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Snake Graphs Arising from Groves with an Application in Coding Theory

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Abstract: *Snake graphs* are connected planar graphs consisting of a finite sequence of adjacent tiles (squares) $T_1, T_2, \ldots, T_n$. In this case, for $1 \leq j \leq n-1$, two consecutive tiles T_j and T_{j+1} share exactly one edge, either the edge at the east (west) of T_j (T_{j+1}) or the edge at the north (south) of T_j (T_{j+1}). Finding the number of *perfect matchings* associated with a given snake graph is one of the most remarkable problems regarding these graphs. It is worth noting that such a number of perfect matchings allows a bijection between the set of snake graphs and the positive *continued fractions*. Furthermore, perfect matchings of snake graphs have also been used to find closed formulas for cluster variables of some cluster algebras and solutions of the Markov equation, which is a well-known Diophantine equation. Recent results prove that snake graphs give rise to some *string modules* over some path algebras, connecting snake graph research with the theory of representation of algebras. This paper uses this interaction to define *Brauer configuration algebras* induced by schemes associated with some multisets called polygons. Such schemes are named Brauer configurations. In this work, polygons are given by some *admissible words*, which, after appropriate transformations, permit us to define sets of binary trees called *groves*. Admissible words generate *codes* whose *energy values* are given by snake graphs. Such energy values can be estimated by using *Catalan numbers*. We include in this paper Python routines to compute admissible words (i.e., codewords), energy values of the generated codes, Catalan numbers and dimensions of the obtained Brauer configuration algebras.

Keywords: binary tree; coding theory; Brauer configuration algebra; Catalan combinatorics; path algebra; snake graph; string modules

MSC: 16G30; 16G60; 05A05; 05E10; 94B65

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1. Introduction

Propp [1] introduced snake graphs in his investigations regarding the Laurent phenomenon associated with cluster algebras. Since then, snake graphs have become a helpful tool for research on different topics in several fields of mathematics. For instance, Schiffler and Çanakçi [2–6] developed a complete calculus for these kinds of graphs, achieving new developments in the continued fraction theory, bearing in mind that there is a bijection between the set of abstract snake graphs and the set of positive continued fractions.

According to this approach, perfect matchings of snake graphs give information on the numerator of a continued fraction. It is worth pointing out that perfect matchings of snake graphs were used by Schiffler et al. [7] to find a formula for cluster variables in cluster algebras from surface types, giving a solution to a well-known conjecture (positivity conjecture) regarding these algebras.

Another interpretation of the snake graph calculus was given by Çanakçi and Schroll [8], who proved that, associated with any snake graph, there is a suitable string module whose submodule lattice is in bijection with the perfect matching lattice of the underlying snake graph.

On the other hand, Brauer configuration algebras (BCAs) were introduced by Green and Schroll [9] as a generalization of Brauer graph algebras [10]. They have been used as a tool in research in different scientific fields. Indecomposable projective modules over appropriated Brauer configuration algebras have been interpreted as shadows in visual secret sharing schemes (VSSS) or as subkeys in the key schedule of the advanced encryption standard (AES) [11].

1.1. Motivations

Currently, snake graphs and Brauer configuration algebras are known to be ubiquitous tools in mathematics and their applications. On the one hand, the snake graph combinatorics gives new advances in the theory of continued fractions and rational knots. On the other hand, the combinatorial properties of Brauer configuration algebras have allowed applications of the theory of representation algebras in cryptography and the theory of graph energy, among others.

This paper uses interactions between the theories of snake graphs and Brauer configuration algebras to give a novel application of snake graphs in the coding theory.

The codes that we are interested in have as codewords n-tuples of q-ary vectors whose coordinates belong to an alphabet Q with q letters—namely, $Q = \{0, 1, 2, \ldots, q-1\}$. In this case, q is not necessarily the power of a prime number. The main problem associated with these kinds of codes is minimizing the potential energy provided by the cardinality $|C|$ of a fixed code C.

We address the coding problem from the Brauer configuration algebras point of view. First, we prove that n-tile snake graphs define Brauer configurations whose polygons are related to some binary trees. Then, such polygons are interpreted as codes whose minimal energy is given by an appropriated snake graph.

In the sequel, we briefly describe the main results presented in this work and how some previous works are used to obtain them.

1.2. Contributions

This work proves that string modules and their corresponding snake graphs define appropriated Brauer configuration algebras, whose indecomposable projective modules can be interpreted as groves of binary trees. It is proven that the dimensions of string modules are given by suitable words (admissible words) whose letters are positive integers.

We define some operations on these words in such a way that sums of Catalan numbers give the dimensions of these algebras and their centers. In particular, the number of indecomposable projective modules over the constructed algebras is another manifestation of Catalan numbers.

Words associated with polygons define codes C_{Γ^j} of type $H(1, j)$. Snake graphs give energy values of these codes. We provide, in the Appendix, Python routines to compute admissible words (associated with indecomposable projective modules over such Brauer configuration algebras), Catalan numbers, energy values of the codes C_{Γ^j}, and dimensions of the generated Brauer configuration algebras.

Figure 1 shows how the results introduced in the Background section regarding snake graphs and Brauer configuration algebras (see blue arrows) are used in this paper to obtain the main results (green arrows).

We define admissible words in Section 3.1, which induce Brauer configuration algebras of type Λ_{Γ^j}. Proposition 2 shows a means of building admissible words, and Proposition 3 proves that snake graphs define admissible words.

Corollary 1 enumerates admissible words, and Theorem 3 gives formulas for the dimension of the Brauer configuration algebras induced by admissible words and their corresponding centers. Theorem 4 estimates the energy values of some codes defined by the Brauer configurations Γ^j. In particular, it is proven that snake graphs give such energy values.

Corollary 2 proves that the distances associated with these codes define the integer partitions of triangular numbers t_{C_j}, where C_j denotes the jth Catalan number.

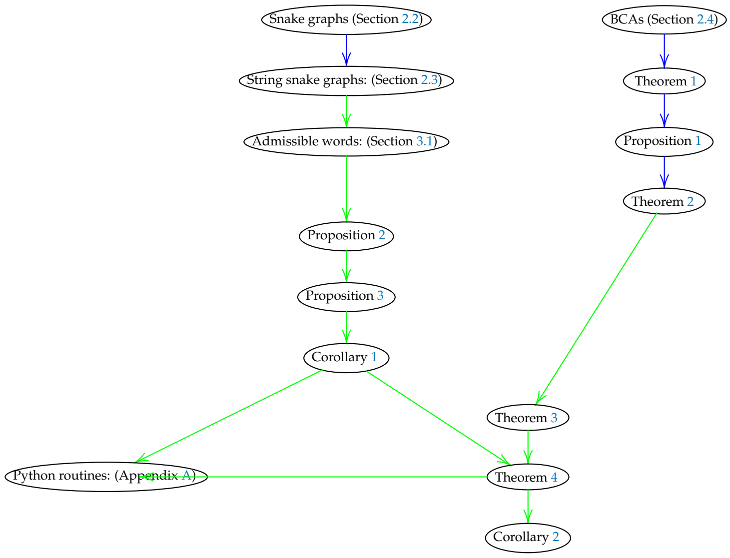


Figure 1. This graph shows how we use topics described in the Background section to obtain the main results presented in this paper.

This paper is distributed as follows. In Section 2, we recall definitions and notations used throughout the paper. In particular, we recall the notions of path algebra, string snake graph, and Brauer configuration algebra. In Section 3, we give our main results. We define and enumerate admissible words (Section 3.1). We also give properties of Brauer configuration algebras defined by admissible words (Section 3.2). In Section 3.3, we give properties of a code defined by the Brauer configurations introduced in Section 3.2. Concluding remarks are given in Section 4. Python routines are included in Appendix A.

2. Background and Related Work

This section introduces some definitions, results, and notations to be used throughout the paper. We also recall some background results, which will allow a better understanding of the main results presented in the next section. The authors refer the interested reader to [8–13] for a detailed study of the treated topics. Henceforth, $\mathbb{F}$ ($\mathbb{N}$) will denote a field (the set of natural numbers).

2.1. Path Algebras

This section recalls some basic notions of path algebras [13].

If $\mathbb{F}$ is an algebraically closed field, then a *path algebra* $\mathbb{F}Q$ is an algebra generated by the paths of a quiver (oriented graph) $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are sets and s, t are maps such that $s, t : Q_1 \to Q_0$, and elements of the set Q_0 (Q_1) are said to be the *vertices* (*arrows*) of the quiver Q.

If $\alpha \in Q_1$, then the vertex $s(\alpha)$ ($t(\alpha)$) is the *source* (*target*) of the arrow α [13].

An *ideal* I of a path algebra $\mathbb{F}Q$ is generated by relations. These relations are nothing but paths with the same starting and ending points. The two-sided ideal generated by the arrows (paths of length greater than or equal to l) of Q is denoted by R_Q (R_Q^l). An ideal I is said to be *admissible*, if there is an integer $m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. R_Q is said to be the *arrow ideal* of $\mathbb{F}Q$.

If I is an admissible ideal of $\mathbb{F}Q$, the pair (Q, I) is said to be a *bound quiver*. The quotient algebra $\mathbb{F}Q/I$ is said to be a *bound quiver algebra* [13]. It is worth noting that any basic algebra is isomorphic to a bound quiver algebra $\mathbb{F}Q/I$ if I is a suitable admissible ideal.

2.2. Snake Graphs

A *tile* G is a square in the plane whose sides are parallel or orthogonal to the elements in the standard orthonormal basis of the plane (as in [5], in this work, a tile G is considered as a graph with four vertices and four edges in the obvious way).

A snake graph is called *straight* if all its tiles lie in one column or row, and a snake graph is called *zigzag* if no three consecutive tiles are straight. Two snake graphs are *isomorphic* if they are isomorphic as graphs (cf. [2–6]).

Each snake graph $\mathcal{G}$ has associated a sign function sgn from the set of edges to the set $\{+,-\}$, which defines an ordered sequence of positive integers $\{a_1, a_2, \dots, a_n\}$. Numbers a_i give rise to the continued fraction

$$[a_1, a_2, \dots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}}. \tag{1}$$

The snake graph $\mathcal{G}[a_1, a_2, \dots, a_n]$ of the positive continued fraction $[a_1, a_2, \dots, a_n]$ is the snake graph with $d = a_1 + a_2 + \cdots + a_n - 1$ tiles determined by the sign function. In particular, $\mathcal{G}[1]$ is a single edge.

Schiffler and Çanakçi [3] proved that if Match$(\mathcal{G})$ denotes the number of perfect matchings of the snake graph $\mathcal{G}$, then $[a_1, a_2, \dots, a_n] = \frac{\text{Match}(\mathcal{G}[a_1,a_2,\dots,a_n])}{\text{Match}(\mathcal{G}[a_2,a_3,\dots,a_n])}$. Here, Match$(\mathcal{G})$ is the set of perfect matchings of $\mathcal{G}$.

As in [12], for positive integers $n_1, n_2, \dots, n_k$, we let $\mathcal{G}_f(n_1, n_2, \dots, n_k)$ denote a snake graph, with $n_1 \geq 2$ tiles in the first row, $n_2 \geq 2$ in the first column, $n_3 \geq 2$ tiles in the second row, and so on up to $n_k \geq 2$. In this case, the last tile in a given row is the first tile in the next column (if it exists) and, vice versa, the last tile in a given column coincides with the first tile in the next row. As an example, in Figure 2, we show the snake graph $\mathcal{G}_f(2,2,2)$.

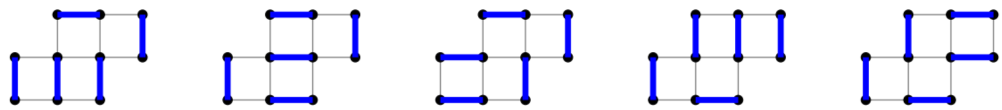


Figure 2. Snake graph $\mathcal{G}_f(2,2,2) = \mathcal{G}[5]$ and its perfect matchings.

Remark 1. *In this paper, we also consider continued fractions associated with rational numbers* $\frac{n}{m}$, *with* $m > n$. *In such a case, the corresponding continued fraction is associated with a special snake graph* $\mathcal{G}[0, cf(\frac{m}{n})]$. *These graphs are given by adding a red point to the first tile of the snake graph associated with the continued fraction* $cf(\frac{m}{n})$ *of* $\frac{m}{n}$.

2.3. String Modules and Snake Graphs

An abstract string is a word of the form $w = a_1 a_2 \dots a_n$, where, for $1 \leq j \leq n$, $a_j \in \{\rightarrow, \leftarrow\}$. $\varnothing$ is also considered an abstract string. If $a_j = \rightarrow$ $(\leftarrow)$, for any j, then w is said to be a *direct string* (*inverse string*).

According to Çanakçi and Schroll [8], the following procedure allows us to build a snake graph with $n+1$ tiles from an abstract string $w = a_1 a_2 \dots a_n$:

1. If $w = \varnothing$, then the corresponding abstract snake graph is given by a single tile.
2. If there is at least one letter, then $a_1, a_2, \dots, a_n$ is a concatenation of a collection of alternating maximal direct and inverse strings w_i such that $w = w_1 w_2 \dots w_k$. Each w_i might be of length 1.
3. For each w_i, we construct a zigzag snake graph $\mathcal{G}_i$ with $l(w_i) + 1$ tiles, where $l(w_i)$ is the number of direct or inverse arrows in w_i. Let $\mathcal{G}_i$ be the zigzag snake graph with tiles $T_1^i, \dots, T_{l(w_i)+1}^i$, such that T_2^i is glued to the right (resp. on top) of T_1^i if w_i is direct (resp. inverse).

4. We now glue $\mathcal{G}_{i+1}$ to $\mathcal{G}_i$, for all i, by identifying the last tile $T^i_{l(w_i)+1}$ of $\mathcal{G}_i$ and the first tile T^{i+1}_1 of $\mathcal{G}_{i+1}$, such that $T^i_{l(w_i)}$, $T^i_{l(w_i)+1}$, T^{i+1}_2 is a straight piece.

Figure 3 shows an orientation of the Dynkin diagram $\mathbb{A}_4$ and its corresponding snake graph. In such a case, w has the following maximal strings:

- $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$, which defines the zigzag snake graph $\mathcal{G}_1$ containing the tiles, $1, 2, 3$, and 4.
- $4 \longleftarrow 5$, which defines the straight snake graph $\mathcal{G}_2$ containing the tiles 4 and 5.

Note that snake graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are glued in a natural way.

$$w = 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longleftarrow 5$$

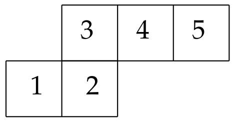


Figure 3. Snake graph $\mathcal{G}_f(2, 2, 3) = \mathcal{G}[4, 2]$ associated with a four-arrow string. The string module $M(w)$ over the corresponding Dynkin algebra of type $\mathbb{A}$ is obtained by replacing every vertex with a copy of a field $\mathbb{F}$. In such a case, arrows correspond to identity morphism.

Çanakçi and Schroll proved that if $A = \mathbb{F}Q/I$ is a bound quiver algebra and $M_{\mathcal{G}}(w)$ is a string module over A with string w and with associated snake graph $\mathcal{G}$, then the perfect matching lattice $\mathcal{L}_{\mathcal{G}}$ of $\mathcal{G}$ is in bijection with the canonical submodule lattice $\mathcal{L}_{\mathcal{G}}(M(w))$ of $M_{\mathcal{G}}(w)$.

We recall that a string module M is given by the orientation of a type $\mathbb{A}$ Dynkin diagram where every vertex is replaced by a copy of $\mathbb{F}$ and the arrows correspond to the identity maps. This paper interprets arrows as operations between suitable words consisting of positive integers.

2.4. Brauer Configuration Algebras

Green and Schroll introduced Brauer configuration algebras as a generalization of Brauer graph algebras [9–12]. Its definition goes as follows:

A Brauer configuration algebra Λ_Γ (or simply Λ if no confusion arises) is a bound quiver algebra induced by a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$, where:

- Γ_0 is a finite set of vertices.
- Γ_1 is a collection of polygons, which are labeled multisets consisting of vertices (vertices repetition allowed). Each polygon contains more than one vertex.
- μ is a map from the set of vertices Γ_0 to the set of positive integers $\mathbb{N}\backslash\{0\} = \mathbb{N}^+$, $\mu : \Gamma_0 \to \mathbb{N}^+$.
- $\mathcal{O}$ is a choice for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which α occurs as a vertex including repetitions (see [9] for more details). For instance, if a vertex $\alpha \in \Gamma_0$ occurs in polygons $U_{i_i}, U_{i_2}, \ldots, U_{i_m}$, for suitable indices $i_1, i_2, \ldots, i_m \in \{1, 2, 3, \ldots, n\}$, then the cyclic order is obtained by linearly ordering the list, say

$$U^{\alpha_1}_{i_1} < U^{\alpha_2}_{i_2} < \cdots < U^{\alpha_m}_{i_m}, \quad \alpha_{i_s} > 0. \tag{2}$$

where $U^{\alpha_s}_{i_s} = U^{(1)}_{i_s} < U^{(2)}_{i_s} < \cdots < U^{(\alpha_s)}_{i_s}$ means that vertex α occurs α_s times in polygon U_{i_s}, denoted $\alpha_s = occ(\alpha, U_{i_s})$. The cyclic order is completed by adding the relation $U_{i_m} < U_{i_1}$. Note that if $U_{i_1} < \cdots < U_{i_t}$ is the chosen ordering at vertex α, then the same ordering can be represented by any cyclic permutation.

The sequence (2) is said to be the successor sequence at vertex α denoted S_α, which is unique up to permutations.

Henceforth, this paper assumes the notation used in [12] for successor sequences and special cycles. Namely, if a vertex $\alpha' \neq \alpha$ belongs to some polygons $U_{j_1}, U_{j_2}, \ldots U_{j_k}$ ordered according to the already defined cyclic ordering associated with the vertex α, then we will assume that, up to permutations, the cyclic ordering associated with the vertex α' is built, taking into account that polygons $U_{j_1}, U_{j_2}, \dots U_{j_k}$ inherit the order given by the successor sequence S_α.

If $\alpha \in \Gamma_0$, then the valency $val(\alpha)$ of α is given by the identity

$$val(\alpha) = \sum_{U \in \Gamma_1} occ(\alpha, U). \tag{3}$$

If $\alpha \in \Gamma_0$ is such that $\mu(\alpha)val(\alpha) = 1$, then α is said to be *truncated* (it occurs once in only one polygon). Otherwise, α is a non-truncated vertex. It is worth pointing out that each polygon in a Brauer configuration has at least one non-truncated vertex. A Brauer configuration without truncated vertices is said to be *reduced*.

Later on, we will assume that successor sequences associated with non-truncated vertices are of the form (2). As Green and Schroll mentioned in [9], if α is a non-truncated vertex and $val(\alpha) = 1$, then there is only one choice for the associated cyclic ordering.

From now on, if no confusion arises, we will assume notations Q, I, and Λ instead of Q_Γ, I_Γ, and Λ_Γ, for a quiver, an admissible ideal, and the Brauer configuration algebra induced by a fixed Brauer configuration Γ.

Since polygons in Brauer configurations are multisets, we will often assume that such polygons are given by words w of the form

$$w = y_1^{f_1} y_2^{f_2} \dots y_{t-1}^{f_{t-1}} y_t^{f_t}, \tag{4}$$

where, for each i, $1 \leq i \leq t$, y_i is an element of the polygon called vertex and f_i is the frequency of the vertex y_i. In other words, f_i is the number of times that a vertex occurs in a polygon [14].

In [11], Cañadas et al. introduced an algorithm to build a Brauer configuration algebra $\Lambda_\Gamma = \mathbb{F}Q_\Gamma / I_\Gamma$. $Q_\Gamma = (Q_0, Q_1, s, t)$ is a quiver, whose set of vertices Q_0 is in bijective correspondence with the set of polygons in Γ_1. Arrows are induced by the orientation $\mathcal{O}$ by identifying each cover $V < W$ in a cyclic ordering with an arrow $\alpha : V \to W$.

The bound quiver algebra $\mathbb{F}Q_\Gamma$ is a path algebra bounded by an admissible ideal I_Γ generated by relations ρ_Γ of the following types:

1. Identify special cycles associated with non-truncated vertices in the same polygon (i.e., if $\delta_1, \delta_2 \in U$ with $U \in \Gamma_1$, then $C_{\delta_1}^{\mu(\delta_1)} - C_{\delta_2}^{\mu(\delta_2)} \in \rho_\Gamma$).
2. If C_δ is a special cycle associated with a non-truncated vertex δ, then a product of the form $C^{\mu(\delta)}a \in \rho_\Gamma$, if a is the first arrow of δ.
3. Quadratic monomial relations of the form ab in $\mathbb{F}Q_\Gamma$, where ab is not a subpath of any special cycle unless $a = b$ and a is a loop associated with a vertex α of valency 1 and $\mu(\alpha) > 1$.

The following Theorem 1 gives some properties of Brauer configuration algebras [9,15].

Theorem 1 ([9], Theorem B)**.** *The following results hold for a Brauer configuration algebra $\Lambda = \mathbb{F}Q/I$ induced by a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$.*

1. *There is a bijection between Γ_1 and the set of indecomposable projective Λ-modules.*
2. *If P is a projective indecomposable Λ-module corresponding to a polygon V in Γ, then* $\mathrm{rad}(P)$ *is a sum of r indecomposable uniserial modules, where r is the number of (non-truncated) vertices of V and where the intersection of any two of the uniserial modules is a simple Λ-module.*

Proposition 1 and Theorem 2 give formulas for the dimensions $\dim_{\mathbb{F}} \Lambda$, and $\dim_{\mathbb{F}} Z(\Lambda)$ of a Brauer configuration algebra Λ and its center $Z(\Lambda)$ [9,15].

Proposition 1 (Proposition 3.13, [9])**.** *Let Λ be a Brauer configuration algebra associated with the Brauer configuration Γ and let $\mathcal{C} = \{C_1, \dots, C_t\}$ be a full set of equivalence class representatives* *of special cycles. Assume that for $i = 1, \ldots, t$, C_i is a special α_i-cycle where α_i is a non-truncated vertex in Γ. Then,*

$$\dim_{\mathbb{F}} \Lambda = 2|Q_0| + \sum_{C_i \in \mathcal{C}} |C_i|(n_i|C_i| - 1),$$

where $|Q_0|$ denotes the number of vertices of Q; $|C_i|$ denotes the number of arrows in the α_i-cycle C_i, and $n_i = \mu(\alpha_i)$.

Theorem 2 (Theorem 4.9, [15])**.** *Let $\Lambda = \mathbb{F}Q/I$ be the Brauer configuration algebra associated with the connected and reduced Brauer configuration Γ. Then,*

$$\dim_{\mathbb{F}} Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#(Loops\ Q) - |\mathcal{C}_\Gamma|,$$

where $\mathcal{C}_\Gamma = \{\alpha \in \Gamma_0 \mid val(\alpha) = 1, \text{ and } \mu(\alpha) > 1\}$.

3. Main Results

In this section, we follow some of Loday's ideas [16] to prove that string snake graphs induce binary trees, Brauer configuration algebras, and codes.

3.1. Admissible Words

An *admissible word* w has positive integers as letters. It can be written in the form:

$$w = \alpha_1\alpha_2 \ldots \alpha_t, \tag{5}$$

where, for each $1 \le i \le t$, α_i is a positive integer, $1 \le \alpha_i \le |w|$, $|w|$ denotes the length of w. If $\alpha_h = |w|$, then $\alpha_i < \alpha_h$, for any $i \ne h$. Moreover, if $w = w_{\alpha_{h-1}}|w|\alpha_{h+1}$, with $w_{h-1} = \alpha_1\alpha_2 \ldots \alpha_{h-1}$, $w_{h+1} = \alpha_{h+1}\alpha_2 \ldots \alpha_t$, then $|w_{h-1}| + |w_{h+1}| + 1 = |w|$. For $n \ge 1$, we let $\mathcal{W}_n$ denote the set of all admissible words of length n, $\mathcal{W} = \bigcup\limits_{n \ge 0} \mathcal{W}_n$, $\mathcal{W}_0 = \varnothing$ is the empty word, $|\mathcal{W}_0| = 0$. For example, $\mathcal{W}_1 = \{1\}$, $\mathcal{W}_2 = \{12, 21\}$, whereas 131 and 4123 are admissible words in $\mathcal{W}_3$ and $\mathcal{W}_4$, respectively.

Any admissible word $w \in \mathcal{W}$ is obtained by applying the operations $\bullet$, $\perp$, $\top$, $\propto$, $\cup$, $\rightarrow$, $\leftarrow$, and $\leftrightarrow$, defined as follows:

1. $\bullet$ is the usual concatenation of words. If no confusion arises, later on, we will write $w \bullet w' = ww'$.
2. If, for $n \ge 1$ fixed $w \in \mathcal{W}_n$, then $\perp(w) = (|w| + 1)w \in \mathcal{W}_{n+1}$. Moreover, $\top(w) = w(|w| + 1) \in \mathcal{W}_{n+1}$.
3. If $w, w' \in \mathcal{W}$, then $w \propto w' = w(|ww'| + 1)w'$. In such a case, we write $(w \propto w')^l = w$, $(w \propto w')^r = w'$. In particular, $\perp(w) = \varnothing \propto w = (|w| + 1)w$, $\top(w) = w \propto \varnothing = w(|w| + 1)$. Thus, any admissible word w can be written in the form $w = (w)^l \propto (w)^r$.
4. If $w, w' \in \mathcal{W}$, then $w \cup w' = \{w, w'\}$. If $w_1, w_2, \ldots, w_k \in \mathcal{W}$, then $\perp(\bigcup\limits_{k=1}^{n} w_k) = \bigcup\limits_{k=1}^{n} \perp(w_k)$. Moreover, $\top(\bigcup\limits_{k=1}^{n} w_k) = \bigcup\limits_{k=1}^{n} \top(w_k)$.
5. $w \leftrightarrow w' = (w \rightarrow w') \cup (w \leftarrow w')$, where $w \rightarrow w' = w^l \propto (w^r \leftrightarrow w')$, and $w \leftarrow w' = (w \leftrightarrow (w')^l) \propto (w')^r$. In particular, if $w = x(|w|)$ $(w' = (|w'|y))$, then $w \rightarrow w' = x(|ww'|)w'$ $(w \leftarrow w' = w(|ww'|)y)$.
6. $\bigcup\limits_{k=1}^{n} w_k \rightarrow w = \bigcup\limits_{k=1}^{n} w_k \rightarrow w, \quad w \leftarrow \bigcup\limits_{k=1}^{n} w_k = \bigcup\limits_{k=1}^{n} w \leftarrow w_k.$

12 (21) is the unique admissible word w for which $w^r = \varnothing$ $(w^l = \varnothing)$.

$$\begin{aligned} (12)^l = (21)^r = 1 = \varnothing \propto \varnothing, \\ (1)^r = (1)^l = \varnothing. \end{aligned} \tag{6}$$

$1 \propto 1 = 131$, $12 \propto 1 = 1241$, $\perp(123) = \varnothing \propto 123 = 4123$. Note that $\rightarrow$ and $\leftarrow$ are not associative. For instance, $1 \rightarrow 21 = 321$, and $21 \rightarrow 1 = \{312, 321\}$. $\leftrightarrow$ is not commutative $1 \leftrightarrow 21 = \{321, 131\}$, $21 \leftrightarrow 1 = \{321, 312, 213\}$.

Proposition 2. *For any $n \geq 1$ fixed, it holds that $\mathcal{W}_n = \bigcup\limits_{i=0}^{n-1} \mathcal{W}_i \propto \mathcal{W}_{n-1-i}$.*

Proof. We note that $\mathcal{W}_1 = 1$, $\mathcal{W}_2 = \{12, 21\}$, $\mathcal{W}_3 = \mathcal{W}_2 \leftrightarrow 1 = \mathcal{W}_2 \leftarrow 1 \cup \mathcal{W}_2 \rightarrow 1$. □

$$\begin{aligned} &\mathcal{W}_2 \leftarrow 1 = \{123, 213\} \\ &\mathcal{W}_2 \rightarrow 1 = 12 \rightarrow 1 \cup 21 \rightarrow 1 = 1 \propto 1 \cup \perp(1 \leftrightarrow 1) = \\ &131 \cup \perp(12 \cup 21) = \{131, 312, 321\}. \end{aligned} \tag{7}$$

If the theorem holds for $1 \leq i < j$, then $\mathcal{W}_j = \mathcal{W}_{j-1} \leftrightarrow 1$. Thus,

$$\begin{aligned} W_j = {} & [\bigcup_{k=0}^{j-2} \mathcal{W}_k \propto W_{j-2-k} \leftarrow 1] \bigcup [\bigcup_{k=0}^{j-2} \mathcal{W}_k \propto W_{j-2-k} \rightarrow 1] = \\ & [(\bigcup_{k=0}^{j-2} \top(\mathcal{W}_k \propto W_{j-2-k}))] \bigcup [\bigcup_{k=0}^{j-2} \mathcal{W}_k \propto (W_{j-2-k} \leftrightarrow 1)] = \\ & \bigcup_{k=0}^{j-1} \mathcal{W}_k \propto W_{j-1-k}. \end{aligned} \tag{8}$$

3.2. Brauer Configuration Algebras Associated with Snake Graphs

This section proves that an n-tile snake graph induces a Brauer configuration whose vertices are positive integers and word polygons consist of admissible words as defined in the previous section.

Proposition 3. *String modules associated with n-tile snake graphs define $\mathcal{W}_n$.*

Proof. (Induction) If $d_n = 1a_1 1a_2 1 \dots 1a_n 1$ is the dimension of a string module associated with an orientation $a_1 a_2 \dots a_n$ for which $a_i \in \{\rightarrow, \leftarrow\}$, then, according to the operations defined in Section 3.1, $d_n = 21a_2 1 \dots 1a_n 1$ if $a_1 = \rightarrow$, $d_n = 12a_2 1 \dots 1a_n 1$, if $a_1 = \leftarrow$, $d_n = \perp(12 \cup 21)a_3 1 \dots 1a_n 1 = 312a_3 1 \dots 1a_n 1 \cup 321a_3 1 \dots 1a_n 1$, if $a_2 = \rightarrow$ has 21 as a source. $d_n = 131a_3 1 \dots 1a_n 1$ if $a_2 = \rightarrow$ has 12 as a source.

$d_n = 123a_3 1 \dots 1a_n 1$ ($d_n = 213a_3 1 \dots 1a_n 1$) if $a_2 = \leftarrow$, $s(a_2) = 1$, $t(a_2) = 12$ ($t(a_2) = 21$).

We note that, for $i \leq n$ fixed, $d_n = (w' \propto (w' \rightarrow w'') \bigcup w' \propto (w' \leftarrow w''))a_i \dots a_n$, if $a_{i-1} = \rightarrow$, $s(a_{i-1}) = w = w' \propto w'' \in \mathcal{W}_i$, $w' \in \mathcal{W}_k$, $w'' \in \mathcal{W}_{i-1-k}$, $k \leq i-1$, $i \leq n$. Meanwhile, $d_n = \top(w)a_i \dots a_n$, if $a_{i-1} = \leftarrow$, $s(a_{i-1}) = 1$ and $t(a_{i-1}) = w = w' \propto w'' \in \mathcal{W}_i$. Thus, for $2 \leq i \leq n$, it holds that the $(i-1)$th arrow of a string module gives rise to $\mathcal{W}_i = \perp(\mathcal{W}_{i-1}) \cup (\mathcal{W}_1 \propto \mathcal{W}_{i-2}) \bigcup \dots \bigcup (\mathcal{W}_k \propto \mathcal{W}_{i-1-k}) \bigcup \top(\mathcal{W}_{i-1})$. We are finished. □

Corollary 1. *For $n \geq 1$, $|\mathcal{W}_n| = C_n$, where C_n is the nth Catalan number.*

Proof. (Induction) Note that $|\mathcal{W}_1| = 1$, $|\mathcal{W}_2| = 2$, $|\mathcal{W}_3| = 5$. If it is assumed that the statement is true for $1 \leq j < i$, then $|\mathcal{W}_i| = |\perp(\mathcal{W}_{i-1})| + |\mathcal{W}_{i-2} \propto \mathcal{W}_1| + \dots + |\mathcal{W}_{i-h} \propto \mathcal{W}_{h-1}| + \dots |\top(\mathcal{W}_{i-1})| = 2 \sum\limits_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} C_k C_{i-1-k} = C_i$. □

Henceforth, for each $i \geq 2$, we will assume that each admissible word $w_{i_k} \in \mathcal{W}_i$ is a multiset U_{i_k} of the form $\pi(1^{f_1} 2^{f_2} 3^{f_3} \dots i^{f_i})$, $f_h \geq 0$ denotes the occurrence of h in U_{i_k}, and $\pi(x)$ denotes a permutation of the $\sum\limits_{k=1}^{f_i} f_k$ letters.

For $n > 1$, the set of n-tile snake graphs defines a Brauer configuration algebra Λ_{Γ^n} induced by a Brauer configuration $\Gamma^n = (\Gamma_0^n, \Gamma_1^n, \mu^n, \mathcal{O}^n)$, where

$$\begin{aligned} \Gamma_0^n &= \{1, 2, \ldots, n-1, n\}, \\ \Gamma_1^n &= \{U_1, U_2, \ldots, U_{C_n} \mid w(U_i) \in \mathcal{W}_n, w(U_i) \neq w(U_j), \text{ if } i \neq j, |\mathcal{W}_n| = |\Gamma_1^n|\}, \\ \mu^n(j) &= 1, \quad 1 \leq j \leq n, \end{aligned} \tag{9}$$

If $j \in \Gamma_0$ belongs to the polygons $U_{j_1}, U_{j_2}, \ldots, U_{j_h}$, where $j_1 < j_2 < \cdots < j_h$ is a subchain of Γ_0, then the corresponding successor sequence S_j has the form $S_j = U_{j_1} \leq U_{j_2} \leq \cdots \leq U_{j_h - 1} \leq U_{j_h}$.

Figure 4 shows the Brauer quiver associated with the Brauer configuration Γ^2, for which $\Gamma_0^2 = \{1, 2\}$, $\Gamma_1^2 = \{U_1 = \{1, 2\}, U_2 = \{1, 2\}\}$, $w_1 = 12$, $w_2 = 21$.

Successor sequence $S_1 = U_1 < U_2$ (associated with the vertex 1) defines the arrow α_1^1. The successor sequence $S_2 = U_1 < U_2$ (associated with the vertex 2) defines the arrow β_2^1. Arrows α_1^2 and β_2^2 complete the special cycles $\alpha_1^1\alpha_1^2$ and $\beta_2^1\beta_2^2$.

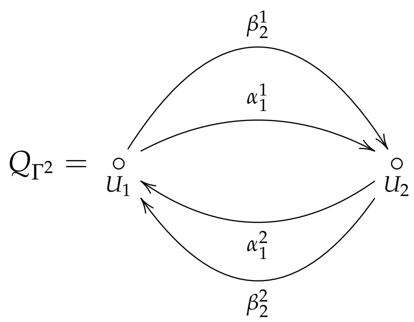


Figure 4. Example of the Brauer quiver defined by the Brauer configuration Γ^2. Relations $\alpha_1^1\beta_2^2$, $\alpha_1^2\beta_2^1$, $\alpha_1^1\alpha_1^2\alpha_1^1$, $\alpha_1^2\alpha_1^1\alpha_1^2$, $\beta_2^1\beta_2^2\beta_2^1$, $\beta_2^2\beta_2^1\beta_2^2$, $L_1^i \sim L_2^i$ (where L_j^i denotes the special cycle associated with the vertex j in polygon U_i, $i = 1, 2$) generate the admissible ideal I_{Γ^2} for which the Brauer configuration algebra $\Lambda_{\Gamma^2} = \mathbb{F}Q_{\Gamma^2}/I_{\Gamma^2}$.

We note that,

$$\begin{aligned} \dim_{\mathbb{F}} \Lambda_{\Gamma^2} &= 4 + 2(2-1) + 2(2-1) = 8. \\ \dim_{\mathbb{F}} Z(\Lambda_{\Gamma^2}) &= 3. \end{aligned} \tag{10}$$

Figure 5 shows the structure of the indecomposable projective modules P_1 (associated with the vertex 1) and P_2 (associated with the vertex 2) over the Brauer configuration algebra Λ_{Γ^2}. Arrows in Q_{Γ^2} define the corresponding composition series.

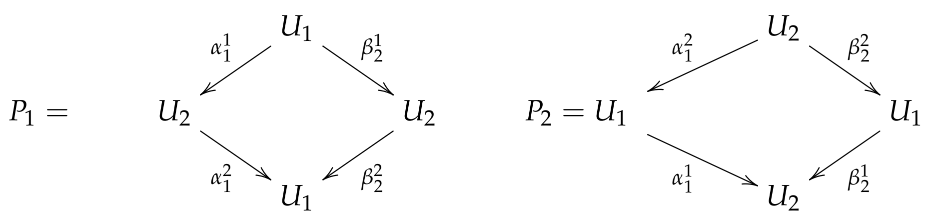


Figure 5. Indecomposable projective Λ_{Γ^2}-modules. Note that the number of composition series equals the number of non-truncated vertices in the corresponding polygon.

The following result regards Brauer configuration algebras of type Λ_{Γ^n}.

Theorem 3. *For $n > 1$ fixed, it holds that*

1. *If P_i is an indecomposable projective module over Λ_{Γ^n} associated with the polygon U_i. Then, the number of summands in* rad P_i *is i, $1 \leq i \leq n$.*

2. $val(i) = C_i C_{n-i}(n-i+1) = e_{in}$,
3. $\dim_{\mathbb{F}} \Lambda^n = 2(C_n + \sum_{i=1}^{n} t_{e_{in}-1})$,
4. $\dim_{\mathbb{F}} Z(\Lambda^n) = 1 + C_n + \sum_{j=1}^{n} \nu_{ij}^n$.

where C_h (t_h) denotes the hth Catalan number (hth triangular number). Moreover,

$$\nu_j^n = \begin{cases} val(j) - 2h, & \text{if } j \geq 2, \ n = j+h, \ 1 \leq h \leq j-1, \\ val(j) - (2j+m), & \text{if } m \geq 0, \ j \geq 2, \ n = 2j+m, \\ val(n) - n, & \text{if } j = n, \\ t_{2^{n-2}-1}, & \text{if } j = 1. \end{cases}$$

Proof. Note that each polygon $U_i \in \Gamma_1^n$ has i vertices. Furthermore, for each vertex $j \in \Gamma_0$, the size $|S_j|$ of the associated successor sequence S_j contains at least two different polygons. Consider the following Table 1.

Table 1. Valencies $val(i;\Gamma_1^j)$ of vertices $i = 1, 2$, and 3 in Γ_1^j, $2 \leq j \leq 6$.

	$val(1)$	$val(2)$	$val(3)$	$\dim_{\mathbb{F}} \Lambda_{\Gamma^j}$
Γ_1^2	2	2		8
Γ_1^3	6	4	5	72
Γ_1^4	20	12	10	812
Γ_1^5	70	40	30	9822
Γ_1^6	252	140	100	124,112

Note that

$$val(1;\Gamma_1^7) = val(1;\bot(\Gamma_1^6)) + \sum_{i=1}^{4} val(1;\Gamma_1^i \propto \Gamma_1^{5-i}) + val(1;\top(\Gamma_1^6)) = \binom{12}{6}.$$
$$val(2;\Gamma_1^7) = val(2;\bot(\Gamma_1^6)) + \sum_{i=1}^{4} val(2;\Gamma_1^i \propto \Gamma_1^{5-i}) + val(2;\top(\Gamma_1^6)) = 2\binom{10}{5}.$$
$$val(3;\Gamma_1^7) = val(3;\bot(\Gamma_1^6)) + \sum_{i=1}^{4} val(3;\Gamma_1^i \propto \Gamma_1^{5-i}) + val(3;\top(\Gamma_1^6)) = 5\binom{8}{4}.$$

In general,

$$\begin{aligned} val(1;\Gamma_1^j) &= C_1\binom{2j-2}{j-1}, \quad j \geq 1, \\ val(2;\Gamma_1^j) &= C_2\binom{2j-4}{j-2}, \quad j \geq 2, \\ val(3;\Gamma_1^j) &= C_3\binom{2j-6}{j-3}, \quad j \geq 3, \\ val(4;\Gamma_1^j) &= C_4\binom{2j-8}{j-4}, \quad j \geq 4, \\ \vdots &= \vdots \\ val(i;\Gamma_1^j) &= C_i\binom{2j-2i}{j-i} = C_i C_{j-i}(j-i+1), \quad j \geq i. \end{aligned} \tag{11}$$

Since $|\Gamma_1^n| = C_n$, then the result follows from item 2 and Proposition 1. By definition, the number $\#loops(j;\Gamma_1^n)$ of loops provided by the vertex j in Γ_1^n is given by ν_j^n. Furthermore, $|\Gamma_1^n| = C_n$. Therefore, the result is a consequence of Theorem 2. □

Following Loday's construction of binary trees [16], it is possible to associate a binary tree with each admissible word defined by a snake graph. In such a case, a word w of the form $w = w^l \propto w^r$ defines a binary tree whose left (right) leaf is given by w^l (w^r).

Figure 6 shows the binary trees defined by the admissible words 0, $1 = \varnothing \propto \varnothing$, $12 = 1 \propto \varnothing$, which has the binary tree 1 at the left leaf. Moreover, $21 = \varnothing \propto 1$, which has the binary tree 1 at the right leaf.

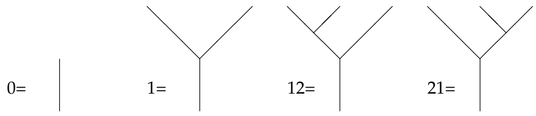


Figure 6. Admissible words and their corresponding binary trees.

The following Figure 7 shows the binary trees associated with three-tile snake graphs.

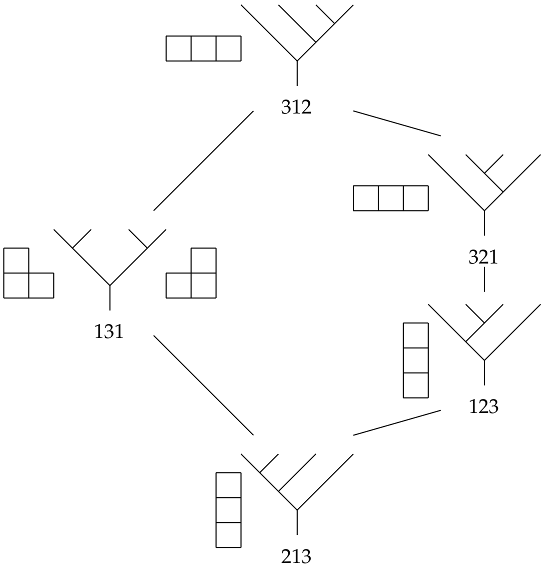


Figure 7. The grove consisting of binary trees associated with three-tile snake graphs.

The Tamari lattice arises from the order $(\preceq, \mathcal{W}_n)$ defined on the set $\mathcal{W}_n$ of admissible words, such that

$$\begin{aligned}(w_1 \propto w_2) \propto w_3 \prec w_1 \propto (w_2 \propto w_3),\\ w_1 \prec w_2 \Longrightarrow w_1 \propto w_3 \prec w_2 \propto w_3,\\ w_1 \prec w_2 \Longrightarrow w_3 \propto w_1 \prec w_3 \propto w_2.\end{aligned} \tag{12}$$

3.3. The Associated Code

Let $\mathcal{Q} = \{0, 1, 2, \ldots, q-1\}$ be the alphabet of q symbols, and $H(n, q) = \{(x_1, x_2, \ldots, x_n) \mid x_j$ is a q-ary vector$\}$.

The Hamming distance $d(x, y)$ between two elements $x, y \in H(n, q)$ equals the number of coordinates in which they differ. The *inner product* $\langle x, y\rangle$ is defined in such a way that $\langle x, y\rangle = 1 - \frac{2d(x,y)}{n}$.

The *h-energy* or *potential energy* $E(n, C; h)$ of a code C is given by the identity

$$E(n, C; h) = \frac{1}{|C|} \sum_{x,y \in C} h(\langle x, y\rangle), \tag{13}$$

where h is a function $h : [-1, 1) \longrightarrow (0, +\infty)$ (for convenience, often, h is considered absolutely monotone). According to Boyvalenkov et al. [17], energy minimizing codes

$C \subset H(n,q)$ for the potential function $h_\alpha(t) = [\frac{2}{n(1-t)}]^\alpha$, $\alpha \to \infty$ are maximizing the minimum distance $d(C) = \min\{d(x,y) \mid x,y \in C, x \neq y\}$.

We recall that the main problem associated with these codes is minimizing the potential energy provided by the cardinality $|C|$ of a fixed code C, i.e., finding $\mathcal{E}(n,M;h) = \min\{E(n,C;h) \mid |C| = M\}$.

To address the coding problem from the Brauer configurations algebras point of view, we note that, by definition, for $j > 1$ fixed, a Brauer configuration Γ^j defines a code $C_{\Gamma^j} \subset H(1,j)$ of admissible words associated with polygons in Γ^j_1. The alphabet is given by $\Gamma^j_0 = \{1,2,3,\ldots,j-1,j\}$. The purpose of this section is to give the potential energy $E(1,C_{\Gamma^j};h_\alpha)$ of a code C_{Γ^j} for any $j > 1$ and $\alpha \geq 1$. Actually, we have the following result:

Theorem 4. *For $1 \leq a \leq \alpha$ and $j > 1$ fixed, the energy $E(1,C_{\Gamma^j};h_a(t) = [\frac{2}{1-t}]^a)$ is given by a snake graph. In particular,* $\min E(1,C_{\Gamma^j};h_a)$ *is given by a zigzag special snake graph $\mathcal{G}[0,2^{\alpha+1}]$ of the form $\mathcal{G}_f(\underbrace{(2,2,\ldots,2)}_{(2^{\alpha+1}-2)-times})$ (see Figure 2). Moreover,*

$$(\frac{1}{2})^{\alpha+1} \leq E(1,C_{\Gamma^j};h_a) < \frac{j-1}{C_j}F_{j-1}(\frac{1}{2})^\alpha, \tag{14}$$

where C_i denotes the ith Catalan number, and $F_i = |\{(w,w') \in C_{\Gamma^n} \times C_{\Gamma^n} \mid d(w,w') = d(w',w) = i\}|$. Furthermore, for $\alpha > 1$ fixed, it holds that

$$\min E(1,C_{\Gamma^2};h_a) = (\frac{1}{2})^{\alpha+1}. \tag{15}$$

Proof. We note that, for all $w,w' \in \mathcal{W}_j$, it holds that $d(w,w') \in \{2,3,\ldots,j\}$. In addition, $h_\alpha(i) = \frac{1}{i^\alpha}$, $\quad 2 \leq i \leq j$. Thus,

$$E(1,C_{\Gamma^j};h_\alpha) = \frac{1}{C_j}\sum_{k=2}^{j} F_k(\frac{1}{k})^\alpha \tag{16}$$

is a rational number. In particular, if $j = 2$, then $\langle 12,21\rangle = -3$, $h_\alpha(-3) = (\frac{1}{2})^\alpha$. Moreover, $E(1,C_{\Gamma^2};h_\alpha) = (\frac{1}{2})^{\alpha+1}$. Since $F_i \leq F_{j-1} < t_{C_j-1}$ (where t_i is the ith triangular number), for any $1 \leq i \leq j$, the result follows. We are finished. □

Corollary 2. *For $j \geq 2$, numbers $F_2, F_3, \ldots, F_{j-1}$, and F_j constitute an integer partition of the t_{C_j-1} triangular number.*

Proof. Since $F_i = |\{(w,w') \in C_{\Gamma^n} \times C_{\Gamma^n} \mid d(w,w') = d(w',w) = i\}|$, then $2\sum_{i=1}^{j} F_i = C_j^2 - C_j$. We are finished. □

Table 2 shows a table giving the number of times F_i, $2 \leq i \leq j$, $2 \leq j \leq 9$, that a distance $d(w,w') \in \{2,\ldots,9\}$ occurs in a code C_{Γ^j}. The entry at the jth row and ith column, $2 \leq i \leq 9$, gives the number of times that the distance i appears in code C_{Γ^j}.

Table 3 shows a table with the energy values of codes C_{Γ^j}, $2 \leq j \leq 9$ by using $h_\alpha(t) \in \{1/i \mid 2 \leq i \leq 9\}, 1 \leq \alpha \leq 3$. Columns are labeled by h_α, whereas rows are labeled by pairs (α, j). In this case, an entry (α, j, h_α) gives the energy value $E(1, C_{\Gamma^j}; h_\alpha)$.

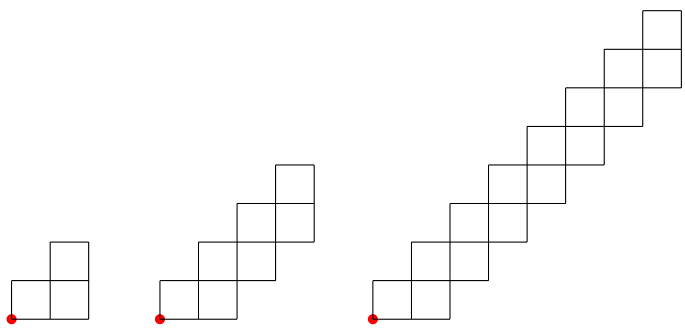
Table 2. Distances associated with a code C_{Γ^j}, $2 \le j \le 9$.

$j \backslash d(w,w')$	2	3	4	5	6	7	8	9
2	1							
3	6	4						
4	25	34	32					
5	100	172	329	260				
6	390	754	1990	3126	2386			
7	1516	3130	9983	21,638	32,481	23,058		
8	5869	12,660	45,872	119,312	251,334	351,506	235,182	
9	22,746	50,570	202,205	589,306	1,519,120	3,001,666	3,944,860	2,486,618

Table 3. Energy values of codes C_{Γ^j}, $2 \le j \le 9$ given by functions of the form $h_\alpha(t) \in \{1/i \mid 2 \le i \le 9\}$, $1 \le \alpha \le 3$.

α	$j \backslash h_\alpha(t)$	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9
1	2	1/2							
	3	3	4/3						
	4	25/2	34/3	8					
	5	50	172/3	329/4	52				
	6	195	754/3	995/2	3126/5	1193/3			
	7	758	3130/3	9983/4	21,638/5	10,827/2	3294		
	8	5869/2	4220	11,468	119,312/5	41,889	351,506/7	117,591/4	
	9	11,373	50,570/3	202,205/4	589,306/5	759,560/3	3,001,666/7	986,215/2	2,486,618/9
2	2	1/4							
	3	3/2	4/9						
	4	25/4	34/9	2					
	5	25	172/9	329/16	52/5				
	6	195/2	754/9	995/8	3126/25	1193/18			
	7	379	3130/9	9983/16	21,638/25	3609/4	3294/7		
	8	5869/4	4220/3	2867	119,312/25	13,963/2	351,506/49	117,591/32	
	9	211,373/2	50,570/9	202,205/16	589,306/25	379,780/9	3,001,666/49	986,215/16	2,486,618/81
3	2	1/8							
	3	3/4	4/27						
	4	25/8	34/27	1/2					
	5	25/2	172/27	329/64	52/25				
	6	195/4	754/27	995/32	3126/125	1193/108			
	7	379/2	3130/27	9983/64	21,638/125	1203/8	3294/49		
	8	5869/8	4220/9	2867/4	119,312/125	13,963/12	351,506/343	117,591/256	
	9	11,373/4	50,570/27	202,205/64	589,306/125	189,890/27	3,001,666/343	986,215/128	2,486,618/729

Figure 8 shows special snake graphs giving $\min E(1, C_{\Gamma j}; h_\alpha)$ for $j = 2$ and $\alpha = 1 (h_\alpha = \frac{1}{4})$, $\alpha = 2 (h_\alpha = \frac{1}{8})$, and $\alpha = 3 (h_\alpha = \frac{1}{16})$.



$\frac{1}{4} = [0,4]$ $\frac{1}{8} = [0,8]$ $\frac{1}{16} = [0,16]$

Figure 8. Examples of special snake graphs giving the minimal energy $\min E(1, C_{\Gamma^j}, h_\alpha)$, for $j = 2$, $\alpha = 1, 2, 3$.

Figure 9 shows examples of snake graphs given the energy $E(1, C_{\Gamma^j}, h_\alpha)$, for $\alpha = 1, j = 4, 5$, and $\alpha = 2, j = 5, 6$. h_α, defined as in Figure 3.

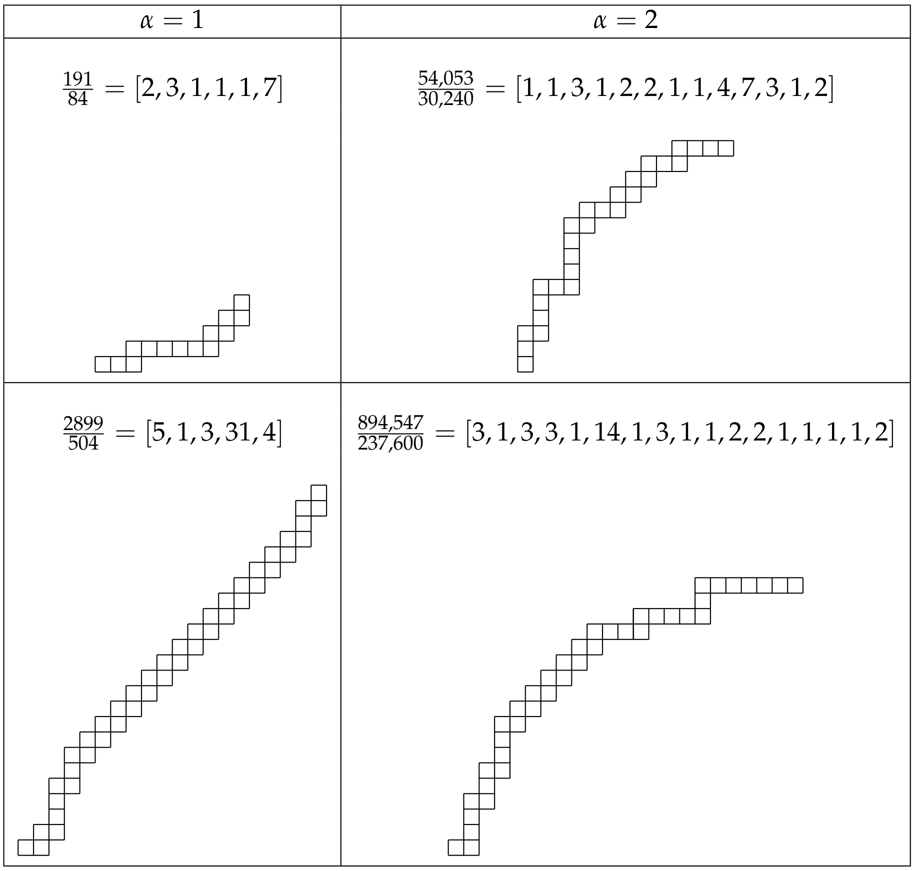


Figure 9. Snake graphs associated with the energy values $\frac{191}{84}$, $\frac{2899}{504}$, $\frac{54{,}053}{30{,}240}$, and $\frac{894{,}547}{237{,}600}$.

4. Concluding Remarks and Future Work

String snake graphs induce admissible words related to binary trees. Such admissible words give rise to Brauer configuration algebras of type Λ_{Γ^j}, $j \geq 2$, with C_j indecomposable projective modules. Catalan numbers are helpful to obtain the dimensions of these algebras and their centers. Brauer configurations Λ_{Γ^j} define codes C_{Γ^j} of type $H(1, j)$, in the sense of Boyvalenkov et al., whose energy values are given by snake graphs. Distances associated with these codes allow integer partitions of the triangular numbers indexed by Catalan numbers.

Future Work

We note that $E(1, C_{\Gamma^j}, h_\alpha) = \sum\limits_{i=2}^{j} F_j(\frac{1}{j})^\alpha$, where $F_i = |\{(w, w') \in C_{\Gamma^n} \times C_{\Gamma^n} \mid d(w, w') = d(w', w) = i\}|$. It is an open problem to give a closed formula for numbers F_i, as well as the number of perfect matchings of the associated snake graphs.

Another interesting task for the future consists of giving a generalization of the presented results to arbitrary $H(n, q)$ codes.

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Abbreviations

The following abbreviations are used in this manuscript:

C_j	(jth Catalan number)
$\dim_{\mathbb{F}} \Lambda_\Gamma$	(Dimension of a Brauer configuration algebra)
$\dim_{\mathbb{F}} Z(\Lambda_\Gamma)$	(Dimension of the center of a Brauer configuration algebra)
$E(1, C_{\Gamma^j}, h_\alpha)$	(Energy of a code C_{Γ^j})
$\mathbb{F}$	(Field)
Γ_0	(Set of vertices of a Brauer configuration Γ)
t_i	(ith triangular number)
$\mathrm{occ}(\alpha, V)$	(Number of occurrences of a vertex α in a polygon V)
$w(V)$	(The word associated with a polygon V)
$V_i^{(\alpha)}$	(Ordered sequence of polygons)
$val(\alpha)$	(Valency of a vertex α)

Appendix A. Python Routines

In this section, we give Python routines to compute the word product $\propto$ (routine [2]), Catalan numbers (routine [3]), admissible words (routine [4]), dimensions of the Brauer configuration algebras Λ_{Γ^j} (routine 5), and energy values of the defined codes C_{Γ^j} (routine [6]).

[1]:
```
from scipy.spatial.distance import hamming
import numpy as np
from collections import Counter
import itertools
```

[2]:
```
def word_prod(A,B):
    if  A.size == 0:
        dimension=len(B[0])+1
        list1 = [np.zeros((len(B),dimension)) for i in range(1)]
        for i in range(1):
            for j in range(len(B)):
                    list1[i][j]=np.concatenate((dimension, B[j]),␣
↪axis=None)
                    C=np.concatenate(list1, axis=0)
        return C

    if  B.size == 0:
        dimension=len(A[0])+1
        list1 = [np.zeros((1,dimension)) for i in range(len(A))]
        for i in range(len(A)):
            for j in range(1):
                    list1[i][j]=np.concatenate((A[i],dimension),␣
↪axis=None)
                    C=np.concatenate(list1, axis=0)
        return C
    else:
        dimension=len(A[0]) +len(B[0])+1
        list1 = [np.zeros((len(B),dimension)) for i in range(len(A))]
        for i in range(len(A)):
            for j in range(len(B)):
```

```
                  list1[i][j]=np.concatenate((A[i],dimension, B[j]),␣
↪axis=None)
                  C=np.concatenate(list1, axis=0)
    return C
```

[3]:
```
def catalan(n):
    # Base Case
    if n <= 1:
        return 1
    res = 0
    for i in range(n):
        res += catalan(i) * catalan(n-i-1)
    return res
```

[4]:
```

def Gamma(n):
    G_0=np.empty((0, 0))
    G_1=np.array([[1]])
    if n==0:
        return G_0
    if n==1:
        return G_1
    else:
        list2 =[np.zeros((catalan(i),i)) for i in range(0,n+1)]
        list2[0]=G_0
        list2[1]=G_1
        for i in range(2,n+1):
            list2[i]=np.
↪concatenate([word_product(list2[j],list2[i-1-j]) for j in␣
↪range(i)], axis=0)
        return list2[n]
```

[5]:
```
def dim_algebra(n):
   ␣
↪b=[catalan(i)*catalan(n-i)*(n-i+1)*(catalan(i)*catalan(n-i)*(n-i+1)
    -1)/
  2  for i in range(1,n+1)]
    a=sum(b)
    res=2*(catalan(n)+a)
    return res
```

[6]:
```
def energy(n,a):
    C=np.zeros((len(Gamma(n)),len(Gamma(n))))
    for i in range (len(C)):
        for j in range(len(C)):
            C[i][j]=hamming(Gamma(n)[i],␣
↪Gamma(n)[j])*len(Gamma(n)[i])
    res = dict(Counter(itertools.chain(*C)))
    energy=sum([res[m]*1/(2*(m**a)) for m in range(2,n+1)])*1/
↪catalan(n)
    return energy
```

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