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Characteristic Sequence of Strongly Minimal Directed Single Graphs of 1-Arity

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Abstract: In this paper, we will classify the strongly minimal directed single graphs of 1-arity by axiomatizing the theory of characteristic sequence of such a graph. Then we will show this theory is complete by using Łos-Vaught test. Complete theory is important to capture all the models of the theory and hence can be applied on mathematical structures which meet such a theory. The theory of algebraically closed fields with a given characteristic is complete. Thus, in this paper we will classify the strongly minimal directed single graphs of 1-arity with given characteristic sequence which can be applied on many mathematical structures not only algebraically closed fields.

Keywords: strongly minimal; complete theory; directed graph; axiomatization; Łos-Vaught test

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1. Introduction and Preliminaries

The theory of algebraically closed fields with given characteristic is complete which means that we can capture all such fields that meat this theory. In this research we will first define directed single graphs of 1-arity and the construction of such graphs by defining the components of these graphs. Hence we can after present the theory of these graphs in general. This theory has many axioms which will be presented. However, not all these axioms can give a complete theory. Thus we will give a characteristic sequence which capture the strongly minimal directed single graphs of 1-arity, and algebraically close fields are strongly minimal [1,2] and show the theory of this sequence is uncountably categorical in all uncountable cardinals κ ans hence is complete via Łos-Vaught test.

We introduce now preliminaries that are needed in this research, ref. [3] is the main source of these preliminaries and recommended for more details.

A sequence of parameters $(a_1, ..., a_n)$ in the graph G is written as $\overline{a} \in G$. We write $\varphi(\overline{v}, \overline{u})$ as formula in the tuples $\overline{v}$ and $\overline{u}$ of vertices. If the formula $\varphi(\overline{a})$ is true in G then we write $G \vDash \varphi(\overline{a})$ The cardinality of the graph G is denoted by $|G|$. Let $F \subseteq G$. Then L_F is the language obtained by adding constant symbols to L for each element in F and $Th_F(G)$ is the set of all L_F-sentences true in G. The notation $\mathbb{N}^+$ stands for the set of positive integers.

Definition 1 (p. 117, [4])**.** *Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. Let $X \subseteq M^n$. We say X is definable with parameters from A if and only if there is an L-formula $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ and elements $a_1, ..., a_m \in A$ such that $X = \{(x_1, ..., x_n) \in M^n : \mathcal{M} \vDash \varphi(x_1, ..., x_n, a_1, ..., a_m)\}$.*

Theorem 1 ([5])**.** *The Łos-Vaught test state that a theory is complete if it is uncountably categorical for each uncountable cardinal κ and it has no finite model.*

Definition 2 (p. 1, [6])**.** *An infinite definable set $X \subseteq M^n$, where X is definable with parameters, is called minimal if every definable (with parameters) subset of X is either finite or cofinite. If $\varphi(\overline{x}, \overline{a})$ is the formula that defines X, then $\varphi(\overline{x}, \overline{a})$ is minimal. We say that X and $\varphi(\overline{x}, \overline{a})$ are strongly minimal if $\varphi(\overline{x}, \overline{a})$ is minimal in any elementary extension $\mathcal{N}$ of $\mathcal{M}$.*

Theorem 2 (p. 211, [3]). *Suppose T is a strongly minimal theory in a countable language. If* $\kappa \geq \aleph_1$ *and* $\mathcal{M}, \mathcal{N} \models T$ *with* $|\mathcal{M}| = |\mathcal{N}| = \kappa$*, then* $\mathcal{M} \cong \mathcal{N}$*.*

Definition 3. *A graph G is a mathematical structure consists of the ordered pair* (V, E) *where V is a set of vertices and E is a relation symbol defined by* $E = \{\{u, v\} | u, v \in V^2\}$*.*

If G is directed graph, then $E = \{(u, v) | u, v \in V^2\}$ *where the edge starts from u to v and we denote such a graph by* $\overrightarrow{G}$*.*

In the above definition, note that v and u in the set E are not necessarily distinct. So loops are allowed which are vertices with an edge from the vertex to itself.

Definition 4. *Let* $\overrightarrow{G} = (V, E)$ *be a directed graph and let* $v, w \in V$*. We define the relation* $vE^{n_i}_{(l)_i}w$ *for* $i \in \mathbf{N}$ *to be the edges directed from the vertex v to vertex w where l is the number of edges starting from v and* n_i *is the length of each* l_i *path from v to w.*

Definition 5. *Let* $\overrightarrow{G} = (V, E)$ *be a directed graph. Then* $\overrightarrow{G}$ *is called a graph of l-arity and denoted by* $\overrightarrow{G}_l$ *if the following holds:*

$$\overrightarrow{G} \models \forall v[\exists^{=l} w \text{ such that } vE^{n_i}_{l_{(i)}} w.$$

If $l = 1$ *then* $\overrightarrow{G}_1$ *is called a graph of* 1*-arity and is satisfies the formula*

$$\forall v \exists^{=1} w \text{ such that } vE^n_1 w.$$

Definition 6. *Let* $\overrightarrow{G}_1 = (V, E)$ *be a directed graph with* 1*-arity. Then* $\overrightarrow{G}_1 = (V, E)$ *is called a single graph if it satisfy the following*

$$\forall v[\exists^{\leq 1} u \text{ such that } uEv],$$

we denote such a graph by $Sing(\overrightarrow{G}_1)$*.*

If $v \in V$ *such that v satisties the formula*

$$\exists^{=0} u \text{ such that } uEv],$$

then v is the zero vertex and denoted by v_0*.*

2. Classification of Directed Single Graphs of 1-Arity

Definition 7. *Let* $C \subseteq V$ *and* $v, w \in V$*. If there is* $n \in \mathbb{N}$ *such that* $vE^n w$ *then* v, w *are connected. Let* $C \subseteq V$*. If any two vertices of C are connected then we say that C is connected. A connected component of* $Sing(\overrightarrow{G}_1)$ *is maximal connected graph.*

Lemma 1. *Every connected component of* $Sing(\overrightarrow{G}_1)$ *is either*

1. *a copy of* $C_0 = \{v_0 E^n w : \text{ for each } n \in \mathbb{N}\}$*,*
2. *a copy of* $C_\omega = \{vE^n w : \text{ for each } n \in \mathbb{Z}\}$*,*
3. *a copy of* $C_r = \{v_i E^r v_i : \text{ for some } r \in \mathbb{N}^+\}$*.*

Proof. Let C be a finite connected component of $Sing(\overrightarrow{G}_1)$. Let $v \in C$. Then

$$vE^0 v, vE^1 v_1, vE^2 v_2, vE^3 v_3, ... \in C.$$

So $vE^n v_i \in C$ for all $n \in \mathbb{N}$. Let r be the smallest number in $\mathbb{N}$ such that $vE^r v_i = vE^q v_i$ for some $q \in \{0, ..., r-1\}$. If $q \neq 0$, $vE^{r-1} v_i = vE^{q-1} v_i$ as $Sing(\overrightarrow{G}_1)$ is a directed single graph with 1-arity. So, $q - 1 \in \{0, ..., r-2\}$, contradicting that r is the smallest such number. So

$q = 0$ which means that C contains a cycle of period r. Since $Sing(\overrightarrow{G}_1)$ is a directed single graph with 1-arity, $C = C_r$.

Now let C be infinite connected component. Let $v \in C$. Then we have

$$vE^0v, vE^1v_1, vE^2v_2, vE^3v_3, ..., vE^nv_i, ... \textit{for } n \in \mathbb{N}.$$

If $vE^rv_i = vE^qv_i$ for some $r, q \in \mathbb{N}$, and $r \neq q$, then C is finite by the previous argument. So all vE^nv_i are distinct for $n \in \mathbb{N}$ and we will have two possibilities:

First, there is $v_0 \in C$ such that v_0 satisfies the formula

$$\exists^{=0} w \textit{ such that } wEv,$$

then C is $\{v_0, v_0E^1v_1, v_0E^2v_2, ..., v_0E^nv_n, ...\}$, i.e, $C = \{v_0E^nw : \textit{ for each } n \in \mathbb{Z}, v\} = C_0$.

Second, choosing any $v \in C$, such that v satisfies the formula

$$\exists^{=1} w \textit{ such that } wEv$$

and w is unique. So we have $vE^nw \in C$ for each $n \in \mathbb{Z}$ and for any $n, m \in \mathbb{Z}$, vE^nw is distinct from vE^mw unless $n = m$. So, $C = C_\omega$. □

Example 1. *1. Consider the structure $(\mathbb{Q}, <)$ where the vertices are $q \in \mathbb{Q}$ and the edges defined regarding the relation $<$ as follows:*

$$\forall q_i [q_i E q_j \iff q_i < q_j \wedge (\neg \exists q_r \textit{ such that } q_i < q_r < q_j)].$$

This formula guarantee there is exactly one edge going from q_i to q_j. This formula is also true in $(\mathbb{Z}, <)$ but not true in $(\mathbb{R}, <)$ as there is always an element $q_r \in \mathbb{R}$ such that $q_i < q_r < q_j$. We can consider $(\mathbb{Q}, <)$ and $(\mathbb{Z}, <)$ regarding the above formula as copies of the connected component C_ω in $Sing(\overrightarrow{G}_1)$. From this example, we learned a mathematical property about $(\mathbb{Q}, <)$, $(\mathbb{Z}, <)$, and $(\mathbb{R}, <)$ which is $(\mathbb{Q}, <)$, $(\mathbb{Z}, <)$ are countable structure and $(\mathbb{R}, <)$ is uncountable.

2. *Consider the structure $(\mathbb{N}, <)$, then we need to the formula above will be:*

$$\forall q_i [q_i E q_j \iff q_i < q_j \wedge (\neg \exists q_r \textit{ such that } q_i < q_r < q_j) \wedge (\exists^{=0} q_k \textit{ such that } q_k < q_0)].$$

We can consider $(\mathbb{N}, <)$ regarding the above formula as copy of the connected component C_0 in $Sing(\overrightarrow{G}_1)$.

3. *Consider the permutation On $A_4 = \{1, 2, 3, 4\}$ defined by*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

It is defined by the formula

$$iEj \iff \sigma(i) = j.$$

This permutation can be considered as a copy of the connected component C_4 in $Sing(\overrightarrow{G}_1)$. Now take another permutation on A_4 defined by

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

It is also defined by the formula

$$iEj \iff \tau(i) = j.$$

By looking to this formula defined on τ we can see we have two connected components of C_2 in $Sing(\overrightarrow{G}_1)$. The permutations σ and τ are in the permutation group S_4 of A_4 but from the formulas above we know what is the algebraic structures of these two permutations.
Such methodology can be applied on many algebraic structures, for example $(\mathbb{Z}_4, +)$ and hence we can study the cyclic subgroups in $\mathbb{Z}_4$.

4. *Ref. [7] Consider the structure $(\mathbb{N}, Succ)$ where Succ is the successor function defined on the natural numbers as $S(n) = n + 1$. It can be defined by the formula*

$$\forall n_i [n_i E n_j \iff S(n_i) = n_j \wedge (\exists^{=0} n_k \text{ such that } S(n_k) = 0)].$$

We can consider $(\mathbb{N}, Succ)$ regarding the above formula as copy of the connected component C_0 in $Sing(\overrightarrow{G}_1)$.

Definition 8. *Let $Sing(\overrightarrow{G}_1) = \langle V, E\rangle$. We define the characteristic sequence for $Sing(\overrightarrow{G}_1)$ to be*

$$\rho(Sing(\overrightarrow{G}_1)) = (\rho_r)_{r\in\mathbb{N}\cup\{\omega\}}$$

a sequence of cardinals such that ρ_0 is the numbers of copies of C_0, ρ_ω is the numbers of copies of C_ω, and ρ_r is the number of copies of C_r for $r \in \mathbb{N}^+$.

Proposition 1. *Let $Sing(\overrightarrow{G}_1) = \langle V, E\rangle$ and $Sing(\overrightarrow{H}_1) = \langle W, E\rangle$. If $\rho(Sing(\overrightarrow{G}_1)) = \rho(Sing(\overrightarrow{H}_1))$ Then $Sing(\overrightarrow{G}_1) \cong Sing(\overrightarrow{H}_1)$.*

2.1. Axiomatization of the Theory of Directed Single Graphs of 1-Arity

Definition 9. *For a sequence $\rho = (\rho_r)_{r\in\mathbb{N}}$ where each $\rho_r \in \mathbb{N}\cup\{\omega\}$, we axiomatize the theory Th_ρ by the following:*

$$\forall v[\exists^{=1} w \text{ such that } vEw].$$
$$\forall v[\exists^{\leq 1} w \text{ such that } wEv].$$

If ρ_0 is finite, then

$$\exists^{=\rho_0} v[\exists^{=0} w[wEv]].$$

If ρ_0 is infinite, then

$$\exists^{\geq n} v[\exists^{=0} w[wEv]] \text{ for each } n \in \mathbb{N}^+.$$

For each $r \in \mathbb{N}^+$, if ρ_r is finite, then

$$\exists^{=n} v[vE^r v \wedge \bigwedge_{q|r, q\neq r} \neg(vE^q v)] \text{ where } n = \rho_r.r.$$

For each $r \in \mathbb{N}^+$, if ρ_r is infinite, then

$$\exists^{\geq n} v[vE^r v \wedge \bigwedge_{q|r, q\neq r} \neg(vE^q v)] \text{ for each } n \in \mathbb{N}^+.$$

If $\forall v[\exists^{=1} w$ such that $wEv]$ and there is $N \in \mathbb{N}$ such that for all $r > N$, $\neg[vE^r v \wedge \bigwedge_{q|r,q\neq r} \neg(vE^q v)]$ and $\sum_{r=1}^{N} \rho_r$ is infinite, then

$$\forall v \bigvee_{i=1}^{N} [vE^i v \wedge \bigwedge_{q|i, q\neq i} \neg(vE^q v)].$$

For each $n \in \mathbb{N}^+$,

$$\exists^{\geq n} v[v = v].$$

Definition 10. *We define the dominant component in $Sing(\overrightarrow{G}_1)$ to be the component C_i for some $i \in \mathbb{N}$ such that the following holds:*

$$|\rho_i C_i| > |\rho_j C_j| \text{ for all } j \in \mathbb{N} \text{ and } j \neq i.$$

If the dominant component in $Sing(\overrightarrow{G}_1)$ is C_i for some $i \in \mathbb{N}^+$, then $Sing(\overrightarrow{G}_1)$ is with cyclic dominant component of length i. Otherwise, $Sing(\overrightarrow{G}_1)$ is with the infinite dominant component C_0.

Definition 11. *We define $char(\rho)(Sing(\overrightarrow{G}_1))$ to be the characteristic sequence for strongly minimal directed single graphs of 1-arity in the language $L = \langle E \rangle$.*

Lemma 2. *If $Sing(\overrightarrow{G}_1) = \langle V, E \rangle$ is strongly minimal, then the characteristic sequence $char(\rho(Sing(\overrightarrow{G}_1))$ satisfies one of the following cases:*

Case 1: There is only finitely many infinite dominant components, that is, $|\rho_0.v_0|$ is finite.

Case 2: If $Sing(\overrightarrow{G}_1)$ is with cyclic dominant component of length i for some $i \in \mathbb{N}^+$, then $|\rho_i C_i|$ is infinite, $|\rho_0 C_0| = 0$, $|\rho_\omega C_\omega| = 0$, and $\sum_{q \neq i} \rho_q.C_q$ is finite.

Proof. Case 1: Let $\theta(v)$ be the formula

$$\exists^{=1} w[vEw]$$

As $Sing(\overrightarrow{G}_1)$ is 1-arity, $|\theta(v), Sing(\overrightarrow{G}_1)|$ is infinite which means that $\theta(v)$ defines an infinite subset of $Sing(\overrightarrow{G}_1)$. Now, $\neg\theta(v)$ will be the formula

$$\neg\exists^{=1} w[vEw]$$

which is equivalent to the formula

$$\exists^{=0} w[vEw].$$

So $\neg\theta(v)$ defines the zero vertices v_0 in $Sing(\overrightarrow{G}_1)$. As $Sing(\overrightarrow{G}_1)$ is strongly minimal, $\neg\varphi(v)$ defines a finite set. Hence, there is only finitly many v_0 in $Sing(\overrightarrow{G}_1)$. So there is only finitely many infinite dominant component.

Case 2: Suppose $Sing(\overrightarrow{G}_1)$ is with cyclic dominant component of length i for some $i \in \mathbb{N}^+$. Then

$$|\rho_i C_i| > |\rho_j C_j| \text{ for all } j \in \mathbb{N} \text{ and } j \neq i.$$

Let $\varphi(\overline{a}, v)$ be the formula $vE^i v \wedge \bigwedge_{q|r, q \neq i} \neg(vE^q v)$. Now, $|\rho_0 C_0| = \rho_0.\aleph_0$ and $|\rho_\omega C_\omega| = \rho_\omega.\aleph_0$. But C_i is a cyclic dominant component. So

$$|\rho_i C_i| > \rho_0.\aleph_0$$

and

$$|\rho_i C_i| > \rho_\omega.\aleph_0.$$

Thus, $|\varphi(\overline{a}, Sing(\overrightarrow{G}_1))|$ is infinite which means there is infinitely many cyclic component of length i and $\varphi(\overline{a}, Sing(\overrightarrow{G}_1))$ defines this infinite subset of $Sing(\overrightarrow{G}_1)$. Since $Sing(\overrightarrow{G}_1)$ is strongly minimal, $\neg(\varphi(\overline{a}, Sing(\overrightarrow{G}_1)))$ defines a finite subset of $Sing(\overrightarrow{G}_1)$. Hence, $|\rho_0 C_0| = 0$, $|\rho_\omega C_\omega| = 0$, and $\sum_{q \neq i} \rho_q.q$ is finite.

□

2.2. Models of the Theory of Strongly Minimal Directed Single Graphs of 1*-Arity*

Proposition 2. *The theory of strongly minimal directed single graphs of* 1*-arity* $Th_{char(\rho)}$ *satisfies the following:*

1. *If the dominant component is cyclic of length i for some* $i \in \mathbb{N}^+$ *then* $Th_{char(\rho)}$ *is totally categorical*
2. *If the dominant component is infinite then* $Th_{char(\rho)}$ *is uncountably categorical*

Proof. 1. Suppose that the dominant component is cyclic of length i for some $i \in \mathbb{N}^+$. Then by Theorem 2, $|\rho_0(C_0)_{char(\rho)}| = 0$, and $|\rho_\omega(C_\omega)_{char(\rho)}| = 0$, and $|\rho_i(C_i)_{char(\rho)}|$ is infinite, and $\sum_{q \neq i} \rho_q.(C_q)_{char(\rho)}$ is finite. This means that there is $N \in \mathbb{N}$ such that if $r > N$ then $|\rho_r.(C_r)_{char(\rho)}| = 0$. Hence,$Th_{char(\rho)}$ satisfy the formula

$$\forall v \bigvee_{j=1}^{N} [vE^j v \wedge \bigwedge_{q|j, q \neq j} \neg(vE^q v)].$$

Suppose $Sing(\overrightarrow{G}_1)_\kappa \models Th_{char(\rho)}$. Then ,$|Sing(\overrightarrow{G}_1)_\kappa|$ will be

$$|\rho_0.(C_0)_{char(\rho)}| + \sum_{q \neq r} q \cdot |\rho_q.(C_q)_{char(\rho)}| + \kappa \cdot r + |\rho_\omega(C_\omega)_{char(\rho)}| = \kappa$$

as $\kappa = |\rho_r.(Sing(\overrightarrow{G}_1))|$. Therefor, $Th_{char(\rho)}$ is totally categorical.

2. Suppose the dominant component is infinite. Then $|\rho_0.(v_0)_{char(\rho)}| > 0$. So $|\rho_q.(C_q)_{char(\rho)}|$ is finite for all $q \in \mathbb{N}^+$. Suppose $Sing(\overrightarrow{G}_1)_\kappa \models Th_{char(\rho)}$. Then ,$|Sing(\overrightarrow{G}_1)_\kappa|$ will be

$$\aleph_0 \cdot |\rho_0.(v_0)_{char(\rho)}| + \sum_{q \in \mathbb{N}^+} q \cdot |\rho_q.(C_q)_{char(\rho)}| + \kappa \cdot \aleph_0 = \aleph_0 + \kappa$$

where $\kappa = |\rho_\omega.(C_\omega)_{(Sing(\overrightarrow{G}_1))}|$.
Therefor, $Th_{char(\rho)}$ is uncountably categorical.
□

Proposition 3. *The theory of strongly minimal directed single graphs of* 1*-arity* $Th_{char(\rho)}$ *is complete.*

Proof. • If $|\rho_0.(v_0)_{char(\rho)}| = 0$ and $\sum_{q \in \mathbb{N}^+} q.|\rho_q.(C_q)_{char(\rho)}|$ is finite for all $q \in \mathbb{N}^+$. So there is $N \in \mathbb{N}$ such that if $q > N$ then $|\rho_q.(C_q)_{char(\rho)}| = 0$. So, $Th_{char(\rho)}$ satisfy the formula

$$\forall v \bigvee_{q=1}^{N} [vE^q v \wedge \bigwedge_{s|q, s \neq q} \neg(vE^s v)].$$

This means that if $Sing(\overrightarrow{G}_1) \models Th_{char(\rho)}$ then $|\rho_q.(C_q)_{Sing(\overrightarrow{G}_1)}|$ is finite. So the complement of this set is not empty as $Sing(\overrightarrow{G}_1)$ is infinite. So there is at least on copy of $(C_\omega)_{Sing(\overrightarrow{G}_1)}$. Suppose $Sing(\overrightarrow{G}_1)_\kappa \models Th_{char(\rho)}$. Then, $|Sing(\overrightarrow{G}_1)_\kappa|$ will be

$$\aleph_0 \cdot |\rho_0.(v_0)_{char(\rho)}| + \sum_{q \in \mathbb{N}^+} q \cdot |\rho_q.(C_q)_{char(\rho)}| + \kappa \cdot \aleph_0 = \aleph_0 + \kappa$$

where $\kappa = |\rho_\omega.(C_\omega)_{(Sing(\overrightarrow{G}_1))}|$.

- If $|\rho_0.(v_0)_{char(\rho)}| = 0$ and $|\rho_q.(C_q)_{char(\rho)}|$ is finite for all $q \in \mathbb{N}^+$ but $\sum_{q\in\mathbb{N}^+} q.|\rho_q.(C_q)_{char(\rho)}|$ is infinite, then $|Sing(\overrightarrow{G}_1)_\kappa|$ will be

$$\aleph_0 \cdot |\rho_0.(v_0)_{char(\rho)}| + \sum_{q\in\mathbb{N}^+} q \cdot |\rho_q.(C_q)_{char(\rho)}| + \kappa \cdot \aleph_0 = \aleph_0 + \kappa$$

where $\kappa = |\rho_\omega.(C_\omega)_{(Sing(\overrightarrow{G}_1))}|$.
So from the above cases and from Proposition 2, $Th_{char(\rho)}$ is uncountably categorical. Also from the axiomatization of $char(\rho)$, any model of Th_{char_ρ} satisfy the axiom

$$\exists^{\geq n} v[v = v] \text{ for each } n \in \mathbb{N}^+.$$

Hence, by by the Łos-Vaught test, $Th_{char(\rho)}$ is complete.
□

3. Conclusions and Future Work

This research started with classifying $Sing(\overrightarrow{G}_1)$ and then introduced all the axioms of the theory of $Sing(\overrightarrow{G}_1)$. Then we proved the conditions of $char(\rho(Sing(\overrightarrow{G}_1))$ for the strongly minimal $Sing(\overrightarrow{G}_1)$ and examined the theory $Th_{char(\rho)}$ and proved is complete. The importance of such theory of such graphs is that we can apply many mathematical structures on it where as algebraically closed fields narrowed only on field theory.For example, we can apply groups, permutation groups, and other mathematical sets with some relation such as ordering relation and the successor function.

The goal of this work is we will show in the future that these structures can be Zariski Geometry. Zariski axioms relate algebra to geometry by topologize strongly minimal structures. When this is done the work will even be extended to directed single graphs with l-arity not only 1-arity and give the complete theory of such graphs so that we can extend the range of the mathematical structure that can be applied on these graphs. Again we will show that these structures are Zariski Geometry. **Hrushovski** and **Zilber** classified strongly minimal structures which are Zariski Geometry are only algebraically closed fields, vector spaces, affine spaces, and pure sets [8] and algebraically closed field are on top of these classification. Thus this research is aiming to be extended in the future to find a wide range of mathematical structures which is not listed in **Hrushovski** and **Zilber** classification.

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