_{10}, \\
 \dot{V}_{11}(t, x_t) &\leq h_1\dot{x}^T(t)W_1\dot{x}(t) - e^{-4\alpha h_1}\int_{-h_1}^0\int_{t+\tau}^t \dot{x}^T(s)W_1\dot{x}(s)dsd\tau - 2\alpha V_{11}, \\
 \dot{V}_{12}(t, x_t) &\leq h_2\dot{x}^T(t)W_2\dot{x}(t) - e^{-4\alpha h_2}\int_{-h_2}^0\int_{t+\tau}^t \dot{x}^T(s)W_2\dot{x}(s)dsd\tau - 2\alpha V_{12}, \\
 \dot{V}_{13}(t, x_t) &\leq h_2\dot{x}^T(t)W_3\dot{x}(t) - e^{-4\alpha h_2}\int_{-h_2}^0\int_{t+\tau}^t \dot{x}^T(s)W_3\dot{x}(s)dsd\tau - 2\alpha V_{13}, \\
 \dot{V}_{14}(t, x_t) &\leq dx^T(t)Q_3x(t) - e^{-2\alpha d}\int_{t-d}^t x^T(s)Q_3x(s)ds - 2\alpha V_{14}.
 \end{aligned}
 \tag{19}$$

By Lemmas 1 and 2, we have

$$\begin{aligned}
 2f^T(x(t))B^T P_1x(t) &\leq x^T(t)F^T P_1U_2^{-1}P_1Fx(t) + x^T(t)B^T U_2Bx(t), \\
 2g^T(x(t-h(t)))C^T P_1x(t) &\leq x^T(t-h(t))G^T C^T U_3CGx(t-h(t)) \\
 &\quad + x^T(t)P_1U_3^{-1}P_1x(t), \\
 2\left(\int_{t-d(t)}^t h(x(s))ds\right)^T D^T P_1x(t) &\leq \frac{e^{-2\alpha d}}{2}\int_{t-d}^t h^T(x(s))Uh(x(s))ds \\
 &\quad + 2de^{2\alpha d}x^T(t)P_1DU^{-1}D^T P_1x(t), \\
 2w^T(t)E^T P_1x(t) &\leq \frac{\gamma}{2}w^T(t)w(t) + \frac{2}{\gamma}x^T(t)P_1E^T EP_1x(t), \\
 2u^T(t-\tau(t))B_2^T P_1x(t) &\leq \frac{e^{-2\alpha \tau}}{4}u^T(t-\tau(t))S_1u(t-\tau(t)) \\
 &\quad + 4e^{2\alpha \tau}x^T(t)P_1B_2S_1^{-1}B_2^T P_1x(t), \\
 2\left(\int_{t-d_1(t)}^t u(s)ds\right)^T B_3^T P_1x(t) &\leq \frac{e^{-2\alpha d_1}}{2}\int_{t-d_1}^t u^T(s)S_2u(s)ds \\
 &\quad + 2d_1e^{2\alpha d_1}x^T(t)P_1B_3S_2^{-1}B_3^T P_1x(t), \\
 dh^T(x(t))Uh(x(t)) &\leq dx^T(t)H^T UHx(t), \\
 d_1u^T(t)S_2u(t) &= d_1x^T(t)P_1^{-1}B_1^T S_2B_1P_1^{-1}x(t), \\
 \tau^2\dot{u}^T(t)S_1\dot{u}(t) &= \tau^2\dot{x}^T(t)P_1^{-1}B_1^T S_1B_1P_1^{-1}\dot{x}(t),
 \end{aligned}
 \tag{20}$$

and the Leibniz–Newton formula gives

$$\begin{aligned}
 -\tau e^{-2\alpha\tau} \int_{t-\tau}^t \dot{u}^T(s) S_2 \dot{u}(s) ds &\leq -\tau(t) e^{-2\alpha\tau} \int_{t-\tau(t)}^t \dot{u}^T(s) S_2 \dot{u}(s) ds \\
 &\leq -e^{-2\alpha\tau} \left( \int_{t-\tau(t)}^t \dot{u}(s) ds \right)^T S_2 \left( \int_{t-\tau(t)}^t \dot{u}(s) ds \right) \\
 &\leq -e^{-2\alpha\tau} u^T(t) S_1 u(t) + 2e^{-2\alpha\tau} u^T(t) S_1 u(t - \tau(t)) \\
 &\quad - e^{-2\alpha\tau} u^T(t - \tau(t)) S_1 u(t - \tau(t)) \\
 &\leq -e^{-2\alpha\tau} u^T(t) S_1 u(t) + 2e^{-2\alpha\tau} u^T(t) S_1 u(t) \\
 &\quad + \frac{e^{-2\alpha\tau}}{2} u^T(t - \tau(t)) S_1 S_1^{-1} S_1 u(t - \tau(t)) \\
 &\quad - e^{-2\alpha\tau} u^T(t - \tau(t)) S_1 u(t - \tau(t)) \\
 &= e^{-2\alpha\tau} u^T(t) S_1 u(t) \\
 &\quad - \frac{e^{-2\alpha\tau}}{2} u^T(t - \tau(t)) S_1 u(t - \tau(t)) \\
 &= e^{-2\alpha\tau} x^T(t) P_1^{-1} B_1^T S_1 B_1 P_1^{-1} x(t) \\
 &\quad - \frac{e^{-2\alpha\tau}}{2} u^T(t - \tau(t)) S_1 u(t - \tau(t)).
 \end{aligned} \tag{21}$$

Denote

$$\sigma_1(t) = \int_{t-h_2}^{t-h(t)} \dot{x}(s) ds, \quad \sigma_2(t) = \int_{t-h(t)}^{t-h_1} \dot{x}(s) ds. \tag{22}$$

Next, when  $0 < h_1 < h(t) < h_2$ , we have

$$\int_{t-h_2}^{t-h_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds = \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) Z_2 \dot{x}(s) ds + \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds.$$

Using Lemma 2, we get

$$h_{12} \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) Z_2 \dot{x}(s) ds \geq \frac{h_{12}}{h_2 - h(t)} \sigma_1^T(t) Z_2 \sigma_1(t),$$

and

$$h_{12} \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds \geq \frac{h_{12}}{h(t) - h_1} \sigma_2^T(t) Z_2 \sigma_2(t),$$

then

$$\begin{aligned}
 h_{12} \int_{t-h_2}^{t-h_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds &\geq \frac{h_{12}}{h_2 - h(t)} \sigma_1^T(t) Z_2 \sigma_1(t) + \frac{h_{12}}{h(t) - h_1} \sigma_2^T(t) Z_2 \sigma_2(t) \\
 &= \sigma_1^T(t) Z_2 \sigma_1(t) + \frac{h(t) - h_1}{h_2 - h(t)} \sigma_1^T(t) Z_2 \sigma_1(t) \\
 &\quad + \sigma_2^T(t) Z_2 \sigma_2(t) + \frac{h_2 - h(t)}{h(t) - h_1} \sigma_2^T(t) Z_2 \sigma_2(t).
 \end{aligned} \tag{23}$$

By reciprocally convex with  $a = \frac{h_2 - h(t)}{h_{12}}$ ,  $b = \frac{h(t) - h_1}{h_{12}}$ , the following inequality holds:

$$\begin{bmatrix} \sqrt{\frac{b}{a}} \sigma_1(t) \\ -\sqrt{\frac{a}{b}} \sigma_2(t) \end{bmatrix}^T \begin{bmatrix} Z_2 & Z_3 \\ Z_3^T & Z_2 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{b}{a}} \sigma_1(t) \\ -\sqrt{\frac{a}{b}} \sigma_2(t) \end{bmatrix} \geq 0, \tag{24}$$

which implies

$$\frac{h(t) - h_1}{h_2 - h(t)} \sigma_1^T(t) Z_2 \sigma_1(t) + \frac{h_2 - h(t)}{h(t) - h_1} \sigma_2^T(t) Z_2 \sigma_2(t) \geq \sigma_1^T(t) Z_3 \sigma_2(t) + \sigma_2^T(t) Z_3^T \sigma_1(t). \quad (25)$$

Then, we can get from (22)–(25) that

$$\begin{aligned} -e^{-2\alpha h_2} h_{12} \int_{t-h_2}^{t-h_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds &\leq -e^{-2\alpha h_2} \left[ \sigma_1^T(t) Z_2 \sigma_1(t) + \sigma_2^T(t) Z_2 \sigma_2(t) \right. \\ &\quad \left. + \sigma_1^T(t) Z_3 \sigma_2(t) + \sigma_2^T(t) Z_3^T \sigma_1(t) \right]. \end{aligned} \quad (26)$$

By using Lemmas 2 and 3, we obtain

$$\begin{aligned} -h_1 e^{-2\alpha h_1} \int_{t-h_1}^t \dot{x}^T(s) Q_1 \dot{x}(s) ds &\leq -e^{-2\alpha h_1} \left( \int_{t-h_1}^t \dot{x}(s) ds \right)^T Q_1 \left( \int_{t-h_1}^t \dot{x}(s) ds \right) \\ &= -e^{-2\alpha h_1} [x(t) - x(t-h_1)]^T \\ &\quad \times Q_1 [x(t) - x(t-h_1)], \\ -h_2 e^{-2\alpha h_2} \int_{t-h_2}^t \dot{x}^T(s) Q_2 \dot{x}(s) ds &\leq -e^{-2\alpha h_2} \left( \int_{t-h_2}^t \dot{x}(s) ds \right)^T Q_2 \left( \int_{t-h_2}^t \dot{x}(s) ds \right) \\ &= -e^{-2\alpha h_2} [x(t) - x(t-h_2)]^T \\ &\quad \times Q_2 [x(t) - x(t-h_2)], \\ -e^{-2\alpha d} \int_{t-d}^t x^T(s) Q_3 x(s) ds &\leq -\frac{e^{-2\alpha d}}{d} \left( \int_{t-d}^t x(s) ds \right)^T Q_3 \left( \int_{t-d}^t x(s) ds \right), \\ -e^{-4\alpha h_1} \int_{-h_1}^0 \int_{t+\tau}^t \dot{x}^T(s) W_1 \dot{x}(s) ds d\tau &\leq -2 \frac{e^{-4\alpha h_1}}{h_1^2} \left( \int_{-h_1}^0 \int_{t+\tau}^t \dot{x}(s) ds d\tau \right)^T \\ &\quad \times W_1 \int_{-h_1}^0 \int_{t+\tau}^t \dot{x}(s) ds d\tau \\ &= -2 \frac{e^{-4\alpha h_1}}{h_1^2} \left( h_1 x(t) - \int_{t-h_1}^t x(s) ds \right)^T \\ &\quad \times W_1 \left( h_1 x(t) - \int_{t-h_1}^t x(s) ds \right), \\ -e^{-4\alpha h_2} \int_{-h_2}^0 \int_{t+\tau}^t \dot{x}^T(s) W_2 \dot{x}(s) ds d\tau &\leq -2 \frac{e^{-4\alpha h_2}}{h_2^2} \left( h_2 x(t) - \int_{t-h_2}^t x(s) ds \right)^T \\ &\quad \times W_2 \left( h_2 x(t) - \int_{t-h_2}^t x(s) ds \right), \\ -e^{-4\alpha h_2} \int_{-h_2}^0 \int_{t+\tau}^t \dot{x}^T(s) W_3 \dot{x}(s) ds d\tau &\leq -2 \frac{e^{-4\alpha h_2}}{h_2^2} \left( h_2 x(t) - \int_{t-h_2}^t x(s) ds \right)^T \\ &\quad \times W_3 \left( h_2 x(t) - \int_{t-h_2}^t x(s) ds \right), \\ -e^{-4\alpha h_2} \int_{-h_2}^{-h_1} \int_{t+\theta}^t \dot{x}^T(u) Z_1 \dot{x}(u) du d\theta &\leq -2 \frac{e^{-4\alpha h_2}}{h_{12}^2} \left( \int_{-h_2}^{-h_1} \int_{t+\theta}^t \dot{x}(u) du d\theta \right)^T \\ &\quad \times Z_1 \left( \int_{-h_2}^{-h_1} \int_{t+\theta}^t \dot{x}(u) du d\theta \right) \\ &\leq -2 \frac{e^{-4\alpha h_2}}{h_{12}^2} \left( h_{12} x(t) - \int_{t-h_2}^{t-h_1} x(s) ds \right)^T \\ &\quad \times Z_1 \left( h_{12} x(t) - \int_{t-h_2}^{t-h_1} x(s) ds \right). \end{aligned} \quad (27)$$

By using the following identity relation:

$$\begin{aligned}
 0 = & -\dot{x}(t) - Ax(t) + Bf(x(t)) + Cg(x(t - h(t))) + Ew(t) + D \int_{t-d(t)}^t h(x(s))ds \\
 & + B_1u(t) + B_2u(t - \tau(t)) + B_3 \int_{t-d_1(t)}^t u(s)ds,
 \end{aligned} \tag{28}$$

we have

$$\begin{aligned}
 0 = & -2\dot{x}^T R_1 \dot{x}(t) - 2\dot{x}^T R_1 Ax(t) + 2\dot{x}^T R_1 Bf(x(t)) + 2\dot{x}^T R_1 Cg(x(t - h(t))) \\
 & + 2\dot{x}^T R_1 D \int_{t-d(t)}^t h(x(s))ds + 2\dot{x}^T R_1 Ew(t) + 2\dot{x}^T R_1 B_1u(t) \\
 & + 2\dot{x}^T R_1 B_2u(t - \tau(t)) + 2\dot{x}^T R_1 B_3 \int_{t-d_1(t)}^t u(s)ds.
 \end{aligned} \tag{29}$$

By Lemmas 1 and 2, we get

$$\begin{aligned}
 2\dot{x}^T R_1 Bf(x(t)) & \leq \dot{x}^T(t) R_1 B U_2^{-1} B^T R_1 \dot{x}(t) + f^T(x(t)) U_2 f(x(t)) \\
 & \leq \dot{x}^T(t) R_1 B U_2^{-1} B^T R_1 \dot{x}(t) + x^T(t) F^T U_2 F x(t), \\
 2\dot{x}^T R_1 Cg(x(t - h(t))) & \leq \dot{x}^T(t) R_1 C U_3^{-1} C^T R_1 \dot{x}(t) \\
 & \quad + g^T(x(t - h(t))) U_3 g(x(t - h(t))) \\
 & \leq \dot{x}^T(t) R_1 C U_3^{-1} C^T R_1 \dot{x}(t) \\
 & \quad + x^T(t - h(t)) G^T U_3 G x(t - h(t)), \\
 2\dot{x}^T R_1 D \int_{t-d(t)}^t h(x(s))ds & \leq 2de^{2\alpha d} \dot{x}^T(t) R_1 D U^{-1} D^T R_1 \dot{x}(t) \\
 & \quad + \frac{e^{-2\alpha d}}{2d} \left( \int_{t-d(t)}^t h(x(s))ds \right)^T U \int_{t-d(t)}^t h(x(s))ds \\
 & \leq 2de^{2\alpha d} \dot{x}^T(t) R_1 D U^{-1} D^T R_1 \dot{x}(t) \\
 & \quad + \frac{e^{-2\alpha d}}{2} \int_{t-d}^t h^T(x(s)) U h(x(s))ds, \\
 2\dot{x}^T R_1 Ew(t) & \leq \frac{2}{\gamma} \dot{x}^T(t) R_1 E^T E R_1 \dot{x}(t) + \frac{\gamma}{2} w^T(t) w(t), \\
 2\dot{x}^T R_1 B_1u(t) & \leq \dot{x}^T(t) R_1 B_1 S_3^{-1} B_1^T R_1 \dot{x}(t) + u^T(t) S_3 u(t) \\
 & = \dot{x}^T(t) R_1 B_1 S_3^{-1} B_1^T R_1 \dot{x}(t) \\
 & \quad + x^T(t) P_1^{-1} B_1^T S_3 B_1 P_1^{-1} x(t), \\
 2\dot{x}^T R_1 B_2u(t - \tau(t)) & \leq 4e^{2\alpha\tau} \dot{x}^T(t) R_1 B_2 S_1^{-1} B_2^T R_1 \dot{x}(t) \\
 & \quad + \frac{e^{-2\alpha\tau}}{4} u^T(t - \tau(t)) S_1 u(t - \tau(t)), \\
 2\dot{x}^T R_1 B_3 \int_{t-d_1(t)}^t u(s)ds & \leq 2d_1 e^{2\alpha d_1} \dot{x}^T(t) R_1 B_3 S_2^{-1} B_3^T R_1 \dot{x}(t) \\
 & \quad + \frac{e^{-2\alpha d_1}}{2d_1} \left( \int_{t-d_1(t)}^t u(s)ds \right)^T S_2 \left( \int_{t-d_1(t)}^t u(s)ds \right) \\
 & \leq 2d_1 e^{2\alpha d_1} \dot{x}^T(t) R_1 B_3 S_2^{-1} B_3^T R_1 \dot{x}(t) \\
 & \quad + \frac{e^{-2\alpha d_1}}{2} \int_{t-d_1}^t u^T(s) S_2 u(s)ds.
 \end{aligned} \tag{30}$$

From (19)–(30), we obtain

$$\begin{aligned}
 \dot{V}(t, x_t) + 2\alpha V(t, x_t) \leq & \gamma w^T(t)w(t) + \zeta^T(t)\mathcal{M}_1\zeta(t) + \dot{x}^T(t)\mathcal{M}_2\dot{x}(t) \\
 & + x^T(t)\mathcal{M}_3x(t) + \dot{x}^T(t)\mathcal{M}_4\dot{x}(t) \\
 & - x^T(t) \left[ A_1^T A_1 + A_1^T C_1 B_1 P_1^{-1} \right] x(t) \\
 & - x^T(t) \left[ P_1^{-1} B_1^T C_1^T A_1 + P_1^{-1} B_1^T C_1^T B_1 P_1^{-1} \right] x(t) \\
 & - 2x^T(t) \left[ A_1^T B_4 + P_1^{-1} B_1^T C_1^T B_4 \right] x(t-h(t)) \\
 & - 2x^T(t) \left[ A_1^T E_1 + P_1^{-1} B_1^T C_1^T E_1 \right] w(t) \\
 & - 2x^T(t) \left[ A_1^T D_1 + P_1^{-1} B_1^T C_1^T D_1 \right] \int_{t-d}^t x(s)ds \\
 & - x^T(t-h(t)) B_4^T B_4 x(t-h(t)) \\
 & - 2x^T(t-h(t)) B_4^T D_1 \int_{t-d}^t x(s)ds \\
 & - 2x^T(t-h(t)) B_4^T E_1 w(t) - w^T(t) E_1^T E_1 w(t) \\
 & - \left( \int_{t-d}^t x(s)ds \right)^T D_1^T D_1 \int_{t-d}^t x(s)ds \\
 & - 2 \left( \int_{t-d}^t x(s)ds \right)^T D_1^T E_1 w(t),
 \end{aligned} \tag{31}$$

where

$$\begin{aligned}
 \mathcal{M}_1 &= \Pi + F^T P_1 U_2^{-1} P_1 F + P_1 U_3^{-1} P_1 + 2de^{2\alpha d} P_1 D U^{-1} D^T P_1 + 4e^{2\alpha\tau} P_1 B_2 S_1^{-1} B_2^T P_1 \\
 &\quad + 2d_1 e^{2\alpha d_1} P_1 B_3 S_2^{-1} B_3^T P_1 + \frac{2}{\gamma} P_1 E^T E P_1, \\
 \mathcal{M}_2 &= -0.4R_1 + R_1 B U_2^{-1} B^T R_1 + R_1 C U_3^{-1} C^T R_1 + 2de^{2\alpha d} R_1 D U^{-1} D^T R_1 \\
 &\quad + 4e^{2\alpha\tau} R_1 B_2 S_1^{-1} B_2^T R_1 + 2d_1 e^{2\alpha d_1} R_1 B_3 S_2^{-1} B_3^T R_1 + R_1 B_1 S_3^{-1} B_1^T R_1 \\
 &\quad + \frac{2}{\gamma} R_1 E^T E R_1, \\
 \mathcal{M}_3 &= -0.5e^{-2\alpha h_2} Q_2 + dP_1^{-1} B_1^T S_2 B_1 P_1^{-1} + e^{-2\alpha\tau} P_1^{-1} B_1^T S_1 B_1 P_1^{-1} + P_1^{-1} B_1 S_3 B_1 P_1^{-1}, \\
 \mathcal{M}_4 &= -0.1R_1 + \tau^2 P_1^{-1} B_1^T S_1 B_1 P_1^{-1}, \\
 \zeta(t) &= \left[ x^T(t) \quad x^T(t-h_1) \quad x^T(t-h(t)) \quad x^T(t-h_2) \quad \int_{t-h_1}^t x^T(s)ds \quad \int_{t-h_2}^t x^T(s)ds \right. \\
 &\quad \left. \int_{t-h_2}^{t-h_1} x^T(s)ds \quad \int_{t-d}^t x^T(s)ds \quad \int_{-h_2}^0 \int_{t+s}^t x^T(\theta)d\theta ds \quad \dot{x}^T(t) \right]^T.
 \end{aligned}$$

Using the Schur complement lemma, pre-multiplying and post-multiplying  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$  and  $\mathcal{M}_4$  by  $P_1$  and  $P_1$  respectively, the inequality  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$  and  $\mathcal{M}_4$  are equivalent to  $\Xi_1 < 0, \Xi_2 < 0, \Xi_3 < 0$  and  $\Xi_4 < 0$  respectively, and from the inequality (31) it follows that

$$\begin{aligned}
 \dot{V}(t, x_t) + 2\alpha V(t, x_t) \leq & \gamma w^T(t)w(t) - x^T(t) \left[ A_1^T A_1 + A_1^T C_1 B_1 P_1^{-1} \right] x(t) \\
 & - x^T(t) \left[ P_1^{-1} B_1^T C_1^T B_1 P_1^{-1} + P_1^{-1} B_1^T C_1^T A_1 \right] x(t) \\
 & - 2x^T(t) \left[ A_1^T B_4 + P_1^{-1} B_1^T C_1^T B_4 \right] x(t - h(t)) \\
 & - 2x^T(t) \left[ A_1^T D_1 + P_1^{-1} B_1^T C_1^T D_1 \right] \int_{t-d}^t x(s) ds \\
 & - 2x^T(t - h(t)) B_4^T D_1 \int_{t-d}^t x(s) ds \\
 & - 2x^T(t) \left[ A_1^T E_1 + P_1^{-1} B_1^T C_1^T E_1 \right] w(t) \\
 & - x^T(t - h(t)) B_4^T B_4 x(t - h(t)) \\
 & - \left( \int_{t-d}^t x(s) ds \right)^T D_1^T D_1 \int_{t-d}^t x(s) ds \\
 & - 2 \left( \int_{t-d}^t x(s) ds \right)^T D_1^T E_1 w(t) \\
 & - 2x^T(t - h(t)) B_4^T E_1 w(t) - w^T(t) E_1^T E_1 w(t).
 \end{aligned} \tag{32}$$

Letting  $w(t) = 0$ , and since

$$\begin{aligned}
 -x^T(t) \left[ A_1^T A_1 \right] x(t) & \leq 0, \\
 -x^T(t) \left[ A_1^T C_1 B_1 P_1^{-1} \right] x(t) & \leq 0, \\
 -x^T(t) \left[ P_1^{-1} B_1^T C_1^T A_1 \right] x(t) & \leq 0, \\
 -x^T(t) \left[ P_1^{-1} B_1^T C_1^T B_1 P_1^{-1} \right] x(t) & \leq 0, \\
 -2x^T(t) \left[ A_1^T B_4 \right] x(t - h(t)) & \leq 0, \\
 -2x^T(t) \left[ P_1^{-1} B_1^T C_1^T B_4 \right] x(t - h(t)) & \leq 0, \\
 -2x^T(t) \left[ A_1^T D_1 \right] \int_{t-d}^t x(s) ds & \leq 0, \\
 -2x^T(t) \left[ P_1^{-1} B_1^T C_1^T D_1 \right] \int_{t-d}^t x(s) ds & \leq 0, \\
 -x^T(t - h(t)) B_4^T B_4 x(t - h(t)) & \leq 0, \\
 -2x^T(t - h(t)) B_4^T D_1 \int_{t-d}^t x(s) ds & \leq 0, \\
 - \left( \int_{t-d}^t x(s) ds \right)^T D_1^T D_1 \int_{t-d}^t x(s) ds & \leq 0,
 \end{aligned}$$

we finally obtain from the inequality (32) that

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \leq 0,$$

we have

$$\dot{V}(t, x_t) \leq -2\alpha V(t, x_t), \forall t \geq 0. \tag{33}$$

Integrating both sides of (33) from 0 to  $t$ , we obtain

$$V(t, x_t) \leq V(0, x_0) e^{-2\alpha t}, \forall t \geq 0.$$

Taking the condition (18) into account, we have

$$\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq V(0, x_0)e^{-2\alpha t} \leq \lambda_2 \|\phi\|_c^2 e^{-2\alpha t}.$$

Then, the solution  $\|x(t, \phi)\|$  of the system (3) satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c e^{-\alpha t}, \quad \forall t \geq 0, \tag{34}$$

which implies that the zero solution of the closed-loop system is  $\alpha$ -stable. To complete the proof of the theorem, it remains to show the  $\gamma$ -optimal level condition (ii). For this, we consider the following relation:

$$\begin{aligned} \int_0^t \|z(s)\|^2 - \gamma \|w(s)\|^2 ds &= \int_0^t [\|z(s)\|^2 - \gamma \|w(s)\|^2 \\ &\quad + \dot{V}(s, x_s)] ds - \int_0^t \dot{V}(s, x_s) ds. \end{aligned}$$

Since  $V(t, x_t) \geq 0$ , we obtain

$$\begin{aligned} - \int_0^t \dot{V}(s, x_s) ds &= V(0, x_0) - V(t, x_t) \\ &\leq V(0, x_0), \quad \forall t \leq 0. \end{aligned}$$

Therefore, for all  $t \leq 0$

$$\int_0^t \|z(s)\|^2 - \gamma \|w(s)\|^2 ds \leq \int_0^t [\|z(s)\|^2 - \gamma \|w(s)\|^2 + \dot{V}(s, x_s)] ds + V(0, x_0). \tag{35}$$

From (32) we obtain that

$$\begin{aligned} \dot{V}(t, x_t) &\leq \gamma w^T(t)w(t) - x^T(t) \left[ A_1^T A_1 + A_1^T C_1 B_1 P_1^{-1} + P_1^{-1} B_1^T C_1^T A_1 \right] x(t) \\ &\quad - x^T(t) P_1^{-1} B_1^T C_1^T B_1 P_1^{-1} x(t) - 2x^T(t) \left[ A_1^T B_4 + P_1^{-1} B_1^T C_1^T B_4 \right] x(t - h(t)) \\ &\quad - 2x^T(t) \left[ A_1^T D_1 + P_1^{-1} B_1^T C_1^T D_1 \right] \int_{t-d}^t x(s) ds \\ &\quad - 2x^T(t) \left[ A_1^T E_1 + P_1^{-1} B_1^T C_1^T E_1 \right] w(t) - x^T(t - h(t)) B_4^T B_4 x(t - h(t)) \\ &\quad - 2x^T(t - h(t)) B_4^T E_1 w(t) - \left( \int_{t-d}^t x(s) ds \right)^T D_1^T D_1 \int_{t-d}^t x(s) ds \\ &\quad - 2 \left( \int_{t-d}^t x(s) ds \right)^T D_1^T E_1 w(t) - 2x^T(t - h(t)) B_4^T D_1 \int_{t-d}^t x(s) ds \\ &\quad - w^T(t) E_1^T E_1 w(t) - 2\alpha V(t, x_t). \end{aligned} \tag{36}$$

Observe that the value of  $\|z(t)\|^2$  is defined as

$$\begin{aligned}
 \|z(t)\|^2 &= z^T(t)z(t) \\
 &= x^T(t) \left[ A_1^T A_1 + A_1^T C_1 B_1 P_1^{-1} + P_1^{-1} B_1^T C_1^T A_1 + P_1^{-1} B_1^T C_1^T C_1 B_1 P_1^{-1} \right] x(t) \\
 &\quad + 2x^T(t) \left[ A_1^T B_4 + P_1^{-1} B_1^T C_1^T B_4 \right] x(t - h(t)) \\
 &\quad + x^T(t - h(t)) B_4^T B_4 x(t - h(t)) \\
 &\quad + 2x^T(t) \left[ A_1^T D_1 + P_1^{-1} B_1^T C_1^T D_1 \right] \int_{t-d}^t x(s) ds \\
 &\quad + 2x^T(t - h(t)) B_4^T D_1 \int_{t-d}^t x(s) ds \\
 &\quad + 2x^T(t) \left[ A_1^T E_1 + P_1^{-1} B_1^T C_1^T E_1 \right] w(t) + 2x^T(t - h(t)) B_4^T E_1 w(t) \\
 &\quad + \left( \int_{t-d}^t x(s) ds \right)^T D_1^T D_1 \int_{t-d}^t x(s) ds + 2 \left( \int_{t-d}^t x(s) ds \right)^T D_1^T E_1 w(t) \\
 &\quad + w^T(t) E_1^T E_1 w(t).
 \end{aligned} \tag{37}$$

Submitting the estimation of  $\dot{V}(t, x_t)$  and  $\|z(t)\|^2$ , we obtain

$$\int_0^t \|z(s)\|^2 - \gamma \|w(s)\|^2 ds \leq \int_0^t \left[ -2\alpha V(t, x_t) \right] ds + V(0, x_0). \tag{38}$$

Hence, from (38) it follows that

$$\begin{aligned}
 \int_0^t \|z(s)\|^2 - \gamma \|w(s)\|^2 ds &\leq V(0, x_0) \\
 &\leq \lambda_2 \|\phi\|_c^2,
 \end{aligned}$$

equivalently,

$$\int_0^t \|z(s)\|^2 dt \leq \int_0^t \gamma \|w(s)\|^2 ds + \lambda_2 \|\phi\|_c^2.$$

Letting  $t \rightarrow \infty$ , and setting  $c_0 = \frac{\lambda_2}{\gamma}$ , we obtain that

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{c_0 \|\phi\|_c^2 + \int_0^\infty \|w(t)\|^2 dt} \leq \gamma,$$

for all non-zero  $w(t) \in L_2([0, \infty], \mathbb{R}^n)$ ,  $\phi(t) \in \mathbb{C}[[-\varrho, 0], \mathbb{R}^n]$ . This completes the proof of the theorem.  $\square$

For neural networks with parameter uncertainties, we consider the following system

$$\begin{aligned}
 \dot{x}(t) &= [-(A + \Delta A) + B_1 K]x(t) + [B + \Delta B]f(x(t)) + [C + \Delta C]g(x(t - h(t))) \\
 &\quad + [D + \Delta D] \int_{t-d(t)}^t h(x(s)) ds + Ew(t) + B_2 Kx(t - \tau(t)) + B_3 K \int_{t-d_1(t)}^t x(s) ds, \\
 z(t) &= [A_1 + C_1 K]x(t) + B_4 x(t - h(t)) + D_1 \int_{t-d(t)}^t x(s) ds + E_1 w(t), \\
 x(t) &= \phi(t), \quad t \in [-\varrho, 0],
 \end{aligned} \tag{39}$$

where  $\Delta A, \Delta B, \Delta C$  and  $\Delta D$  are the unknown matrices, denoting the uncertainties of the concerned system and satisfying the following equation:

$$[\Delta A \ \Delta B \ \Delta C \ \Delta D] = N \tilde{F}(t) [E_A \ E_B \ E_C \ E_D], \tag{40}$$



where  $E_A, E_B, E_C$  and  $E_D$  are known matrices,  $\tilde{F}(t)$  is an unknown, real and possibly time-varying matrix with Lebesgue measurable elements and satisfies

$$\tilde{F}^T(t)\tilde{F}(t) \leq I. \tag{41}$$

Then, we have the following theorem.

**Theorem 2.** Given  $\alpha > 0$ , The  $H_\infty$  control of system (39) has a solution if there exist symmetric positive definite matrices  $Q_1, Q_2, Q_3, R_1, R_2, S_1, S_2, S_3, W_1, W_2, W_3, Z_1, Z_2, Z_3$ , diagonal matrices  $U > 0, U_2 > 0, U_3 > 0$ , and matrices  $P_1 = P_1^T, P_3 = P_3^T, P_6 = P_6^T, P_2, P_4, P_5$  such that the following LMI hold:

$$\Omega_1 = \begin{bmatrix} \tilde{\Xi}_1 & P_1N & P_1N & P_1N & 4dP_1N \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -e^{-2\alpha d}I \end{bmatrix} < 0, \tag{42}$$

$$\Omega_2 = \begin{bmatrix} \Xi_2 & R_1N & R_1N & R_1N & 4dR_1N \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -e^{-2\alpha d}I \end{bmatrix} < 0, \tag{43}$$

$$\Xi_3 = [ -0.5e^{-2\alpha h_2}Q_2 + \mathcal{N} ] < 0, \tag{44}$$

$$\Xi_4 = [ -0.1R_1 + \tau^2B_1^TS_1B_1 ] < 0, \tag{45}$$

where

$$\tilde{\Xi}_1 = \begin{bmatrix} \tilde{\Pi} & F^TP_1 & P_1 & 4dP_1D & 4P_1B_2 & 2d_1P_1B_3 & P_1E \\ * & -U_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -U_3 & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{1(4,4)} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{1(5,5)} & 0 & 0 \\ * & * & * & * & * & \Xi_{1(6,6)} & 0 \\ * & * & * & * & * & * & -0.5\gamma \end{bmatrix} < 0,$$

$$\tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} \\ * & \tilde{\Pi}_{22} \end{bmatrix} < 0$$

$$\tilde{\Pi}_{11} = \begin{bmatrix} \tilde{\Pi}_{1,1} & \tilde{\Pi}_{1,2} & 0 & \tilde{\Pi}_{1,4} & \tilde{\Pi}_{1,5} \\ * & \tilde{\Pi}_{2,2} & \tilde{\Pi}_{2,3} & \tilde{\Pi}_{2,4} & 0 \\ * & * & \tilde{\Pi}_{3,3} & \tilde{\Pi}_{3,4} & 0 \\ * & * & * & \tilde{\Pi}_{4,4} & 0 \\ * & * & * & * & \tilde{\Pi}_{5,5} \end{bmatrix}$$

$$\tilde{\Pi}_{12} = \begin{bmatrix} \tilde{\Pi}_{1,6} & \tilde{\Pi}_{1,7} & 0 & \tilde{\Pi}_{1,9} & -A^TR_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -P_3 & 0 & 0 & -P_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\Pi}_{22} = \begin{bmatrix} \tilde{\Pi}_{6,6} & 0 & 0 & \tilde{\Pi}_{6,9} & P_2 & 0 \\ * & \tilde{\Pi}_{7,7} & 0 & 0 & 0 & 0 \\ * & * & \tilde{\Pi}_{8,8} & 0 & 0 & 0 \\ * & * & * & -2\alpha P_6 & P_4 & 0 \\ * & * & * & * & \tilde{\Pi}_{10,10} & 0 \\ * & * & * & * & * & \tilde{\Pi}_{11,11} \end{bmatrix}$$

$$\begin{aligned}
 \tilde{\Pi}_{1,1} &= -AP_1 - P_1A^T + B_1B_1 + B_1^TB_1^T + B^TU_2B + P_2 + P_2^T + h_2P_4 + h_2P_4^T \\
 &\quad - e^{-2\alpha h_1}Q_1 + dQ_3 - 0.5e^{-2\alpha h_2}Q_2 + F^TU_2F - e^{-4\alpha h_1}(W_1 + W_1^T) \\
 &\quad - e^{-4\alpha h_2}(Z_1 + Z_1^T + W_2 + W_2^T + W_3 + W_3^T) + E_A^TE_A + F^TE_B^TE_B F \\
 &\quad + E_A^TE_A + F^TE_B^TE_B F + R_1 + R_2 + dH^T UH - 2\alpha P_1, \\
 \tilde{\Pi}_{1,2} &= e^{-2\alpha h_1}Q_1, \quad \tilde{\Pi}_{1,4} = -P_2 + e^{-2\alpha h_2}Q_2, \quad \tilde{\Pi}_{1,5} = 2h_1^{-1}e^{-2\alpha h_1}W_1, \\
 \tilde{\Pi}_{1,6} &= P_3 - P_4 + h_2P_5^T + 2h_2^{-1}e^{-4\alpha h_2}(W_2 + W_3) - 2\alpha P_2 \quad \tilde{\Pi}_{1,7} = 2h_{12}^{-1}e^{-4\alpha h_2}Z_1, \\
 \tilde{\Pi}_{1,9} &= P_5 + h_2P_6 - 2\alpha P_4, \quad \tilde{\Pi}_{2,2} = -e^{-2\alpha h_1}(R_1 + Q_1) - e^{-2\alpha h_2}Z_2, \\
 \tilde{\Pi}_{2,3} &= e^{-2\alpha h_2}(Z_2 - Z_3), \quad \tilde{\Pi}_{2,4} = e^{-2\alpha h_2}Z_3, \\
 \tilde{\Pi}_{3,3} &= -e^{-2\alpha h_2}(Z_2 + Z_2^T) + e^{-2\alpha h_2}(Z_3 + Z_3^T) + G^TC^TU_3CG + G^TU_3G \\
 &\quad + G^TE_C^TE_CG + G^TE_C^TE_CG, \\
 \tilde{\Pi}_{3,4} &= e^{-2\alpha h_2}(Z_2 - Z_3), \quad \tilde{\Pi}_{4,4} = -e^{-2\alpha h_2}(Z_2 + R_2 + Q_2), \\
 \tilde{\Pi}_{5,5} &= -h_1^{-2}e^{-4\alpha h_1}(W_1 + W_1^T), \\
 \tilde{\Pi}_{6,6} &= -P_5 - P_5^T - 2\alpha P_3 - h_2^{-2}e^{-4\alpha h_2}(W_2 + W_2^T + W_3 + W_3^T), \\
 \tilde{\Pi}_{7,7} &= -h_{12}^{-2}e^{-4\alpha h_2}(Z_1 + Z_1^T), \quad \tilde{\Pi}_{8,8} = -d^{-1}e^{-2\alpha d}Q_3, \\
 \tilde{\Pi}_{10,10} &= -1.5R_1 + h_1^2Q_1 + h_2^2Q_2 + h_2^2Z_2 + h_{12}h_2Z_1 + h_1W_1 + h_2W_2 + h_2W_3, \\
 \tilde{\Pi}_{11,11} &= -\frac{e^{-2\alpha d}}{2}U + \frac{e^{-2\alpha d}}{4d}E_D^TE_D + \frac{e^{-2\alpha d}}{4d}E_D^TE_D, \\
 \Xi_{1(4,4)} &= -2de^{-2\alpha d}U, \quad \Xi_{1(5,5)} = -4e^{-2\alpha \tau}S_1, \quad \Xi_{1(6,6)} = -2d_1e^{-2\alpha d_1}S_2, \\
 \Xi_{2(4,4)} &= -2de^{-2\alpha d}U, \quad \Xi_{2(5,5)} = -4e^{-2\alpha \tau}S_1, \quad \Xi_{2(6,6)} = -2d_1e^{-2\alpha d_1}S_2,
 \end{aligned}$$

$$\mathcal{N} = e^{-2\alpha \tau}B_1^TS_1B_1 + dB_1^TS_2B_1 + B_1^TS_3B_1.$$

Moreover, stabilizing feedback control is given by

$$u(t) = B_1P_1^{-1}x(t), \quad t \geq 0,$$

and the solution of the system satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|_c e^{-\alpha t}, \quad t \geq 0.$$

**Proof.** We choose the similar Lyapunov–Krasovskii functional in Theorem 1, where matrices  $A, B, C$  and  $D$  in (19) and (29) are replaced by  $A + N\tilde{F}(t)E_A, B + N\tilde{F}(t)E_B, C + N\tilde{F}(t)E_C$  and  $D + N\tilde{F}(t)E_D$ , respectively. By Lemmas 1 and 2, we have

$$\begin{aligned}
 -2x^T(t)E_A^T\tilde{F}^T(t)N^T P_1x(t) &\leq x^T(t)E_A^T E_Ax(t) + x^T(t)P_1NN^T P_1x(t), \\
 2f^T(x(t))E_B^T\tilde{F}^T(t)N^T P_1x(t) &\leq x^T(t)E_B^T E_Bx(t) + x^T(t)P_1NN^T P_1x(t), \\
 2g^T(x(t-h(t)))E_C^T\tilde{F}^T(t)N^T P_1x(t) &\leq x^T(t-h(t))G^T E_C^T E_CGx(t-h(t)) \\
 &\quad + x^T(t)P_1NN^T P_1x(t), \\
 2\left(\int_{t-d(t)}^t h(x(s))ds\right)^T E_D^T\tilde{F}^T(t)N^T P_1x(t) &\leq \frac{e^{-2\alpha d}}{4d} \int_{t-d}^t h^T(x(s))ds \\
 &\quad \times E_D^T E_D \int_{t-d}^t h(x(s))ds \\
 &\quad + 4de^{2\alpha d}x^T(t)P_1NN^T P_1x(t), \\
 2\left(\int_{t-d(t)}^t h(x(s))ds\right)^T D^T P_1x(t) &\leq \frac{e^{-2\alpha d}}{4} \int_{t-d}^t h^T(x(s))Uh(x(s))ds \\
 &\quad + 4de^{2\alpha d}x^T(t)P_1DU^{-1}D^T P_1x(t), \\
 2\dot{x}^T(t)R_1N\tilde{F}(t)E_D \int_{t-d(t)}^t h(x(s))ds &\leq 4de^{2\alpha d}\dot{x}^T(t)R_1NN^T R_1\dot{x}(t) \\
 &\quad + \frac{e^{-2\alpha d}}{4d} \int_{t-d}^t h^T(x(s))ds \\
 &\quad \times E_D^T E_D \int_{t-d}^t h(x(s))ds, \\
 2\dot{x}^T(t)R_1D \int_{t-d(t)}^t h(x(s))ds &\leq 4de^{2\alpha d}\dot{x}^T(t)R_1DU^{-1}D^T R_1\dot{x}(t) \\
 &\quad + \frac{e^{-2\alpha d}}{4} \int_{t-d}^t h^T(x(s))Uh(x(s))ds, \\
 -2\dot{x}^T(t)R_1N\tilde{F}(t)E_Ax(t) &\leq \dot{x}^T(t)R_1NN^T R_1\dot{x}(t) + x^T(t)E_A^T E_Ax(t), \\
 2\dot{x}^T(t)R_1N\tilde{F}(t)E_Bf(x(t)) &\leq \dot{x}^T(t)R_1NN^T R_1\dot{x}(t) \\
 &\quad + x^T(t)F^T E_B^T E_BFx(t), \\
 2\dot{x}^T(t)R_1N\tilde{F}(t)E_Cg(x(t-h(t))) &\leq \dot{x}^T(t)R_1NN^T R_1\dot{x}(t) \\
 &\quad + x^T(t-h(t))G^T E_C^T E_CGx(t-h(t)). \tag{46}
 \end{aligned}$$

From (46), we get

$$\begin{aligned}
 \zeta^T(t)\Xi_1\zeta(t) + x^T(t)P_1NN^T P_1x(t) + x^T(t)P_1NN^T P_1x(t) \\
 + x^T(t)P_1NN^T P_1x(t) + 4de^{2\alpha d}x^T(t)P_1NN^T P_1x(t) \leq 0, \tag{47}
 \end{aligned}$$

$$\begin{aligned}
 \dot{x}^T(t)\Xi_2\dot{x}(t) + \dot{x}^T(t)R_1NN^T R_1\dot{x}(t) + \dot{x}^T(t)R_1NN^T R_1\dot{x}(t) \\
 + \dot{x}^T(t)R_1NN^T R_1\dot{x}(t) + 4de^{2\alpha d}x^T(t)R_1NN^T R_1x(t) \leq 0, \tag{48}
 \end{aligned}$$

where  $\zeta^T(t) = \left[ \zeta^T(t) \int_{t-d}^t h^T(x(s))ds \right]$ .

By using the Schur complement lemma, the inequality (47) and (48) are equivalent to  $\Omega_1 < 0$  and  $\Omega_2 < 0$  respectively. By the similar proof of Theorem 1, so the proof is completed.  $\square$

**Remark 4.** The time delay in this paper is identified as a continuous function which serve on a given interval that the lower and upper bounds for the time-varying delay exist. Moreover, the time delay function is not necessary to be differentiable. In some previous works, the time delay function needs to be differentiable which are shown in [24–26,33–37].

### 4. Numerical Examples

In this section, we provide two numerical examples with their simulations to demonstrate the effectiveness of our results.

**Example 1.** Consider neural networks (3) with parameters as follows:

$$\begin{aligned}
 A &= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} -0.7 & 0.2 \\ 0.4 & -0.1 \end{bmatrix}, C = \begin{bmatrix} 0.7 & -0.8 \\ 0.5 & -0.9 \end{bmatrix}, \\
 D &= \begin{bmatrix} 0.7 & -0.7 \\ -0.1 & -0.4 \end{bmatrix}, E = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.4 \end{bmatrix}, F = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \\
 G &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, H = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}, A_1 = \begin{bmatrix} -0.2 & 0.3 \\ 0 & -0.4 \end{bmatrix}, \\
 B_1 &= \begin{bmatrix} -0.4 & 0 \\ 0 & -0.4 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \\
 B_4 &= \begin{bmatrix} -0.4 & 0 \\ -0.1 & -0.5 \end{bmatrix}, C_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.5 \end{bmatrix}, D_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.4 \end{bmatrix}, \\
 E_1 &= \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.3 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
 f(\cdot) &= g(\cdot) = 0.2 \left[ \frac{|x_1(t) + 1| - |x_1(t) - 1|}{|x_2(t) + 1| - |x_2(t) - 1|} \right], h(\cdot) = \tanh(\cdot).
 \end{aligned}$$

From the conditions (14)–(17) of Theorem 1, we let  $\alpha = 0.01, h_1 = 0.1, h_2 = 0.3, d = 0.3, d_1 = 0.5,$  and  $\tau = 0.4.$  By using the LMI Toolbox in MATLAB, we obtain  $\gamma = 1.7637,$

$$\begin{aligned}
 P_1 &= \begin{bmatrix} 0.9248 & -0.1581 \\ -0.1581 & 0.7921 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0948 & -0.0501 \\ -0.0521 & 0.1187 \end{bmatrix}, \\
 P_3 &= \begin{bmatrix} -0.3225 & 0.0156 \\ 0.0156 & -0.3867 \end{bmatrix}, P_4 = \begin{bmatrix} 0.0011 & -0.0006 \\ -0.0006 & 0.0013 \end{bmatrix}, \\
 P_5 &= \begin{bmatrix} -0.0020 & -0.0002 \\ -0.0001 & -0.0028 \end{bmatrix}, P_6 = \begin{bmatrix} 0.0146 & -0.0020 \\ -0.0020 & 0.0161 \end{bmatrix}, \\
 Q_1 &= \begin{bmatrix} 0.5107 & -0.0462 \\ -0.0462 & 0.5004 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.4714 & -0.0328 \\ -0.0328 & 0.5166 \end{bmatrix}, \\
 Q_3 &= \begin{bmatrix} 0.4332 & 0.0043 \\ 0.0043 & 0.4422 \end{bmatrix}, R_1 = \begin{bmatrix} 0.1392 & -0.0521 \\ -0.0521 & 0.1790 \end{bmatrix}, \\
 R_2 &= \begin{bmatrix} 0.4409 & -0.0481 \\ -0.0481 & 0.4248 \end{bmatrix}, S_1 = \begin{bmatrix} 0.2493 & -0.0177 \\ -0.0177 & 0.2606 \end{bmatrix}, \\
 S_2 &= \begin{bmatrix} 0.8199 & -0.0161 \\ -0.0161 & 0.8399 \end{bmatrix}, S_3 = \begin{bmatrix} 0.5445 & -0.0353 \\ -0.0353 & 0.6032 \end{bmatrix}, \\
 U &= \begin{bmatrix} 1.5450 & 0 \\ 0 & 1.5450 \end{bmatrix}, U_2 = \begin{bmatrix} 1.0492 & 0 \\ 0 & 1.0492 \end{bmatrix}, \\
 U_3 &= \begin{bmatrix} 0.9085 & 0 \\ 0 & 0.9085 \end{bmatrix}, W_1 = 10^{-3} \begin{bmatrix} 7.4622 & -0.0421 \\ -0.0421 & 7.5146 \end{bmatrix}, \\
 W_2 &= \begin{bmatrix} 0.0518 & -0.0083 \\ -0.0083 & 0.0582 \end{bmatrix}, W_3 = \begin{bmatrix} 0.0203 & -0.0007 \\ -0.0007 & 0.0207 \end{bmatrix}, \\
 Z_1 &= \begin{bmatrix} 0.0349 & -0.0002 \\ -0.0002 & 0.0353 \end{bmatrix}, Z_2 = \begin{bmatrix} 0.8015 & -0.1874 \\ -0.1874 & 0.8118 \end{bmatrix}, \\
 Z_3 &= \begin{bmatrix} 0.2777 & 0.0028 \\ 0.0028 & 0.3139 \end{bmatrix}.
 \end{aligned}$$

The feedback control is given by

$$u(t) = B_1 P_1^{-1} x(t) = \begin{bmatrix} -0.4478 & -0.0894 \\ -0.0894 & -0.5228 \end{bmatrix} x(t), \quad t \geq 0.$$

Moreover, the solution  $x(t, \phi)$  of the system satisfies

$$\|x(t, \phi)\| \leq 1.2300e^{-0.01t} \|\phi\|_c.$$

Figure 1 shows the response solution  $x(t)$  of the neural network system (3) where  $w(t) = 0$  and the initial condition  $\phi(t) = [-0.1 \ 0.1]^T$ .

Figure 2 shows the response solution  $x(t)$  of the neural network system (3) with the initial condition  $\phi(t) = [-0.1 \ 0.1]^T$ .

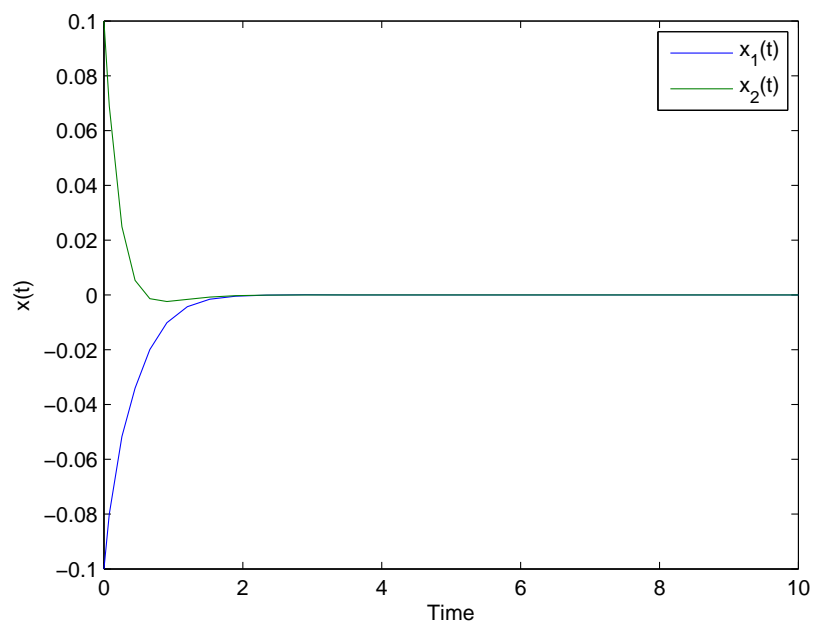


Figure 1. Response solution of the system (3) where  $w(t) = 0$ .

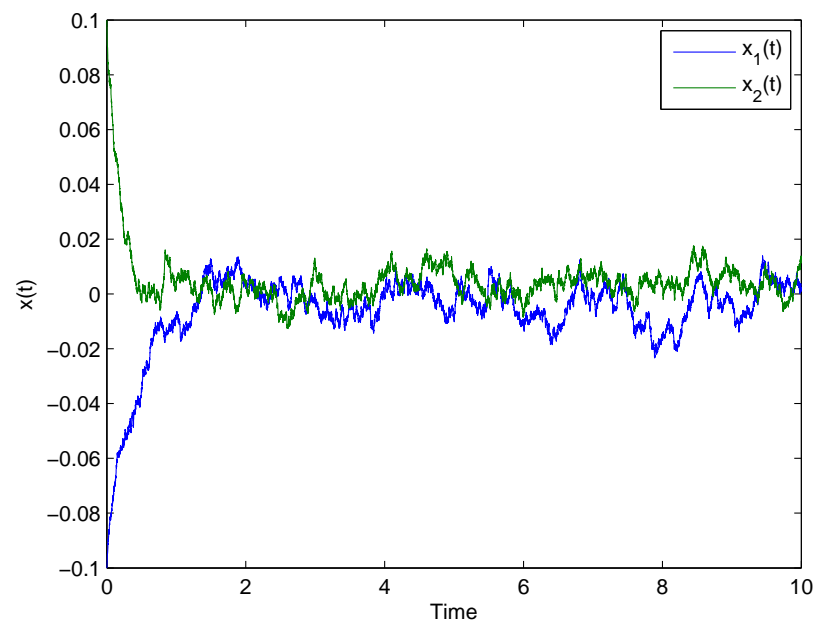


Figure 2. Response solution of the system (3).

**Example 2.** Consider neural networks (39) with parameters as follows:

$$\begin{aligned}
 A &= \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} -0.7 & 0.2 \\ 0.4 & -0.1 \end{bmatrix}, C = \begin{bmatrix} 0.7 & -0.8 \\ 0.5 & -0.9 \end{bmatrix}, \\
 D &= \begin{bmatrix} 0.7 & -0.7 \\ -1 & -0.4 \end{bmatrix}, E = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.4 \end{bmatrix}, F = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \\
 G &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, H = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}, A_1 = \begin{bmatrix} -0.2 & 0.3 \\ 0 & -0.4 \end{bmatrix}, \\
 B_1 &= \begin{bmatrix} 0.3 & -0.1 \\ 0.2 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} -0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.3 \end{bmatrix}, \\
 B_4 &= \begin{bmatrix} -0.4 & 0 \\ -0.1 & -0.5 \end{bmatrix}, C_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.5 \end{bmatrix}, \\
 D_1 &= \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.4 \end{bmatrix}, E_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.3 \end{bmatrix}, \\
 N &= I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_A = E_B = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, \\
 E_C &= \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}, E_D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \\
 f(\cdot) &= g(\cdot) = 0.2 \begin{bmatrix} |x_1(t) + 1| - |x_1(t) - 1| \\ |x_2(t) + 1| - |x_2(t) - 1| \end{bmatrix}, \\
 h(\cdot) &= \tanh(\cdot), \tilde{F}(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(t) \end{bmatrix}.
 \end{aligned}$$

From the conditions (42)–(45) of Theorem 2, we let  $\alpha = 0.01$ ,  $h_1 = 0.1$ ,  $h_2 = 0.3$ ,  $d = 0.3$ ,  $d_1 = 0.5$ , and  $\tau = 0.4$ . By using the LMI Toolbox in MATLAB, we obtain

$$\begin{aligned}
 P_1 &= \begin{bmatrix} 0.7130 & -0.0418 \\ -0.0418 & 0.6099 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0036 & 0.0030 \\ 0.0040 & 0.0074 \end{bmatrix}, \\
 P_3 &= \begin{bmatrix} 0.0842 & 0.0047 \\ 0.0047 & 0.0822 \end{bmatrix}, P_4 = 10^{-4} \begin{bmatrix} 3.9360 & 2.9290 \\ 4.1062 & 7.8329 \end{bmatrix}, \\
 P_5 &= 10^{-3} \begin{bmatrix} 7.7772 & 0.7065 \\ 0.7274 & 7.8126 \end{bmatrix}, P_6 = \begin{bmatrix} 0.0107 & 0.0088 \\ 0.0088 & 0.0175 \end{bmatrix}, \\
 Q_1 &= \begin{bmatrix} 0.3415 & -0.0824 \\ -0.0824 & 0.2748 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.0819 & 0.0527 \\ 0.0527 & 0.1249 \end{bmatrix}, \\
 Q_3 &= \begin{bmatrix} 0.1095 & 0.0935 \\ 0.0935 & 0.1220 \end{bmatrix}, R_1 = \begin{bmatrix} 0.0417 & -0.0025 \\ -0.0025 & 0.0410 \end{bmatrix}, \\
 R_2 &= \begin{bmatrix} 0.6112 & -0.1130 \\ -0.1130 & 0.4407 \end{bmatrix}, S_1 = \begin{bmatrix} 0.2316 & 0.0486 \\ 0.0486 & 0.0856 \end{bmatrix}, \\
 S_2 &= \begin{bmatrix} 0.2987 & 0.2759 \\ 0.2759 & 0.5621 \end{bmatrix}, S_3 = \begin{bmatrix} 0.1103 & 0.0690 \\ 0.0690 & 0.1068 \end{bmatrix}, \\
 U &= \begin{bmatrix} 6.8799 & 0 \\ 0 & 6.8799 \end{bmatrix}, U_2 = \begin{bmatrix} 0.3898 & 0 \\ 0 & 0.3898 \end{bmatrix}, \\
 U_3 &= \begin{bmatrix} 0.4546 & 0 \\ 0 & 0.4546 \end{bmatrix}, W_1 = 10^{-3} \begin{bmatrix} 0.0052 & 0.0047 \\ 0.0047 & 0.0084 \end{bmatrix}, \\
 W_2 &= \begin{bmatrix} 0.0016 & 0.0015 \\ 0.0015 & 0.0027 \end{bmatrix}, W_3 = \begin{bmatrix} 0.0030 & 0.0029 \\ 0.0029 & 0.0053 \end{bmatrix},
 \end{aligned}$$

$$Z_1 = \begin{bmatrix} 0.0086 & 0.0079 \\ 0.0079 & 0.0140 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.4969 & -0.1357 \\ -0.1357 & 0.4097 \end{bmatrix},$$

$$Z_3 = \begin{bmatrix} 0.0085 & 0.0078 \\ 0.0078 & 0.0137 \end{bmatrix}.$$

The feedback control is given by

$$u(t) = B_1 P_1^{-1} x(t) = \begin{bmatrix} -0.4224 & -0.0290 \\ -0.0386 & -0.6584 \end{bmatrix} x(t), \quad t \geq 0.$$

Moreover, the solution  $x(t, \phi)$  of the system satisfies

$$\|x(t, \phi)\| \leq 1.2049e^{-0.1t} \|\phi\|_c.$$

Figure 3 shows the response solution  $x(t)$  of the neural network system (39) where  $w(t) = 0$  the initial condition  $\phi(t) = [-0.15 \ 0.15]^T$ .

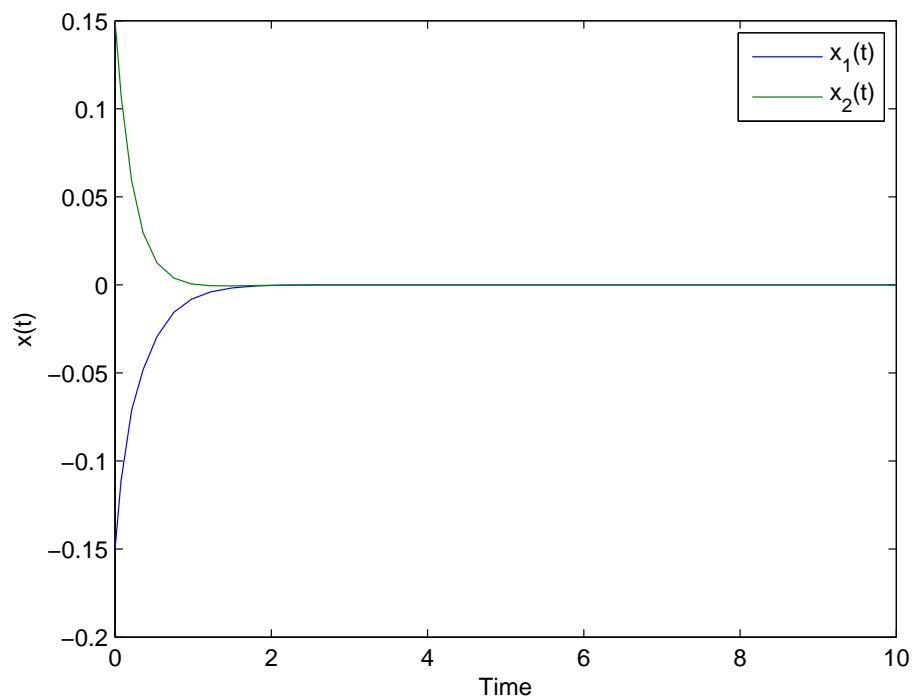


Figure 3. Response solution of the system (39) where  $w(t) = 0$ .

Figure 4 shows the response solution  $x(t)$  of the neural network system (39) with the initial condition  $\phi(t) = [-0.15 \ 0.15]^T$ .

**Remark 5.** The advantages of Examples 1 and 2 are the lower bound of the delay  $h_1 \neq 0$  and interval time-varying delay and distributed time-varying delay are non-differentiable. Moreover, in these examples we still investigate various activation functions and mixed time-varying delays in state and feedback control. Thus, the neural network conditions derived in [23] cannot be applied to these examples.

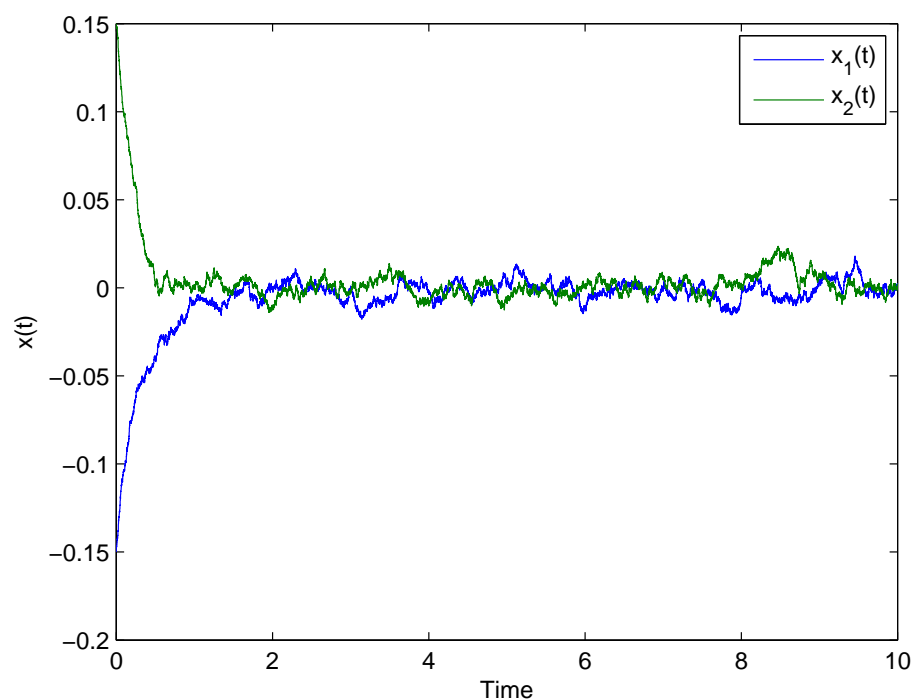


Figure 4. Response solution of the system (39).

## 5. Conclusions

In this paper, the problem of a robust  $H_\infty$  control for a class of uncertain systems with interval and distributed time-varying delays was investigated. It is assumed that the interval and distributed time-varying delays are not necessary to be differentiable. Firstly, we considered an  $H_\infty$  control for exponential stability of neural network with interval and distributed time-varying delays via hybrid feedback control and a robust  $H_\infty$  control for exponential stability of uncertain neural network with interval and distributed time-varying delays via hybrid feedback control. Secondly, by using a novel Lyapunov–Karsovskii functional that the Lyapunov matrix  $P_i$  ( $i = 1, 2, \dots, 6$ ) do not need to be positive definiteness, the employment of a tighter bounding technique, some slack matrices and newly introduced convex combination condition in the calculation, improved delay-dependent sufficient conditions for the robust  $H_\infty$  control with exponential stability of the system are obtained. Finally, numerical examples have been given to illustrate the effectiveness of the proposed method. The results in this paper improve the corresponding results of the recent works. In the future work, the derived results and methods in this work are expected to be applied to other systems, for example,  $H_\infty$  state estimation of neural networks, exponential passivity of neural networks, neutral-type neural networks, stochastic neural networks, T-S fuzzy neural networks, and so on [24,38–41].

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