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# Superconductors without Symmetry Breaking

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**Abstract:** We review the main features of type-III superconductivity. This is a new type of superconductivity that exists in both 2 and 3 spatial dimensions. The main characteristics are emergent granularity and the superconducting gap being opened by a topological mechanism, with no Higgs field involved. Superconductivity is destroyed by the proliferation of vortices and not by the breaking of Cooper pairs, which survive above the critical temperature. The hallmark of this superconductivity mechanism, in 3 spatial dimensions (3D), is the Vogel–Fulcher–Taman scaling of the resistance with temperature.

**Keywords:** topological mass; topological phase transition; superconductivity

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## 1. Introduction

Traditional superconductors (see, e.g., [1]) are characterized by two length scales, the coherence length $\xi$ and the London penetration depth $\lambda$. The corresponding energies describe the gaps for two excitations: the Higgs mode, corresponding to amplitude oscillations of the order parameter, and the photon, which becomes massive by absorbing the phase of the order parameter via the Anderson–Higgs mechanism. The Higgs gap coincides with the superconducting gap (for a recent comprehensive review, see [2]). However, since its decay to single-particle excitations is suppressed by inverse powers of the temperature [3], and it is unaffected by Coulomb interactions [4], at low temperatures, it can be considered a well-defined, undamped mode.

Since the Higgs does not couple linearly to external electromagnetic probes, it is difficult to detect by standard techniques. In fact, it has been observed only recently by [5] using monocycle THz pulses able to excite it. THz lasers became available only in the last decade or so.

The two scales of superconductors determine the shape and behavior of Abrikosov vortices, the solitons of the Ginzburg–Landau theory (for a review, see [1]). These have two cores of sizes $\xi$ and $\lambda$, respectively. Type-I superconductors are realized when $\xi > \sqrt{2}\lambda$. In this case, the Higgs core exceeds the gauge core, and magnetic fields $H$ cannot penetrate the superconductor. When it exceeds critical value $H_c$, superconductivity is destroyed. Type-II superconductors are realized when $\xi < \sqrt{2}\lambda$. The Higgs core is the smallest one, and magnetic fields can penetrate in the form of a vortex lattice between two lower and upper critical fields $H_{c1}$ and $H_{c2}$, superconductivity being destroyed only above the upper critical field $H_{c2}$ [1].

Type-I superconductors [1,6] are characterized by a complete Meissner effect and are, thus, perfectly diamagnetic. The critical magnetic field $H_C$ separates the superconducting and non-superconducting states of the materials. Type-I superconductors have, in general, a critical temperature below 10 K and are referred to as low-temperature superconductors. They also have a low critical magnetic field, below $1T$. Type-I superconductors consist, in general, of elementary metals such as mercury, lead, or aluminum, which become superconductors below their critical temperature. They were discovered in mercury in 1911 by H.K. Onnes [7]. Type-II superconductors are characterized by two critical values of the magnetic field, $H_{C1}$ and $H_{C2}$. The Meissner effect is not complete between the two critical magnetic fields. Here, there is the formation of a mixed state with a magnetic field that penetrates in the form of Abrikosov vortices [8]. In type-II superconductors, the critical temperature at which superconductivity is destroyed is, in general, greater than 10 K. This type of superconductivity is, in general, realized in metal alloys or complex oxide ceramics. For critical temperatures below 30–40 K, type-I and type-II superconductivity are successfully explained by the BCS theory [9].

In [10,11], a new type of superconductivity, later called type-III superconductivity [12], was proposed. In type-III superconductors, there is only a gauge scale $\lambda$ and no Higgs core. At zero temperature, they are characterized by the vanishing of the lower critical field $H_{c1}$ [12]. This is a crucial difference with respect to type-I superconductors, in which magnetic flux never penetrates the sample, and type-II supeconductors, in which magnetic flux penetrates above $H_{c1} \neq 0$. At high temperatures, superconductivity is destroyed by the liberation of vortices, which are logarithmically confined in the superconducting state. The fact that superconductivity is destroyed by the proliferation of vortices means that the systems undergo a Berezinskii–Kosterlitz–Thouless (BKT) transition [13,14] in (2+1) dimensions as experimentally confirmed in [1,15,16]. The BKT transition is the paradigm of a topological phase transition. It is not due to spontaneous symmetry breaking; it cannot be described by a local order parameter, and there is no true long-range order. It describes the phase transition in the XY model: vortex-antivortex pairs, which are topological defects with logarithmic interaction, break at a critical temperature. The system has, thus, a phase transition between a quasi-long-range ordered state and a disordered one. Planar superconductivity arises, thus, via a topological phase transition and is not destroyed by the breaking of Cooper pairs like in type-I and type-II superconductors. Cooper pairs survive above the critical temperature in type-III superconductors. Moreover, as we showed in [17], this is the only possible mechanism in (2+1) dimensions. This superconductivity mechanism can be generalized to any dimension [10,11]. In (3+1) dimensions, the topological phase transition is the Vogel–Fulcher–Tamman (VFT) transition [18], which has been experimentally confirmed in NbTin sample of 86 nm thickness [12]. BKT scaling in (2+1) dimensions, VFT scaling in (3+1) dimensions, and the vanishing of $H_{c1}$ are, thus, the hallmarks of type-III superconductivity.

In type-III superconductors, the superconducting gap is not opened through the Higgs mechanism but through a topological mechanism [19]. As it has been shown in [10–12], the infrared-dominant term in the effective action that describes type-II superconductors is the topological Chern-Simons (CS) [19] term in 2D and the BF [20] term in 3D. In the effective action for the electromagnetic field, the presence of these topological terms generates a gauge-invariant mass for the photon, as we will show. The topological mass generation mechanism is the generalization of the topologically massive gauge theories proposed first in [19] to the coupling of two gauge fields (mixed CS term) in (2+1) dimensions and to the coupling of a gauge field and a tensor field in (3+1) dimensions (BF term).

In the resistive state, due to the presence of the topological term, the interactions between charges and vortices become screened Yukawa interactions. The superconducting state is realized when the charges condense. The physics of systems that realize type-III superconductivity is characterized by the competition of two quantum orders, charges and vortices, and is realized in materials that exhibit the superconductor-to-insulators transition (SIT) like Tin and NbTin superconducting films (for a review see: [21]). As shown in [22,23], when charges condense, the materials become type-III superconductors, while when vortices proliferate, the materials behave like superinsulators [22]. An intermediate bosonic topological insulating state is possible between these two phases [24] when both excitations are frozen. Topological insulators are a topological phase of matter characterized by a new type of order not due to spontaneous symmetry breaking, called symmetry-protected topological order [25,26]. They are materials that are insulating in bulk and have conducting edge states protected by symmetries. The longitudinal conductance is mediated by symmetry-protected $U(1) \rtimes Z_2^T$ edge modes, where $Z_2^T$ denotes time-reversal symmetry.

Their effective gauge theory description contains a topological term: a mixed CS term in (2+1) dimensions and a BF term in (3+1) dimensions [27,28].

Type-III superconductors are also characterized by an emergent granularity. In 2D, this emergent granularity can be explained in the following way. The electromagnetic coupling is $e^2_{\rm eff} = e^2/d$ and has canonical dimension [mass] in natural units $c = 1$, $\hbar = 1$, $\varepsilon_0 = 1$. In the Ginzburg–Landau model, the coupling with electric and magnetic fluctuations cannot be neglected for very thin materials ($d \to 0$ ), and the theory that describes these superconductors becomes scalar QED [17], which is plagued by infrared divergences [29], since the perturbative expansion parameter is $\propto \log\left(e^2 L/d\right)$, where $L$ is the system size, representing the necessary infrared cutoff. As a consequence, when decreasing $d$, a traditional superconductor will break up in islands of condensate of typical dimension $O(d)$ [17], detected in superconducting films in [30–32]. Granularity in superconducting films is now a generally accepted paradigm [33–35], and the films are modeled by Josephson junction arrays (JJAs) [36,37] with random couplings (see, e.g., [38]). In 3D, type-III superconductors are also characterized by emergent granularity. Bulk superconductors with emergent granularity have been experimentally detected in [39], and emergent granularity is a characteristic of high-$T_c$ superconductors, especially in the underdoped regime [40].

Here, in Section 1, we will review the role of vortices in type-III superconductors and show that the liberation of vortices leads, in 3D, to the VFT scaling of the resistance, which is the hallmark of this new superconductivity. Section 2 is dedicated to reviewing the effective gauge theory of these superconductors, which cannot be described by the Ginzuburg-Landau model of superconductivity.

## 2. Vortices in Type-III Superconductors

What is the fundamental difference between homogeneous and granular superconductors in 3D? A granular superconductor can be modeled as an emergent 3D JJA, where the individual grains, the superconducting islands, are local superconductors. As in JJA, global superconductivity is realized when global phase coherence between the order parameters on the islands is established. Type-III superconductors do not possess a global order parameter, and we refer to them as "Higgsless superconductors" [11], the only relevant scale being the typical distance $\ell$ between grains, which plays the role of the ultraviolet cutoff in the model. Vortex degrees of freedom appear when the phases on adjacent droplets realize a non-trivial $2\pi$ circulation, as shown in Figure 1. Contrary to Abrikosov vortices, these vortices have no dissipative core. Like for the vortices in the XY model, they are point excitations associated with a site of the dual array [16]. In principle, there could also be Abrikosov vortices within the individual islands, associated with local superconductivity, but if the islands are small enough, local Abrikosov vortices are not even stable [41,42].

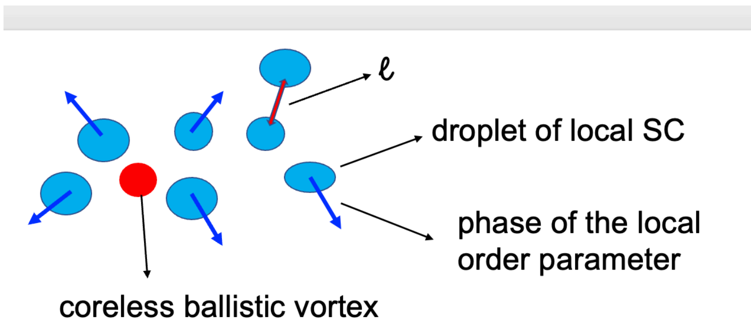


**Figure 1.** Formation of a vortex in a granular film.

The main characteristic of these coreless, point-like XY vortices is their extreme quantum mobility. This is a crucial aspect that is regularly neglected when focusing only on the classical statistical mechanics of the XY model. The point is that vortices are not Noether charges but are topological ones, and, as such, at very low temperatures, they can freely tunnel between different topological sectors. Let us consider, in particular, a quantum event in which all phases on a line ending in a particular island simultaneously flip by $2\pi$, with alternating signs [43]. No phase degrees of freedom, neither small fluctuations (like spin waves) nor vortices, are involved in such an event, except at the endpoint, where it corresponds to the tunneling of one vortex from one plaquette to the neighboring one. This is the 2D equivalent of the usual 1D quantum phase slips [44,45]. As a consequence of quantum phase slips, XY vortices can tunnel over the dual array exactly as Cooper pairs can tunnel between grains of the original array, as was originally pointed out in [27]. Again, we stress that no dissipative core is involved in this process, and the vortices that are tunneling are the point excitations of the XY model. However, one might think that the phase flip on one island might excite quasi-particles out of the local condensate there. However, this is not so; at low temperatures, dissipation due to quasi-particle excitation is negligible since the typical tunneling frequency is much smaller than the excitation gap for quasi-particles. The dissipation picture of [46] is also not valid since it is fully predicated on a vortex liquid in the Bose metal state, and the latest experiments have now confirmed the original picture [24,27] in which vortices are mostly frozen in this state, and not in a liquid phase [47]. Of course, however, vortex tunneling creates quantum dissipation for charges in the orthogonal direction and vice versa. But these effects are represented by effective Coulomb potentials (giving rise to related effective "electric fields") for both charges and vortices and can thus be incorporated dynamically into a gauge theory as Gauss law constraints, giving rise exactly to the frozen topological state [27,48].

Global superconductivity is lost by a proliferation of vortices and not by the breaking of Cooper pairs. In 2D, this leads to the BKT scaling, typical of the XY model, of the sheet resistance:

$$R \propto \mathrm{e}^{-\sqrt{\frac{b}{|T-T_{\mathrm{BKT}}|}}} \tag{1}$$

where $T_{\mathrm{BKT}}$ is the transition temperature and $b$ is a constant with dimension temperature [13,14]. In 3D, the scaling of the resistance is not the scaling of the 3D XY model. Vortices are 1D-extended string-like objects, and vortex deconfinement is characterized by the string becoming loose. The corresponding critical scaling of the resistance at the transition is given by the VFT scaling [18],

$$R \propto \mathrm{e}^{-\frac{b'}{|T-T_{\mathrm{BKT}}|}} \,. \tag{2}$$

This behavior has now been clearly detected in bulk samples of NbN and NbTiN [12] as shown in Figure 2.

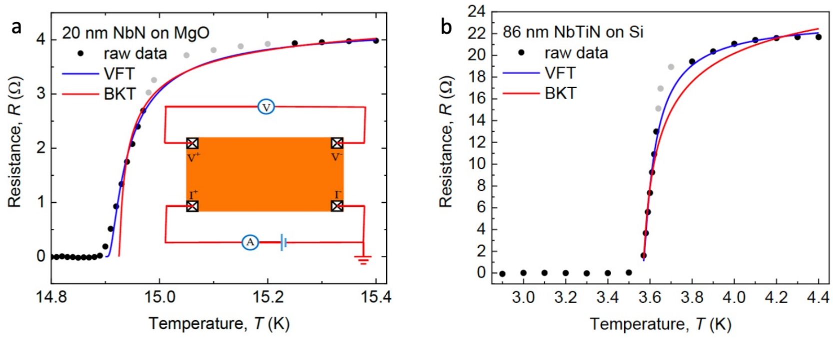


**Figure 2.** Scaling of the resistance. (**a**) Four-terminal dc resistance measurements resistance measurements of the 20-nm-thick NbN film. The red and blue curves correspond to the BKT fitting and VFT fitting of the experimental data, respectively. The gray points mark experimental data deviating from the fits. The subfigure illustrates a sketch of the four-terminal dc resistance measurements of NbN and NbTiN films. (**b**) Four-terminal DC resistance measurements of a 20-nm-thick NbN film. The red and blue curves correspond to the BKT fitting and VFT fitting of the experimental data, respectively. The three data points marked gray show a noticeable deviation from the fits. From Advance Science, accessed on 25 March 2023, https://doi.org/10.1002/advs.202206523.

While in 2D, as we already explained, all superconducting films are type-III, the real experimental challenge is in 3D. Granular superconductors are the ideal candidates for this new type of superconductivity. The thickness of the sample will play a crucial role in the experiments, as in the case of NbN and NbTiN. The hallmarks are the presence of Cooper pairs above the transition, the vanishing of $H_{C1}$, and the VFT scaling of the resistance when the system is thick enough to be 3D.

## 3. Effective Gauge Theory Description of Type-III Superconductors

In this section, we will review the effective gauge theory description of type-III superconductors derived in [12]. We will concentrate on the 3D case. Granular superconductors can be modeled by a cubic lattice, a 3D JJA, but, in what follows, for simplicity, we will keep a continuous notation and use natural units $c = 1$, $\hbar = 1$, $\varepsilon_0 = 1$.

The key point of this derivation is, as we said, the fact that the relevant degrees of freedom in the resistive state of type-III superconductors are charges and vortices, which are subject to mutual Aharonov–Bohm–Casher (ABC) statistical interactions [49–51]. Charge degrees of freedom are described by a conserved current $Q_\mu$, representing the world-line of charges (Greek letters standing for the space–time indices), with $\partial_\mu Q_\mu = 0$. The zero component $Q_0$ represents integer $2e$ charges located in the droplets. In 2D, vortices are point-like objects with integer charge in units $2\pi/2e$, described by a current $M_\mu$, while, in 3D, they are 1D extended objects, represented by an antisymmetric tensor $M_{\mu\nu}$ which describes the world-surface of vortices. We will be interested in vortices that form closed loops $\partial_\mu M_\mu = 0$ in 2D, and $\partial_\mu M_{\mu\nu} = \partial_\nu M_{\mu\nu} = 0$ in 3D (open vortices with magnetic monopoles at their ends [52,53] are not relevant for what follows).

The ABC statistical interactions are given by the Gaussian linking of closed loops in three Euclidean dimensions and closed loop and closed surfaces in four Euclidean dimensions [54]. As shown by Wilczek [49], they can be written in local form by introducing a vector gauge field $a_\mu$ coupling to $Q_\mu$ and either a pseudovector gauge field $b_\mu$ coupling to the vortex current, in 2D, or, in 3D, a pseudotensor antisymmetric gauge field $b_{\mu\nu}$ coupling to $M_{\mu\nu}$ [55]. These emergent gauge fields interact through a topological term: the CS term [19] in 2D and the BF term [21] in 3D:

$$
\begin{aligned}
S &= \int d^3x \, \frac{1}{2\pi} a_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha b_\nu - a_\mu Q^\mu - b_\mu M^\mu \, , \qquad 2D \, , \\
S &= \int d^4x \, \frac{1}{4\pi} a_\mu \epsilon^{\mu\alpha\nu\rho} \partial_\alpha b_{\nu\rho} - a_\mu Q^\mu - \frac{1}{2} b_{\mu\nu} M^{\mu\nu} \, , \; 3D \, .
\end{aligned} \tag{3}
$$

We would like to stress that although a CS term is present in the effective action of type-III superconductors in (2+1) dimensions, there is no relation between type-III superconductivity and anyon superconductivity [56]. The idea of anyon superconductivity, as it was originally proposed as a mechanism for high $T_C$ superconductors, was based on the possibility of fractional statistics in 2 space dimensions. The interaction between a statistical CS gauge field and fermions turns these into anyons by attaching magnetic flux to their charge density. Anyon superconductivity is thus confined to (2+1) dimensions and is P and T breaking since the CS term involving just one gauge field breaks these symmetries. In type-III superconductivity, the CS term in (2+1) dimensions and its generalization to (3+1) dimensions, the BF term couples a vector and a pseudo vector (pseudo tensor) gauge fields and describes the ABC interactions between charges and vortices as described in Equation (4) and the theory is P and T invariant.

We will now concentrate on the 3D case (four Euclidean dimensions) The actions in Equation (4) need a regularization [49]. This is accomplished by adding the next-order gauge-invariant terms for the two gauge fields, which are nothing else than the kinetic terms: the usual Maxwell term for $a_\mu$, quadratic in $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and the Kalb–Ramond term for the tensor gauge field $b_{\mu\nu}$, whose field strength is a three-tensor:

$$h_{\mu\nu\alpha} = \partial_\mu b_{\nu\alpha} + \partial_\nu b_{\alpha\mu} + \partial_\alpha b_{\mu\nu} \ . \tag{4}$$

The regularized Euclidean effective action becomes:

$$S = \int d^4x \frac{1}{4f^2} f_{\mu\nu} f_{\mu\nu} + \frac{i}{4\pi} a_\mu \epsilon_{\mu\nu\alpha\beta} \partial_\nu b_{\alpha\beta} + \frac{1}{12\Lambda^2} h_{\mu\nu\alpha} h_{\mu\nu\alpha} + i a_\mu Q_\mu + i \frac{1}{2} b_{\mu\nu} M_{\mu\nu} \ , \tag{5}$$

and is invariant under the (generalized) gauge transformations:

$$\begin{aligned} a_\mu &\to a_\mu + d_\mu \xi \ , \\ b_{\mu\nu} &\to b_{\mu\nu} + d_\mu \lambda_\nu - d_\nu \lambda_\mu \ . \end{aligned} \tag{6}$$

In 2D, the kinetic terms for both emergent gauge fields are infrared-irrelevant since both coupling constants have the dimensions of [mass]. They must, however, be added due to an anomaly for the CS term [57], and, for the description of physical systems, the pure topological limit (topological mass $\to \infty$) must be taken starting from the action with kinetic terms. In 3D, instead, in Equation (5), the Maxwell term for the gauge field $a_\mu$ is not simply a regularization: the coupling constant $f$ is dimensionless, so the Maxwell term is marginal and must be added to the action, it co-determines the infrared behavior of the theory with the BF term. Since $\Lambda$ has the dimensions of a mass, the kinetic term for the Kalb–Ramond tensor field is infrared-irrelevant, and it serves only as an ultraviolet regulator for Gaussian integrals describing the massive modes on a scale:

$$\frac{1}{m} \ll \frac{\pi}{f\Lambda} \ .$$

It can be removed taking the limit $\Lambda \to \infty$ after the integration. The BF term has the effect of generating a topological mass for the emergent fields [20]:

$$m = \frac{f\Lambda}{2\pi} \ , \tag{7}$$

so that, in the resistive state, the interactions between charges and vortices become Yukawa interactions. As shown in [12,17], when charges condense in the superconducting phase, the vortex interactions become long-range Coulomb interactions, and vortices are logarithmically confined [27]. The liberation of vortices destroys superconductivity with a VFT scaling for the resistance [12].

In conventional superconductivity, due to spontaneous symmetry breaking, the photon acquires a mass via the Anderson–Higgs mechanism, while in type-III superconductors, it acquires a mass due to the BF mechanism. To see this, we will use the Julia and Toulouse mechanism [58]. We start by noticing that $Q_\mu$ and $M_{\mu\nu}$ in Equation (4) have the effect of making the BF term periodic (they are integers on the lattice) and can be seen as topological defects arising from the compactness of the emergent gauge symmetries. Following [59], they can be interpreted as singularities in the field strength. In a nutshell, the Julia–Toulouse mechanism tells us that the condensation of topological defects, in our case Cooper pairs and vortices, $Q_\mu$ and $M_{\mu\nu}$, in solid-state media generates new hydrodynamical modes for the low-energy effective theory. These hydrodynamical modes represent long wavelength fluctuations over the continuous distribution of topological defects. In solid-state systems, it is, in general, very difficult to derive the action for the phase with condensation of topological defects, one of the causes being the lack of gauge invariance. In [60], it has been shown, however, that in the case of compact antisymmetric field theories, as in our case, the Julia–Toulouse prescription is sufficient to fully determine the low-energy action due to the condensation of topological defects. To see this we couple Equation (5) to the real electromagnetic field $A_\mu$

$$S \longrightarrow S+ = ie \int d^4x A_\mu j_\mu \,, \tag{8}$$

with $j_\mu = \frac{1}{2\pi}\epsilon^{\mu\alpha\nu\rho}\partial_\alpha b_{\nu\rho}$. Integrating over the emergent gauge fields and taking the limit $\Lambda \to \infty$, we obtain:

$$S = \int d^4x \, \frac{1}{4} F_{\mu\nu}F_{\mu\nu} + ieI_{\mu\alpha}\epsilon^{\mu\alpha\nu\rho}F_{\nu\rho} \,, \tag{9}$$

where $Q_\mu = \epsilon^{\mu\alpha\nu\rho}\partial_\alpha I_{\nu\rho}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Following the Julia–Toulouse prescription, due to the Cooper pair condensation in the superconducting phase, the proliferating point charge singularities carrying the delta-like current densities $I_{\nu\rho}$ are promoted to a continuous two-form antisymmetric field $B_{\mu\nu}$ defined on the whole space, without singularities: $I_{\nu\rho} \to B_{\mu\nu}$. Following [60], the action for this new mode is obtained by adding the kinetic term for $B_{\mu\nu}$, the square of the three-form field strength: $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$

$$S_{\text{eff}} = \int d^4x \, \frac{1}{4} F_{\mu\nu}F_{\mu\nu} + i\pi B_{\mu\alpha}\epsilon^{\mu\alpha\nu\rho}F_{\nu\rho} + \frac{1}{12\tilde{\Lambda}^2} H_{\mu\nu\rho}H_{\mu\nu\rho} \,, \tag{10}$$

where $\tilde{\Lambda}$ is the new emergent mass scale describing the average density of condensed charges. Like Equation (5), Equation (10) contains a BF term which, in this case, generates a topological mass for the photon:

$$m = 4\pi\tilde{\Lambda} \,.$$

Due to gauge invariance, the antisymmetric Kalb–Ramond field embodies a single scalar degree of freedom. This scalar degree of freedom is "eaten" by the original photon, which becomes massive. No symmetry-breaking is involved.

The essence of type-III superconductivity is that one single massless superfluid mode can always be described in terms of a generalized gauge field in any dimension. When vortices are structureless point particles, as in granular media, they couple directly to this gauge field, and, as always in gauge theories, a Coulomb interaction is generated by the Gauss law constraint. Furthermore, the vortex coupling automatically generates a photon gap as a gauge-invariant topological mass. A single generalized gauge theory accommodates all superconducting phenomena and replaces the Ginzburg–Landau theory, which does not need a scalar field and the associated coherence length.

Type-III superconductors are granular materials with Cooper pairs that are localized in granules [12]. We believe, thus, that they can be relevant for high $T_C$ superconductivity, where emergent granularity appears [40,61]. The pairing mechanism, in this case, cannot be the BCS one since Cooper pairs are localized, and a possible mechanism to account for pairing in type-III superconductors is described in [62].

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