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Approximating Fixed Points Using a Faster Iterative Method and Application to Split Feasibility Problems

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Abstract: In this article, the recently introduced iterative scheme of Hassan et al. (*Math. Probl. Eng.* 2020) is re-analyzed with the connection of Reich–Suzuki type nonexpansive (RSTN) maps. Under mild conditions, some important weak and strong convergence results in the context of uniformly convex Banach spaces are provided. To support the main outcome of the paper, we provide a numerical example and show that this example properly exceeds the class of Suzuki type nonexpansive (STN) maps. It has been shown that the Hassan et al. iterative scheme of this example is more useful than the many other iterative schemes. We provide an application of our main results to solve split feasibility problems in the setting of RSTN maps. The presented outcome is new and compliments the corresponding results of the current literature.

Keywords: Hassan et al. iteration; Reich–Suzuki type operator; convergence; speed of convergence; Banach space

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1. Introduction and Preliminaries

Different kinds of numerical schemes, especially iteration schemes were successfully applied for finding the solutions of many different kinds of functional, differential and integral operators (see e.g., [1–3] and others). Very recently, we have observed the effectiveness of the iterative approximation schemes, in the finding of such solutions on different Banach spaces. Fixed point existence theorem suggests that under certain assumptions, any given operator has a fixed point and, in fact, this fixed point is the solution for the original underlying problem.

It is well-known that in the year 1922, Banach [4] presented a famous result, which provides the requirements for the existence and approximation of a unique fixed point for contraction operators. Precisely, the Banach result essentially states that if $\mathcal{T}$ is a self contraction operator of a closed subset $\mathcal{D}$ of a complete normed space, that is, ($||\mathcal{T}r - \mathcal{T}r'|| \leq \xi||r - r'||$ for all $r, r' \in \mathcal{D}$ and $\xi \in [0,1)$ is a fixed real number), then $\mathcal{T}$ attains a unique fixed point say x_0, that is, $\mathcal{T}x_0 = x_0$, and the iterative scheme of Picard [5], $r_{m+1} = \mathcal{T}r_m$ is strongly convergent to this x_0 for every choice of a starting point. This result is extensively used for finding the solution of many numerical problems that are available in various areas of applied mathematics and science. A mapping $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ is called a nonexpansive selfmap in the case if one has $||\mathcal{T}r - \mathcal{T}r'|| \leq ||r - r'||$ for every choice of $r, r' \in \mathcal{D}$. It has been known for many years that every nonexpansive selfmap admits a fixed point (which may not be unique) when one considers $\mathcal{M}$ being a uniformly convex Banach space (UCBS) and the set $\mathcal{D}$ closed convex and bounded (cf. [6–8] and others). In 2008, Suzuki [9] observed another class of selfmaps that admits a condition (C). Notice that a selfmap $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ is said to admit a condition (C) (also known as STN map) iff for $r, r' \in \mathcal{D}$, the nonexpansiveness requirement, that is, $||\mathcal{T}r - \mathcal{T}r'|| \leq ||r - r'||$ holds whenever the condition $\frac{1}{2}||r - \mathcal{T}r|| \leq ||r - r'||$ is satisfied. Obviously a nonexpansive selfmap is STN. Just like nonexpansive selfmaps, Suzuki [9] showed that STN selfmaps also admit a fixed point in the setting of UCBS. By providing an example, he noted that every STN selfmap is not necessarily nonexpansive. Consequently, we conclude that the theory of STN selfmaps is essentially generalized compared to the theory of contractions and nonexpansive selfmaps. In the year 2019, Pandey et al. [10] proposed the notion of RSTN selfmaps in the following way: a selfmap $\mathcal{T}$ defined on a subset $\mathcal{D}$ of any Banach space is called RSTN provided that for all $r, r' \in \mathcal{D}$, it follows that

$$\frac{1}{2}||r - \mathcal{T}r|| \leq ||r - r'|| \Rightarrow ||\mathcal{T}r - \mathcal{T}r'|| \leq w||r - \mathcal{T}r|| + w||r' - \mathcal{T}r'|| + (1 - 2w)||r - r'||,$$

where $w \in [0, 1)$ is any fixed real constant.

We note that the following facts are not hard to establish; however, for the sake of completeness, we include some details.

Proposition 1. *If $\mathcal{D}$ is any nonempty subset of a Banach space and consider a selfmap $\mathcal{T}$ of $\mathcal{D}$ with $F_{\mathcal{T}} = \{x_0 \in \mathcal{D} : x_0 = \mathcal{T}x_0\} \neq \emptyset$. We show that the following hold.*

(i) If $\mathcal{T}$ is RSTN then for every choice of $r \in \mathcal{D}$ and $x_0 \in F_{\mathcal{T}}$, it follows that $||\mathcal{T}r - \mathcal{T}x_0|| \leq ||r - x_0||$.

(ii) If $\mathcal{T}$ is STN then $\mathcal{T}$ is RSTN .

Proof. For (i), since $\mathcal{T}$ is RSTN, one has a $w \in [0, 1)$ such that

$$\begin{aligned} ||\mathcal{T}r - \mathcal{T}x_0|| &= w||r - \mathcal{T}r|| + w||x_0 - \mathcal{T}x_0|| + (1 - 2w)||r - x_0|| \\ &= w||r - \mathcal{T}r|| + (1 - 2w)||r - x_0|| \\ &\leq w||r - x_0|| + w||x_0 - \mathcal{T}r|| + (1 - 2w)||r - x_0|| \\ &= w||\mathcal{T}r - \mathcal{T}x_0|| + (1 - w)||r - x_0||. \end{aligned}$$

Hence $(1 - w)||\mathcal{T}r - \mathcal{T}x_0|| \leq (1 - w)||r - x_0||$. However, $w \in [0, 1)$, hence (i) is proved. For (ii), since $\mathcal{T}$ is STN, put $w = 0$, we obtain (ii). □

The converse of the Proposition 1(ii) does not hold, in general, as shown by the following example.

Example 1. *Suppose $\mathcal{D} = [6, 8]$ and set $\mathcal{T}$ by the following rule*

$$\mathcal{T}r = \begin{cases} \frac{1}{6}(r + 30) & \text{if } r < 8 \\ 5 & \text{if } r = 8. \end{cases}$$

We choose $w = \frac{1}{2}$ and consider the following cases.

(i): *If we choose $r, r' < 8$. Then $\mathcal{T}r = \frac{1}{6}(r + 30)$ and $r' = \frac{1}{6}(r' + 30)$. Using triangle inequality, we have*

$$\begin{aligned} w|r - \mathcal{T}r| + w|r' - \mathcal{T}g'| + (1 - 2w)|r - r'| &= \frac{1}{2}|\frac{5r - 30}{6}| + \frac{1}{2}|\frac{5r' - 30}{6}| \\ &\geq \frac{1}{2}|(\frac{5r - 30}{6}) - (\frac{5r' - 30}{6})| \\ &= \frac{1}{2}|\frac{5r - 5r'}{6}| = \frac{5}{12}|r - r'| \\ &\geq \frac{1}{6}|r - r'| = |\mathcal{T}r - \mathcal{T}r'|. \end{aligned}$$

(ii): *If we choose* $r < 8$ *and* $r' = 8$. *Then* $\mathcal{T}r = \frac{r+30}{6}$ *and* $\mathcal{T}r' = 5$. *Now*

$$\begin{aligned} w|r-\mathcal{T}r| + w|r'-\mathcal{T}r'| + (1-2w)|r-r'| &= \frac{1}{2}|\frac{5r-30}{6}| + \frac{3}{2} \geq \frac{3}{2} \\ &> \frac{8}{6} \geq |\frac{r}{6}| = |\mathcal{T}r - \mathcal{T}r'|. \end{aligned}$$

(iii): *If we choose* $r' < 8$ *and* $r = 8$. *Then* $\mathcal{T}r' = \frac{r'+30}{6}$ *and* $\mathcal{T}r = 5$. *Now*

$$\begin{aligned} w|r-\mathcal{T}r| + w|r'-\mathcal{T}r'| + (1-2w)|r-r'| &= \frac{3}{2} + \frac{1}{2}|\frac{5r'-30}{6}| \geq \frac{3}{2} \\ &> \frac{8}{6} \geq |\frac{r'}{6}| = |\mathcal{T}r - \mathcal{T}r'|. \end{aligned}$$

(iv): *If we choose* $r = 8 = r'$. *Then* $\mathcal{T}r = 5 = \mathcal{T}r'$. *Now*

$$w|r-\mathcal{T}r| + w|r'-\mathcal{T}r'| + (1-2w)|r-r'| \geq 0 = |\mathcal{T}r - \mathcal{T}r'|.$$

Keeping above cases in mind, one can conclude that $\mathcal{T}$ *is RSTN. On the other hand,* $\mathcal{T}$ *is not STN. Because, for* $r = 7$ *and* $r' = 8$, *as* $\frac{1}{2}|r-\mathcal{T}r| < 1 = |r-r'|$, *and* $|\mathcal{T}r - \mathcal{T}r'| > 1 = |r-r'|$.

The discussion suggests that the class of RSTN maps properly includes the class of STN mappings. In 1955, Krasnoselskii [11] showed that the sequence of Picard iterates $r_{m+1} = \mathcal{T}r_m$ fails to converge to a fixed point when one replaces the class of contractions by the wider class of mappings, so-called, nonexpansive mappings. For investigation of fixed points for nonexpansive and at the same time for generalized nonexpansive mappings, some authors introduced different types of iterative schemes as follows. Notice that $\mathcal{D}$ is a nonempty subset of a Banach space and $a_m, b_m, c_m, d_m \in (0,1)$.

The Mann [12] iteration process is stated as follows:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ r_{m+1} = (1-a_m)r_m + a_m\mathcal{T}r_m. \end{cases} \tag{1}$$

The Ishikawa [13] iterative process may be viewed as a two-step Mann iteration, which is given by:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ s_m = (1-b_m)r_m + b_m\mathcal{T}r_m, \\ r_{m+1} = (1-a_m)r_m + a_m\mathcal{T}s_m. \end{cases} \tag{2}$$

In 2000, Noor [14] provided a three-step iteration method, which includes both of the Mann and Ishikawa iteration processes as:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ p_m = (1-c_m)r_m + c_m\mathcal{T}r_m, \\ s_m = (1-b_m)r_m + b_m\mathcal{T}p_m, \\ r_{m+1} = (1-a_m)r_m + a_m\mathcal{T}s_m. \end{cases} \tag{3}$$

In 2007, Agarwal et al. [15] suggested a new iteration meethod and noted that its rate of convergence is good as compared to the Mann iteration for contractions in Banach spaces:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ s_m = (1-b_m)r_m + b_m\mathcal{T}r_m, \\ r_{m+1} = (1-a_m)\mathcal{T}r_m + a_m\mathcal{T}s_m \end{cases} \tag{4}$$

In 2014, Abbas and Nazir [1] proposed a new three-step iterative method, which converges better than all of the Picard, Mann, Ishikawa and Agarwal iterative methods for nonexpansive selfmaps, as follows:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ p_m = (1 - c_m)r_m + c_m \mathcal{T} r_m, \\ s_m = (1 - b_m)\mathcal{T} r_m + b_m \mathcal{T} p_m, \\ r_{m+1} = (1 - a_m)\mathcal{T} s_m + a_m \mathcal{T} p_m. \end{cases} \tag{5}$$

In the year 2016, Thakur et al. [16] constructed one of the well-known effective iterative processes as compared to the above iterative processes in the setting of STN mappings:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ p_m = (1 - b_m)r_m + b_m \mathcal{T} r_m, \\ s_m = \mathcal{T}[(1 - a_m)r_m + a_m p_m], \\ r_{m+1} = \mathcal{T} s_m. \end{cases} \tag{6}$$

In 2018, Ullah and Arshad [17] suggested the following scheme for STN mappings as follows, and showed that it has better speed of convergence than all of the above iterative schemes:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ p_m = (1 - b_m)r_m + b_m \mathcal{T} p_m, \\ s_m = \mathcal{T}[(1 - a_m)r_m + a_m \mathcal{T} p_m], \\ r_{m+1} = \mathcal{T} s_m. \end{cases} \tag{7}$$

Very recently in 2020, Hassan et al. [18] introduced a new iterative scheme for STN mappings as follows:

$$\begin{cases} r_1 = r \in \mathcal{D}, \\ q_m = \mathcal{T}[(1 - d_m)r_m + d_m \mathcal{T} r_m], \\ p_m = \mathcal{T}[(1 - c_m)q_m + b_m \mathcal{T} q_m], \\ s_m = \mathcal{T}[(1 - b_m)p_m + b_m \mathcal{T} p_m], \\ r_{m+1} = \mathcal{T}[(1 - a_m)s_m + a_m \mathcal{T} s_m]. \end{cases} \tag{8}$$

They observed that iterative scheme (8) essentially converges in the weak as well as in the strong sense to the fixed point of a given self STN map in the case of some restrictions of the whole selfmap or on its domain. Moreover, they proved by providing a numerical example of contraction mappings that this scheme is more efficient than all of the above iterative schemes. However, we may note that they did not provide an example of STN mapping, which is not nonexpansive. In this research article, first we improve and extend the main convergence results of Hassan et al. [18] from the context of STN maps to the more general framework of RSTN mappings. We then use the non-trivial Example 1 of RSTN maps, which exceeded the corresponding setting of STN maps. We connect the iterative scheme (8) and some other prominent iterative schemes with this example to show the effectiveness of this research. In this way, we improve all other results of Hassan et al. [18] from the class of STN maps to the more general setting of RSTN maps.

We now provide some definitions.

Definition 1 ([19])**.** *Let a Banach space $\mathcal{M}$ be given. If one assumes that for any weakly convergent sequence $\{r_m\} \subseteq \mathcal{M}$ is endowed with the weak limit $v_0 \in \mathcal{M}$, such that*

$$\limsup_{m\to\infty} ||r_m - v_0|| < \limsup_{m\to\infty} ||r_m - u_0|| \text{ for every choice of } u_0 \neq v_0.$$

Then, in such a case, we called $\mathcal{M}$ a Banach space with Opial's property.

Definition 2 ([20])**.** *We say that a selfmap $\mathcal{T}$ of a subset $\mathcal{D}$ of any Banach space admits a condition I if there is a $\eta : [0,\infty) \to [0,\infty)$ such that η satisfies $\eta(0) = 0$, $\eta(\nu) > 0$ for $\nu \in [0,\infty) - \{0\}$*

and $||r - \mathcal{T}r|| \geq \eta(d(r, F_{\mathcal{T}}))$ *for any element* $r \in \mathcal{D}$*. Note that* $d(r, F_{\mathcal{T}})$ *represents the distance between* r *and the set* $F_{\mathcal{T}}$*.*

Definition 3. *Suppose* $\mathcal{M}$ *is any given Banach space and* $\{r_m\} \subseteq \mathcal{M}$ *is bounded. Let* $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ *be closed and convex. Then the asymptotic radius of* $\{r_m\}$ *associated with* $\mathcal{D}$ *is* $r(\mathcal{D}, \{r_m\}) = \inf\{\limsup_{m\to\infty} ||r_m - z|| : z \in \mathcal{D}\}$*. Moreover, the asymptotic center of* $\{r_m\}$ *with respect to* $\mathcal{D}$ *is given by* $A(\mathcal{D}, r_m\}) = \{z \in \mathcal{D} : \limsup_{m\to\infty} ||r_m - z|| = r(\mathcal{D}, r_m)\}$*.*

Remark 1. *In the setting of UCBS [21], the property that each set* $A(\mathcal{D}, \{r_m\})$ *has a unique element is now well-known. We also know that* $A(\mathcal{D}, \{r_m\})$ *is essentially nonempty and convex if one can show that* $\mathcal{D}$ *is weakly compact and convex [22,23].*

Lemma 1 ([10])**.** *Consider a Banach space* $\mathcal{M}$ *and* $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$*. Then every RSTN mapping* $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ *with a real constant* w *has the following property*

$$||r - \mathcal{T}r'|| \leq \frac{(3+w)}{(1-w)}||r - \mathcal{T}r|| + ||r - r'||.$$

Another important property of RSTN maps is the following.

Lemma 2 ([10])**.** *Consider a Banach space* $\mathcal{M}$ *and* $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ *and* $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ *to be RSTN . If* $\mathcal{M}$ *has the Opial property and* $\{r_m\} \subseteq \mathcal{D}$ *is weakly convergent to* r_0 *such that* $\lim_{m\to\infty} ||r_m - \mathcal{T}r_m|| = 0$*, then* $r_0 \in F_{\mathcal{T}}$*.*

The following key property of a UCBS $\mathcal{M}$ was proved in [24].

Lemma 3. *Assume that* $0 < i \leq a_m \leq j < 1$ *and* $\mathcal{M}$ *is a UCBS. Then for some* $y \geq 0$*, if* $\{r_m\}$ *and* $\{s_m\}$ *in* $\mathcal{M}$ *are such that* $\limsup_{m\to\infty} ||r_m|| \leq y$*,* $\limsup_{m\to\infty} ||s_m|| \leq y$ *and* $\lim_{m\to\infty} ||(1 - a_m)r_m + a_m s_m|| = y$*. Then* $\lim_{m\to\infty} ||r_m - s_m|| = 0$*.*

2. Main Results

We now establish several convergence results for RSTN maps under the iterative scheme (8), which will extend and improve the corresponding results of Hassan et al. [18] from the framework of STN maps to the more general setting of RSTN maps. The section begins by providing a crucial lemma as follows.

Lemma 4. *Consider a UCBS* $\mathcal{M}$ *and let* $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ *be closed and convex. Suppose a selfmap* $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ *is RSTN endowed with* $F_{\mathcal{T}} \neq \emptyset$ *and* $\{r_m\}$ *is a sequence obtained from the iterative scheme (8). Then* $\lim_{m\to\infty} ||r_m - x_0||$ *exists for all* x_0 *in the set* $F_{\mathcal{T}}$*.*

Proof. Let $x_0 \in F_{\mathcal{T}}$. Then using (8) along with Proposition 1(i), we have

$$\begin{aligned} ||q_m - x_0|| &= ||\mathcal{T}[(1-d_m)r_m + d_m\mathcal{T}r_m] - x_0|| \\ &\leq ||(1-d_m)r_m + d_m\mathcal{T}r_m - x_0|| \\ &\leq (1-d_m)||r_m - x_0|| + d_m||\mathcal{T}r_m - x_0|| \\ &\leq (1-d_m)||r_m - x_0|| + d_m||r_m - x_0|| \\ &\leq ||r_m - x_0||. \end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned}||p_m - x_0|| &= ||\mathcal{T}[(1-c_m)q_m + c_m\mathcal{T}q_m] - x_0|| \\ &\leq ||(1-c_m)q_m + c_m\mathcal{T}q_m - x_0|| \\ &\leq (1-c_m)||q_m - x_0|| + c_m||\mathcal{T}q_m - x_0|| \\ &\leq (1-c_m)||q_m - x_0|| + c_m||q_m - x_0|| \\ &\leq ||q_m - x_0||. \end{aligned} \tag{10}$$

Furthermore

$$\begin{aligned}||s_m - x_0|| &= ||\mathcal{T}[(1-b_m)p_m + b_m\mathcal{T}p_m] - x_0|| \\ &\leq ||(1-b_m)p_m + b_m\mathcal{T}p_m - x_0|| \\ &\leq (1-b_m)||p_m - x_0|| + b_m||\mathcal{T}p_m - x_0|| \\ &\leq (1-b_m)||p_m - x_0|| + b_m||p_m - x_0|| \\ &\leq ||p_m - x_0||. \end{aligned} \tag{11}$$

Now (9)–(11) imply that

$$\begin{aligned}||r_{m+1} - x_0|| &= ||\mathcal{T}[(1-a_m)s_m + a_m\mathcal{T}s_m] - x_0|| \\ &\leq ||(1-a_m)s_m + a_m\mathcal{T}s_m - v_0|| \\ &\leq (1-a_m)||s_m - x_0|| + a_m||\mathcal{T}s_m - x_0|| \\ &\leq (1-a_m)||s_m - x_0|| + a_m||s_m - x_0|| \\ &\leq ||s_m - x_0|| \leq ||p_m - x_0|| \leq ||q_m - x_0|| \\ &\leq ||r_m - x_0||. \end{aligned} \tag{12}$$

Consequently, for every fixed point x_0 of $\mathcal{T}$, we have obtained $\{||r_m - x_0||\}$ is non-increasing and bounded. It follows that $\lim_{m\to\infty} ||r_m - x_0||$ exists for each element x_0 of $F_{\mathcal{T}}$. □

Theorem 1. *Consider a UCBS $\mathcal{M}$ and $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ as closed and convex. Suppose a selfmap $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ is RSTN and $\{r_m\}$ is a sequence obtained from the iterative scheme (8). Then, $F_{\mathcal{T}} \neq \emptyset$ if and only if the iterative sequence $\{r_m\}$ is bounded and fulfils $\lim_{m\to\infty} ||r_m - \mathcal{T}r_m|| = 0$.*

Proof. To prove the required result, we consider the set $F_{\mathcal{T}} \neq \emptyset$. Then for any $x_0 \in F_{\mathcal{T}}$, Lemma 4 suggests that $\{r_m\}$ is bounded and $\lim_{m\to\infty} ||r_m - x_0||$ exists. Thus, we may put

$$\lim_{m\to\infty} ||r_m - x_0|| = y. \tag{13}$$

It is now our target to show $\lim_{m\to\infty} ||r_m - \mathcal{T}r_m|| = 0$. From (9), we see that

$$||q_m - v_0|| \leq ||r_m - x_0||,$$

$$\Rightarrow \limsup_{m\to\infty} ||q_m - x_0|| \leq \limsup_{m\to\infty} ||r_m - x_0|| = y. \tag{14}$$

However, the element x_0 is in the set $F_{\mathcal{T}}$, so applying Proposition 1(i), one concludes that

$$||\mathcal{T}r_m - x_0|| \leq ||r_m - x_0||,$$

$$\Rightarrow \limsup_{m\to\infty} ||\mathcal{T}r_m - x_0|| \leq \limsup_{m\to\infty} ||r_m - x_0|| = y. \tag{15}$$

Now from (12), we have

$$||r_{m+1} - x_0|| \leq ||q_m - x_0||.$$

Using this together with (13), we obtain

$$y \leq \liminf_{m\to\infty} ||q_m - x_0||. \tag{16}$$

From (14) and (16), we obtain

$$\lim_{m\to\infty} ||q_m - x_0|| = y. \tag{17}$$

Using (17), we get

$$\begin{aligned}
y &= \lim_{m\to\infty} ||q_m - x_0|| = \lim_{m\to\infty} ||\mathcal{T}[(1-d_m)r_m + d_m\mathcal{T}r_m] - x_0|| \\
&\leq \lim_{m\to\infty} ||(1-d_m)r_m + d_m\mathcal{T}r_m - x_0|| \\
&= \lim_{m\to\infty} ||(1-d_m)(r_m - x_0) + d_m(\mathcal{T}r_m - x_0)|| \\
&\leq \lim_{m\to\infty} (1-d_m)||r_m - x_0|| + \lim_{m\to\infty} d_m||\mathcal{T}r_m - x_0|| \\
&\leq \lim_{m\to\infty} (1-d_m)||r_m - x_0|| + \lim_{m\to\infty} d_m||r_m - x_0|| \\
&= \lim_{m\to\infty} ||r_m - x_0|| \\
&= y.
\end{aligned}$$

Consequently, we have

$$y = \lim_{m\to\infty} ||(1-d_m)(r_m - x_0) + d_m(\mathcal{T}r_m - x_0)||. \tag{18}$$

By using (13), (15) and (18) and applying Lemma 3, the following facts are obtained

$$\lim_{m\to\infty} ||r_m - \mathcal{T}r_m|| = 0.$$

Conversely, we consider the sequence $\{r_m\}$ to be bounded and $\lim_{m\to\infty} ||r_m - \mathcal{T}r_m|| = 0$. We try to show $F_{\mathcal{T}} \neq \emptyset$. We may select any element $x_0 \in A(\mathcal{D}, \{r_m\})$. Applying Lemma 1, we have

$$\begin{aligned}
A(\mathcal{T}x_0, \{r_m\}) &= \limsup_{m\to\infty} ||r_m - \mathcal{T}x_0|| \\
&\leq \frac{(3+w)}{(1-w)} \limsup_{m\to\infty} ||r_m - \mathcal{T}r_m|| + \limsup_{m\to\infty} ||r_m - x_0|| \\
&= \limsup_{m\to\infty} ||r_m - x_0|| \\
&= A(x_0, \{r_m\}).
\end{aligned}$$

Accordingly, we obtained $\mathcal{T}x_0 \in A(\mathcal{D}, \{r_m\}$. By the singletoness property of $A(\mathcal{D}, \{r_m\}$ the element $\mathcal{T}x_0$ is equal to the element x_0, that is, $x_0 \in F_{\mathcal{T}}$. This shows that $F_{\mathcal{T}}$ is nonempty. □

First, we provide a weak convergence theorem.

Theorem 2. *Consider a UCBS $\mathcal{M}$ and let $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ be closed and convex. Suppose a selfmap $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ is RSTN endowed with $F_{\mathcal{T}} \neq \emptyset$ and $\{r_m\}$ is a sequence obtained from the iterative scheme (8). Then $\{r_m\}$ converges weakly to a point of $F_{\mathcal{T}}$ provided that $\mathcal{M}$ has Opial's property.*

Proof. As $\mathcal{M}$ is UCBS, it is reflexive. By Theorem 1, the sequence $\{r_m\}$ is bounded. Hence, the sequence $\{r_m\}$ must have a weakly convergent subsequence $\{r_{m_t}\}$ endowed with a weak limit, namely, r_0. Theorem 1 suggests that $\lim_{t\to\infty} ||r_{m_t} - \mathcal{T}r_{m_t}|| = 0$. Applying Lemma 2, one can conclude that $r_0 \in F_{\mathcal{T}}$. The purpose is that the element r_0 is also the weak limit of the original sequence $\{r_m\}$. To succeed in the purpose, we assume that r_0 is not the weak limit of $\{r_m\}$, that is, $\{r_m\}$ also has a subsequence, namely, $\{r_{m_s}\}$ with a weak limit, namely, $r'_0 \neq r_0$. According to Theorem 1, $\lim_{s\to\infty} ||r_{m_s} - \mathcal{T} r_{m_s}|| = 0$. Hence, using Lemma 2, we get $r'_0 \in F_{\mathcal{T}}$. Now using Lemma 4 and Opial's property, we have

$$\begin{aligned}
\lim_{m\to\infty} ||r_m - r_0|| &= \lim_{t\to\infty} ||r_{m_t} - r_0|| < \lim_{t\to\infty} ||r_{m_t} - r'_0|| \\
&= \lim_{m\to\infty} ||r_m - r'_0|| = \lim_{s\to\infty} ||r_{m_s} - r'_0|| \\
&< \lim_{s\to\infty} ||r_{m_s} - r_0|| = \lim_{m\to\infty} ||r_m - r_0||.
\end{aligned}$$

Consequently, one can conclude that $\lim_{m\to\infty} ||r_m - r_0|| < \lim_{m\to\infty} ||r_m - r_0||$, which is a contradiction. It follows that the element r_0 in the domain $\mathcal{D}$ essentially becomes the weak limit for $\{r_m\}$ too. □

After the weak convergence, we are now interested in the strong convergence theorems. First, we want to provide the following facts.

Theorem 3. *Let $\mathcal{M}$ be a UCBS, $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ be compact and convex, and $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ be RSTN such that $F_{\mathcal{T}} \neq \emptyset$. Suppose $\{r_m\}$ is a sequence of the iterative scheme (8). Then $\{r_m\}$ converges strongly to an element of $F_{\mathcal{T}}$*

Proof. Since $\{r_m\} \subseteq \mathcal{D}$ and $\mathcal{D}$ are compact, we can find a subsequence, namely, $\{r_{m_k}\}$ of $\{r_m\}$ such that $\lim_{k\to\infty} ||r_{m_k} - y_0|| = 0$ for some element $y_0 \in \mathcal{D}$. Moreover, since $F_{\mathcal{T}} \neq \emptyset$, according to the Theorem 1, $\lim_{k\to\infty} ||r_{m_k} - \mathcal{T} r_{m_k}|| = 0$. Applying Lemma 1, we get

$$||r_{m_k} - \mathcal{T} y_0|| \leq \frac{(3+w)}{(1-w)} ||r_{m_k} - \mathcal{T} r_{m_k}|| + ||r_{m_k} - y_0||.$$

Now we conclude that $r_{m_k} \to \mathcal{T} y_0$ when $k \to \infty$. In the case of Banach spaces, a convergent sequence has only one limit point. It follows that $\mathcal{T} y_0 = y_0$, that is, y_0 is fixed point for $\mathcal{T}$. Furthermore, applying Lemma 4 on this y_0, means that $\lim_{m\to\infty} ||r_m - y_0||$ exists. One now concludes that y_0 also becomes the strong limit element for the original sequence $\{r_m\}$. □

We have noted in the above result that the compactness of the domain played an important roll in establishing the proof. Now we suggest a statement of a strong convergence theorem in which we do not use the compactness of the domain.

Theorem 4. *Suppose $\mathcal{M}$ is a UCBS, $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ is compact convex and $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ is a RSTN such that $F_{\mathcal{T}} \neq \emptyset$. If $\{r_m\}$ is a sequence of the iterative scheme (8), then $\{r_m\}$ converges strongly to a point $F_{\mathcal{T}}$ whenever* $\liminf_{m\to\infty} d(r_m, F_{\mathcal{T}}) = 0$.

Proof. We neglect the proof due to the fact it is elementary. □

At last, we impose a condition on selfmap and in this way, we shall drop the compactness of the domain.

Theorem 5. *Let $\mathcal{M}$ be a UCBS, $\emptyset \neq \mathcal{D} \subseteq \mathcal{M}$ be closed and convex, and $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ be RSTN such that $F_{\mathcal{F}} \neq \emptyset$. Suppose $\{r_m\}$ is a sequence of the iterative scheme (8) and $\mathcal{T}$ has condition (I). Then $\{r_m\}$ converges strongly to elements of $F_{\mathcal{T}}$.*

Proof. According to Theorem 1, one can conclude that $\liminf_{m\to\infty} ||r_m - \mathcal{T} r_m|| = 0$. Applying the condition I of $\mathcal{T}$, one obtains $\liminf_{m\to\infty} d(r_m, F_{\mathcal{T}}) = 0$. It now follows from Theorem 4 that $\{r_m\}$ is strongly convergent in the set $F_{\mathcal{T}}$. □

3. Application

In some cases, a problem has a solution, but it is possible that the ordinary analytical methods may fail to find it. Thus, fixed point theory suggests in such cases an alternative technique, that is, such solutions can be found by finding a fixed point of the fixed point equation $r = \mathcal{T}r$, where $\mathcal{T}$ is an appropriate operator. When $\mathcal{T}$ is nonexpansive, it is always uniformly continuous. Now we consider a split feasibility problem (SFP) and the iterative scheme (8). We have proved the main results for RSTN maps, which are often discontinuous on their domains. We shall apply these results on SFPs. Here, we may consider two complete inner product spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ and take a $\emptyset \neq C \subseteq \mathcal{M}_1$ and $\emptyset \neq Q \subseteq \mathcal{M}_2$ and assume that both of these subsets are convex and closed. We now consider a bounded linear map $\mathcal{F} : \mathcal{M}_1 \to \mathcal{M}_2$ and we set an SFP by the formula

$$\text{Compute } r \in C \text{ in the way that } \mathcal{F}r \in Q. \tag{19}$$

It should be noted that here we shall consider the solution set $\mathbb{S}$ endowed with the SFP (19) nonempty. We further assume

$$\mathbb{S} = \{r \in C : \mathcal{F}r \in Q\} = C \cap \mathcal{F}^{-1}Q.$$

It is not very hard to observe the convexity and closeness of the set $\mathbb{S}$. Censor and Elfving [25] fruitfully solved an inverse problem (IP) by using the technique of an SFP. However, Byrne [26] then constructed a well-known CQ-algorithm in order to solve a SFP. Consider a suitable scaler η, and suppose P_C and P_Q are the projections onto the subsets C and Q, respectively, and assume further that $\mathcal{F}^* : \mathcal{M}_2^* \to \mathcal{M}_1^*$ is an associated adjoint map for $\mathcal{F}$. In this case, the sequence $\{r_m\}$ obtained from the CQ-algorithm is given by the formula:

$$r_{m+1} = P_C[I - \eta\mathcal{F}^*(I - P_Q)\mathcal{F}]r_m, \ m \geq 1. \tag{20}$$

Remark 2. *The operator $\mathcal{T} = P_C[I - \eta\mathcal{F}^*(I - P_Q)\mathcal{F}]$, for $\eta \in (0, \frac{2}{||\mathcal{F}||^2})$ is nonexpansive (see e.g., [27] and others) and the CQ-algorithm converges in the weak sense to the solution of the SFP.*

Now the set $\mathbb{S}$ endowed with the SFP is nonempty; thus, a point, namely, $r^* \in C$ must be a solution for SFP if and only if it can solve the following equation

$$P_C[I - \eta\mathcal{F}^*(I - P_Q)\mathcal{F}]r = r, \ r \in C.$$

Notice that the set $\mathbb{S}$ is the same as the set $F_{\mathcal{T}}$, that is, $F_{\mathcal{T}} = \mathbb{S} = C \cap \mathcal{F}^{-1}Q \neq \emptyset$. For details, one can refer to [28,29].

Now we are interested in the weak and strong convergence using an alternative to the previous approaches, by RSTN maps, which are generally discontinuous on their domains (as shown by the Example 1), instead of nonexpansive maps, which are essentially continuous.

First, we discuss the weak convergence.

Theorem 6. *Suppose $\mathcal{T} = P_C[I - \eta\mathcal{F}^*(I - P_Q)\mathcal{F}]$ is RSTN and $\{r_m\}$ is a sequence produced from (8). In this case, $\{r_m\}$ converges in the weak sense to a solution of the SFP (19).*

Proof. From the assumption, $\mathcal{T}$ is an RSTN operator. Thus, applying Theorem 2, we obtain the required weak convergence. □

The desirable strong convergence is proved as follows.

Theorem 7. *Suppose $\mathcal{T} = P_C[I - \eta\mathcal{F}^*(I - P_Q)\mathcal{F}]$ is RSTN and $\{r_m\}$ is a sequence produced from (8). In this case, $\{r_m\}$ converges in the strong sense to a solution of the SFP (19) whenever* $\liminf_{m\to\infty} d(r_m, \mathbb{S}) = 0$.

Proof. From the assumption, $\mathcal{T}$ is an RSTN operator. Thus, applying Theorem 4, we obtain the required strong convergence. □

4. Comparison of Prominent Iterative Processes

Consider the RSTN mapping $\mathcal{T}$, which is not STN as defined in the Example 1. Tables 1 and 2 and Figure 1 suggest that the iteration of Hassan et al. (8) is better than the basic iterations like Mann (1), Ishikawa (2) and Noor (3). In addition, it is better than the Agarwal (4), Abbas (5), Thakur (6), Ullah (7) iterative processes in the general setting of RSTN maps.

Table 1. Sequences generated by the different iterative processes for the RSTN selfmap $\mathcal{T}$ given in Example 1.

	Hassan et al. (8)	Ullhah and Arshad (7)	Thakur et al. (6)	Abbas and Nazir (5)
1	6.6	6.6	6.6	6.6
2	6.000103051	6.004947917	6.008854167	6.030338542
3	6.000000015	6.000040803	6.000130660	6.001534045
4	6.000000000	6.000000336	6.000001928	6.000077568
5	6.000000000	6.000000003	6.000000028	6.000003922
6	6.000000000	6.000000000	6.000000000	6.000000198
7	6.000000000	6.000000000	6.000000000	6.000000010
8	6.000000000	6.000000000	6.000000000	6.000000001
9	6.000000000	6.000000000	6.000000000	6.000000000
10	6.000000000	6.000000000	6.000000000	6.000000000
11	6.000000000	6.000000000	6.000000000	6.000000000

Table 2. Sequences generated by the different iterative processes for the RSTN selfmap $\mathcal{T}$ given in Example 1.

	Agarwal et al. (4)	Noor (3)	Ishikawa (2)	Mann (1)
1	6.6	6.6	6.6	6.6
2	6.053125000	6.172265625	6.178125000	6.225000000
3	6.004703776	6.049459076	6.052880859	6.084370000
4	6.000416480	6.014200164	6.015699005	6.031640625
5	6.000036876	6.004077000	6.004660642	6.011865234
6	6.000003265	6.001170545	6.001383628	6.004449463
7	6.000000289	6.000336074	6.000410765	6.001668549
8	6.000000026	6.000096490	6.000121946	6.000625706
9	6.000000002	6.000027703	6.000036203	6.000234640
10	6.000000000	6.000007954	6.000010748	6.000087990
11	6.000000000	6.000002284	6.000003191	6.000032996
12	6.000000000	6.000000656	6.000000947	6.000012374
13	6.000000000	6.000000188	6.000000281	6.000004640
14	6.000000000	6.000000054	6.000000083	6.000001740
15	6.000000000	6.000000016	6.000000025	6.000000653
16	6.000000000	6.000000004	6.000000007	6.000000245
17	6.000000000	6.000000001	6.000000002	6.000000092
18	6.000000000	6.000000000	6.000000001	6.000000034
19	6.000000000	6.000000000	6.000000000	6.000000013
20	6.000000000	6.000000000	6.000000001	6.000000005
21	6.000000000	6.000000000	6.000000000	6.000000002
22	6.000000000	6.000000000	6.000000000	6.000000001
23	6.000000000	6.000000000	6.000000000	6.000000000

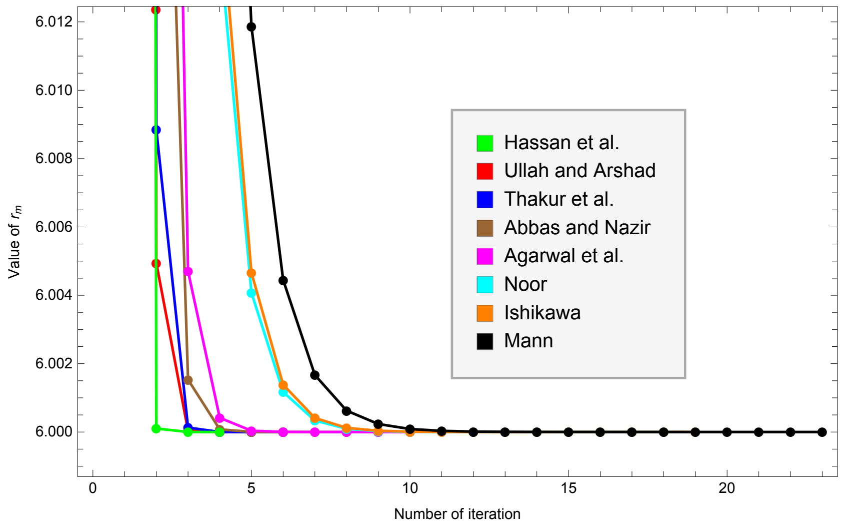

Figure 1. Behaviors of different iterative procedures in the class of RSTN mappings using $\mathcal{T}$ in Example 1.

Remark 3. *The main outcome of this article improved and extended the main outcome of the article of Hassan et al. [18] from the class of STN maps to the setting of RSTN maps. We have seen in Tables 1 and 2 and in Figure 1 that the Hassan iterative scheme (8) is still more effective than the other iterative schemes even in the general setting of RSTN maps. We have applied the main results on SFP in the context of discontinuous maps.*

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References

1. Abbas, M.; Nazir, T. A new faster iteration process applied to constrained minimization and feasibility problems. *Math. Vesnik* **2014**, *66*, 223–234.
2. Garodia, C.; Uddin, I. A new fixed point algorithm for finding the solution of a delay differential equation. *AIMS Math.* **2020**, *5*, 3182–3200. [CrossRef]
3. Ullah, K.; Ahmad, J.; de la Sen, M. Some new results on a three-step iteration process. *Axioms* **2020**, *9*, 110. [CrossRef]
4. Banach, S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
5. Picard, E. Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. *J. Math. Pures Appl.* **1880**, *6*, 145–210.
6. Browder, F.E. Nonexpansive nonlinear operators in a Banach space. *Proc. Natl. Acad. Sci. USA* **1965**, *54*, 1041–1044. [CrossRef]
7. Gohde, D. Zum Prinzip der Kontraktiven Abbildung. *Math. Nachr.* **1965**, *30*, 251–258. [CrossRef]
8. Kirk, W.A. A fixed point theorem for mappings which do not increase distance. *Am. Math. Mon.* **1965**, *72*, 1004–1006. [CrossRef]
9. Suzuki, T. Fixed point theorems and convergence theorems for some generalized non-expansive mapping. *J. Math. Anal. Appl.* **2008**, *340*, 1088–1095. [CrossRef]
10. Pandey, R.; Pant, R.; Rakocevic, V.; Shukla, R. Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications. *Results Math.* **2019**, *74*, 1–24. [CrossRef]

11. Krasnoselskii, M.A. Two remarks on the method of succesive approximations. *Uspekhi Math. Nauk.* **1955**, *10*, 123–127. (In Russian)
12. Mann, W.R. Mean value methods in iteration. *Proc. Am. Math. Soc.* **1953**, *4*, 506–510. [CrossRef]
13. Ishikawa, S. Fixed points by a new iteration method. *Proc. Am. Math. Soc.* **1974**, *44*, 147–150. [CrossRef]
14. Noor, M.A. New approximation schemes for general variational inequalities. *J. Math. Anal. Appl.* **2000**, *251*, 217–229. [CrossRef]
15. Agarwal, R.P.; O'Regon, D.; Sahu, D.R. Iterative construction of fixed points of nearly asymtotically non-expansive mappings. *J. Nonlinear Convex Anal.* **2007**, *8*, 61–79.
16. Thakur, B.S.; Thakur, D.; Postolache, M. A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings. *Appl. Math. Comput.* **2016**, *275*, 147–155. [CrossRef]
17. Ullah, K.; Arshad, M. New iteration process and numerical reckoning fixed point in Banach spaces. *UPB Sci. Bull.* **2017**, *79*, 113–122.
18. Hassan, S.; de la Sen, M.; Agarwal, P.; Ali, Q.; Hussain, A. A new faster iterative scheme for numerical fixed points estimation of Suzuki's generalized nonexpansive mappings. *Math. Probl. Eng.* **2020**, *2020*, 3863819. [CrossRef]
19. Opial, Z. Weak and strong convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597. [CrossRef]
20. Senter, H.F.; Dotson, W.G. Approximating fixed points of nonexpansive mappings. *Proc. Am. Math. Soc.* **1974**, *44*, 375–380. [CrossRef]
21. Clarkson, J.A. Uniformly convex spaces. *Trans. Am. Math. Soc.* **1936**, *40*, 396–414. [CrossRef]
22. Agarwal, R.P.; O'Regan, D.; Sahu, D.R. *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*; Topological Fixed Point Theory and Its Applications; Springer: New York, NY, USA, 2009; Volume 6.
23. Takahashi, W. *Nonlinear Functional Analysis*; Yokohoma Publishers: Yokohoma, Japan, 2000.
24. Schu, J. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bull. Austral. Math. Soc.* **1991**, *43*, 153–159. [CrossRef]
25. Censor,Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algor.* **1994**, *8*, 221–239. [CrossRef]
26. Byrne, C. Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse Prob.* **2002**, *18*, 411–453. [CrossRef]
27. Feng, M.; Shi, L.; Chen, R. A new three-step iterative algorithm for solving the split feasibility problem. *Univ. Politeh. Buch. Ser. A* **2019**, *81*, 93–102.
28. Xu, H.K. A variable Krasnoselskii-Mann algorithm and the multiple-set split feasibility problem. *Inverse Probl.* **2006** *22*, 2021–2034. [CrossRef]
29. Xu, H.K. Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. *Inverse Probl.* **2010**, *26*, 105018. [CrossRef]