computer sciences and mathematics forum

MDPI

Proceeding Paper

Abelian Groups of Fractional Operators †

Anthony Torres-Hernandez [1,2,*], Fernando Brambila-Paz [3] and Rafael Ramirez-Melendez [2]

1 Department of Physics, Faculty of Science, Universidad Nacional Autónoma de México, Mexico City 04510, Mexico
2 Music and Machine Learning Lab, Department of Information and Communication Technologies, Universitat Pompeu Fabra, 08018 Barcelona, Spain
3 Department of Mathematics, Faculty of Science, Universidad Nacional Autónoma de México, Mexico City 04510, Mexico
* Correspondence: anthony.torres@ciencias.unam.mx

† Presented at the 5th Mexican Workshop on Fractional Calculus, Tecnológico de Monterrey (TEC), Monterrey, Mexico, 5–7 October 2022.

Abstract: Taking into count the large number of fractional operators that have been generated over the years, and considering that their number is unlikely to stop increasing at the time of writing this paper due to the recent boom of fractional calculus, everything seems to indicate that an alternative that allows to fully characterize some elements of fractional calculus is through the use of sets. Therefore, this paper presents a recapitulation of some fractional derivatives, fractional integrals, and local fractional operators that may be found in the literature, as well as a summary of how to define sets of fractional operators that allow to fully characterize some elements of fractional calculus, such as the Taylor series expansion of a scalar function in multi-index notation. In addition, it is presented a way to define finite and infinite Abelian groups of fractional operators through a family of sets of fractional operators and two different internal operations. Finally, using the above results, it is shown one way to define commutative and unitary rings of fractional operators.

Keywords: fractional operators; set theory; group theory; fractional calculus of sets

check for updates

Citation: Torres-Hernandez, A.; Brambila-Paz, F.; Ramirez-Melendez, R. Abelian Groups of Fractional Operators. *Comput. Sci. Math. Forum* **2022**, *4*, 4. https://doi.org/10.3390/cmsf2022004004

Academic Editors: Jorge M. Cruz-Duarte and Porfirio Toledo-Hernández

Published: 19 December 2022

1. Introduction

Fractional calculus is a branch of mathematics that uses derivatives of non-integer order that originated around the same time as conventional calculus due to Leibniz's notation for derivatives of integer order

$$\frac{d^n}{dx^n}.$$

Therefore, thanks to this notation, L'Hopital could ask in a letter to Leibniz about the interpretation of taking $n = 1/2$ in a derivative. Since at that moment Leibniz could not give a physical or geometrical interpretation of this question, he simply answered to L'Hopital in a letter, "... is an apparent paradox of which, one day, useful consequences will be drawn" [1]. The name of fractional calculus comes from a historical question since, in this branch of mathematical analysis, the derivatives and integrals of a certain order α are studied, with $\alpha \in \mathbb{R}$. Currently, fractional calculus does not have a unified definition of what is considered a fractional derivative. As a consequence, when it is not necessary to explicitly specify the form of a fractional derivative, it is usually denoted as follows

$$\frac{d^\alpha}{dx^\alpha}.$$

The fractional operators have many representations, but one of their fundamental properties is that they allow retrieving the results of conventional calculus when $\alpha \to n$.

For example, let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L^1_{loc}(a, b)$, where $L^1_{loc}(a, b)$ denotes the space of locally integrable functions on the open interval $(a, b) \subset \Omega$. One of the fundamental operators of fractional calculus is the operator Riemann–Liouville fractional integral, which is defined as follows [2,3]:

$$ {}_aI_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{1} $$

where Γ denotes the Gamma function. It is worth mentioning that the above operator is a fundamental piece to construct the operator Riemann-Liouville fractional derivative, which is defined as follows [2,4]:

$$ {}_aD_x^\alpha f(x) := \begin{cases} {}_aI_x^{-\alpha} f(x), & \text{if } \alpha < 0 \\ \dfrac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x)\right), & \text{if } \alpha \geq 0 \end{cases}, \tag{2} $$

where $n = \lceil \alpha \rceil$ and ${}_aI_x^0 f(x) := f(x)$. On the other hand, let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function n-times differentiable such that $f, f^{(n)} \in L^1_{loc}(a, b)$. Then, the Riemann–Liouville fractional integral also allows constructing the operator Caputo fractional derivative, which is defined as follows [2,4]:

$$ {}_a^CD_x^\alpha f(x) := \begin{cases} {}_aI_x^{-\alpha} f(x), & \text{if } \alpha < 0 \\ {}_aI_x^{n-\alpha} f^{(n)}(x), & \text{if } \alpha \geq 0 \end{cases}, \tag{3} $$

where $n = \lceil \alpha \rceil$ and ${}_aI_x^0 f^{(n)}(x) := f^{(n)}(x)$. Furthermore, if the function f fulfills that $f^{(k)}(a) = 0 \; \forall k \in \{0, 1, \cdots, n-1\}$, the Riemann–Liouville fractional derivative coincides with the Caputo fractional derivative, that is,

$$ {}_aD_x^\alpha f(x) = {}_a^CD_x^\alpha f(x). \tag{4} $$

Therefore, applying the operator (2) with $a = 0$ to the function x^μ, with $\mu > -1$, we obtain the following result:

$$ {}_0D_x^\alpha x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{5} $$

where if $1 \leq \lceil \alpha \rceil \leq \mu$, it is fulfilled that ${}_0D_x^\alpha x^\mu = {}_0^CD_x^\alpha x^\mu$. To illustrate a bit the diversity of representations that fractional operators may have, we proceed to present a recapitulation of some fractional derivatives, fractional integrals, and local fractional operators that may be found in the literature [5–7]:

1. Grünwald-Letnikov fractional derivative:

$$ {}_a^{GL}D_x^\alpha f(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} f(x-kh), \quad n = \lfloor (x-a)/h \rfloor. $$

2. Marchaud fractional derivative:

$$ {}_{-\infty}^{Ma}D_x^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^x (x-t)^{-\alpha-1}(f(x)-f(t))dt, \quad 0 < \alpha < 1. $$

3. Hadamard fractional derivative:

$$ {}_a^{Ha}D_x^\alpha f(x) = \frac{x}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (\ln(x) - \ln(t))^{2-\alpha} \frac{f(t)}{t} dt, \quad 0 < \alpha < 1. $$

4. Chen fractional derivative:

$$ {}_{a}^{Ch}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}(x-t)^{-\alpha}f(t)dt, \quad 0<\alpha<1. $$

5. Caputo-Fabrizio fractional derivative:

$$ {}_{a}^{CF}D_{x}^{\alpha}f(x) = \frac{M(\alpha)}{1-\alpha}\int_{a}^{x}\exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)f^{(1)}(t)dt, \quad 0<\alpha<1, \quad M(0)=M(1)=1. $$

6. Atangana-Baleanu-Caputo fractional derivative:

$$ {}_{a}^{ABC}D_{x}^{\alpha}f(x) = \frac{M(\alpha)}{1-\alpha}\int_{a}^{x}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(x-t)^{\alpha}\right)f^{(1)}(t)dt, \quad 0<\alpha<1, \quad M(0)=M(1)=1. $$

7. Canavati fractional derivative:

$$ {}_{a}^{Ca}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(1+\alpha-n)}\frac{d}{dx}\int_{a}^{x}(x-t)^{n-\alpha}\frac{d^{n}}{dt^{n}}f(t)dt, \quad n=\lfloor\alpha\rfloor. $$

8. Jumarie fractional derivative:

$$ {}_{a}^{Ju}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{a}^{x}(x-t)^{n-\alpha-1}(f(t)-f(a))dt, \quad n=\lceil\alpha\rceil. $$

9. Hadamard fractional integral:

$$ {}_{a}^{Ha}I_{x}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(\ln(t)-\ln(x))^{\alpha-1}\frac{f(t)}{t}dt. $$

10. Weyl fractional integral:

$$ {}_{x}W_{\infty}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}f(t)dt. $$

11. Conformable fractional operator:

$$ T_{\alpha}f(x) = \lim_{h\to 0}\frac{f\left(x+hx^{1-\alpha}\right)-f(x)}{h}. $$

12. Katugampola fractional operator:

$$ D^{\alpha}f(x) = \lim_{h\to 0}\frac{f(x\exp(hx^{-\alpha}))-f(x)}{h}. $$

13. Deformable fractional operator:

$$ \mathcal{D}^{\alpha}f(x) = \lim_{h\to 0}\frac{(1+h\beta)f(x+h\alpha)-f(x)}{h}, \quad \alpha+\beta=1. $$

Before continuing, it is worth mentioning that the applications of fractional operators have spread to different fields of science, such as finance [8,9], economics [10,11], number theory through the Riemann zeta function [12,13], in engineering with the study for the manufacture of hybrid solar receivers [14,15], and in physics and mathematics to solve nonlinear algebraic equation systems [16–25], which is a classical problem in mathematics, physics and engineering that consists of finding the set of zeros of a function $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$, that is,

$$\{\xi\in\Omega\ :\ \|f(\xi)\|=0\},$$

where $\|\cdot\|:\mathbb{R}^n\to\mathbb{R}$ denotes any vector norm, or equivalently,

$$\{\xi\in\Omega\ :\ [f]_k(\xi)=0\ \forall k\geq 1\},$$

where $[f]_k:\mathbb{R}^n\to\mathbb{R}$ denotes the k-th component of the function f.

2. Sets of Fractional Operators

Before continuing, it is worth mentioning that due to the large number of fractional operators that exist [5–7,26–41], it seems that the most natural way to fully characterize the elements of the fractional calculus is by using sets, which is the main idea behind of the methodology known as fractional calculus of sets [42,43]. Therefore, considering a scalar function $h:\mathbb{R}^m\to\mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k\geq 1}$, it is feasible to define the following fractional operator of order α using Einstein's notation

$$o_x^\alpha h(x):=\hat{e}_k o_k^\alpha h(x). \tag{6}$$

Therefore, denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator, it is feasible to define the following set of fractional operators

$$O_{x,\alpha}^n(h):=\left\{o_x^\alpha\ :\ \exists o_k^\alpha h(x)\ \text{ and }\ \lim_{\alpha\to n}o_k^\alpha h(x)=\partial_k^n h(x)\ \forall k\geq 1\right\}, \tag{7}$$

which corresponds to a nonempty set since it contains the following sets of fractional operators

$$O_{0,x,\alpha}^n(h):=\left\{o_x^\alpha\ :\ \exists o_k^\alpha h(x)=(\partial_k^n+\mu(\alpha)\partial_k^\alpha)h(x)\ \text{ and }\ \lim_{\alpha\to n}\mu(\alpha)\partial_k^\alpha h(x)=0\ \ \forall k\geq 1\right\}. \tag{8}$$

As a consequence, it is feasible to obtain the following result:

$$\text{If }\ o_{i,x}^\alpha,o_{j,x}^\alpha\in O_{x,\alpha}^n(h)\ \text{ with }\ i\neq j\ \Rightarrow\ \exists o_{k,x}^\alpha=\frac{1}{2}\left(o_{i,x}^\alpha+o_{j,x}^\alpha\right)\in O_{x,\alpha}^n(h). \tag{9}$$

On the other hand, the complement of the set (7) may be defined as follows

$$O_{x,\alpha}^{n,c}(h):=\left\{o_x^\alpha\ :\ \exists o_k^\alpha h(x)\ \forall k\geq 1\ \text{ and }\ \lim_{\alpha\to n}o_k^\alpha h(x)\neq\partial_k^n h(x)\ \text{ in at least one value } k\geq 1\right\}, \tag{10}$$

with which it is feasible to obtain the following result:

$$\text{If }\ o_{i,x}^\alpha=\hat{e}_k o_{i,k}^\alpha\in O_{x,\alpha}^n(h)\ \Rightarrow\ \exists o_{j,x}^\alpha=\hat{e}_k o_{i,\sigma_j(k)}^\alpha\in O_{x,\alpha}^{n,c}(h), \tag{11}$$

where $\sigma_j:\{1,2,\cdots,m\}\to\{1,2,\cdots,m\}$ denotes any permutation different from the identity. Before continuing, it is necessary to mention that set (7) allows generalizing elements of conventional calculus. For example, let $\mathbb{N}_0$ be the set $\mathbb{N}\cup\{0\}$. If $\gamma\in\mathbb{N}_0^m$ and $x\in\mathbb{R}^m$, then it is feasible to define the following multi-index notation:

$$\left\{\begin{array}{c}\gamma!:=\prod_{k=1}^m[\gamma]_k!,\quad|\gamma|:=\sum_{k=1}^m[\gamma]_k,\quad x^\gamma:=\prod_{k=1}^m[x]_k^{[\gamma]_k}\\ \dfrac{\partial^\gamma}{\partial x^\gamma}:=\dfrac{\partial^{[\gamma]_1}}{\partial[x]_1^{[\gamma]_1}}\dfrac{\partial^{[\gamma]_2}}{\partial[x]_2^{[\gamma]_2}}\cdots\dfrac{\partial^{[\gamma]_m}}{\partial[x]_m^{[\gamma]_m}}\end{array}\right.. \tag{12}$$

Therefore, considering a function $h : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ and the fractional operator

$$s_x^{\alpha\gamma}(o_x^\alpha) := o_1^{\alpha[\gamma]_1} o_2^{\alpha[\gamma]_2} \cdots o_m^{\alpha[\gamma]_m}, \tag{13}$$

it is feasible to define the following set of fractional operators

$$\mathrm{S}_{x,\alpha}^{n,\gamma}(h) := \left\{ s_x^{\alpha\gamma} = s_x^{\alpha\gamma}(o_x^\alpha) \ : \ \exists s_x^{\alpha\gamma} h(x) \ \text{ with } \ o_x^\alpha \in \mathrm{O}_{x,\alpha}^s(h) \ \forall s \leq n^2 \ \text{ and } \ \lim_{\alpha \to k} s_x^{\alpha\gamma} h(x) = \frac{\partial^{k\gamma}}{\partial x^{k\gamma}} h(x) \ \forall \alpha, |\gamma| \leq n \right\}, \tag{14}$$

from which it is feasible to obtain the following results:

$$\text{If } s_x^{\alpha\gamma} \in \mathrm{S}_{x,\alpha}^{n,\gamma}(h) \ \Rightarrow \ \begin{cases} \lim\limits_{\alpha \to 0} s_x^{\alpha\gamma} h(x) = o_1^0 o_2^0 \cdots o_m^0 h(x) = h(x) \\ \lim\limits_{\alpha \to 1} s_x^{\alpha\gamma} h(x) = o_1^{[\gamma]_1} o_2^{[\gamma]_2} \cdots o_m^{[\gamma]_m} h(x) = \dfrac{\partial^{\gamma}}{\partial x^{\gamma}} h(x) \ \forall |\gamma| \leq n \\ \lim\limits_{\alpha \to q} s_x^{\alpha\gamma} h(x) = o_1^{q[\gamma]_1} o_2^{q[\gamma]_2} \cdots o_m^{q[\gamma]_m} h(x) = \dfrac{\partial^{q\gamma}}{\partial x^{q\gamma}} h(x) \ \forall q|\gamma| \leq qn \\ \lim\limits_{\alpha \to n} s_x^{\alpha\gamma} h(x) = o_1^{n[\gamma]_1} o_2^{n[\gamma]_2} \cdots o_m^{n[\gamma]_m} h(x) = \dfrac{\partial^{n\gamma}}{\partial x^{n\gamma}} h(x) \ \forall n|\gamma| \leq n^2 \end{cases}. \tag{15}$$

On the other hand, using little-o notation, it is feasible to obtain the following result:

$$\text{If } x \in B(a;\delta) \ \Rightarrow \ \lim_{x \to a} \frac{o((x-a)^\gamma)}{(x-a)^\gamma} \to 0 \ \forall |\gamma| \geq 1, \tag{16}$$

with which it is feasible to define the following set of functions

$$R_{\alpha\gamma}^n(a) := \left\{ r_{\alpha\gamma}^n \ : \ \lim_{x \to a} \left\| r_{\alpha\gamma}^n(x) \right\| = 0 \ \forall |\gamma| \geq n \ \text{ and } \ \left\| r_{\alpha\gamma}^n(x) \right\| \leq o\left(\|x-a\|^n \right) \ \forall x \in B(a;\delta) \right\}, \tag{17}$$

where $r_{\alpha\gamma}^n : B(a;\delta) \subset \Omega \to \mathbb{R}$. Therefore, considering the previous set and some $B(a;\delta) \subset \Omega$, it is feasible to define the following sets of fractional operators

$$\mathrm{T}_{x,\alpha,p}^{n,q,\gamma}(a,h) := \left\{ t_x^{\alpha,p} = t_x^{\alpha,p}(s_x^{\alpha\gamma}) \ : \ s_x^{\alpha\gamma} \in \mathrm{S}_{x,\alpha}^{M,\gamma}(h) \ \text{ and } \ t_x^{\alpha,p} h(x) := \sum_{|\gamma|=0}^{p} \frac{1}{\gamma!} s_x^{\alpha\gamma} h(a)(x-a)^\gamma + r_{\alpha\gamma}^p(x) \ \begin{matrix} \forall \alpha \leq n \\ \forall p \leq q \end{matrix} \right\}, \tag{18}$$

$$\mathrm{T}_{x,\alpha}^{\infty,\gamma}(a,h) := \left\{ t_x^{\alpha,\infty} = t_x^{\alpha,\infty}(s_x^{\alpha\gamma}) \ : \ s_x^{\alpha\gamma} \in \mathrm{S}_{x,\alpha}^{\infty,\gamma}(h) \ \text{ and } \ t_x^{\alpha,\infty} h(x) := \sum_{|\gamma|=0}^{\infty} \frac{1}{\gamma!} s_x^{\alpha\gamma} h(a)(x-a)^\gamma \right\}, \tag{19}$$

which allow generalizing the Taylor series expansion of a scalar function in multi-index notation [22], where $M = \max\{n, q\}$. As a consequence, it is feasible to obtain the following results:

$$\text{If } t_x^{\alpha,p} \in \mathrm{T}_{x,\alpha,p}^{1,q,\gamma}(a,h) \text{ and } \alpha \to 1 \ \Rightarrow \ t_x^{1,p} h(x) = h(a) + \sum_{|\gamma|=1}^{p} \frac{1}{\gamma!} \frac{\partial^\gamma}{\partial x^\gamma} h(a)(x-a)^\gamma + r_\gamma^p(x), \tag{20}$$

$$\text{If } t_x^{\alpha,p} \in \mathrm{T}_{x,\alpha,p}^{n,1,\gamma}(a,h) \text{ and } p \to 1 \ \Rightarrow \ t_x^{\alpha,1} h(x) = h(a) + \sum_{k=1}^{m} o_k^\alpha h(a)[(x-a)]_k + r_{\alpha\gamma}^1(x). \tag{21}$$

Finally, it is worth mentioning that the set (7) may be considered as a generating set of sets of fractional tensor operators. For example, considering $\alpha, n \in \mathbb{R}^d$ with $\alpha = \hat{e}_k[\alpha]_k$ and $n = \hat{e}_k[n]_k$, it is feasible to define the following set of fractional tensor operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_x^\alpha h(x) \ \text{ and } \ o_x^\alpha \in \mathrm{O}_{x,[\alpha]_1}^{[n]_1}(h) \times \mathrm{O}_{x,[\alpha]_2}^{[n]_2}(h) \times \cdots \times \mathrm{O}_{x,[\alpha]_d}^{[n]_d}(h) \right\}. \tag{22}$$

3. Groups of Fractional Operators

Considering a function $h:\Omega\subset\mathbb{R}^m\to\mathbb{R}^m$, it is feasible to define sets of fractional operators for a vector function in the following way:

$$ {}_m\mathrm{O}^{n}_{x,\alpha}(h):=\left\{o^\alpha_x\ :\ o^\alpha_x\in \mathrm{O}^{n}_{x,\alpha}([h]_k)\ \forall k\leq m\right\}, \tag{23}$$

$$ {}_m\mathrm{O}^{n,c}_{x,\alpha}(h):=\left\{o^\alpha_x\ :\ o^\alpha_x\in \mathrm{O}^{n,c}_{x,\alpha}([h]_k)\ \forall k\leq m\right\}, \tag{24}$$

$$ {}_m\mathrm{O}^{n,u}_{x,\alpha}(h):={}_m\mathrm{O}^{n}_{x,\alpha}(h)\cup{}_m\mathrm{O}^{n,c}_{x,\alpha}(h), \tag{25}$$

where $[h]_k:\Omega\subset\mathbb{R}^m\to\mathbb{R}$ denotes the k-th component of the function h. Therefore, using the above sets, it is feasible to construct the following family of fractional operators

$$ {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h):=\bigcap_{k\in\mathbb{Z}}{}_m\mathrm{O}^{k,u}_{x,\alpha}(h). \tag{26}$$

Before continuing, it should be noted that the above family of fractional operators fulfills the following property with respect to the classical Hadamar product:

$$ o^0_x\circ h(x):=h(x)\quad \forall o^\alpha_x\in{}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h). \tag{27}$$

Furthermore, for each operator $o^\alpha_x\in{}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h)$, it is feasible to define the following fractional matrix operator [44]:

$$ A_\alpha(o^\alpha_x)=\left([A_\alpha(o^\alpha_x)]_{jk}\right):=(o^\alpha_k). \tag{28}$$

On the other hand, defining the following modified Hadamard product [42]:

$$ o^{p\alpha}_{i,x}\circ o^{q\alpha}_{j,x}:=\begin{cases} o^{p\alpha}_{i,x}\circ o^{q\alpha}_{j,x}, & \text{if } i\neq j\ \text{(Hadamard product of type horizontal)}\\ o^{(p+q)\alpha}_{i,x}, & \text{if } i=j\ \text{(Hadamard product of type vertical)}\end{cases}, \tag{29}$$

for each operator $o^\alpha_x\in{}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h)$, it is feasible to define an Abelian group of fractional operators isomorphic to the group of integers under the addition, as shown by the following theorem [43,44]:

Theorem 1. *Let o^α_x be a fractional operator such that $o^\alpha_x\in{}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h)$ and let $(\mathbb{Z},+)$ be the group of integers under the addition. Therefore, considering the modified Hadamard product given by (29), it is feasible to define the following set of fractional matrix operators*

$$ {}_m\mathrm{G}(A_\alpha(o^\alpha_x)):=\left\{A^{\circ r}_\alpha=A_\alpha(o^{r\alpha}_x)\ :\ r\in\mathbb{Z}\ \text{and}\ A^{\circ r}_\alpha=\left([A^{\circ r}_\alpha]_{jk}\right):=(o^{r\alpha}_k)\right\}, \tag{30}$$

which corresponds to the Abelian group generated by the operator $A_\alpha(o^\alpha_x)$ isomorphic to the group $(\mathbb{Z},+)$, that is,

$$ {}_m\mathrm{G}(A_\alpha(o^\alpha_x))\cong(\mathbb{Z},+). \tag{31}$$

Proof. It should be noted that due to the way the set (30) is defined, just the Hadamard product of type vertical is applied among its elements. So, $\forall A^{\circ p}_\alpha, A^{\circ q}_\alpha\in{}_m\mathrm{G}(A_\alpha(o^\alpha_x))$ it is fulfilled that

$$ A^{\circ p}_\alpha\circ A^{\circ q}_\alpha=\left([A^{\circ p}_\alpha]_{jk}\right)\circ\left([A^{\circ q}_\alpha]_{jk}\right)=\left(o^{(p+q)\alpha}_k\right)=\left([A^{\circ(p+q)}_\alpha]_{jk}\right)=A^{\circ(p+q)}_\alpha. \tag{32}$$

So, from the previous result, it is feasible to prove that the set ${}_m\mathrm{G}(A_\alpha(o^\alpha_x))$ is a semigroup since it fulfills the following property:

$$ \forall A^{\circ p}_\alpha, A^{\circ q}_\alpha, A^{\circ r}_\alpha\in{}_m\mathrm{G}(A_\alpha(o^\alpha_x))\ \text{ it is fulfilled that }\left(A^{\circ p}_\alpha\circ A^{\circ q}_\alpha\right)\circ A^{\circ r}_\alpha=A^{\circ p}_\alpha\circ\left(A^{\circ q}_\alpha\circ A^{\circ r}_\alpha\right). \tag{33}$$

Furthermore, it follows from the definition of the set (30) that it contains a neutral element, with which it is feasible to prove from the previous result that the set ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ is also a monoid since it fulfills the following property:

$$\exists A_\alpha^{\circ 0} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \;\text{ such that }\; \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \;\text{ it is fulfilled that }\; A_\alpha^{\circ 0} \circ A_\alpha^{\circ p} = A_\alpha^{\circ p}. \tag{34}$$

It should be noted that due to the way in which the set (30) is defined, for each element contained in the set its symmetric element is also defined, with which from the previous result the set ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ is also a group since it fulfills the following property:

$$\forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha))\; \exists A_\alpha^{\circ -p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \;\text{ such that }\; A_\alpha^{\circ p} \circ A_\alpha^{\circ -p} = A_\alpha^{\circ 0}. \tag{35}$$

Finally, observing that the order in which the elements of the sets are operated does not influence the final result, it is obtained that the set ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ is also an Abelian group since it fulfills the following property:

$$\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \;\text{ it is fulfilled that }\; A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = A_\alpha^{\circ q} \circ A_\alpha^{\circ p}. \tag{36}$$

Once proven that the set (30) defines an Abelian group, to finish the proof of the theorem it is enough to define a bijective homomorphism between the sets ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ and $(\mathbb{Z},+)$. So, defining the following functions

$$\begin{array}{c} \psi : {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \to (\mathbb{Z},+) \\ \psi(A_\alpha^{\circ r}) = r \end{array} \quad \text{and} \quad \begin{array}{c} \psi^{-1} : (\mathbb{Z},+) \to {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \\ \psi^{-1}(r) = A_\alpha^{\circ r} \end{array}, \tag{37}$$

it is feasible to prove that the function ψ defines a homeomorphism between the sets ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ and $(\mathbb{Z},+)$ through the following result:

$$\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \;\text{ it is fulfilled that }\; \psi\left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) = \psi\left(A_\alpha^{\circ (p+q)}\right) = p+q = \psi\left(A_\alpha^{\circ p}\right) + \psi\left(A_\alpha^{\circ q}\right), \tag{38}$$

and analogously it is proved that the function ψ^{-1} defines a homeomorphism between the sets $(\mathbb{Z},+)$ and ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ through the following result:

$$\forall p,q \in (\mathbb{Z},+) \;\text{ it is fulfilled that }\; \psi^{-1}(p+q) = A_\alpha^{\circ (p+q)} = A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \psi^{-1}(p) \circ \psi^{-1}(q). \tag{39}$$

Therefore, from the previous results, it follows that the function ψ defines an isomorphism between the sets ${}_m\mathrm{G}(A_\alpha(o_x^\alpha))$ and $(\mathbb{Z},+)$. $\square$

Therefore, from the previous theorem, it is feasible to obtain the following corollary:

Corollary 1. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and let $(\mathbb{Z},+)$ be the group of integers under the addition. Therefore, considering the modified Hadamard product given by (29) and some subgroup $\mathbb{H}$ of the group $(\mathbb{Z},+)$, it is feasible to define the following set of fractional matrix operators*

$${}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{H}) := \left\{ A_\alpha^{\circ r} = A_\alpha(o_x^{r\alpha}) \;:\; r \in \mathbb{H} \;\textit{ and }\; A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := (o_k^{r\alpha}) \right\}, \tag{40}$$

which corresponds to a subgroup of the group generated by the operator $A_\alpha(o_x^\alpha)$, that is,

$${}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{H}) \leq {}_m\mathrm{G}(A_\alpha(o_x^\alpha)). \tag{41}$$

Example 1. *Let $\mathbb{Z}_n$ be the set of residual classes less than a positive integer n. Therefore, considering a fractional operator $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and the set $\mathbb{Z}_{14}$, it is feasible to define, under the modified Hadamard product given by (29), the following Abelian group of fractional matrix operators*

$${}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{Z}_{14}) = \left\{ A_\alpha^{\circ 0}, A_\alpha^{\circ 1}, A_\alpha^{\circ 2}, A_\alpha^{\circ 3}, A_\alpha^{\circ 4}, A_\alpha^{\circ 5}, A_\alpha^{\circ 6}, A_\alpha^{\circ 7}, A_\alpha^{\circ 8}, A_\alpha^{\circ 9}, A_\alpha^{\circ 10}, A_\alpha^{\circ 11}, A_\alpha^{\circ 12}, A_\alpha^{\circ 13} \right\}. \tag{42}$$

Furthermore, all possible combinations of the elements of the group are summarized in the following Cayley table:

$\circ$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$
$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$
$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$
$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$
$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$
$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$
$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$
$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$
$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$
$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$
$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$
$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$
$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$
$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 12}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$
$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 13}$	$A_\alpha^{\circ 0}$	$A_\alpha^{\circ 1}$	$A_\alpha^{\circ 2}$	$A_\alpha^{\circ 3}$	$A_\alpha^{\circ 4}$	$A_\alpha^{\circ 5}$	$A_\alpha^{\circ 6}$	$A_\alpha^{\circ 7}$	$A_\alpha^{\circ 8}$	$A_\alpha^{\circ 9}$	$A_\alpha^{\circ 10}$	$A_\alpha^{\circ 11}$	$A_\alpha^{\circ 12}$

It is worth mentioning that the Corollary 1 allows generating groups of fractional operators under other operations. For example, considering the following operation

$$A_\alpha^{\circ r} * A_\alpha^{\circ s} = A_\alpha^{\circ rs}, \tag{43}$$

it is feasible to obtain the following corollaries:

Corollary 2. *Let $\mathbb{M}_n$ be the set of positive residual classes corresponding to the coprimes less than a positive integer n. Therefore, for each fractional operator $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$, it is feasible to define the following Abelian group of fractional matrix operators under the operation (43):*

$$ {}_m\mathrm{G}^*(A_\alpha(o_x^\alpha), \mathbb{M}_n) := \left\{ A_\alpha^{\circ r} = A_\alpha(o_x^{r\alpha}) \ : \ r \in \mathbb{M}_n \ \textit{and} \ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk} \right) := (o_k^{r\alpha}) \right\}. \tag{44}$$

Example 2. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$. Therefore, considering the set $\mathbb{M}_{14}$, it is feasible to define, under the operation (43), the following Abelian group of fractional matrix operators*

$$ {}_m\mathrm{G}^*(A_\alpha(o_x^\alpha), \mathbb{M}_{14}) = \left\{ A_\alpha^{\circ 1}, A_\alpha^{\circ 3}, A_\alpha^{\circ 5}, A_\alpha^{\circ 9}, A_\alpha^{\circ 11}, A_\alpha^{\circ 13} \right\}. \tag{45}$$

Furthermore, all possible combinations of the elements of the group are summarized in the following Cayley table:

$*$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 13}$
$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 13}$
$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 13}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 11}$
$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 13}$	$A_{\alpha}^{\circ 9}$
$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 13}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 5}$
$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 13}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 3}$
$A_{\alpha}^{\circ 13}$	$A_{\alpha}^{\circ 13}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 1}$

Corollary 3. *Let $\mathbb{Z}_p^+$ be the set of positive residual classes less than p, with p a prime number. Therefore, for each fractional operator $o_x^{\alpha} \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$, it is feasible to define the following Abelian group of fractional matrix operators under the operation (43):*

$$ {}_m\mathrm{G}^*\left(A_{\alpha}(o_x^{\alpha}), \mathbb{Z}_p^+\right) := \left\{A_{\alpha}^{\circ r} = A_{\alpha}(o_x^{r\alpha}) \ : \ r \in \mathbb{Z}_p^+ \ \text{ and } \ A_{\alpha}^{\circ r} = \left([A_{\alpha}^{\circ r}]_{jk}\right) := (o_k^{r\alpha})\right\}. \tag{46} $$

Example 3. *Let o_x^{α} be a fractional operator such that $o_x^{\alpha} \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$. Therefore, considering the set $\mathbb{Z}_{13}^+$, it is feasible to define, under the operation (43), the following Abelian group of fractional matrix operators*

$$ {}_m\mathrm{G}^*\left(A_{\alpha}(o_x^{\alpha}), \mathbb{Z}_{13}^+\right) = \left\{A_{\alpha}^{\circ 1}, A_{\alpha}^{\circ 2}, A_{\alpha}^{\circ 3}, A_{\alpha}^{\circ 4}, A_{\alpha}^{\circ 5}, A_{\alpha}^{\circ 6}, A_{\alpha}^{\circ 7}, A_{\alpha}^{\circ 8}, A_{\alpha}^{\circ 9}, A_{\alpha}^{\circ 10}, A_{\alpha}^{\circ 11}, A_{\alpha}^{\circ 12}\right\}. \tag{47} $$

Furthermore, all possible combinations of the elements of the group are summarized in the following Cayley table:

$*$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 12}$
$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 12}$
$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 11}$
$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 10}$
$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 9}$
$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 8}$
$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 7}$
$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 6}$
$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 5}$
$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 4}$
$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 3}$
$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 1}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 2}$
$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 12}$	$A_{\alpha}^{\circ 11}$	$A_{\alpha}^{\circ 10}$	$A_{\alpha}^{\circ 9}$	$A_{\alpha}^{\circ 8}$	$A_{\alpha}^{\circ 7}$	$A_{\alpha}^{\circ 6}$	$A_{\alpha}^{\circ 5}$	$A_{\alpha}^{\circ 4}$	$A_{\alpha}^{\circ 3}$	$A_{\alpha}^{\circ 2}$	$A_{\alpha}^{\circ 1}$

Finally, it should be noted that when n is a prime number, the following result is obtained:

$$ {}_m\mathrm{G}^*\left(A_{\alpha}(o_x^{\alpha}), \mathbb{M}_n\right) = {}_m\mathrm{G}^*\left(A_{\alpha}(o_x^{\alpha}), \mathbb{Z}_n^+\right). \tag{48} $$

4. Conclusions

Although this article presents one way to define groups of fractional operators using sets related to the set of integer numbers, it would be feasible to extend the results using other sets of numbers that allow defining Abelian groups, as is the case of the set of rational numbers and the set of real numbers, being feasible to define the following groups:

$$ {}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{Q}) \quad \text{and} \quad {}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{R}). \tag{49}$$

Furthermore, from the groups generated by the equation (30), it is feasible define the following group of fractional matrix operators [42,44]:

$$ {}_m\mathrm{G}_{FIM}(\alpha) := \bigcup_{o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)} {}_m\mathrm{G}(A_\alpha(o_x^\alpha)), \tag{50}$$

in which it is assumed that through combinations of the horizontal and vertical type of the modified Hadamard product given by the equation (29), the fractional operators are reduced to their minimum expression, allowing to obtain $\forall A_{i,\alpha}^{\circ p}, A_{j,\alpha}^{\circ q}, A_{j,\alpha}^{\circ r} \in {}_m\mathrm{G}_{FIM}(\alpha)$, with $i \neq j$, the following result:

$$ \left(A_{i,\alpha}^{\circ p} \circ A_{j,\alpha}^{\circ q}\right) \circ A_{j,\alpha}^{\circ r} = A_{i,\alpha}^{\circ p} \circ \left(A_{j,\alpha}^{\circ q} \circ A_{j,\alpha}^{\circ r}\right) = A_{k,\alpha}^{\circ 1} := A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{(q+r)\alpha}\right), \quad p,q,r \in \mathbb{Z} \setminus \{0\}. \tag{51}$$

As a consequence, the following result is obtained:

$$ \forall A_{k,\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FIM}(\alpha) \text{ such that } A_{k,\alpha}\left(o_{k,x}^{\alpha}\right) = A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right) \exists A_{k,\alpha}^{\circ r} = A_{k,\alpha}^{\circ (r-1)} \circ A_{k,\alpha}^{\circ 1} = A_{k,\alpha}\left(o_{i,x}^{rp\alpha} \circ o_{j,x}^{rq\alpha}\right). \tag{52}$$

Therefore, if Φ_{FIM} denotes the iteration function of some fractional iterative method [43,44], it is feasible to obtain the following result:

$$ \text{Let } \alpha_0 \in \mathbb{R} \setminus \mathbb{Z} \Rightarrow \forall A_{\alpha_0}^{\circ 1} \in {}_m\mathrm{G}_{FIM}(\alpha) \ \exists \Phi_{FIM} = \Phi_{FIM}(A_{\alpha_0}) \ \therefore \ \forall A_{\alpha_0} \ \exists\{\Phi_{FIM}(A_\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Z}\}. \tag{53}$$

Finally, it is worth mentioning that it is feasible to develop more complex algebraic structures of fractional operators using the presented results. For example, without loss of generality, considering the modified Hadamard product (29) and the operation (43), a commutative and unitary ring of fractional operators may be defined as follows

$$ {}_m\mathrm{R}(A_\alpha(o_x^\alpha),\mathbb{R}) := ({}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{R}),\circ,*), \tag{54}$$

in which it is not difficult to verify the following properties:

1. The pair $({}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{R}),\circ)$ is an Abelian group.
2. The pair $({}_m\mathrm{G}(A_\alpha(o_x^\alpha),\mathbb{R}),*)$ is a commutative monoid.
3. $\forall A_\alpha^{\circ p}, A_\alpha^{\circ q}, A_\alpha^{\circ r} \in {}_m\mathrm{R}(A_\alpha(o_x^\alpha),\mathbb{R})$, the operation $*$ is distributive with respect to the operation $\circ$, that is,

$$ \begin{cases} A_\alpha^{\circ p} * \left(A_\alpha^{\circ q} \circ A_\alpha^{\circ r}\right) = \left(A_\alpha^{\circ p} * A_\alpha^{\circ q}\right) \circ \left(A_\alpha^{\circ p} * A_\alpha^{\circ r}\right) \\ \left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) * A_\alpha^{\circ r} = \left(A_\alpha^{\circ p} * A_\alpha^{\circ r}\right) \circ \left(A_\alpha^{\circ q} * A_\alpha^{\circ r}\right) \end{cases}. \tag{55}$$

Author Contributions: Conceptualization, Methodology, Formal Analysis, Investigation, Writing—Original Draft Preparation, Writing—Review and Editing, A.T.-H.; Formal Analysis, Validation, Supervision, Project Administration, F.B.-P. and R.R.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley-Interscience: Hoboken, NJ, USA, 1993; pp. 1–125.
2. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2000; pp. 3–73.
3. Oldham, K.; Spanier, J. *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*; Elsevier: Amsterdam, The Netherlands, 1974; Volume 111, pp. 25–121.
4. Kilbas, A.; Srivastava, H.; Trujillo, J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; pp. 69–132.
5. De Oliveira, E.C.; Tenreiro Machado, J.A. A review of definitions for fractional derivatives and integral. *Math. Probl. Eng.* **2014**, *2014*, 238459. [CrossRef]
6. Teodoro, G.S.; Machado, J.T.; De Oliveira, E.C. A review of definitions of fractional derivatives and other operators. *J. Comput. Phys.* **2019**, *388*, 195–208. [CrossRef]
7. Valério, D.; Ortigueira, M.D.; Lopes, A.M. How Many Fractional Derivatives Are There? *Mathematics* **2022**, *10*, 737. [CrossRef]
8. Safdari-Vaighani, A.; Heryudono, A.; Larsson, E. A radial basis function partition of unity collocation method for convection–diffusion equations arising in financial applications. *J. Sci. Comput.* **2015**, *64*, 341–367. [CrossRef]
9. Torres-Hernandez, A.; Brambila-Paz, F.; Torres-Martínez, C. Numerical solution using radial basis functions for multidimensional fractional partial differential equations of type Black–Scholes. *Comput. Appl. Math.* **2021**, *40*, 245. [CrossRef]
10. Traore, A.; Sene, N. Model of economic growth in the context of fractional derivative. *Alex. Eng. J.* **2020**, *59*, 4843–4850. [CrossRef]
11. Tejado, I.; Pérez, E.; Valério, D. Fractional calculus in economic growth modelling of the group of seven. *Fract. Calc. Appl. Anal.* **2019**, *22*, 139–157. [CrossRef]
12. Guariglia, E. Fractional calculus, zeta functions and Shannon entropy. *Open Math.* **2021**, *19*, 87–100. [CrossRef]
13. Torres-Henandez, A.; Brambila-Paz, F. An Approximation to Zeros of the Riemann Zeta Function Using Fractional Calculus. *Math. Stat.* **2021**, *9*, 309–318. [CrossRef]
14. De-la Vega, E.; Torres-Hernandez, A.; Rodrigo, P.M.; Brambila-Paz, F. Fractional derivative-based performance analysis of hybrid thermoelectric generator-concentrator photovoltaic system. *Appl. Therm. Eng.* **2021**, *193*, 116984. [CrossRef]
15. Torres-Hernandez, A.; Brambila-Paz, F.; Rodrigo, P.M.; De-la-Vega, E. Reduction of a nonlinear system and its numerical solution using a fractional iterative method. *J. Math. Stat. Sci.* **2020**, *6*, 285–299.
16. Erfanifar, R.; Sayevand, K.; Esmaeili, H. On modified two-step iterative method in the fractional sense: Some applications in real world phenomena. *Int. J. Comput. Math.* **2020**, *97*, 2109–2141. [CrossRef]
17. Cordero, A.; Girona, I.; Torregrosa, J.R. A variant of chebyshev's method with 3αth-order of convergence by using fractional derivatives. *Symmetry* **2019**, *11*, 1017. [CrossRef]
18. Gdawiec, K.; Kotarski, W.; Lisowska, A. Newton's method with fractional derivatives and various iteration processes via visual analysis. *Numer. Algorithms* **2021**, *86*, 953–1010. [CrossRef]
19. Gdawiec, K.; Kotarski, W.; Lisowska, A. Visual analysis of the Newton's method with fractional order derivatives. *Symmetry* **2019**, *11*, 1143. [CrossRef]
20. Akgül, A.; Cordero, A.; Torregrosa, J.R. A fractional Newton method with 2αth-order of convergence and its stability. *Appl. Math. Lett.* **2019**, *98*, 344–351. [CrossRef]
21. Torres-Hernandez, A.; Brambila-Paz, F. Fractional Newton-Raphson Method. *Appl. Math. Sci. Int. J. (MathSJ)* **2021**, *8*, 1–13. [CrossRef]
22. Torres-Hernandez, A.; Brambila-Paz, F.; Iturrarán-Viveros, U.; Caballero-Cruz, R. Fractional Newton-Raphson Method Accelerated with Aitken's Method. *Axioms* **2021**, *10*, 47. [CrossRef]
23. Torres-Hernandez, A.; Brambila-Paz, F.; De-la-Vega, E. Fractional Newton-Raphson Method and Some Variants for the Solution of Nonlinear Systems. *Appl. Math. Sci. Int. J. (MathSJ)* **2020**, *7*, 13–27. [CrossRef]
24. Candelario, G.; Cordero, A.; Torregrosa, J.R. Multipoint Fractional Iterative Methods with ($2\alpha+ 1$) th-Order of Convergence for Solving Nonlinear Problems. *Mathematics* **2020**, *8*, 452. [CrossRef]
25. Candelario, G.; Cordero, A.; Torregrosa, J.R.; Vassileva, M.P. An optimal and low computational cost fractional Newton-type method for solving nonlinear equations. *Appl. Math. Lett.* **2022**, *124*, 107650. [CrossRef]
26. Osler, T.J. Leibniz rule for fractional derivatives generalized and an application to infinite series. *SIAM J. Appl. Math.* **1970**, *18*, 658–674. [CrossRef]
27. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]
28. Fu, H.; Wu, G.C.; Yang, G.; Huang, L.L. Continuous time random walk to a general fractional Fokker–Planck equation on fractal media. *Eur. Phys. J. Spec. Top.* **2021**, *230*, 3927–3933. [CrossRef]
29. Fan, Q.; Wu, G.C.; Fu, H. A note on function space and boundedness of the general fractional integral in continuous time random walk. *J. Nonlinear Math. Phys.* **2022**, *29*, 95–102. [CrossRef]
30. Abu-Shady, M.; Kaabar, M.K. A Generalized Definition of the Fractional Derivative with Applications. *Math. Probl. Eng.* **2021**, *2021*, 9444803. [CrossRef]
31. Saad, K.M. New fractional derivative with non-singular kernel for deriving Legendre spectral collocation method. *Alex. Eng. J.* **2020**, *59*, 1909–1917. [CrossRef]

32. Rahmat, M.R.S. A new definition of conformable fractional derivative on arbitrary time scales. *Adv. Differ. Equ.* **2019**, *2019*, 354. [CrossRef]
33. Sousa, J.V.d.C.; De Oliveira, E.C. On the ψ-Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [CrossRef]
34. Jarad, F.; Uğurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 247. [CrossRef]
35. Atangana, A.; Gómez-Aguilar, J. A new derivative with normal distribution kernel: Theory, methods and applications. *Phys. A Stat. Mech. Appl.* **2017**, *476*, 1–14. [CrossRef]
36. Yavuz, M.; Özdemir, N. Comparing the new fractional derivative operators involving exponential and Mittag-Leffler kernel. *Discret. Contin. Dyn. Syst.-S* **2020**, *13*, 995. [CrossRef]
37. Liu, J.G.; Yang, X.J.; Feng, Y.Y.; Cui, P. New fractional derivative with sigmoid function as the kernel and its models. *Chin. J. Phys.* **2020**, *68*, 533–541. [CrossRef]
38. Yang, X.J.; Machado, J.T. A new fractional operator of variable order: Application in the description of anomalous diffusion. *Phys. A Stat. Mech. Its Appl.* **2017**, *481*, 276–283. [CrossRef]
39. Atangana, A. On the new fractional derivative and application to nonlinear Fisher's reaction–diffusion equation. *Appl. Math. Comput.* **2016**, *273*, 948–956. [CrossRef]
40. He, J.H.; Li, Z.B.; Wang, Q.L. A new fractional derivative and its application to explanation of polar bear hairs. *J. King Saud Univ.-Sci.* **2016**, *28*, 190–192. [CrossRef]
41. Sene, N. Fractional diffusion equation with new fractional operator. *Alex. Eng. J.* **2020**, *59*, 2921–2926. [CrossRef]
42. Torres-Hernandez, A.; Brambila-Paz, F. Sets of Fractional Operators and Numerical Estimation of the Order of Convergence of a Family of Fractional Fixed-Point Methods. *Fractal Fract.* **2021**, *5*, 240. [CrossRef]
43. Torres-Hernandez, A.; Brambila-Paz, F.; Montufar-Chaveznava, R. Acceleration of the order of convergence of a family of fractional fixed point methods and its implementation in the solution of a nonlinear algebraic system related to hybrid solar receivers. *Appl. Math. Comput.* **2022**, *429*, 127231. [CrossRef]
44. Torres-Hernandez, A. Code of a multidimensional fractional quasi-Newton method with an order of convergence at least quadratic using recursive programming. *Appl. Math. Sci. Int. J. (MathSJ)* **2022**, *9*, 17–24. [CrossRef]