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Pólya's Methodology for Strengthening Problem-Solving Skills in Differential Equations: A Case Study in Colombia

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Abstract: The formation of students is integral to education. Strengthening critical thinking and reasoning are essential for the professionals that today's world needs. For this reason, the authors of this article applied Pólya's methodology, an initiative based on observing students' difficulties when facing mathematical problems. The present study is part of the qualitative and quantitative research paradigm and the action research methodology. In this study, the inquiry process was inductive, the sample is non-probabilistic, and the data interpretation strategy is descriptive. As a case study, six students were enrolled onto a differential equations course at the Universidad Autónoma de Bucaramanga. A didactic process was designed using information and communication technologies (ICTs) in five sequences that address first-order differential equation applications. As a result of the pedagogical intervention, problem-solving skills were strengthened. All this was based on asking the right questions, repeated reading, identifying and defining variables, mathematization, communication, and decomposing the problem into subproblems. This research study seeks to set a precedent in the Latin American region that will be the basis for future studies.

Keywords: didactic strategy; differential equations; ICT; problem solving; Pólya method

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Citation: Chacón-Castro, M.; Buele, J.; López-Rueda, A.D.; Jadán-Guerrero, J. Pólya's Methodology for Strengthening Problem-Solving Skills in Differential Equations: A Case Study in Colombia. *Computers* **2023**, *12*, 239. https://doi.org/10.3390/computers12110239

Academic Editors: Antonio Sarasa Cabezuelo and Covadonga Rodrigo San Juan

Received: 30 September 2023
Revised: 5 November 2023
Accepted: 7 November 2023
Published: 18 November 2023

1. Introduction

Mathematics comprehension is a fundamental pillar in the education of young people in modern society, as indicated by the Organization for Economic Cooperation and Development (OECD) [1]. Mathematics is not just an academic subject but an essential tool to address a wide range of challenges in everyday life, as well as in professional, scientific, and natural contexts [2–4]. According to the results of the PISA tests in 2018, which assess critical reading, science, and mathematics, Colombia ranks 58th out of 79 countries. This is a concern, given the poor results and the impact on the level of competence in mathematics required for entry into education. In Colombia, the Ministry of National Education (MEN) states that students nationwide need to strengthen their ability to justify, assert, and express criteria in situations involving quantitative information or mathematical objects based on mathematical considerations or conceptualizations [5].

The Colombian Association of Engineering Schools (ACOFI) administered a test to engineering students from affiliated public and private universities. The purpose was to assess the set of fundamental knowledge that should be incorporated through courses related to basic sciences (Mathematics, Statistics, Physics, Chemistry, and Biology). According to the results [6], ACOFI states that 55% of the evaluated students need to improve their ability to interpret, represent, and synthesize presented problems. This is evidenced by their limited capacity to recognize the necessary mathematical model for a specific situation, even if it is of low complexity [7,8]. Due to this issue, this article will focus on designing a strategy that allows students to manipulate mathematical objects, activate their mental capacity, exercise their creativity, and enhance the development of thinking processes [9]. In this line of thought, it is essential to bridge theory and practice, connecting students to real-world problems to foster conceptual understanding, because a significant portion of classroom time has traditionally been focused on repetitive exercises, limiting opportunities for practical applications that give meaning to what is learned [10–12]. To address this, students need competency in problem solving that relates to their daily experiences [13].

In response to these challenges, this study was conducted within the context of a mathematics course during the fourth semester of an engineering course at the Universidad Autónoma de Bucaramanga (UNAB). The primary research question that guided this study was: "How can problem solving competency in students taking a Differential Equations course at UNAB be enhanced through the application of mathematics didactics?" The answer to this question was sought through the implementation of a didactic strategy designed to foster contextualized and innovative learning practices. Problem solving stands as a fundamental pillar within mathematics education. From an epistemological perspective, it involves the application of mathematical logic to analyze statements and mathematical objects, ultimately leading to the formulation of estimates and hypotheses [14]. This skill is considered indispensable, serving as the foundation for the exploration of techniques and the development of self-study skills.

In 1945, George Pólya introduced "How to Solve It" [15], a groundbreaking work that outlines a structured approach to teaching and learning mathematics centered on the problem solving process. Pólya's method comprises four essential phases: understanding the problem, devising a plan, executing the plan, and critically evaluating the results. These phases provide a systematic and effective framework for approaching mathematical problems, emphasizing deep comprehension, careful planning, meticulous execution, and critical reviewing as foundational principles. The choice of methodology plays a pivotal role in achieving effective results in education and in sustaining student engagement and motivation throughout courses. The use of active methodologies is particularly effective in improving the teaching–learning process and reshaping any negative perceptions some students may hold about mathematics. By constructing mathematical concepts through problem solving, students can establish meaningful connections between the subject matter and their daily lives, fostering a deep understanding as opposed to mere rote memorization [16].

The classification of mathematical problems plays a pivotal role in education and research. This classification serves as a crucial tool for understanding the diversity of the mathematical challenges that arise. Consequently, it becomes an essential element for systematically and efficiently addressing problems. In this regard, Figure 1, developed by Foong [17], represents a classification of mathematical problems that holds significant relevance in comprehending the nature and diversity of the mathematical challenges faced by students and mathematicians. This classification eases the identification of problems and one's approach to them, contributing to the reinforcement of learning and advances in mathematical research.

According to Piñeiro et al. [18], there exists a classification of mathematical problems: (i) Closed structure problems—these are well-formulated problems in which the answer can be found within the problem statement itself. This type of closed structure is further divided into routine and non-routine problems. In routine problems, one studies the mathematics of a specific topic to solve this type of problem. In non-routine problems, the subject matter is not inherently specific due to several assumptions that need to be considered. (ii) Open structure problems—these problems are poorly formulated, resulting in a lack of clarity. This category encompasses real applied problems, mathematical investigations, and short open-ended problems. In real applied problems, the focus is on reality, and a mathematical solution is sought. In mathematical investigation problems, students engage in various exploratory and testing activities to find the solution. Short open-ended problems offer different possible solutions.

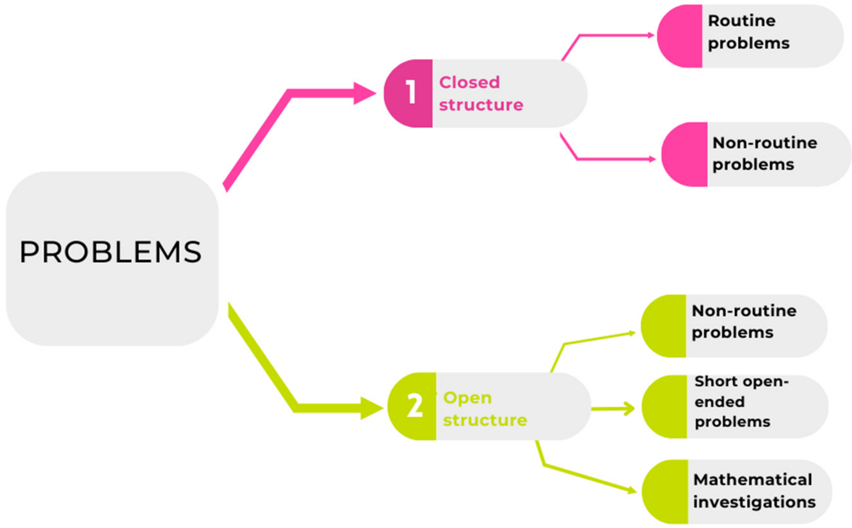

Figure 1. Mathematical problem classification.

To date, the teaching of differential equations (DEs) courses has predominantly emphasized the presentation of analytical procedures for tackling abstract mathematical problems. However, this overreliance on analytical methods, rather than incorporating numerical and qualitative approaches, has resulted in only partial learning outcomes for students. This presents a challenge, as according to both national and international standards, the memorization of analytical procedures no longer suffices for the comprehensive education of future professionals [19]. Today's students need to develop competencies that extend beyond the mere acquisition of mathematical problem-solving knowledge; they must also be able to apply these skills in real and specific professional contexts.

In this context, Pólya's proposal has had a significant influence on problem-solving instruction, yet it is also criticized for its rigidly defined stages. Some authors argue that problem solving is a more intricate process than simply applying predefined phases. As Schoenfeld [20] suggests, "problem-solving entails the use of heuristics and the management of the solver's cognitive resources", indicating that effective problem solving relies not only on applying specific phases but also on cognitive strategies and metacognitive control. Additionally, Santos Trigo [21] states, "problem-solving is an interactive process, where phases are not always linear; they can overlap or alternate". Furthermore, it underscores that "factors such as the solver's beliefs, attitudes, and emotions play a significant role". From a more contextual perspective, Campistrous and Rizo [22] argue that "problem-solving is a cyclical process, with constant evaluation between phases, rather than a simple sequential application". They also emphasize "the importance of considering the context and meaning of the problem for the solver" [23].

We deemed it essential to address the Pólya approach, which has proven valuable for problem-solving training but has also faced criticism [24]. While this method is often deemed suitable for routine problems, it may not be as effective when dealing with non-routine, real-world, or ill-defined mathematical challenges. Therefore, considering the incorporation of the Pólya methodology, we must explore ways to adapt it to address broader and more diverse mathematical challenges [25].

Furthermore, information and communication technologies (ICTs), though not a definitive solution to educational challenges in mathematics, have significantly transformed the teaching landscape. These technologies offer various ways to represent mathematical problems, allowing students to develop problem-solving strategies and a deeper understanding of the mathematical concepts they are studying [26]. Notably, ICTs, particularly simulators and applications, have gained increasing importance in mathematical education. Computational simulators can represent processes or phenomena, whether natural or artificial, through computational models. Users can interact with these simulators, adjusting model parameters to gain a better understanding of the system under study.

The combination of the Pólya approach with modern technology has empowered students to actively and collaboratively engage in solving complex mathematical problems, particularly in the context of differential equations. The use of simulators and applications, in conjunction with appropriate instruction, not only motivates and involves students but also provides them with a deeper and more practical understanding of a subject that might otherwise seem overwhelming. These technological tools enable students to explore and experiment, thereby strengthening their cognitive and mental processes in mathematical problem solving [27].

2. Materials and Methods

The research work conducted for the creation of this paper was qualitative. It involved directly engaging with students in their contextualized academic environment. The inductive method was employed. An analysis of the findings in each intervention facilitated the observation of the progression from the specific to the general. This paper is a descriptive study. Each action and process were interpreted from both individual and group perspectives [28]. The study followed the dynamics of action research, a methodology of inquiry and reflection conducted through collaboratively developed cycles involving teachers and students [29,30]. The pedagogical intervention was structured into didactic sequences, aiming for meaningful learning. The structuring of mathematical thinking to enhance problem-solving competence, specifically in the application of first-order differential equations, is conceptually rooted in Pólya's method. The application of its phases facilitated the exploration and comprehension of the method through critical and reflective progress within each sequence [15]. This type of proposed task serves as a potential tool for fostering logical reasoning. It utilizes data and information to formulate, summarize, strategize, and comprehend mathematical concepts.

Classes are generally taught in a traditional manner, without the incorporation of technological tools such as computers, simulators, or applications. Class dynamics do not involve group work; instead, students complete quizzes and assignments individually. Thus, during classes, the teacher adopts a traditional approach, and students remain in their seats, participating in the class passively. Our research was carried out using the research activity method and didactic sequencing. The information gathered allowed us to analyze the combination of Pólya's methods and technologies to develop applications for solving differential equations. This proposed combination is a potential tool for developing logical reasoning using data and information to formulate, summarize, strategize, and understand mathematical concepts.

To facilitate the development of this study, an initial test was formulated and validated by two experts in the fields of mathematics education and pedagogy. The primary objective of this test was to assess students' prior knowledge concerning problem solving in differential calculus and integral calculus. The outcomes of this assessment enabled the formulation of five sets of dynamic activities: radioactive decay, Newton's Law of Cooling, mixing problems, orthogonal trajectories, and Newton's second law. These sequences were meticulously crafted to promote active participation and enhance student engagement throughout their learning journey. They are presented as follows:

- Radioactive decay: This sequence commences by introducing the concept of radioactive decay, which is fundamental in solving differential equations. During this phase, students explore energy-related equations that depict this phenomenon. The sequence culminates with practical exercises that challenge students to solve differential equations pertaining to radioactive decay, thereby solidifying their comprehension of this phenomenon.
- Newton's Law of Cooling: The behavior in heat transfer is essential in modeling temperature as a function of time. Therefore, this sequence allows students to relate the law with differential equations by solving heat transfer problems in different systems. In this way, exercises are proposed to apply the cooling law in situations contextualized to the level of the course.

- Mixing problems: Within the realm of mixing problems, different concepts involving the variation of quantities of substances are employed to modify variables. In this subject, changes in quantities of substances as a function of time are applied, enabling an understanding of dynamic systems in real situations. As a result, participants engage in solving mixture-related problems pertinent to engineering and chemistry.
- Orthogonal trajectories: Orthogonal trajectories allow for an understanding of the relationship between solutions of differential equations and their behavioral interactions, facilitating an analysis within vector fields and dynamical systems. Thus, the development of orthogonal trajectories within vector fields and analysis in dynamical systems is proposed within this phase.
- Newton's second law: Within the Law of Force and Acceleration, the relationship between the force applied to an object and its resultant acceleration is discovered. As part of this phase, applications of the concepts related to the law are proposed under situations of motion, particle dynamics, and rigid bodies. Finally, the sequences conclude with exercises applied to complex systems, reflecting the academic challenges in the field.

2.1. Participants

The research period was from August to November during an academic cycle, and the study population consisted of 18 fourth-semester students of the Biomedical Engineering program at UNAB. The sample was a non-probabilistic sample of six students enrolled onto the Differential Equations course.

2.2. Techniques and Instruments

For this section, a mixed empirical approach that involved employing observation techniques, interviews, questionnaires, and educational interventions to gather information was used [30]. First and foremost, a diagnostic test was administered to identify prior knowledge, establish connections with new knowledge, and assess the strengths and weaknesses in students' problem solving processes. A field diary was maintained to record the most relevant events that significantly contributed to the research. The observations and the analytical methodology developed allowed for inferences to be made about students' behavior in each phase of Pólya's method. Furthermore, video recordings were utilized to obtain detailed observations of each educational intervention workshop construction session. These recordings assisted in identifying the researcher's behavior, interactions between the researcher and the participants, and the role of the students in the course and their performance in approaching problem solutions following Pólya's Problem phases [28]. Lastly, didactic sequences were employed as a source of pedagogical intervention, enabling the identification of students' actions, the researcher's role, and interactions between the researcher and the students. Throughout these interventions, applications such as GeoGebra, Matlab, and Wolfram Alpha, among others, were incorporated.

2.3. Procedure

Each of the suggested didactic sequences was planned according to Pólya's theory in such a way that it led to its conceptual and procedural empowerment. Action research methods were proposed by Kemmis and McTaggart [29], and they conceptualize research as a circular spiral process with four stages: observation, planning, action, and reflection. Figure 2 shows the procedure followed in this study (derived from this theoretical approach), where long-term reflection embodies the construction of knowledge and enriches practice.

Figure 2. Stages of action research according to Kemmis and McTaggart [29].

2.3.1. Planning: Diagnostic or Preparatory

The first action research cycle is enacted when the problem becomes evident [31]. From there, the approach focuses on questions, concerns, and objectives that indicate the progress of the work. This phase begins with identifying the problem to be investigated [29].

2.3.2. Action

In this stage, we applied didactic strategies through a series of steps focusing on using the Pólya model proposed above. During this phase, we collected information through observations and recorded the most relevant facts for the present research paper.

2.3.3. Observation: Redesign and Verification

During this phase, activities that were part of a didactic strategy designed to enhance problem-solving skills were analyzed, and data were collected through other tools, as reported by [31].

2.3.4. Reflection

This is the cycle's final stage in which the information is organized for presentation and evaluation according to categories of analysis relevant to the problem posed and the proposed purpose. Conclusions are drawn from the results, and the triangulation exercise is carried out for validation.

2.4. Methodology

Thanks to the design of didactic sequences and evaluation instruments, students were evaluated before and after the training. Once the diagnostic test was applied, the development of the problem-solving strategy was carried out using Pólya's method and ICT tools.

Each of the five didactic sequences focused on making the application of mathematics visible in solving problems requiring a first-order differential equation (FDE) approach. In this case, based on the categories/subcategories—problem solving (problem posing, mathematical processes, problem analysis, problem relationship, and review), perception (level of satisfaction, model recognition, benefits, retrospective view), implementation of the didactic strategy (understanding, mastery of the methodology, analysis)—elaborated through the theoretical elements of content, algebraic thinking, operations and algorithms, visualization and graphical representation, and problem solving, notes on each category were taken synchronously in a Google Drive (Word) document to facilitate interactions among students and the use of the method of Pólya, who defined a series of steps that lead to guiding questions for problem solving.

2.4.1. Understanding the Problem

Based on the numerous studies on the theory of the Pólya method, it is clear that this stage is necessary to make manipulating objects in mathematics more understandable to students. For this, students should ask and answer questions such as the following: Are you familiar with the problem? Is this a student problem? What data can solve the problem? Is the information complete, or is there a redundancy of data? What variables or unknowns should be configured to solve the problem? Do you need a drawing or diagram to solve your problem? What is your doubt? What is the theoretical operation or rules that allow you to ask the question? What units does it present, or does it not have dimensions?

2.4.2. Conceiving the Plan

This stage attempts to find similar or analogous problems. In this stage, we used questions to identify simple problems for use as a starting point for more complex problems, such as the following: We must solve the minor problem in question, but have you solved a similar problem? Can you restate the problem in your own words? Are all the data or conditions used correctly? Are all the basic concepts, theorems, or theoretical problem-solving elements emphasized? After you organize and structure the problem solution into a series of steps, do you have a single model? What is your plan or strategy for solving the problem? How else could you solve this problem?

2.4.3. Executing the Plan

During the development of this phase, mathematical operations are performed step by step to verify the stated results. Thus, the crucial questions in this phase are as follows: What has been obtained in these processes? Are there answers to the questions?

2.4.4. Retrospective View

This stage allows students to examine each step taken in each previous stage and to cross-check the questions, data, and information presented by the problem situation. This step is necessary to check or verify the results obtained, provide a way to generalize the problem, and answer the following questions: Does the result match the sentence? Do you have a problem? Is the solution correct? Is it possible to validate the solution? Is the data used part of the information problem? Can the problem be generalized, and can the results or processes be used to solve similar problems, and if so, how?

3. Results

This study was carried out for the module on applications of first-order ODEs; it was executed throughout the academic semester using five didactic sequences that include the following topics: radioactive disintegration, Newton's Law of Cooling, mixing problems, orthogonal trajectories, and Newton's second law. The results obtained regarding each of the proposed activities are discussed below.

3.1. Educational Interventions

3.1.1. Diagnostic Test

The tests show the need to implement methods that allow students to solve problems and relate theory to practice using concepts and procedures. We obtained the correspondence between the questions and the answers. In the answers, there needed to be more clarity when reading the contextualized question. In these cases, the situation had to be modeled from a function. In this regard, Arias-Rueda et al. [32] point out that machine learning represents the process that students prefer to use to solve problems and that most students substitute numbers in formulas and results, choosing to limit counting because it does not mean anything to them. For example, in one of the diagnostic tests, the students were expected to easily calculate the rate at which the volume increased concerning the radius of a spherical globe. Answers to this question are shown in Figure 3 below.

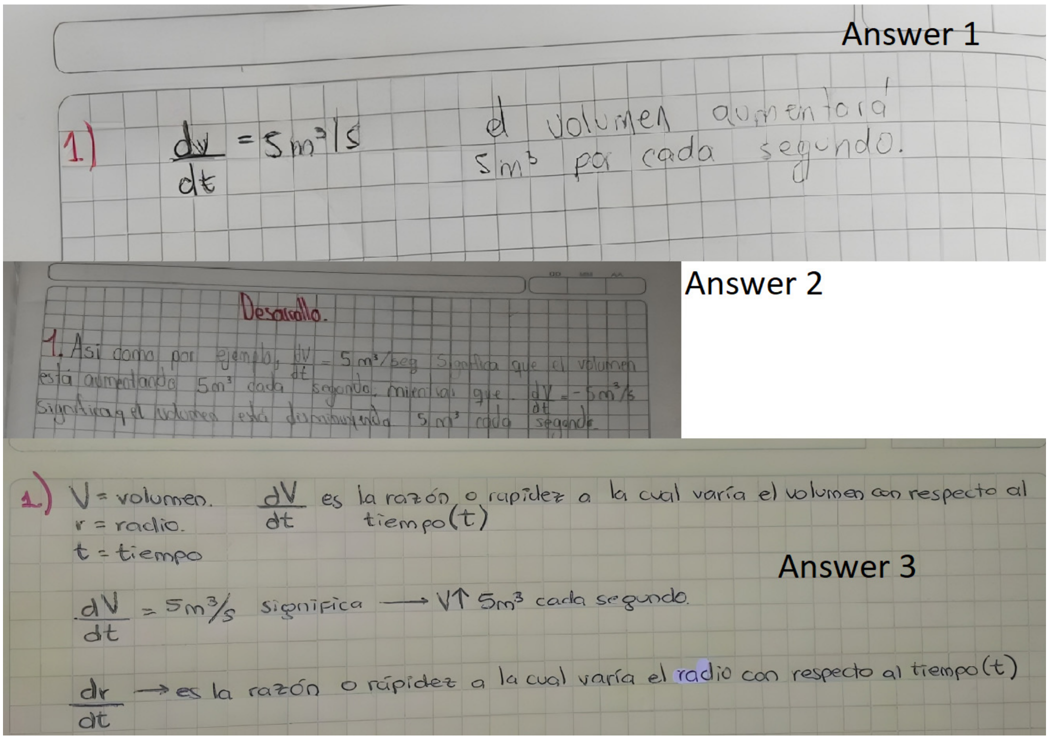

Figure 3. Diagnostic test results of three students to question 1. **Note:** The translation of answer 1 says, "the volume will increase by 5 m^3 for each second". The translation of answer 2 says, "So for example, dv/dt = 5 m^3/s means that the volume is increasing 5 m^3 every second, while dv/dt = −5 m^3/s means that the volume is decreasing 5 m^3 every second".

The student writes the formula for the volume change rate shown in the problem statement. In addition to taking III the same ratio, he uses the words increase and decrease to see if they are negative instead of positive. Student E31 shows relevant variables, which represent changes in the volume–radius ratio. We observed that the participants had an intuitive idea of the rate of change. However, the students could not determine the volume function of the sphere and failed to establish the necessary relationship from the derivative of the variables concerning time. All three students left a blank and responded that they did not remember the topic when asked about their condition. Here are some basic actions to solve this problem.

3.1.2. Didactic Strategy

Once we completed the diagnostic test, we began implementing the five didactic strategies, starting with explaining Pólya's method as a strategy for solving mathematical problems, the results are shown below.

3.1.3. Radioactive Disintegration

In the didactic sequence, radioactive disintegration, and following the respective phases, it is evident that the students needed to comprehensively read the problem statement. The expression of the data, the unknowns, and the initial conditions show that they still need to gain the competence to develop and execute a plan that leads to the solution to the situation posed. Figure 4 describes this stage.

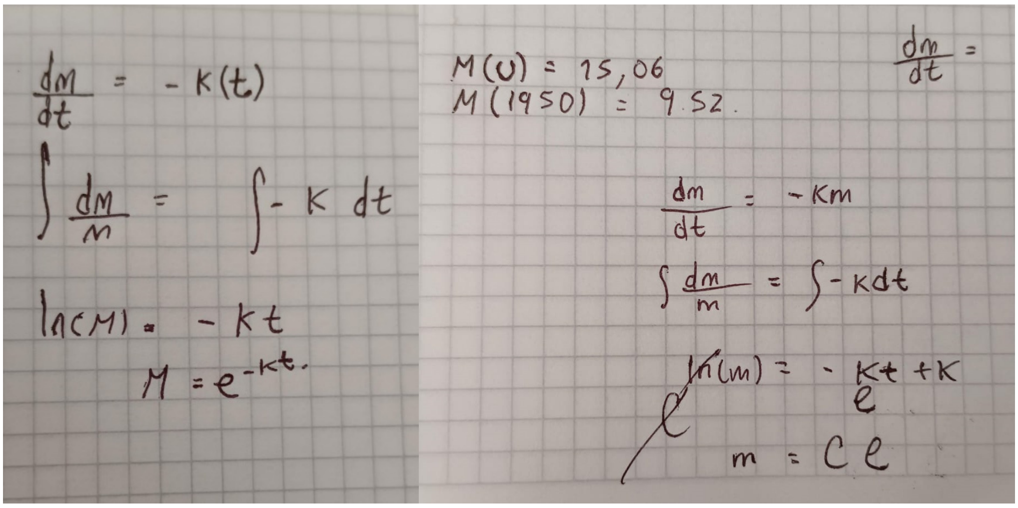

Figure 4. Mathematical model approach.

Group 1 showed that students could not establish an algebraic relationship between the data and the initial conditions of the problem. Group 2, on the other hand, did not use initial conditions to find the parameters "c" and "k". Due to the data required for specific ODE solutions, both groups left the problem's solution unfinished. Finally, no student successfully compared the results during the retrospective period.

3.1.4. Newton's Law of Cooling

We observed that the students gradually appropriated the methodology, identifying the variables (temperature and time); they used their excel skills to represent the situation. However, as expected, they did not analyze the graphical representation of the time–temperature relationship. There was no correspondence between the data presented, as shown in Figure 5.

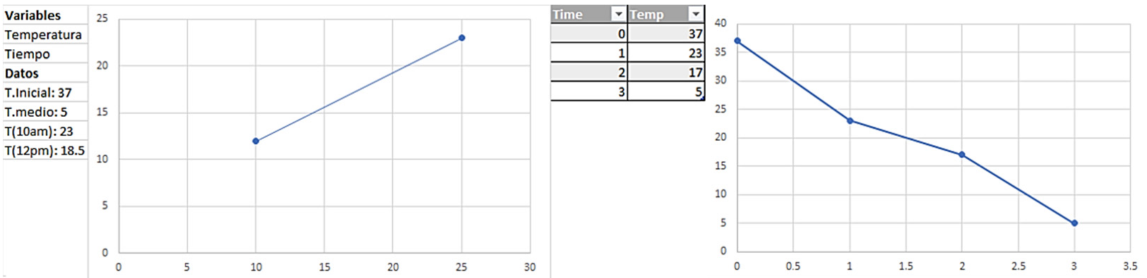

Figure 5. Graphical representation of the interpretation of the exercise. **Note:** Figure translation: "Variables: Temperature vs. Time, Data: Initial temperature: 37, Average time: 5, Temperature at 10 am: 23, Temperature at 12 pm: 18.5".

When observing group work, students read the problem in detail to find relevant variables and data and show progress using their prior knowledge. However, they did not use their graphs to reflect on the magnitudes of the problem and did not relate the data to the given situation (Figure 6). In the next phase of plan development and execution, students identified the cooling models and Newton's laws and applied their knowledge of integrals to find the mathematical model corresponding to the differential equation closely related to the problem statement.

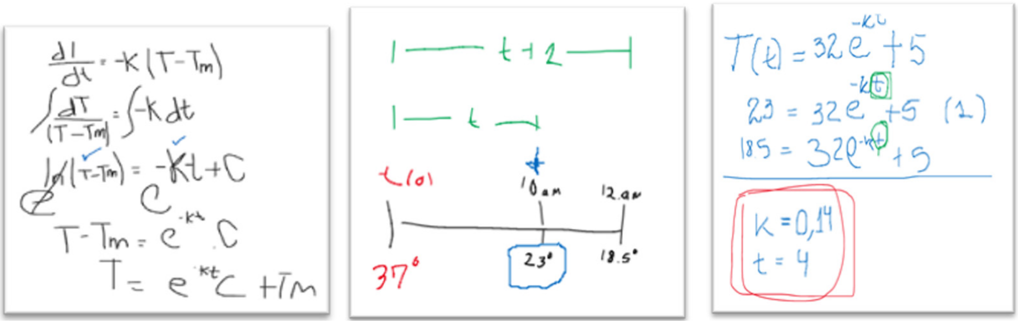

Figure 6. Elaboration of the plan: Newton's Law of Cooling.

In the process, the students presented errors in identifying the initial conditions and the approach to the problem situation, as described in Figure 7. This situation required the intervention of the research teacher; we posed theoretical questions and gave effective feedback to facilitate the student's progress and learning. In this phase, we observed an evolutionary advance in the use of the methodology in the students; despite not achieving the respective solution, they reached the retrospective phase using applications such as Symbolab, Wolfram Alpha, and GeoGebra to verify what had been performed and achieved in the previous steps.

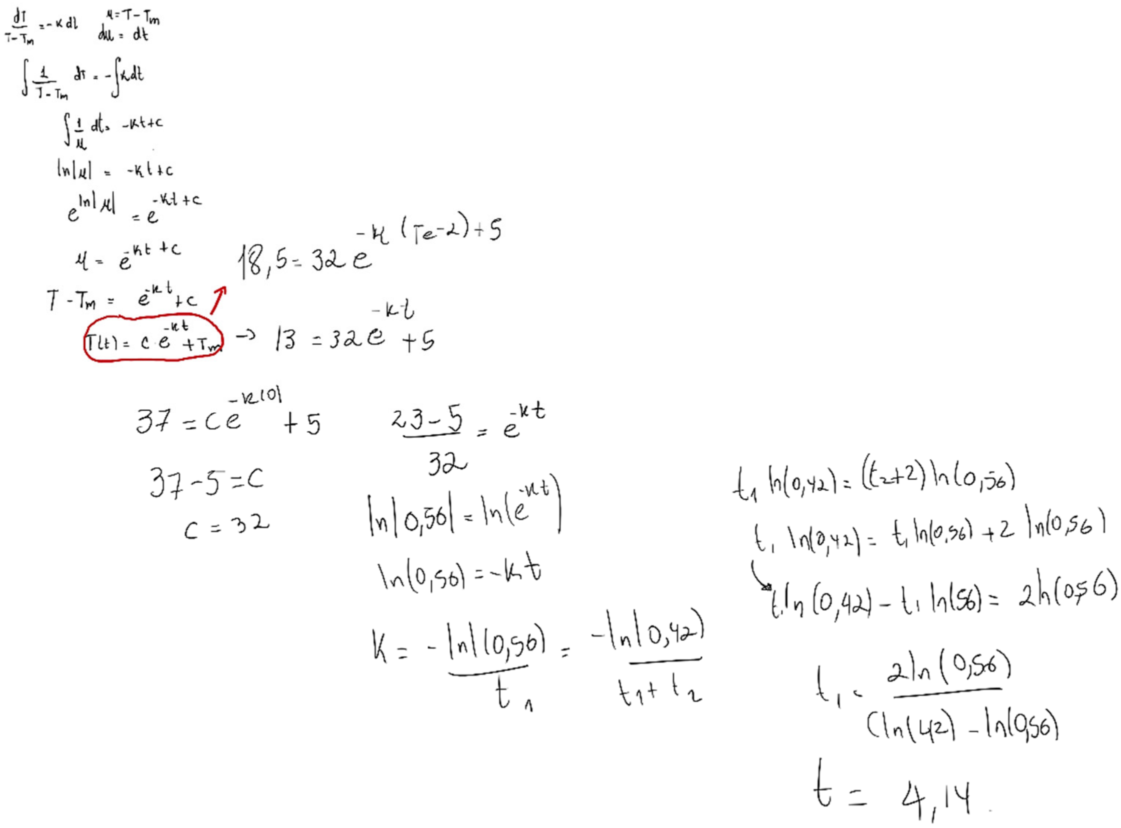

Figure 7. Plan execution and problem review.

3.1.5. Mixing Problem

This workshop exhibited a simulation where the liquid in the tank mixes with the contents and changes color. Therefore, differential equation models are built intuitively through rate of change and chain rules. The resolution is shown in Figure 8.

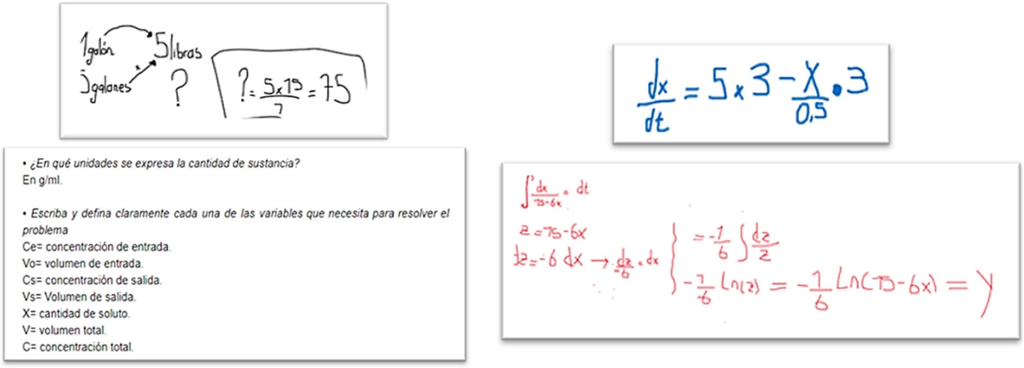

Figure 8. Relationship between the simulation and the guiding questions. **Note:** Translation of each response. Answer 1: 1 gallon, 5 gallons, 5 pounds. Answer 3: "What units is the amount of substance expressed in? In g/mL. Write and clearly define each of the variables needed to solve the problem. Ce = Inlet concentration. Vo = Inlet volume. Cs = Outlet concentration. Vs = Outlet volume. X = Amount of solute. V = Total volume. C = Total concentration".

In the execution of this didactic sequence and from the dynamizing action of the learning process that has been achieved with the two previous experiences, the students showed significant progress in using the methodology. In addition, with the implementation of the proposed guiding questions, they broke down the problem with repeated reading, identified the variables and data, answered the guiding questions, and analyzed the problem. An example of the progress described is shown in Figure 9.

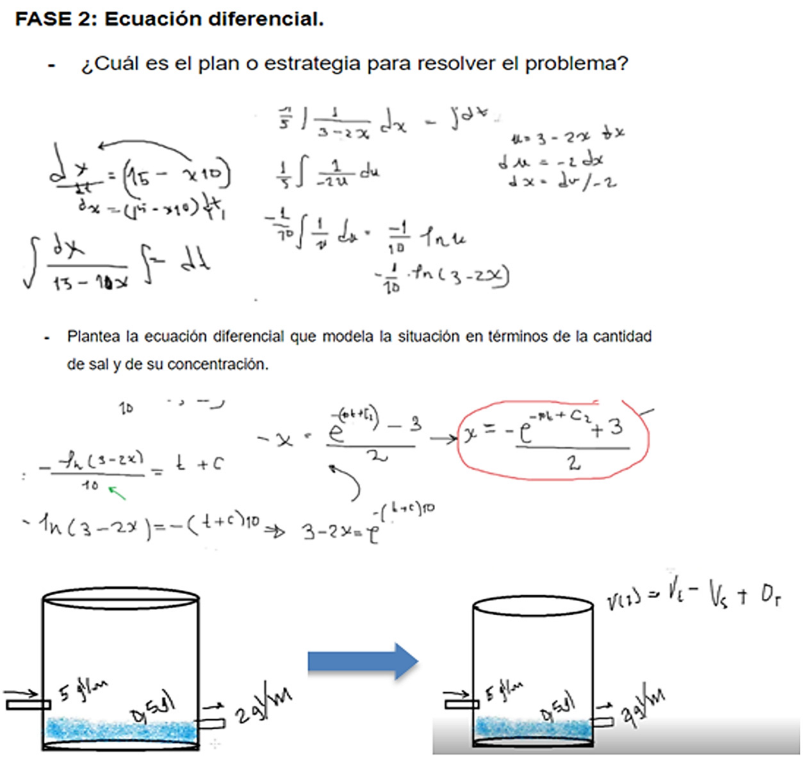

Figure 9. Volume variation modeling method. **Note:** Figure translation: "Phase 2. Differential Equation", "What is the plan or strategy to solve the problem?", "Formulate the differential equation that models the situation in terms of the quantity of salt and its concentration".

The students were able to create an outline of the problem situation, working as a team and in continuous dialogue to analyze the planned situation. This dialogue was led by the teacher–researcher, posing questions that allowed the students to understand and argue the solution to the question posed. Finally, in the retrospective view, the students, when checking the answer, discovered that they had made a mistake in one of the data points, which allowed them to rethink the mathematical process (Figure 10).

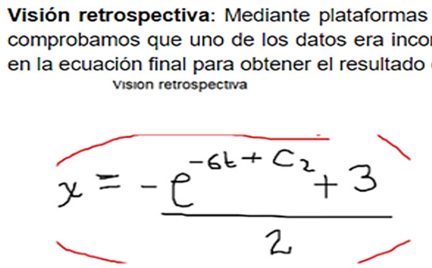

Figure 10. Problem solution used to compare with the Symbolab calculator result.

During the execution of this didactic sequence, we employed the GeoGebra tool to conduct blending exercises. This tool offered an active and dynamic alternative that sustained the students' interest and attention in the learning process. Through the GeoGebra simulation, we presented a scenario in which the tank's liquid mixed with its contents, resulting in a color change that piqued the students' curiosity, as depicted in Figure 11.

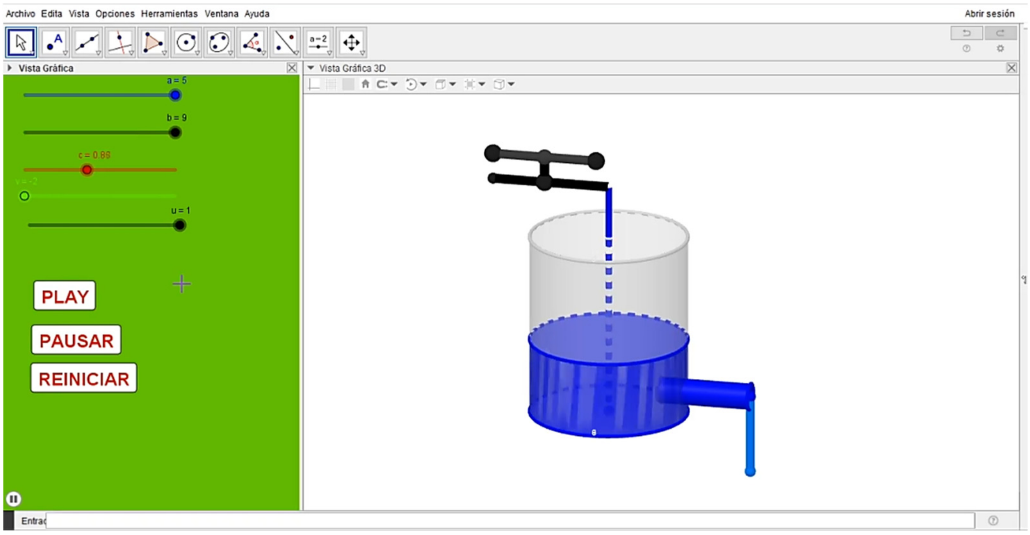

Figure 11. Activity carried out using the GeoGebra tool.

In this context, differential equation models were intuitively developed by examining rates of change and chain rules. Students actively engaged in problem decomposition, identified relevant variables and data, responded to guiding questions, and engaged in problem analysis. This approach fostered collaboration and an ongoing dialogue among the students, who worked together to analyze the planned situation.

The teacher–researcher played a crucial role in leading the dialogue and posing questions that prompted students to understand and articulate their solutions. In a reflective process, students, while verifying their response, identified an error in one of the data points, providing them with the opportunity to reconsider the mathematical process and correct their approach. The use of GeoGebra as an active and dynamic tool, coupled with the didactic sequence and guiding questions, enabled students to advance their competence in addressing mathematical problems and enhance their understanding of the concepts involved in the blending problem.

3.1.6. Orthogonal Trajectories

In this workshop, the process of strengthening knowledge and applying the mathematical problem solving method according to Pólya's theory continued. The aim was to find the family of orthogonal curves belonging to a given family of curves via the application of first-order ordinary differential equations. The pedagogical activity prompted the use of a graphical environment to visualize the orthogonality, all through drawing a pair of tangent lines to the orthogonal curves, respectively, at a given point. For this, a simulation created in GeoGebra was used. Figure 12 shows the result of the process that involved repeated reading, using graphical representations in GeoGebra, and following up on guiding questions. It is important to mention that Geogebra and Winplot were employed for these activities to visualize the slope fields and the orthogonal families of curves, respectively.

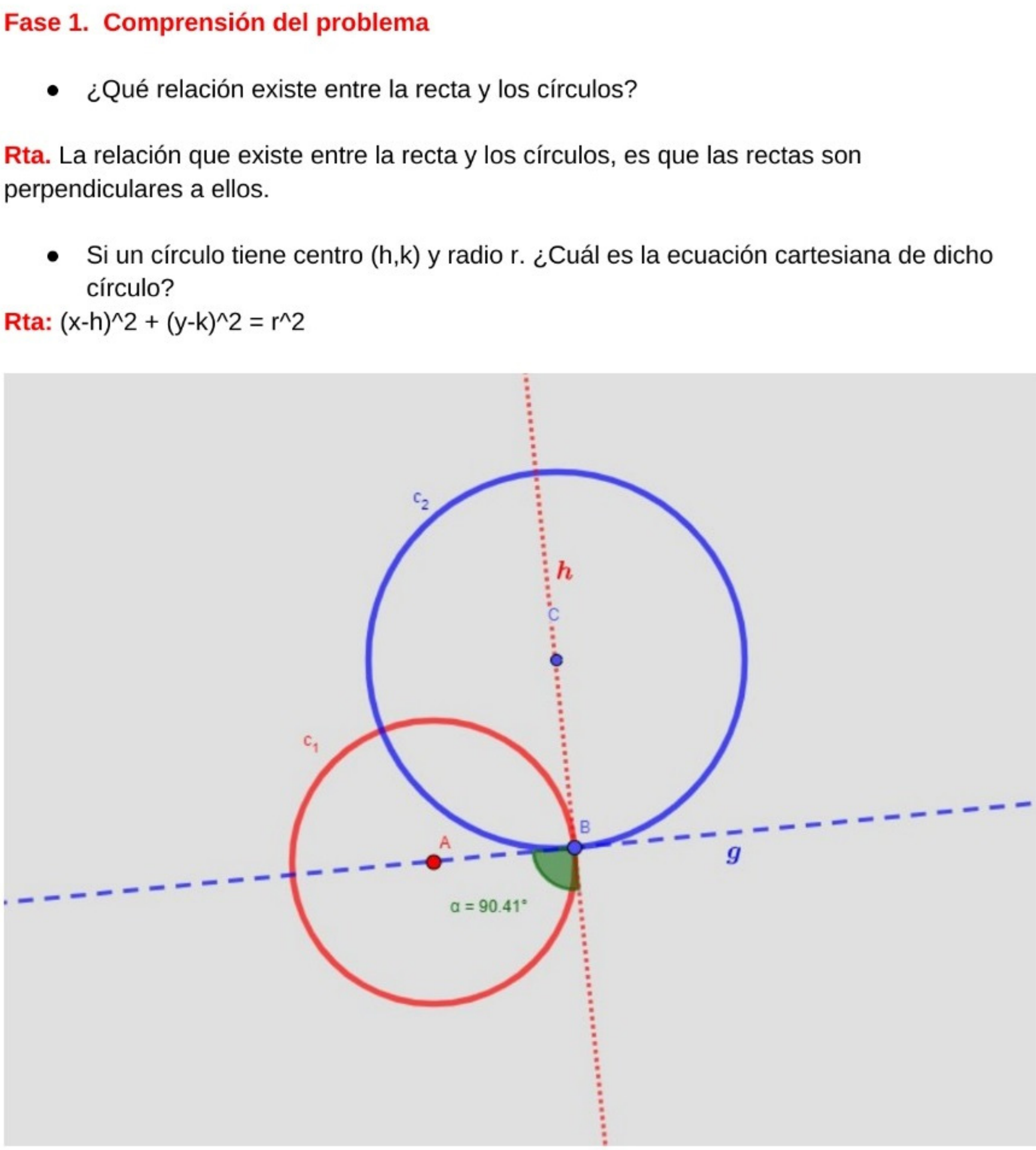

Figure 12. Relationship between the simulation and the guiding questions. **Note:** Figure translation. Phase 1. "Understanding the problem", "What relationship exists between the line and the circles?", "Answer: The relationship that exists between the line and the circles is that the lines are perpendicular to them", "If a circle has a center (h, k) and a radius r, what is the Cartesian equation of that circle?".

The student's progress in using and formulating guiding questions is fundamental to concretize the learning of the methodology and the application of the ODEs. The use of the GeoGebra application allowed the students to contrast the graphical representation with the algebraic one and to make their assessments of the questions posed. In summary, the analysis of the product made by the group that integrated the reading questions given as a fundamental element of the development of new questions to define data and variables clearly and previous link knowledge showed a significant improvement in the students' learning. This activity allowed for them to reflect on what happens when teachers use simulated guided questions to establish dialogues and face situations in a fun way; it is interesting to develop strategies that lead to the realization of a planned position.

3.1.7. Newton's Second Law

After the accompanying process, the students became aware of the methodological support that George Pólya's theory represents as a didactic strategy for solving problems. Here, it is worth noting that the students of the differential equations course improved significantly in their ability to solve problems, particularly those related to first-order ordinary differential equations. This is because the process led to the development of aspects of identifying variables or data and connecting existing knowledge in their cognitive structure with the new contents to calculate and analyze the effect of algebraic processes and consider how to solve situations in mathematics. In addition, with the help of previously formulated questions, the role of the teacher as a facilitator of the proposed problem is vital. The teacher becomes a spectator and active participant in the dialogue of the students (organizers) and builds meaningful learning. The dynamic classroom exercise encourages students to understand, reason, discuss, establish relationships, work in groups, reflect on and validate the results obtained, and facilitate critical thinking and research.

3.2. Analysis of Educational Intervention

For this study, interviews with a group of six university students were performed to carry out an analysis of the educational interventions. The aim was to qualitatively evaluate the impact of the pedagogical strategies implemented to enhance the problem-solving competence of the students. This analysis of educational interventions was centered around comprehending how the student responded to the pedagogical strategies designed to bolster their problem-solving skills. Throughout the interviews, the individual's perceptions, experiences, and challenges in the learning process were probed. Below are the most precise and recurring responses collected during these interviews.

Before participating in this research, had you used Pólya's methodology in problem solving?

The students generally stated that they had not seen this problem solving methodology throughout their education. Only one student commented that he had met a secondary education teacher who had used similar steps. He did not know of the Pólya methodology but had used some methodologies with many similarities.

Briefly explain the phases recommended by Pólya for effective problem solving.

The participating students explained that they should focus on reading the problem and understood the need to review the data (i.e., what are the unknowns or variables of the problem). In this way, they relate the data to elaborate a plan (how are they going to do it?) and take into account the strategy, execute the plan, solve it, and verify that the solution is correct. One of the students emphasized that it is necessary to read the problem very carefully and identify the required variables and how to organize it before developing the equation that can be used to solve the problem. The last step is to receive feedback on whether the result answers the question and whether the methods used were effective.

This process made one of the students recall the methodology of his high school physics teacher. The stages of this methodology were very similar to that of Pólya's, but this student emphasized that it was not practiced in other subjects, obstructing its consolidation within the learning process.

What mistakes did you make before working with Pólya's method?

Each student stated they needed to understand the problem, elaborate a plan, and review the solution (i.e., concentrating only on execution). It was identified that high-speed reading and not having a broad vision of the problem makes it difficult to solve it correctly. One student explained that he needed to learn how to differentiate between the variables that are necessary and those that are not. Using variables that do not contribute to the resolution is a distractor that could confuse the student. Another strategy used was to perform a process and start again with another option when not finding a solution. This strategy involves investing a lot of time and guarantees good results. In general, each student was able to significantly strengthen their ability to relate numerical data to their respective variables and identify the appropriate model needed to solve the problem. It was also validated as a tool used by the teacher to change the conventional process, motivating them.

Regarding student E11, he stated, "Before using this method, I was accustomed to following a step-by-step process for each exercise, like following a sequence rather than asking myself which variables are relevant and which are not. I used all the given variables, but I couldn't differentiate which ones weren't useful. So, this method helped me to address these kinds of doubts".

What new skills did you achieve using the Pólya methodology in your academic semester?

The students commented that it allowed them to approach each of the phases—understanding the problem, conceiving and executing the plan, and the retrospective review—in a more orderly and structured way. Now that they are familiar with it, they highly recommend using this methodology in other scenarios.

What recommendations do you propose for the solution of mathematical situations?

The practice of a more critical analysis allows us to identify what we have and how to orient it to the results. One of the students recommends using animations, drawings, and graphics in the classroom to learn visually; the teacher must implement this in their daily work. In physics, understanding the concept under study allows for a more straightforward resolution of exercises since physical phenomena need, in the first instance, a general knowledge of their conception for later treatment.

How was your academic performance in the differential equations course when participating in the pedagogical strategy proposed in this research?

In general, the students stated that they were delighted with the process used to strengthen their problem-solving skills, since their academic performance in the differential equations course was superior to that of previous semesters in the mathematics courses.

A knowledge assessment was conducted concurrently with the intervention, and its results are presented in Table 1. The diagnostic assessment results indicate a mean of 0.57, with a standard deviation of 0.18, out of a possible 5 points. This reflects a very low level of performance in solving first-degree differential equation problems. Following the implementation of the five sequences, the scores experienced a substantial increase; starting from the fourth sequence, the scores are higher than 4 out of 5. In the post-test, the mean score was 4.57 (with a standard deviation of 0.2), demonstrating a noticeable improvement. To analyze these data, the nonparametric Wilcoxon test was employed; with a p-value of 0.024, the existence of a statistically significant increase was confirmed. The results demonstrate an enhancement in problem-solving capabilities, with improved performance evident from the initial sequences. Nevertheless, the outcomes for the remaining sequences displayed a more modest improvement, indicating the effectiveness of this methodology for such exercises. However, there is still unexplored data that warrant analysis. The utilization of alternative methodologies could offer a deeper understanding of these phenomena, which should be corroborated through future experiments.

Table 1. Results of the application of the didactic sequences.

	Test 1	Intervention					Test 2
Participants	**Pretest**	**Sequence 1**	**Sequence 2**	**Sequence 3**	**Sequence 4**	**Sequence 5**	**Post-Test**
1	0.5	0.8	3.8	3.8	4.2	4.5	4.5
2	0.7	0.8	3.8	3.8	4.6	4.8	4.8
3	0.3	0.8	3.3	3.3	4	4.3	4.3
4	0.5	0.8	3.3	3.3	4.2	4.5	4.5
5	0.8	1.7	3.8	3.8	4.6	4.8	4.8
6	0.6	1.7	3.8	3.8	4.2	4.5	4.5
Average	0.57	1.1	3.63	3.63	4.3	4.57	4.57

4. Discussion

The practices learned in secondary education are carried over to higher education, which shows that the methodologies used will affect students in the future. The bibliography indicates that some proposals use this method in primary education [15,26] and high school [33,34]. There is a need-to-know new alternative to improve learning processes. The implemented sequence allowed for the strengthening of problem-solving competencies. The phases of Pólya's method prioritize repeated reading, the identification and definition of variables, graphic productions, materialization, dialogue between students and teachers, and decomposition into subproblems. This process began with a diagnostic evaluation to assess the subjects' prior knowledge of differential and integral calculus. The results showed a reduced understanding of linear functions and derivative concepts. In addition to this were problems of factoring and differential equations that obstruct the development of differential equations. These results highlight the need to strengthen problem-solving skills. The difficulty in interpreting mathematical problems should be contemplated as part of a comprehensive academic policy.

As the process progressed, students improved their levels of understanding. In terms of mathematization, prior knowledge is essential for establishing relationships and devising planning. Despite this, some differential equations were challenging and required specific algorithms. Collaboration among students and a more significant connection with the teacher–researcher are also encouraged [35]. This allows for the generation of questions and provision of correct feedback through pertinent discussion spaces. Group activities allow for better argumentation, relationships, reflection, and teamwork. The didactic sequences allow for rigorous work that strengthens the students' critical and investigative thinking. In addition, the incorporation of field diaries, videos, transcripts, and interviews serve to enhance this methodology [36].

The systematization of this method is an essential tool in the engineering field since it improves the organization of procedures. This strengthens the relationship between previous and developing knowledge. Mnemonic skills are also involved in remembering the necessary algorithms to follow the path to resolution. This can be enhanced using technological tools such as GeoGebra, MATLAB, Symbolab, and Wolfram Alpha, among others.

The results of this study are satisfactory from an academic point of view; however, the interviews allowed us to identify other considerations that are only sometimes considered. Students commented that the benefits are based on solving everyday life problems modeled using a differential equation in a more analytical, argumentative way and asking key questions that are not usually considered. They consider that they can recall similar problems to know what should happen, relate it to the current problem, and use mathematical theory to obtain the correct solution. It is also important to note that there was active and constant participation on the part of the students, strengthening their capacity to analyze daily life problems.

According to "*Reconstructing School Mathematics*" by Stephen Brown [34], the teaching of mathematics should be approached from broader perspectives, where students are educated as critical thinkers and well-rounded individuals, as he suggests, by solving problems in a more analytical and argumentative manner. Brown proposes that mathematics can be approached through metaphors or analogies that allow us to understand the world and ourselves simultaneously. Thus, the humanistic approach proposed by Brown, in which students are not only trained in technical skills but are also educated comprehensively as reflective, ethical individuals prepared for life's challenges, was of great interest and utility when preparing and carrying out this research study. Therefore, within this study, his approach was used under the question "What if not?" not only after solving a problem but even before attempting to solve it, where teaching is carried out with the expectation that errors, generated questions, and resulting doubts will enable the student to change the way they view any aspect of knowledge or experience. In this way, Brown suggests that by providing a problem and its solution to the student, they should be asked questions such as the following: "Why might this solution have been difficult to reach even though it is quite easy to follow?" "Have you found other mathematical ideas that could be described that way?" "Are there similar phenomena in your life (non-mathematical)?" In this way, students will become more engaged in problem solving.

In the context of Pólya's methodology for mathematical problem solving, Stephen Brown's proposal offers an alternative approach that promotes critical thinking and reflection, with an emphasis on the relationship between the problem and its context. Both authors agree on the importance of questioning both the problem itself and its relevance in the surrounding environment. This encourages students to explore various approaches and methods to solve the problem, even considering those that may lead to incorrect answers. This freedom in exploration aligns with Brown's approach, which recognizes errors, questions, and doubts as essential elements for learning and developing the student's perspective. Furthermore, metacognitive reflection is promoted, urging students to analyze not only the final solution but the entire problem solving process, encouraging them to identify the strengths and weaknesses of their approach and to establish connections between what they have learned during the process and its application in everyday situations. Therefore, the integration of Brown's methodology into Pólya's approach enriches mathematical problem solving by providing a broader and more reflective perspective.

During the diagnostic or preparatory planning phase, a diagnostic test, didactic sequences, and a review of technological tools were designed. The instruments used were validated by the researchers in the company of two experts to review the reliability of the theoretical references addressed, including the appropriate adjustments. During the action phase (design or implementation), the active participation of the students was evident, promoting their autonomous development, which was facilitated by the teacher. The development of group activities favors work through brainstorming, debates, and creativity in problem solving. In the observation phase (redesign and verification), the significant advances in the process of problem solving demonstrated by the students were considered. The last step of reflection, the presentation of results, emphasizes the improved academic performance of the participants. They combined theory with practice during the development of first-order ordinary differential equations. However, these questions were not enough since the teacher's guidance was essential. The methodological support discussed in the present research study paves the way for improving the teaching practice and incorporating digital tools that facilitate the teaching process. The promotion of spaces for strengthening pre-knowledge and the incorporation of new knowledge increases the confidence of students, who will then apply what they have learned in real-life situations [11,37].

As mentioned by the authors of [38], teaching mathematics has become a challenge for teachers in recent years. Not only does one's teaching approach need to be attractive to students, but it needs to factor in learnings that can also be applied in real life. The student's anxiety while studying the different branches of mathematics is an emotional response that can be considered typical, but it is not expected [39]. This may be a product of outdated teaching processes that do not allow the student to visualize the applications of mathematics and become passionate about learning them. This should be a wake-up call to teachers around the world, but especially in Colombia, where it has been observed that there are low levels of performance in regional evaluations. This type of tool builds new concepts that can enhance the problem-solving skills of future professionals; it is an aspect that goes beyond a knowledge test.

While George Pólya's model pioneered problem solving in education with well-defined phases, it has faced substantial criticism. Critics argue that it exhibits limitations due to its sequential and rigid nature, failing to encapsulate the true complexity and multi-dimensionality of the problem solving process. Theoretically, it has been acknowledged that effective mathematical problem solving involves intricate cognitive processes such as heuristic strategies and metacognitive skills on the resolver's part. Additionally, emphasis is placed on the interactive and recursive nature of the involved actions, which do not rigidly adhere to a predefined sequence but fluidly overlap and alternate. It is worth mentioning that not only prior knowledge influences problem-solving ability; attitudinal, motivational, and emotional factors also play significant roles in how students approach problem solving, and the importance of context in particular, as well as the meaning and significance that the problem holds for the resolver, given its impact on the process, should also be considered.

Therefore, it is crucial for students to approach a problem without prior knowledge of the solution process, provoking a conscious and intentional cognitive search for the solution. Consequently, not every exercise or task qualifies as a genuine problem unless it elicits this cognitive search activity from the student. In other words, problem solving is a multifaceted process that encompasses multiple factors. Thus, it is necessary to move beyond mechanistic views based solely on predefined steps and instead consider the cognitive processes and genuine searching involved in a problem's resolution, coupled with the contextual and subjective dimensions that mediate their approach. A multidimensional understanding offers a more comprehensive view of this fundamental mathematical competence.

Techniques must be applied to allow future generations to face an increasingly demanding world that requires rapid adaptation [40,41]. The results presented in this study align with those of other publications in the literature, showing how a methodology from the last century can be adapted to contexts worldwide. Despite this, there is still limited information on how students can apply this knowledge in the real world, which shows that there is still much to study in this field. Following what is stated in [42], it can be discerned that the planning students achieve can be transferred to non-mathematical contexts. Problem solving constitutes one of the essential skills in professional life. This type of tool, which does not include linear processes, should be simplified to be generalized according to the student's needs and situation. All of this will improve their critical thinking, as reflected via an increase in their efficiency and, consequently, in their productivity [43].

Modeling and representations play a crucial role in problem solving and deepening mathematical comprehension. Mathematical models, abstract constructions used to represent real-world situations and mathematical concepts, offer students a more profound and practical understanding. These models are pivotal in structured and logical problem solving, and as noted by Appelbaum [42], the mathematical modeling process may give rise to "ignorances". These gaps in knowledge or understanding should be seen as learning opportunities, motivating students to overcome them in their problem solving journey. Therefore, Appelbaum argues that the role of mathematical modeling in problem solving is of the utmost importance. Through models, students have the ability to address and resolve issues in a structured and evidence-based manner. These models provide a solid framework for reasoning and making informed decisions, essential aspects in contexts related to engineering applied to real situations, where the effectiveness of solutions is critical. By adopting this practice, students develop skills in asking questions, exploring novel perspectives, and challenging prior assumptions, which not only contributes to a deeper understanding of mathematics but also encourages creativity and critical thinking.

Regarding the limitations of this study, although Pólya's method is a support tool in problem solving, it does not have the greater innovation and appropriation of mathematical processes. Pólya's method and its implications for the teaching of differential equations and, in general, in higher education has presented few advances in the incorporation of technological tools. This is reflected in the few changes in the curriculum and by the fact that the methods of problem solving have not been significantly updated. This represents an obstacle from an epistemological and ontological point of view in the teaching–learning process of students. Mastering mathematical problem solving is a fundamental requirement for the development of learning from a historical–cultural perspective, and previous research has identified that the deficiencies begin in the teacher training model. These deficiencies are observed both in the deployment of this skill during the learning processes of teachers and undergraduate students, as well as in the pedagogical practices of these professionals in the school environment [44].

However, these criticisms should not be perceived as insurmountable obstacles but as valuable opportunities to improve and adjust the approach, especially when we take advantage of the technological tools provided by ICT [45]. Therefore, teachers should undergo an initial assessment to gauge their prior knowledge of mathematics, pedagogy, and technology. Once weaknesses have been identified, training programs focused on strengthening the teacher's most obvious areas of weakness could be implemented. Then, a joint intervention with teachers and students could be carried out to obtain better learning outcomes [46]. Students in the era of artificial intelligence deserve renewed plans, activities, and evaluations to improve the class environment through the use of new emerging technologies [47]. In this vein, we urge that mathematical problem solving should also benefit from other models such as TPACK and the use of gamification, given that it incorporates content knowledge, pedagogical knowledge, and technological knowledge [48]. In addition, future research could consider implementing mixed-method approaches that provide greater clarity and rigor in the measurement of outcomes. This would allow for a more accurate assessment of the impact of the applied pedagogical strategies in terms of improving students' academic performance and their effectiveness in improving problem-solving skills.

Furthermore, it is recommended to conduct a more comprehensive analysis of the strategies employed in each phase of the process, considering possible adjustments and supplements that could further improve results. The incorporation of additional strategies at different stages of the methodology might contribute to enriching the students' learning experience and increasing the transferability of their skills to non-mathematical contexts. This approach aligns with the metacognitive challenge posed to students, offering a more holistic view of their learning and problem-solving abilities.

Author Contributions: Conceptualization, M.C.-C., J.J.-G. and A.D.L.-R.; methodology, M.C.-C., J.B. and A.D.L.-R.; software, M.C.-C.; validation, M.C.-C. and J.J.-G.; formal analysis, M.C.-C. and J.B.; investigation, M.C.-C. and A.D.L.-R.; resources, M.C.-C. and J.J.-G.; data curation, M.C.-C., A.D.L.-R. and J.B; writing—original draft preparation, M.C.-C., J.J.-G. and A.D.L.-R.; writing—review and editing, J.B.; visualization, M.C.-C., J.J.-G. and A.D.L.-R.; supervision, J.B.; project administration, M.C.-C., J.B., J.J.-G. and A.D.L.-R.; funding acquisition, J.J.-G. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Universidad Indoamérica through the 305.254.2022 project, titled "Tecnología Educativa para discapacidad-TEDI".

Data Availability Statement: Data are contained within the article.

Acknowledgments: To Universidad Indoamérica and Universidad Autónoma de Bucaramanga for providing the resources necessary for developing this research study.

Conflicts of Interest: The authors declare no conflict of interest.

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