Proceding Paper

Generic Riemannian Maps from Nearly Kaehler Manifolds †

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Abstract: In order to generalise semi-invariant Riemannian maps, Sahin first introduced the idea of “Generic Riemannian maps”. We extend the idea of generic Riemannian maps to the case in which the total manifold is a nearly Kaehler manifold. We study the integrability conditions for the horizontal distribution although vertical distribution is always integrable. We also study the geometry of foliations of two distributions and obtain the necessary and sufficient condition for generic Riemannian maps to be totally geodesic. Additionally, we study the generic Riemannian map with umbilical fibers.

Keywords: nearly Kaehler manifold; Riemannian maps; generic Riemannian maps; anti-invariant; semi-invariant Riemannian maps and product manifolds

1. Introduction

The idea of a Riemannian map between Riemannian manifolds plays a key role in differential geometry, and this idea was first introduced by Fischer [1] as a generalization of the notions of an isometric immersion and a Riemannian submersion.

Let us consider the smooth map \( F : (\mathcal{M}, g_{\mathcal{M}}) \rightarrow (\mathcal{B}, g_{\mathcal{B}}) \) between Riemannian manifolds \((\mathcal{M}, g_{\mathcal{M}})\) and \((\mathcal{B}, g_{\mathcal{B}})\). Then, the tangent bundle of \( \mathcal{M} \) has the following decomposition:

\[
\mathcal{T} \mathcal{M} = (\ker F_*) \oplus (\ker F_*)^\perp,
\]

where the kernal space of \( F_* \) is denoted by \( \ker F_* \) and its orthogonal complement is denoted by \( (\ker F_*)^\perp \). We denote a range space of \( F_* \) by \( \text{rang} F_* \) and its orthogonal complement by \( (\text{rang} F_*)^\perp \). Then, the tangent bundle \( \mathcal{T} \mathcal{B} \) of \( \mathcal{B} \) has following decomposition:

\[
\mathcal{T} \mathcal{B} = (\text{rang} F_*) \oplus (\text{rang} F_*)^\perp.
\]

There are many articles on the geometry of a Riemannian map [1,2]. In this paper, we introduce and study generic Riemannian maps from nearly Kaehler manifolds to Riemannian manifolds.

2. Preliminaries

In this section, we recall some fundamentals of almost Hermitian manifolds, Kaehler manifolds, and nearly Kaehler manifolds, and give a brief review of Riemannian maps and generic Riemannian maps.

Let \( \mathcal{M} \) be an almost complex manifold with an almost complex structure \( J \) and a Riemannian metric \( g_{\mathcal{M}} \) satisfying the condition

\[
g_{\mathcal{M}}(JX, JY) = g_{\mathcal{M}}(X, Y),
\]

(1)
for all $X, Y \in \Gamma(TM)$. Then, $\mathcal{M}$ is called an almost Hermitian manifold. Let $\nabla$ be the Levi-civita connection on an almost Hermitian manifold $\mathcal{M}$; then, $\mathcal{M}$ is called a Kaehler manifold if

$$(\nabla_X J)Y = 0,$$  

and $\mathcal{M}$ is called a nearly Kaehler manifold if the tensor field $\nabla J$ is skew symmetric, i.e.,

$$(\nabla_X J)Y + (\nabla_Y J)X = 0,$$  

for all $X, Y \in \Gamma(TM)$.

Let $F : (M, g_M) \rightarrow (B, g_B)$ be a Riemannian map between Riemannian manifolds. Then, the geometry of Riemannian maps is characterized by the tensor fields $T$ and $A$, which are B.O’Neils fundamental tensor fields defined for the Riemannian submersion. For arbitrary vector fields $E$ and $F$, the tensor fields $T$ and $A$ is defined as follows:

$$A_EF = H\nabla_{HE}VF + V\nabla_{HE}HF,$$  

$$T_EF = H\nabla_{VE}VF + V\nabla_{VE}HF.$$  

Using (4) and (5), we have the following Lemma

**Lemma 1** ([3]). Let $X, Y \in \Gamma(kerF^\perp)$ and $U, V \in \Gamma(kerF)$; then, we have

$$\nabla_U V = T_U V + \hat{\nabla}_U V,$$  

$$\nabla_U X = H\nabla_{UX}X + T_U X,$$  

$$\nabla_X U = A_X U + V\nabla_X U,$$  

$$\nabla_X Y = H\nabla_{XY}Y + A_X Y,$$  

where $\hat{\nabla}_U V = V\nabla_{EU} V$.

3. Generic Riemannian Maps

In this section, we define generic Riemannian maps. We investigate the integrability of the leaves of distribution and obtain the necessary and sufficient conditions for such maps to be totally geodesic. For such maps, we also obtain a decomposition theorem for total manifolds.

First, we recall the following definition [4].

**Definition 1.** Let us consider a Riemannian map $F$ from an almost Hermitian manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$. If the dimension $D_p$ is constant along $M$ and it defines a differentiable distribution on $M$, then we say that $F$ is a generic Riemannian map, where $D_p$ is the complex subspace of the vertical space where $p \in M$.

For a generic Riemannian map,

$$kerF_s = D \oplus D^\perp,$$  

where $D^\perp$ is the orthogonal complementary distribution of $D$ in $\Gamma(kerF_s)$. For any $U \in \Gamma(kerF)$, by the definition of a generic Riemannian map, we write

$$FU = \phi U + \omega U,$$
where \( \phi U \in (\ker F) \) and \( \omega U \in \Gamma(\ker F) \).

We denote the orthogonal complementary distribution of \( \omega D^\perp \) in \( \Gamma(\ker F) \) by \( \mu \). Thus, for any \( X \in \Gamma(\ker F) \), we write
\[
JX = BX + CX,
\]
where \( BX \in \Gamma(D)^\perp \) and \( CX \in \Gamma(\mu) \).

Using (10), for \( U \in \Gamma(\ker F) \), we set
\[
JU = P_1U + P_2U + \omega U,
\]
where the orthogonal projections from \( \ker F \) to \( D \) and \( D^\perp \) are \( P_1 \) and \( P_2 \), respectively.

The covariant derivative of a \((1,1)\) tensor field \( J \) was firstly defined by Ali and Fatima [5]. For arbitrary tangent vector fields \( E \) and \( F \) on \( M \), we set
\[
(\nabla_E J)F = P_E F + Q_E F,
\]
where \( P_E F \) and \( Q_E F \) denote the horizontal and the vertical parts of \( (\nabla_E J)F \), respectively. If \( M \) is a nearly Kaehler manifold, then
\[
P_E F = -P_I E, \quad Q_E F = -Q_I E.
\]

Now, we investigate the integrability of distribution.

**Theorem 1.** Let \( F : (M, g_M, J) \rightarrow (B, g_B) \) be a generic Riemannian map from a nearly Kaehler manifold \( (M, g_M, J) \) to a Riemannian manifold \( (B, g_B) \). Then, the distribution \( D^\perp \) is integrable if and only if
\[
\hat{\nabla} V P_2 U - \hat{\nabla} U P_2 V + T_V \omega U - T_U \omega V + 2Q_V U \in \Gamma(D)^\perp,
\]
for any \( U, V \in \Gamma(D)^\perp \).

**Proof.** For any \( U, V \in \Gamma(D)^\perp \), using Lemma 1 and Equations (10), (11), and (13)–(15), we obtain
\[
[U, V] = \phi(\hat{\nabla} V P_2 U - \hat{\nabla} U P_2 V + T_V \omega U - T_U \omega V + 2Q_V U)
+ \omega(\hat{\nabla} V P_2 U - \hat{\nabla} U P_2 V + T_V \omega U - T_U \omega V + 2Q_V U)
+ B(T_V P_2 U - T_U P_2 V + \mathcal{H} V \omega U - \mathcal{H} U \omega V + 2P_V U)
+ C(T_V P_2 U - T_U P_2 V + \mathcal{H} V \omega U - \mathcal{H} U \omega V + 2P_V U).
\]

Since \( \Gamma(\ker F) \) is integrable, \( [U, V] \in \Gamma(\ker F) \). Comparing the vertical part in (16), we obtain the result. \( \square \)

In a similar vein, we prove the following.

**Theorem 2.** Let \( F : (M, g_M, J) \rightarrow (B, g_B) \) be a proper generic Riemannian map from a nearly Kaehler manifold \( (M, g_M, J) \) to a Riemannian manifold \( (B, g_B) \). Then, the distribution \( D \) is integrable if and only if
\[
\hat{\nabla} U P_2 V - \hat{\nabla} V P_2 U - 2Q_U V \in \Gamma(D),
\]
and
\[
T_U P_2 V - T_V P_2 U - 2P_{U} V \in \Gamma(\mu),
\]
for \( U, V \in \Gamma(D) \).

We now study the geometry of the leaves of distributions \( D \) and \( D^\perp \), and we have following propositions.
Proposition 1. Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \rightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then, the distribution $\mathcal{D}$ defines a totally geodesic foliation in $\mathcal{M}$ if and only if

(i) $\nabla_X p_Z + T_X \omega Z - Q_X Z$ has no component in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D})^\perp$.

(ii) $\nabla X BW + T_X CW - Q_X Z$ has no component in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $W \in \Gamma(\ker F_{\ast})^\perp$.

**Proof.** For $X, Y \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D})^\perp$, using Equations (1) and (13)–(15) and Lemma 1, we obtain

$$g_{\mathcal{M}}(\nabla_X Y, Z) = -g_{\mathcal{M}}(\nabla_X p_Z + T_X \omega Z - Q_X Z, JY).$$

(17)

Now, for $X, Y \in \Gamma(\mathcal{D})$ and $W \in \Gamma(\ker F_{\ast})^\perp$, again using Equations (1) and (13)–(15) and Lemma 1, we obtain

$$g_{\mathcal{M}}(\nabla_X Y, W) = -g_{\mathcal{M}}(\nabla_X BW + T_X CW - Q_X W, JY).$$

(18)

From Equations (17) and (18), we obtain the required result. $\square$

Proposition 2. Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \rightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then, the distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation if and only if

(i) $\nabla U p_V + T_U \omega V - Q_U V = 0$,

(ii) $CT_U p_V + CH U \omega V - CP_U V$ has no components in $\mu$, for $U, V \in \Gamma(\mathcal{D})^\perp$.

From Propositions 1 and 2, we have the following decomposition theorem.

Theorem 3. Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \rightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then, the fibers are a locally product Riemannian manifold of the form $\mathcal{M}_D \times \mathcal{M}_\mu$, if and only if

(i) $\nabla Y p_U + T_Y \omega U - Q_Y U = 0$ for $Y \in \Gamma(\ker F_{\ast})$ and $U \in \Gamma(\mathcal{D})^\perp$.

(ii) $CT_U p_V + CH U \omega V - CP_U V$ has no component in $\mu$, for $U, V \in \Gamma(\mathcal{D})^\perp$.

(iii) $\nabla X BW + T_X CW - Q_X Z$ has no component in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $W \in \Gamma(\ker F_{\ast})^\perp$.

(iv) $\nabla X p_Z + T_X \omega Z - Q_X Z$ has no component in $\mathcal{D}$ for $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proposition 3. Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \rightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then, the distribution $(\ker F_{\ast})$ defines a totally geodesic foliation in $\mathcal{M}$ if and only if

$$T_V \phi W + H \nabla \omega W + P_W V \in \Gamma(\omega \mathcal{D}^\perp),$$

and $$\nabla V \phi W + T_V \omega W + Q_W V \in \Gamma(\mathcal{D}),$$

for any $V, W \in \Gamma(\ker F_{\ast})$.

Proposition 4. Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \rightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then, the distribution $(\ker F_{\ast})^\perp$ defines a totally geodesic foliation in $\mathcal{M}$ if and only if

$$BA_X BY + B \nabla_X CY + BP_Y X = -\phi \nabla_X BY - \phi A_X CY - \phi Q_Y X,$$

for $X, Y \in (\ker F_{\ast})^\perp$. 


Proof. Let $X, Y \in (\ker F_\ast)^\perp$. Using Equations (11), (12), (14), and (15) and Lemma 1, we obtain

$$\nabla_X Y = -B(A_X B Y + H\nabla_X C Y + P_Y X) - C(A_X B Y + H\nabla_X C Y + P_Y X) - \phi(\hat{\nabla}_X B Y + A_X C Y + Q_Y X) - \omega(\hat{\nabla}_X B Y + A_X C Y + Q_Y X).$$

From Equation (19), we obtain the result. \qed

We recall a Riemannian map with totally umbilical fibers if

$$T_U V = g_M(U, V)H,$$

for all $U, V \in \Gamma(\ker F_\ast)$, where $H$ is the mean curvature vector of the fibers.

Therefore, we have the following:

**Theorem 4.** Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a generic Riemannian map with totally umbilical fibers from a nearly Kaehler manifold $(M, g_M, J)$ onto a Riemannian manifold $(B, g_B)$. Then, $H \in \Gamma(\omega_D)^\perp$.

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