The Effective Potential of Scalar Pseudo-Quantum Electrodynamics in (2 + 1)D

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Abstract: The description of the electron–electron interactions in two-dimensional materials has a dimensional mismatch, where electrons live in (2 + 1)D while photons propagate in (3 + 1)D. In order to define an action in (2 + 1)D, one may perform a dimensional reduction of quantum electrodynamics in (3 + 1)D (QED4) into pseudo-quantum electrodynamics (PQED). The main difference between this model and QED4 is the presence of a pseudo-differential operator in the Maxwell term. However, besides the Coulomb repulsion, electrons in a material are subjected to several microscopic interactions, which are inherent in a many-body system. These are expected to reduce the range of the Coulomb potential, leading to a short-range interaction. Here, we consider the coupling to a scalar field in PQED for explaining such a mechanism, which resembles the spontaneous symmetry breaking (SSB) in Abelian gauge theories. In order to do so, we consider two cases: (i) by coupling the quantum electrodynamics to a Higgs field in (3 + 1)D and, thereafter, performing the dimensional reduction; and (ii) by coupling a Higgs field to the gauge field in PQED and, subsequently, calculating its effective potential. In case (i), we obtain a model describing electrons interacting through the Yukawa potential and, in case (ii), we show that SSB does not occur at one-loop approximation. The relevance of the model for describing electronic interactions in two-dimensional materials is also addressed.

Keywords: pseudo-quantum electrodynamics; Coleman–Weinberg potential; spontaneous symmetry breaking

1. Introduction

The experimental realization of two-dimensional materials [1–4] in condensed matter physics has attracted the interest of the community in high energy physics, due to the emergence of Dirac cones and the possibility of observing some interacting effects [5], such as the Fermi velocity renormalization [6–9] and mass renormalization [10–12]. When considering the effects of electronic interactions in these materials, it is useful to consider a dimensional reduction of QED4, namely, PQED, which provides the physical Coulomb potential among static particles in the spatial plane [13]. Within this approach, several results, including strong and weak interactions, have been obtained and compared to experimental data, see Refs. [10,14] and the references therein. Furthermore, the model has also been coined as reduced quantum electrodynamics in Refs. [15–17]. Although the Coulomb potential has been successful in describing these effects, we also expect that this electron–electron interaction becomes screened due to the microscopic interactions within the two-dimensional material [18]. These interactions should include impurities and phonons, which would make the model even more complicated and, for the best of our knowledge, a complete solution is yet to be known [19]. On the other hand, because the long-range Coulomb interaction is related to a massless photon; hence, the simplest method to generate a short-range potential is to obtain a massive bosonic field, which is...
The Higgs mechanism for QED in (2 + 1)D (QED3) would provide the simplest solution for obtaining a mass term for the gauge field and, therefore, a short-range interaction. However, as discussed before, this mechanism must be considered in PQED rather than in QED3. In PQED, the power of the gauge-field propagator is \( \propto (p^2)^{-1/2} \), while in QED, it is \( p^{-2} \), where \( p \) is the external momentum. Hence, in order to obtain a Yukawa potential in PQED, one must obtain a scalar propagator given by \( 1 / \sqrt{p^2 + m^2} \). This, on the other hand, is obtained from the dimensional reduction of QED4 in the presence of a mass term, as it has been shown in Ref. [21].

Here, we consider the coupling of the gauge field in PQED with a scalar field. This is meant to effectively describe the effects of the many-body system and, essentially, to generate a mass-like term for PQED and to reduce the range of the Coulomb interaction.

Firstly, we review the dimensional reduction, which generates reduced models in both (2 + 1)D and (1 + 1)D. In order to do so, we apply the dimensional reduction in the two-point Green functions of the classical fields, through the equation of motion. Thereafter, we calculate the effective potential for a reduced scalar model in (2 + 1)D and discuss SSB. This quantity shows a behavior similar to the effective potential in a self-interacting scalar theory in (3 + 1)D. However, the reduced model does not allow for SSB due to its asymptotic limit, leading to a stable ground state in a symmetric phase. Thereafter, we consider effective models for describing the electronic interactions for Dirac-like quasiparticles in (2 + 1)D in terms of a gauge field interacting with a scalar field. The realization of SSB, however, may occur either in (3 + 1)D or (2 + 1)D, providing two different cases for investigation. In case (i), we conclude that the Yukawa interaction is obtained whenever SSB occurs in (3 + 1)D, and derive an effective action for describing such interaction in (2 + 1)D. In case (ii), which we call the Abelian Higgs PQED (HPQED) in comparison to the scalar QED3 [22], we show that the quantum correction does not provide an SSB; hence, the system remains in its symmetric phase. These results are obtained in the one-loop approximation, using the so-called background field method for calculating the effective potential [20].

This paper is organized as follows. In Section 2, we review the concept of reduced models, using the classical equation of motion. In Section 3, we calculate the effective potential for a reduced version of the Klein–Gordon theory in (2 + 1)D. In Sections 4 and 5, we consider the effects of considering both the SSB and the dimensional reduction in QED4 plus a Higgs field. In Appendices A–C, we show some details of the calculations.

2. The Reduced Models

In this section, we derive a reduced model that describes the Yukawa interaction in (2 + 1) dimensions at the classical level. In order to do so, we start with the Yukawa action in (3 + 1) dimensions, whose Euclidean action is given by

\[
L_{4D} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{m^2 \phi^2}{2} + g \phi \bar{\psi} \psi + L_M[\psi],
\]

where \( g \) is a dimensionless coupling constant, \( \phi \) is a real and massive Klein–Gordon field, \( \psi \) is the Dirac field, and \( L_M[\psi] \) is the matter action. This model will be relevant, because it will work as a simple example for calculating the dimensional reduction in theories with SSB.

The equation of motion for \( \phi \) is promptly obtained from Equation (1), and reads

\[
(-\Box + m^2) \phi(x) = J(x),
\]
where \( \Box \) is the d’Alembertian operator and \( f(x) \equiv -g \bar{\psi} \psi(x) \) works as an external source for the scalar field. The solutions of Equation (2) are obtained by inverting the differential operator, i.e.,

\[
\phi(x) = \int d^4y \, G_{4D}(x-y) f(y),
\]

where

\[
G_{4D}(x-y) = \int \frac{d^4k_E}{(2\pi)^4} \, \frac{e^{ik(x-y)}}{k_E^2 + m^2} \tag{4}
\]
is the Fourier transform of the scalar-field propagator in momentum space. After using these conditions in Equation (3), we find

\[
\phi = \int d^4y \, G_{4D}(x-y) f(y),
\]

where the inverse of \( m \) is the interaction length of the model and \( r \) is the interaction length of the model and \( r = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 \), as expected.

From \((3 + 1)D\) to \((2 + 1)D\)

For calculating a reduced model, we consider that the dynamics of the matter field is constrained to the \((2 + 1)\)-dimensional spacetime. This is obtained by assuming that \( \phi(x) \to \phi_{3D}(x) \equiv \phi(x_0, x_1, x_2) = \phi(x_0, x_1, x_2, x_3 = 0) \). Note that by setting \( x_3 = 0 \), we are considering a prescription to ensure that the scalar field is confined to the \( x_1 - x_2 \) plane. Furthermore, we also adopt the notation, where the \( x \) in \( \phi(x) \) always represent the coordinates in the spacetime where the field is defined; hence, for \( \phi(x) \), we have \( x = (x_0, x_1, x_2) \). On the other hand, for \( \phi_{3D}(x) \), it follows that \( x = (x_0, x_1, x_2, x_3) \). The same holds for loop integrals and propagators in momentum space. After using these conditions in Equation (3), we find

\[
\phi_{3D}(x) = \int d^3y \, G_{3D}(x-y) f_{3D}(y),
\]

where

\[
G_{3D}(x-y) = \int \frac{d^3k_E}{(2\pi)^3} \, \frac{e^{ik(x-y)}}{2\sqrt{k_E^2 + m^2}} \tag{7}
\]
is the Fourier transform of the scalar-field propagator in \((2 + 1)\) dimensions. This also provides the Yukawa potential in Equation (5). Indeed, after using \( k_E^2 = k_0^2 + k_1^2 + k_2^2 \) with \( k_0 \to 0 \) in Equation (7), we find

\[
V(r) = \int \frac{d^2k_E}{(2\pi)^2} \, \frac{e^{ikr}}{2\sqrt{k_E^2 + m^2}} = \frac{e^{-mr}}{4\pi r}, \quad r = (x_1 - y_1)^2 + (x_2 - y_2)^2 \tag{8}
\]

where \( r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 \).

The main idea of a reduced model is to obtain a theory that yields the reduced propagator in Equation (7). This may be obtained from

\[
\mathcal{L}_{3D} = \frac{\partial \mu \phi K_E(\Box) \partial \mu \phi}{2} + g \bar{\psi} \psi + \mathcal{L}_M[\psi] \tag{9}
\]
This model describes the dynamical effects of the Yukawa interaction in the plane, and some of its quantum effects have been discussed in Ref. [21] when coupled to the Dirac field. Obviously, the Coulomb interaction is promptly obtained by using $m \to 0$ within these calculations. The case of dimensional reduction to (1 + 1)D is discussed in Appendix A.

The derivation of the reduced models is more convenient to be performed in the Euclidean spacetime [13]. However, for calculating the Coleman–Weinberg potential, we rather consider the action in the Minkowski space for the sake of comparison with well-known results in the literature.

3. The One-Loop Effective Potential of a Reduced-Scalar Model in (2 + 1)D

In this section, we discuss the Coleman–Weinberg effective potential of a general scalar action in a $D$-dimensional spacetime with a pseudo-differential operator. When considering an interaction term $U[\phi]$, our toy model action reads

$$L_D = \frac{\partial^\mu \phi K[\square] \partial_\mu \phi}{2} + U[\phi],$$

(11)

where the kernel $K[\square]$ is an arbitrary pseudo-differential operator in the Minkowski spacetime. The calculations using Equation (11) shall be useful for our purposes in the next sections, where we discuss the SSB for two-dimensional materials in light of the PQED formalism.

The partition function of the model is defined as usual and, therefore, reads

$$Z = \int D\phi \exp \left\{ i \int d^Dx L_D[\phi] \right\}.$$

(12)

The main idea of the one-loop expansion is to expand the action in Equation (12), using Equation (11), up to an order of $\bar{\hbar}$; see Appendix B for more details. Thereafter, we replace the quantum field by its vacuum expectation value (VEV), i.e., $\phi(x) \to \langle \phi(x) \rangle = \rho$, which is acceptable for describing the ground state of the model through its effective potential $V_{\text{eff}}(\rho)$. Therefore, we have

$$Z = e^{-iV_{\text{eff}}(\rho)\Omega},$$

(13)

where $\Omega = \int d^3x$ is a constant factor, namely, the spacetime volume. Having these steps in mind, after some algebra, one finds

$$V_{\text{eff}}(\rho) = -U(\rho) - \frac{i}{2} \int \frac{d^Dk}{(2\pi)^D} \ln[k^2 K(k^2) + U''(\rho)],$$

(14)

where $U''(\rho)$ means the second derivative of $U[\rho]$. Note that the first term in the rhs of Equation (14) is the classical potential, and the second term is the quantum correction (in order of $\hbar$). Equation (14) is how far we may go for an arbitrary kernel $K$.

3.1. The Case $D = 4$ and $K = 1$

This is the most standard case, and the effective potential reads [20]

$$V_{\text{eff}}^{KG}(\rho, \mu) = -U(\rho) - \frac{(U'')^2}{64\pi^2} \ln \left[ \frac{4\pi\mu^2}{-U} \right],$$

(15)

which is derived through the dimensional regularization scheme. Note that $\mu$ is a renormalization point, and that $V_{\text{eff}}^{KG}(\rho)$ is expected to be unchanged by a scale transformation in $\mu$. For a classical potential $U[\rho] = \lambda \rho^4 / 4!$ with $\lambda < 0$, we conclude that the VEV of the
field does not vanish, and it is double degenerated. Hence, the discrete symmetry $\phi \to -\phi$ is broken by the ground state [20]. Our main goal is to understand what the effects of considering a pseudo-differential operator in a scalar action are, regarding the mechanism of SSB. Thereafter, we apply these results to PQED with a Higgs field.

3.2. The Case $D = 3$ and $\mathcal{K} = 2/\sqrt{-k^2}$

This is the case where $\mathcal{K}[\Box] = 2(\Box)^{-1/2}$ in Equation (11). This corresponds to a reduced Klein–Gordon model. Here, we calculate its effective potential, using both the dimensional regularization and the cutoff regularization. As shall be clear later, both of these methods provide the same logarithmic term for the effective potential.

3.2.1. The Dimensional Regularization Scheme

Within this regularization scheme, Equation (14) is written as

$$V_{\text{eff}}(\rho, \mu) = -U(\rho) - \frac{i \mu^{3-D}}{2} \int \frac{d^Dk}{(2\pi)^D} \ln[k^2\mathcal{K}[k^2] + U''(\rho)]$$

$$\equiv -U(\rho) + V_1(\rho, \mu),$$  \hspace{1cm} (16)

where $\mu$ is an energy scale included in order to preserve the units of the integral and $V_1$ is the quantum correction. In Appendix C, we calculate this correction and obtain

$$V_{\text{eff}}(\rho, \mu) = -U(\rho) + \left(\frac{1/2}{\Gamma[D/2]}\right)^{3-D} \Gamma[D/2]$$

$$x \left(\frac{\mu}{-U''}\right)^{(3-D)} \frac{1}{\Gamma[1-D]}.$$  \hspace{1cm} (17)

Expanding Equation (17) around $D \to 3$, we have

$$V_{\text{eff}}(\rho, \mu) = -U(\rho) - \frac{(U'')^3}{48\pi^2}$$

$$x \left[\frac{1}{2(3-D)} + \frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2} \ln\left(\frac{\mu}{-U''}\right)\right],$$  \hspace{1cm} (18)

where $\gamma \approx 0.58$ is the Euler’s constant. Finally, we use the minimal subtraction scheme (where the pole of the Gamma function is neglected as well as some extra constants) in Equation (18), and find the renormalized effective potential, namely,

$$V_{\text{eff}}(\rho, \mu) = -U(\rho) - \frac{(U'')^3}{96\pi^2} \ln\left(\frac{\mu}{-U''}\right).$$  \hspace{1cm} (19)

Note that Equation (19) resembles the effective potential of the scalar model in $(3 + 1)D$, given by Equation (15). In particular, it has an energy-scale term which depends on $\mu$. This is a surprising conclusion, because in a usual scalar theory in $(2 + 1)D$, such a term does not exist. Obviously, the pseudo-differential operator plays a central role for this result. Next, let us consider the cutoff regularization scheme, which shall be useful when we include a finite lattice regulator, as it occurs for effective models in condensed matter physics [20].

3.2.2. The Cutoff Regularization Scheme

In this case, we also perform the Wick rotation to the Euclidean spacetime, i.e., $d^Dk \to i d^Dk_E$ and $k^2 \to -k_E^2$. Thereafter, we include an ultraviolet cutoff $\Lambda$ in the in-
integral, where $0 \leq k_E \leq \Lambda$ in Equation (14). Furthermore, for solving this integral, we use that

$$
\int k_E^2 dk_E \ln[2k_E - U''] = -\frac{1}{12} \left( k_E^2 U'' + k_E U'^2 \right) - \frac{U'^3}{24} \ln[2k_E - U''] + \frac{k_E^3}{3} \ln[2k_E - U''] - \frac{k_E^3}{9}.
$$

Next, using Equation (20) in Equation (14) and expanding this for $\Lambda \gg U''$, we find

$$
V_{\text{eff}}(\rho, \Lambda) = -U(\rho) - \frac{U'^3}{288\pi^2} - \frac{U'^2}{32\pi^2} - \frac{U''}{16\pi^2} - \frac{(U'')^3}{96\pi^2} \ln \left( \frac{2\Lambda}{-U''} \right).
$$

Note that the coefficient of the logarithmic term, i.e., $-(U'')^3/96\pi^2$, is the same as in Equation (19). This is a relevant feature for calculating the beta functions of the theory in Equation (11), when considering the continuum limit, i.e., $\Lambda \to \infty$ [20]. Here, because the dimensional reduction is related to two-dimensional materials, we consider the case when $\Lambda \propto 1/a$ is a large but finite constant, where $a \approx 10^{-10}$ m is the lattice parameter for a typical crystal. For example, in graphene and other two-dimensional materials, we have $\Lambda \approx 1$ eV [1].

3.2.3. The Vacuum Stability of the Reduced-Scalar Model with a $\phi^4$-Self-Interaction

For a more straightforward application, let us consider a classical potential given by

$$
U[\phi] = \frac{M^2 \phi^2}{2} + \frac{\lambda \phi^4}{4!},
$$

where $M$ is the bare mass term and $\lambda$ is the bare coupling constant. The analysis of the vacuum stability of Equation (21) is similar to the stability given by the effective potential in Equation (15), where we consider $\lambda < 0$ in order to have a real-valued effective potential.

In Figure 1, we plot Equation (21), using Equation (22). From this, we may conclude that the ground state of the system remains symmetric at $\rho = 0$; hence, no SSB occurs.

![Figure 1](image-url)

Figure 1. The effective potential of the reduced-scalar model. We plot Equation (21) with $M = 0$, $\lambda = -0.5$, and $\Lambda = 1.0$. Note that the local minimum in $\rho = 0$ is the only acceptable ground state, whether we assume that $\rho$ is always much less than $\Lambda$.

Although it is interesting that the reduced model has a similar structure for the effective potential, in comparison to the scalar field in $(3 + 1)$D, a few comments have to be addressed.

(i) Our example holds for the $Z_2$ symmetry $\phi \to -\phi$; however, it would be interesting to
discuss the possibility of breaking a continuous symmetry, which are connected to quantum states of matter, for example, the superconductivity [24–26]. This only can be realized by considering a complex scalar field; (ii) the effective potential of the scalar field in (2 + 1)D does not have the logarithmic term [22], which is a striking difference in comparison to both the higher dimensional model (see Equation (15)) and our reduced-scalar model (see Equation (21)); and (iii) because we are assuming a finite Λ, our effective potential is also finite, and we do not need to deal with divergences.

Next, we shall consider a scalar version of PQED with a complex scalar field. This also provides a more physical situation, allowing us to make predictions regarding the interactions between quasiparticles in two-dimensional materials.

4. From SSB to the Dimensional Reduction

The static approximation of PQED describes a system of electrons interacting through the Coulomb potential. However, due to the screening effects within the two-dimensional material, it is expected that the Coulomb potential becomes a short-range interaction [27]. This mechanism, however, must be considered either before or after the dimensional reduction performed in Sec. II. This also may be performed through the generating functional of the order solution, given by φ of the scalar field, i.e, electric charge. In the two-dimensional material [10], and ψ is its field strength tensor. Furthermore, α is the gauge fixing parameter. Because of charge conservation, this parameter is not relevant for our purposes; hence, we simply neglect this term. φ is a massive complex scalar field coined Higgs field, and U[φ] represents its self-interacting potential. ψ is the Dirac field, which is meant to describe the quasi-particles in the two-dimensional material [10], and m is its bare mass while e is the dimensionless electric charge.

The standard approach for SSB in Equation (23) is to consider a polar representation of the scalar field, i.e, φ(x) = ξ(x)e|θ(x)|, Furthermore, we assume that U[φ] = μ2ξ2/2 + λξ4/4!, where (μ, λ) are known constants. After some algebra, we conclude that the scalar field θ(x) may be removed from the model by a gauge transformation, namely Aμ → Aμ − ∂μθ/g [20]. Hence, we find

\[ \mathcal{L}_{HQED4} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\kappa} (\partial_\mu A_\mu)^2 + |D_\mu \phi|^2 + U(|\phi|) + \bar{\psi}(i\sla{\partial} - m)\psi + e\bar{\psi}\gamma_\mu\psi A^\mu, \] (23)

where D_μ = ∂_μ + igA_μ, with μ = 0, 1, 2, 3, is the covariant derivative and g is the dimensionless charge of the complex scalar field φ. A_μ is the gauge field and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is its field strength tensor. Furthermore, \( \alpha \) is the gauge fixing parameter. Because of charge conservation, this parameter is not relevant for our purposes; hence, we simply neglect this term. φ is a massive complex scalar field coined Higgs field, and U[φ] represents its self-interacting potential. ψ is the Dirac field, which is meant to describe the quasi-particles in the two-dimensional material [10], and m is its bare mass while e is the dimensionless electric charge.

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\[ \mathcal{L}_{HQED4} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \xi \partial^\mu \xi + U[\xi] + g^2 \xi^2 \bar{A}_\mu A^\mu + \bar{\psi}(i\sla{\partial} - m)\psi + e\bar{\psi}\gamma_\mu\psi A^\mu. \] (24)

Finally, we consider that the Higgs field is in the broken phase, and take its lowest-order solution, given by \( \xi(x) \rightarrow \rho_0 \). Therefore, after neglecting the constant U(ρ0), we have

\[ \mathcal{L}_{HQED4} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g^2 \rho_0^2 \bar{A}_\mu A^\mu + \bar{\psi}(i\sla{\partial} - m)\psi + e\bar{\psi}\gamma_\mu\psi A^\mu. \] (25)

Next, we use Equation (25) for calculating the dimensional reduction. Note that whenever \( \rho_0 \neq 0 \), we may conclude that the Gauge field has acquired a mass-like term. The dimensional reduction, on the other hand, will follow exactly the same steps as we performed in Sec. II. This also may be performed through the generating functional of the
current–current correlation functions, as it has been performed in Ref. [13] for PQED (see Appendix D for more details). Hence, it follows that the reduced version of Equation (25) is

\[
\mathcal{L}_{HRQED} = \frac{1}{4} F_{\mu\nu} K_E [\Box] \tilde{F}_{\mu\nu} + \psi (i \not{\partial} - m) \psi \\
+ \frac{1}{2\alpha} A^\mu \partial_\mu K_E [\Box] \partial_\nu A^\nu + e \psi \gamma_\mu \phi A^\mu, \tag{26}
\]

which is fully defined in \((2 + 1)D\)

\[
K_E [\Box] = \frac{2 \sqrt{-\Box + 2 \delta^2 \rho_0^2}}{-\Box}. \tag{27}
\]

Despite our notation, the gauge field in Equation (26) is not the same as in Equation (23). This model describes the Yukawa interaction between Dirac-like electrons that are confined to move, for example, in a two-dimensional material [21]. Furthermore, because of the SSB, the interaction become short-ranged and given by

\[
V(r) = e^{-\sqrt{2}g\rho_0/r} \frac{4\pi r}{4\pi r}, \tag{28}
\]

which is obtained similarly to the calculations in Sec. II. Note that, because of the dimensional reduction, the gauge field in \((2 + 1)D\) also has the same exponential decay as the gauge field in \((3 + 1)D\). This is a signature of the Meissner effect, where an external magnetic field is repelled by a superconductor [19]. The mechanism we use in this section does not allow us to discuss further about SSB. Indeed, it occurs in the higher dimensional model, and we simply performed the dimensional reduction. Nevertheless, we could also consider that the Higgs mechanism is driven in \((2 + 1)D\) instead of in \((3 + 1)D\). In this new case, we should also ask whether the effective potential is minimized in the broken phase or not.

5. From Dimensional Reduction to SSB

In this section, we consider a different path for discussing SSB in PQED. Here, we start with PQED, and then include a Higgs field, in the Minkowski spacetime, for calculating the SSB. In this case, we may use some results in Sec. III regarding the loop integrals. Therefore, we have

\[
\mathcal{L}_{HPQED} = -\frac{1}{2} F_{\mu\nu} (\Box)^{-1/2} F_{\mu\nu} - |D_\mu \phi|^2 - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 + U[|\phi|], \tag{29}
\]

where the subscript HPQED stands for Higgs-PQED, \(D_\mu \rightarrow \partial_\mu + ieA_\mu\), with \(\mu = 0, 1, 2\), is the covariant derivative, and \(e\) is the dimensionless electric charge of the complex scalar field \(\phi\). \(A_\mu\) is the gauge field of PQED, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is its field strength tensor, and \(\alpha\) is the gauge-fixing parameter. \(\phi\) is a massive complex scalar field, coined Higgs field, and \(U[|\phi|]\) represents its symmetry-breaking potential. We neglect the Dirac term, because it does not play any role regarding the SSB and may be included later.

The model in Equation (29) has been investigated in the perturbation theory in Ref. [27], within the symmetric phase. In Ref. [28], the coupling of PQED with a nonrelativistic scalar field has been applied for describing thin superconducting films. Here, we shall discuss a relativistic version of this coupling and discuss its broken phase in the light of the effective potential. For the sake of simplicity, we assume the Feynman gauge, where the longitudinal part of the gauge-field propagator vanishes. As matter of fact, the effective potential in scalar QED4 is also gauge-dependent. Nevertheless, the physical predictions, such as the ratio between the masses of both gauge and scalar fields, are expected to independent on \(\alpha\) [20]. Equation (29) may be expanded as

\[
\mathcal{L}_{HPQED} = A^\mu (\Box)^{1/2} A_\mu - ieA_\mu \phi^* (\partial^\mu \phi) - (\partial^\mu \phi^*) \phi \\
- \phi^* \Box \phi + e^2 A_\mu A^\mu |\phi|^2 + U[|\phi|]. \tag{30}
\]
Next, we consider the one-loop approximation for calculating the effective potential. Similar to the calculation in Section 3, we assume that the scalar field may have a nonzero VEV, such that \( \langle \phi \rangle = \rho \) and \( \langle A_\mu \rangle = 0 \). Using these assumptions in Equation (30), and after neglecting some irrelevant constants, we find

\[
V_{\text{eff}}^{\text{HPQED}}(\rho) = -U(\rho) - i \int \frac{d^3k}{(2\pi)^3} \ln[|k^2 + U''|] - \frac{3i}{2} \int \frac{d^3k}{(2\pi)^3} \ln[(-k^2)^{1/2} + \epsilon^2 \rho^2]. \tag{31}
\]

Equation (31) is the main result of this section, and should be compared to the standard scalar-QED in (2 + 1)D [22]. Therefore, let us briefly discuss the result for this case. Firstly, one must replace \((-k^2)^{1/2} \rightarrow (-k)^2\) for describing the Maxwell propagator. It turns out that the ultraviolet divergence is reduced and, because of this, the author in Ref. [22] has concluded that the effective potential in scalar QED3 is independent on both \(\mu\) and \(\Lambda\). For scalar PQED, nevertheless, the third term in rhs of Equation (31) provides a logarithmic term, such as in the previous case of the reduced-scalar model. This term is clearly related to the pseudo-differential operator.

5.1. The Higgs Term

The second term in the rhs of Equation (31) is the Higgs-field contribution to the effective potential. Within the cutoff regularization, after going to the Euclidean space, it yields

\[
\frac{1}{2\pi^2} \int_0^\Lambda k_E^2 dk_E \ln \left[ k_E^2 - U'' \right]. \tag{32}
\]

After solving the integral over \(k_E\), keeping only the \(\rho\)-dependent terms, and expanding for \(\Lambda \gg U''\), we find

\[
- \frac{(U'' \Lambda)}{2\pi^2} - \frac{(-U'')^{3/2}}{6\pi}. \tag{33}
\]

This Higgs contribution is the same for both scalar QED3 and PQED, because it only depends on the scalar-field propagator.

5.2. The PQED Term

The third term in the rhs of Equation (31) is the PQED contribution. Using the same assumptions as before, it yields

\[
\frac{3}{4\pi^2} \int_0^\Lambda k_E^2 dk_E \ln \left[ k_E + \epsilon^2 \rho^2 \right]. \tag{34}
\]

Solving the integral over \(k_E\) and expanding for \(\Lambda \gg \epsilon^2 \rho^2\) (see Equation (20)), we have

\[
\frac{(\epsilon^2 \rho^2)^3}{12\pi^2} - \frac{3\Lambda^2 \rho^4}{8\pi^2} + \frac{3\Lambda^2 \epsilon^2 \rho^2}{8\pi^2} + \frac{3(\epsilon^2 \rho^2)^3}{24\pi^2} \ln \left( \frac{\Lambda^2}{(\epsilon^2 \rho^2)^2} \right). \tag{35}
\]

Note that due to the pseudo-differential operator, we have obtained a logarithmic term that resembles the standard effective potential in the scalar QED in (3 + 1)D [20].

5.3. The Effective Potential

Finally, the effective potential, after taking the two terms, reads

\[
V_{\text{eff}}^{\text{HPQED}}(\rho, \Lambda) = -U(\rho) - \frac{(U'' \Lambda)}{2\pi^2} - \frac{(-U'')^{3/2}}{6\pi} + \frac{(\epsilon^2 \rho^2)^3}{12\pi^2} - \frac{3\Lambda^2 \rho^4}{8\pi^2} + \frac{3\Lambda^2 \epsilon^2 \rho^2}{8\pi^2} + \frac{3(\epsilon^2 \rho^2)^3}{24\pi^2} \ln \left( \frac{\Lambda^2}{(\epsilon^2 \rho^2)^2} \right). \tag{36}
\]
Next, let us discuss the possibility of finding a symmetry-breaking solution in Equation (36). Let us consider a classical potential, given by Equation (22) with $M^2 = 0$. In this case, we find a stable ground state at $\rho = 0$ and the effective potential resembles Figure 1. Otherwise, when $\lambda > 0$, we do not have a real-valued potential.

Although our results indicate that HPQED does not admit a SSB at one-loop approximation, we could also imagine that the scalar field has a nonzero VEV at tree level, i.e., an explicit breaking of symmetry. In this case, the interaction between the electrons is given by a two-dimensional Fourier transform of $1/(2\sqrt{p^2 + 2e^2}\rho_0^2)$. Indeed, such a result has been considered in Ref. [29] in light of the superconductivity driven by a topological phase transition. Clearly, this is quite different in comparison to the Yukawa potential we have discussed in Sec. IV. Indeed, it is not hard to conclude that the resulting potential behaves as $\ln(r)$ for $r \to 0$, and goes to zero as $r \to \infty$. This only proves the need of taking care between the order of the dimensional reduction and SSB.

6. Summary

The major relevance of SSB is in describing quantum states of matter, such as ferromagnetism and superconductivity. This is typically made by considering the Landau–Ginzburg theory [24–26]. For field theories, scalar QED4 is a relativistic generalization of the Landau–Ginzburg theory, where the Dirac field of QED4 is replaced by a scalar field with a self-interacting term [20]. However, when considering a $(2 + 1)$D version of scalar QED, one concludes that the effective potential behaves quite different in comparison with the higher-dimensional version of the model. For example, its effective potential trivially obeys the renormalization group equation [22]. Indeed, because the spatial dimension is reduced by one, the ultraviolet divergences are reduced and all of the beta functions vanish.

In order to have a proper description of the electronic interactions between electrons in a condensed-matter system, we must consider the reduced version of QED4, namely PQED [13]. This dimensional reduction also may be performed to preserve the Yukawa interaction in the plane [21]. Here, we show that the SSB does not occur in PQED, using the one-loop approximation within the background field method. Although we have considered a real scalar field, the generalization for the complex field is straightforward. Thereafter, we consider the Abelian version of scalar PQED in $(2 + 1)$D. In this case, we must be careful about the order in which we realize both the SSB and the dimensional reduction. When the SSB is considered in scalar QED4, after the dimensional reduction, we obtain a version of PQED that describes the Yukawa interaction between electrons in $(2 + 1)$D. Nevertheless, starting with PQED and coupling it to a scalar field, we realize that SSB is not realized and the system remains in its symmetric phase. Obviously, our approach is not the only path for describing screening effects, and it only has the virtue that it follows from a quite simple and well-known method, namely the coupling of the gauge and scalar fields. Indeed, screening effects due to the fermionic loop are also considered in the literature of quantum field theory [20], and may also be relevant for two-dimensional materials.

It would be interesting to investigate the SSB in the Abelian PQED where both the gauge and scalar fields have a pseudo-differential operator, as well as the effect of including a thermal bath. Furthermore, the possibility of dynamical mass generation using the Schwinger–Dyson equations [23] is also an important step for describing the ground state of the scalar PQED. We shall discuss this elsewhere.

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Appendix A. From (3 + 1)D to (1 + 1)D

In this case, we have to consider $J(y) = J(y_0, y_1, y_2)\delta(x_3)$ and $\varphi(x) \rightarrow \varphi_{2D}(x; a) = \Theta(x_0, x_1; a) = \varphi(x_0, x_1, x_2 = a, x_3 = 0)$. Note that, for the sake of convenience, we have introduced a scale parameter $x_2 = a$. Similarly to the previous case, we obtain

$$\varphi_{2D}(x; a) = \int d^2y \; G_{2D}(x - y; a) \varphi(y), \quad (A1)$$

where

$$G_{2D}(x - y; a) = \int \frac{d^2k}{(2\pi)^2} e^{ik(x - y)} \frac{K_0(\sqrt{k^2 + m^2}a)}{2\pi} \quad (A2)$$

is the Fourier transform of the scalar-field propagator in (1 + 1) dimensions. $K_0$ is the modified Bessel function of second kind. Obviously, this also produces the Yukawa potential in Equation (5), but the derivation is more subtle.

The static potential may be obtained by using $k^2 = k_0^2 + k_1^2$ with $k_0 \rightarrow 0$ in Equation (A2); hence,

$$V(x_1; a) = \int_{-\infty}^{+\infty} \frac{dk_1}{(2\pi)} e^{ik_1 x_1} K_0(\sqrt{k_1^2 + m^2}a) \frac{K_0(\sqrt{k_1^2 + m^2}a)}{2\pi}. \quad (A3)$$

Next, we use a parameterization in $\xi$, given by

$$K_0(\sqrt{k_1^2 + m^2}a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi}{2\sqrt{\xi^2 + k_1^2 + m^2}}. \quad (A4)$$

Using Equation (A4) in Equation (A3), and changing the variables to $\rho^2 = k_1^2 + \xi^2$ and $\tan \theta = k_1/\xi$, after some algebra, we find

$$V(x_1; a) = \int_0^{\infty} \frac{d\rho}{2\pi} \frac{J_0(\rho R)}{\sqrt{\rho^2 + m^2}}, \quad (A5)$$

where $R^2 = x_1^2 + a^2$. Finally, solving the integral over $\rho$ in Equation (A5), and using $a \rightarrow 0$, we have

$$V(x_1) = e^{-|mx_1|} \frac{J_0(|x_1|)}{4\pi|x_1|}, \quad (A6)$$

which is the Yukawa potential in (1 + 1) dimensions.

Similarly to the previous case in Equation (9), we may define a full action in (1 + 1)D that generates the scalar-field propagator in Equation (A2), given by

$$L_{2D} = \frac{\partial^\mu \Theta}{2} \Box \partial_\mu \Theta + g\bar{\psi}\psi + \mathcal{L}_M[\psi], \quad (A7)$$
where

\[
L[\Box] \equiv \frac{2\pi}{(-\Box)K_0(\sqrt{-\Box} + m^2)} = \int \frac{d^2k_E}{(2\pi)^2} \frac{2\pi}{k_0^2 K_0(\sqrt{k_0^2 + m^2})}.
\]  
(A8)

This concludes our general example of reduced models. Our main results are easily generalized for gauge theories, such as QED4.

Appendix B. The One-Loop Approximation

In this appendix, we derive Equation (14). We start with Equation (12), and perform an expansion for \(\phi(x) \approx \phi_{cl}(x)\), given by

\[
L_D[\phi] = L_D[\phi_{cl}] + \frac{\delta L_D}{\delta \phi} \mid_{\phi = \phi_{cl}} (\phi - \phi_{cl}) + \frac{1}{2} \frac{\delta^2 L_D}{\delta \phi \delta \phi} \mid_{\phi = \phi_{cl}} (\phi - \phi_{cl})^2 + ..., 
\]  
(A9)

where the classical field \(\phi_{cl}(x)\) is just a solution of the equation of motion. In particular, from Equation (11), note that

\[
\frac{\delta^2 L_D}{\delta \phi \delta \phi} = (-\Box)K + U'' = G^{-1}_\phi(-\Box) + U'', 
\]  
(A10)

where \(G^{-1}_\phi(-\Box) \equiv (-\Box)K\) is, essentially, the inverse of the free propagator of the scalar field. Using Equation (A9) and Equation (A10) in Equation (12), we find

\[
Z = e^{iS_{cl}[\phi_{cl}]} \int D\phi \exp \left\{ \frac{i}{2} \bar{\phi} [G^{-1}_\phi(-\Box) + U''] \phi \right\}, 
\]  
(A11)

where \(\bar{\phi} \rightarrow \phi - \phi_{cl}\). Solving the integral over \(\bar{\phi}\) in Equation (A11) yields

\[
Z = e^{iS_{cl}[\phi_{cl}]} \det \{-i[G^{-1}_\phi(-\Box) + U'']\}^{-1/2}. 
\]  
(A12)

Next, we use that \(\det A = \exp \{\text{Tr} \ln A\}\), where \(\text{Tr}\) is a trace operation over the space of the arbitrary matrix \(A\), i.e., \(\text{Tr} \ln A = \int d^Dx \ln A(x)\). Therefore,

\[
\begin{align*}
- \frac{1}{2} & \int d^Dx \ln[G^{-1}_\phi(-\Box) + U'']|x| = \\
- \frac{\Omega}{2} & \int \frac{d^Dk}{(2\pi)^D} \ln[G^{-1}_\phi(k) + U''], 
\end{align*}
\]  
(A13)

where \(\Omega = \int d^Dx\) and an irrelevant \(-i\) factor has been eliminated in the first line of Equation (A13). The rhs of Equation (A13) is the one-loop quantum correction, proportional to \(\hbar\), and we write this as \(\Omega V_1[\phi_{cl}]\). Hence, using Equation (A13) in Equation (A12), we have

\[
Z = e^{iS_{cl}[\phi_{cl}]} + \Omega V_1[\phi_{cl}], 
\]  
(A14)

Finally, having in mind the constant field configuration where \(\phi_{cl}(x) \rightarrow \rho\), we find that \(S_{cl}[\phi_{cl}] \rightarrow \Omega U(\rho)\). Therefore, after comparing Equation (A14) and Equation (A13) with Equation (13), it follows Equation (14).
Appendix C. The D-Dimensional Integral for the Reduced Model

In this appendix, we calculate Equation (17). Firstly, we must convert the D-dimensional integral to the Euclidean spacetime, which is easily performed by using $k^2 \to -k_E^2$ and $d^D k \to i d^D k_E$. Hence, we find

$$V_1(\rho, \mu) = \frac{\mu^{3-D}}{2} \int \frac{d^D k_E}{(2\pi)^D} \ln[2\sqrt{k_E^2} - U''(\rho)].$$

(A15)

Next, we derive Equation (14) in respect to $U''$; hence, we find

$$\frac{dV_1}{dU''} = -\frac{\mu^{3-D}}{2} \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{2\sqrt{k_E^2} - U''(\rho)}.$$  

(A16)

For solving the integral in Equation (A16), we use the following identity:

$$\int \frac{d^D k_E}{(2\pi)^D} \frac{1}{\sqrt{k_E^2 + A}} = \frac{2\pi^{D/2}}{\Gamma[D/2]} \frac{A^{D-1}}{(2\pi)^D} \Gamma[D]\Gamma[1-D],$$

(A17)

where $A$ is an arbitrary constant. Using Equation (A17) in Equation (A16) and integrating over $U''$, one finds Equation (17).

It is interesting to notice that Equation (A17) is not the same as the $D$-dimensional integral we would find for the usual scalar theory, where the denominator would be like $k_E^2 + A$. Hence, let us briefly derive Equation (A17). Let us call $I_D$ as the lhs of Equation (A17) and use $d^D k_E = k_E^{D-1} dk_E S_D$, where $S_D = 2\pi^{D/2}/\Gamma[D/2]$. Thereafter, we make a variable change to $x \equiv A/(k + A)$ and, after some algebra, we find

$$I_D = \frac{S_D}{(2\pi)^D} A^{D-1} \int_0^1 dxx^{-D}(1-x)^{D-1}.$$  

(A18)

The beta function $B(a, b)$ reads

$$B(a, b) = \frac{\Gamma[a]\Gamma[b]}{\Gamma[a+b]} = \int_0^1 dxx^{a-1}(1-x)^{b-1},$$

(A19)

where $a$ and $b$ are real positive constants. After using the identity in Equation (A19) into Equation (A18), we find Equation (A17).

Appendix D. The Dimensional Reduction of the QED4 Theory Coupled with Higgs Field

In this appendix, we calculate the dimensional reduction of Equation (23). Firstly, we expand

$$|D\mu\varphi|^2 = \partial^\mu \varphi^* \partial_\mu \varphi - igA^\mu [\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*] + g^2 \bar{A}_\mu A^\mu \varphi^2.$$  

(A20)

Next, the scalar field is written as $\varphi(x) = \xi(x)e^{i\theta(x)}$, where $(\xi, \theta)$ are real-valued functions. Hence, it follows that

$$|D\mu\varphi|^2 = \partial^\mu \xi \partial_\mu \xi + \xi^2 (\partial^\mu \theta \partial_\mu \theta + 2g^2 \xi^2 \bar{A}_\mu \partial^\mu \theta + g^2 \bar{A}_\mu \xi \partial_\mu \bar{A}^\mu \xi^2).$$  

(A21)

After using the gauge transformation $\bar{A}_\mu \to \bar{A}_\mu - \partial^\mu \theta / g$, we can eliminate the $\theta(x)$ dependence on the model and obtain

$$|D\mu\varphi|^2 = \partial^\mu \xi \partial_\mu \xi + g^2 \xi^2 \bar{A}_\mu \bar{A}_\mu \xi.$$  

(A22)
This is essentially given in Equation (24). Furthermore, we consider that the Higgs field is in the broken phase and take its lowest-order solution, given by $\xi(x) \rightarrow \rho_0$, and find Equation (25).

In order to perform the dimensional reduction, we define the generating functional of the Green functions for $A_\mu$, namely

$$Z = Z_0 \int D\tilde{A}_\mu e^{-\int d^4x \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu \tilde{A}_\nu)^2 + g_3^2 \rho_0^2 \tilde{A}_\mu \tilde{A}^\mu + e f_\mu \right]}.$$  \hspace{1cm} (A23)

The first two terms of the gauge field action are conveniently written as

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu \tilde{A}_\nu)^2 = -\frac{1}{2} \tilde{A}_\mu \left[ -\delta_{\mu\nu} \Box - \left( 1 + \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right] \tilde{A}_\nu.$$  \hspace{1cm} (A24)

Nevertheless, because the theory is gauge invariant, we may neglect the gauge-dependent term, for when it is coupled to a current, it vanishes. Next, we use the following identity:

$$\int D\varphi e^{-\frac{1}{2} \int dx [\varphi B + \int f dx \varphi J]} = \det \{ K \}^{-1/2} e^{\frac{1}{2} \int dx [K^{-1} J]}.$$

which holds for any bosonic field $\varphi$. Furthermore, $K$ is an arbitrary matrix and $J$ is an external source. Therefore, we have

$$Z = \frac{Z_0}{e} \int D(e \tilde{A}_\mu) e^{-\frac{1}{2} \int d^4x \left[ \frac{1}{2} e (e A^\rho) \frac{-\Box e A^\rho + 2\rho_0^2 |e A^\rho|}{|e A^\rho|^2} (e A^\rho) + J e (e A^\rho) \right]} = \frac{Z_0}{e} \det \left\{ \left( -\Box + 2\rho_0^2 / e^2 \right) / e^2 \right\}^{-1/2} \int d^4x \left[ e^{2 \mu} / \left( -\Box + 2\rho_0^2 e^2 \right) / e^2 \right].$$  \hspace{1cm} (A26)

Obviously, we may use the fact that the physics is unchanged by modifying the constant $Z_0$, and obtain

$$Z[J] = e^{-\int d^4x \left[ e^{2 \mu} / \left( -\Box + 2\rho_0^2 e^2 \right) / e^2 \right]}.$$  \hspace{1cm} (A27)

where it is defined that $Z[0] = 1$. Now, we confine the charges to move in a two-dimensional space, using the condition

$$J_\mu = \begin{cases} \tilde{f}_\mu^{3D}(x_0, x_1, x_2) \delta(x_3), & \mu = 0, 1, 2 \\ 0, & \mu = 3 \end{cases}.$$  \hspace{1cm} (A28)

Therefore,

$$\left. \frac{1}{-\Box + 2\rho_0^2} \right|_{x_3 = y_3 = 0} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + 2\rho_0^2} \bigg|_{x_3 = y_3 = 0} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik_{3D}(x-y)}}{k^2_3 + k^2_{3D} + 2\rho_0^2} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik_{3D}(x-y)}}{2\sqrt{k^2_3 + k^2_{3D} + 2\rho_0^2}} = \frac{1}{2} \frac{1}{(-\Box_{3D} + 2\rho_0^2)^{1/2}}.$$  \hspace{1cm} (A29)
Hence, we obtain the generating functional, given by

\[ Z = e^{-\int d^4x \left[ -\frac{\partial^2}{\partial x^2} + \frac{1}{(\pi^2)^2 + M^2} \right]} \].

(A30)

Finally, it is fair to conclude that the effective action associated with this generating functional is given by Equation (26).

References


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