Existence and Uniqueness of a Solution to a Wentzell’s Problem with Non-Linear Delays

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Abstract: In this work, we study the existence and uniqueness of the solution to a wave equation with dynamic Wentzell-type boundary conditions on a part of the boundary \( \Gamma_1 \) of the domain \( \Omega \) with non-linear delays in non-linear dampings in \( \Omega \) and on \( \Gamma_1 \), using the Faedo–Galerkin method.

Keywords: wave equation; Wentzell boundary conditions; non-linear dampings; non-linear delays; Faedo–Galerkin method

1. Introduction

We consider the following coupled system wave/Wentzell:

\[
\begin{aligned}
\partial_t u - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_2(v_t) &= 0, & & \text{in } \Omega \times (0, \infty), \\
\partial_t v + \partial_n u - \Delta_T v + \mu'_1 g'_1(v_t) + \mu'_2 g'_2(v_t) &= 0, & & \text{on } \Gamma_1 \times (0, \infty), \\
u &= v, & & \text{on } \Gamma_0 \times (0, \infty), \\
\mu \partial_n u &= 0, & & \text{on } \Gamma_0 \times (0, \infty), \\
(u(0), v(0)) &= (u_0, v_0), & & \text{in } \Omega \times \Gamma, \\
(u_1(0), v_1(0)) &= (u_1, v_1), & & \text{in } \Omega \times \Gamma, \\
u_1(x, t - \tau) &= f_0(x, t - \tau), & & \text{on } \Omega \times (0, \tau), \\
v_1(x, t - \tau) &= f_0(x, t - \tau), & & \text{on } \Gamma_1 \times (0, \tau),
\end{aligned}
\] (1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), with smooth boundary \( \Gamma = \partial \Omega \), divided into two closed and disjoint subsets \( \Gamma_0 \) and \( \Gamma_1 \), such that \( \Gamma_0 \cap \Gamma_1 = \emptyset \) and \( \Gamma_0 \cup \Gamma_1 = \Gamma \). We denote by \( \nabla_T \) the tangential gradient on \( \Gamma \), by \( \Delta_T \) the tangential Laplacian on \( \Gamma \) and by \( \partial_n \), the normal derivative where \( n \) represents the unit outward normal to \( \Gamma \).

\( \mu_1, \mu_2, \mu'_1 \) and \( \mu'_2 \) are positive real numbers, the two functions \( g_1(u_t) \) and \( g_2(v_t) \) describe the delays on the non-linear frictional dissipations \( g_1(u_t) \) and \( g_2(v_t) \), on \( \Omega \) and \( \Gamma_1 \), respectively. \( \tau > 0 \) is a time delay and \( u_0, v_0, u_1, v_1, f_0 \) and \( f_0 \) are the initial data in some suitable (Sobolev) function space.

Throughout history, the wave equation has known a great deal of research. In our work, we are particularly interested in the wave equation with Wentzell-type boundary conditions, characterized by the presence of differential operators (\( \Delta_T u \)) of the same order as the main operator.

These problems are involved in the modelling of many phenomena: mechanical-like elasticity, diffusion processes or wave propagation physics.

Wentzell-type conditions are obtained by asymptotic methods from transmission problems, (see Lemrabet K. [1]).

The following condition:

\[ \partial_n u - \Delta_T u = g, \text{ on } \Gamma \]
for this equation
\[-\triangle u + u = f, \quad \text{in } \Omega\]
was first introduced by Wentzell (Ventcel) in 1959, (see [2]), for diffusion processes. It models the heat exchange of the body \( \Omega \) with the surrounding environment in the presence of a thin film, a very good conductor, on the surface of the body.

Delay is the property of a physical system by which the response to an applied force is slowed in its effect. Whenever material, information, or energy is physically transmitted from one place to another, there is a delay present in the law of feedback modelling the mechanical shift over time.

Delays often occur in many ways: physical problems, chemical, biological and economic phenomena.

The system (1) describes the vibrations of a flexible body with a thin boundary layer of high rigidity on its boundary \( \Gamma \).

Our goal is to show that this problem is well posed, and that a unique solution exists.

1.1. Assumptions on the Damping and Delay Functions \( g_i \) for \( i = 1, 2 \)

We pose the following assumptions on the damping and delay functions:

(A1) \( g_i : \mathbb{R} \rightarrow \mathbb{R} \) is an odd non-decreasing function of the class \( C^1(\mathbb{R}) \), such that there exist \( r \) (sufficiently small), \( c_i, C_i, c_1, c_2 \) and \( \alpha_1, \alpha_2 > 0 \) for \( i = 1, 2 \), and a convex, increasing function \( H : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) of the class \( C^1(\mathbb{R}_+) \cap C^2([0, \infty[) \) that satisfies:

\[ H(0) = 0 \text{ and } H \text{ is linear on } [0, r] \text{ or } (H'(0) = 0 \text{ and } H'' > 0 \text{ on } [0, r]), \]

\[ c_i |s| \leq |g_i(s)| \leq C_i |s| \quad \text{if } |s| \geq r, \quad (2) \]

\[ s^2 + g_i^2(s) \leq H^{-1}(sg_i(s)) \quad \text{if } |s| \leq r, \quad (3) \]

\[ |g'_i(s)| \leq c, \quad (4) \]

\[ \alpha_1 sg_i(s) \leq G_i(s) \leq \alpha_2 sg_i(s), \quad (5) \]

where
\[ G_i(s) = \int_0^s g_i(y)dy. \]

(A2) \( \alpha_2 \mu_1 < \alpha_1 \mu_1 \) and \( \alpha_2 \mu_2 < \alpha_1 \mu_1' \).

1.2. Transformation of Problem (1)

Now, as in [3], we introduce the new variables:

\[
\begin{cases}
  z_1(x, \rho, t) = u_t(x, t - \rho \tau), & x \in \Omega, \ \rho \in (0, 1), \ t > 0,
  \\
  z_2(x, \rho, t) = v_t(x, t - \rho \tau), & x \in \Gamma_1, \ \rho \in (0, 1), \ t > 0,
\end{cases}
\]

where \( \tau > 0 \) is a time delay.

Then, we have

\[
\begin{cases}
  \tau(z_1)_t(x, \rho, t) + (z_1)_\rho(x, \rho, t) = 0, & \text{on } \Omega \times (0, 1) \times (0, \infty),
  \\
  \tau(z_2)_t(x, \rho, t) + (z_2)_\rho(x, \rho, t) = 0, & \text{on } \Gamma_1 \times (0, 1) \times (0, \infty),
\end{cases}
\]

where \( (z_1)_t = \frac{\partial z_1}{\partial t} \) and \( (z_i)_\rho = \frac{\partial z_i}{\partial \rho} \) for \( i = 1, 2 \).

Therefore, problem (1) is equivalent to
\[
\begin{align*}
&u_t - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_2(z_1(x,1,t)) = 0, &\text{in} & &\Omega \times (0,\infty), \\
v_t + \partial_x u - \Delta v + \mu_1' g_2(v_t) + \mu_2' g_2(z_2(x,1,t)) = 0, &\text{on} & &\Gamma_1 \times (0,\infty), \\
\tau(z_1)_1(x,\rho, t) + (z_1)_2(x,\rho, t) = 0, &\text{in} & &\Omega \times (0,1) \times (0,\infty), \\
\tau(z_2)_1(x,\rho, t) + (z_2)_2(x,\rho, t) = 0, &\text{on} & &\Gamma_1 \times (0,1) \times (0,\infty), \\
u = v, &\text{in} & &\Omega \times (0,\infty), \\
u = 0, &\text{on} & &\Gamma_1 \times (0,\infty), \\
z_1(x,0,t) = u_t(x,t), &\text{in} & &\Omega \times (0,\infty), \\
z_2(x,0,t) = v_t(x,t), &\text{on} & &\Gamma_1 \times (0,\infty), \\
(u(0),v(0)) = (u_0,v_0), &\text{in} & &\Omega \times \Gamma, \\
(u_t(0),v_t(0)) = (u_1,v_1), &\text{in} & &\Omega \times \Gamma, \\
z_1(x,\rho, 0) = f_0(x,-\rho\tau), &\text{in} & &\Omega \times (0,1), \\
z_2(x,\rho, 0) = f_0(x,-\rho\tau), &\text{on} & &\Gamma_1 \times (0,1).
\end{align*}
\]

1.3. Energy of System (6)

Let \(\xi\) and \(\zeta\) be strictly positive constants, such that
\[
\tau \frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau \frac{\mu_1-\alpha_2\mu_2}{\alpha_2},
\]
\[
\tau \frac{\mu_2'(1-\alpha_1)}{\alpha_1} < \zeta < \tau \frac{\mu_1'-\alpha_2\mu_2'}{\alpha_2}.
\]

We define the energy associated with the solution to problem (6) by
\[
E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|v_t\|^2_{\Gamma_1^1} + \frac{1}{2} \|\nabla v\|^2_{\Gamma_1^1} + \xi \int_{\Omega} \left( \int_0^1 G_1(z_1(x,\rho,t))d\rho \right) dx + \zeta \int_{\Gamma_1} \left( \int_0^1 G_2(z_2(x,\rho,t))d\rho \right) d\sigma,
\]
where \(\|\cdot\| = (\_,\_)^{\frac{1}{2}}\) and \(\|\cdot\|_{\Gamma_1^1} = (\_,\_)^{\frac{1}{2}}\), (the norms associated with the inner products in \(L^2(\Omega)\) and \(L^2(\Gamma_1)\), respectively).

1.4. Energy Decay

Therefore, we have the following lemma on the dissipation of energy \(E(t)\):

**Lemma 1.** Let \((u,v,z_1,z_2)\) be a solution to problem (6). Then, the energy functional defined by (9) satisfies
\[
E'(t) \leq -a_1 \int_{\Omega} u_t g_1(u_t)dx - a_2 \int_{\Gamma_1} v_t g_2(v_t) d\sigma - a_3 \int_{\Omega} z_1(z_1(x,1,t))g_1(z_1(x,1,t))dx - a_4 \int_{\Gamma_1} z_2(z_2(x,1,t))g_2(z_2(x,1,t))d\sigma \leq 0, \quad \forall t \geq 0,
\]

where \(a_1 = (\mu_1 - \xi \alpha_2 - \mu_2 \alpha_2)\), \(a_2 = (\mu_1' - \zeta \alpha_2 - \mu_2' \alpha_2)\), \(a_3 = (a_1 \xi - \mu_2(1-\alpha_1))\) and \(a_4 = (a_1 \zeta - \mu_2'(1-\alpha_1))\).

For the proof of Lemma 1, see [4].
2. The Main Results

We introduce the following set

\[ H^1_{Γ_0}(Ω) = \{ u ∈ H^1(Ω), u|_{Γ_0} = 0 \}, \]

which is given the Hilbert structure induced by \( H^1(Ω) \).

Then, we consider the canonical norms of \( H^1_{Γ_0}(Ω) \) and \( H^1(Γ_1) \)

\[ \|u\|_{H^1_{Γ_0}(Ω)}^2 = \|∇u\|^2, \quad \|v\|_{H^1(Γ_1)}^2 = \|∇TV\|^2. \]

Now, we state the following unique result exists:

**Theorem 1.** Let \( (u_0,u_1,v_0,v_1) ∈ [H^2(Ω) ∩ H^1_{Γ_0}(Ω)] × [H^1_{Γ_0}(Ω)] × [H^2(Γ_1)] × [H^1(Γ_1)] \), \( f_0 ∈ H^1_{Γ_0}(Ω; H^1(0,1)) \) and \( f_0 ∈ H^1(Γ_1; H^1(0,1)) \) satisfying the following compatibility condition:

\[ \begin{aligned}
   & \partial_ν u_0 - \Delta TV_0 + μ_1g_2(v_1) = 0, & \text{on} & \ Γ_1, \\
   & f_0(.,0) = u_t, & \text{in} & \ Ω, \\
   & f_0(.,0) = v_t, & \text{on} & \ Γ_1.
\end{aligned} \]  

(11)

Assume that (A1) and (A2) hold, then problem (6) possesses a unique globally weak solution verifying \( T > 0 \):

\[ (u,u_t,u_{tt}) ∈ L^∞(0,T; [H^1_{Γ_0}(Ω)]^2 × L^2(Ω)), \]

\[ (v,v_t,v_{tt}) ∈ L^∞(0,T; [H^1(Γ_1)]^2 × L^2(Γ_1)). \]

The proof of Theorem 1 is given below using the Faedo-Galerkin approximation.

**Proof of Theorem 1.** Throughout this proof, assume \( (u_0,v_0) ∈ (H^2(Ω) ∩ H^1_{Γ_0}(Ω)) × (H^2(Γ_1) ∩ H^1(Γ_1)), (u_1,v_1) ∈ H^1_{Γ_0}(Ω) × H^1(Γ_1), f_0 ∈ H^1_{Γ_0}(Ω; H^1(0,1)) \) and \( f_0 ∈ H^1(Γ_1; H^1(0,1)). \)

For any \( n ∈ N \), we denote by \( U_n \) and \( V_n \) the two finite dimensional spaces defined by, respectively, \( U_n = span \{ w_1, w_2, ..., w_n \} \) and \( V_n = span \{ ˜w_1, ˜w_2, ..., ˜w_n \} \), where \( \{ w_i \}_{1 ≤ i ≤ n} \) and \( \{ ˜w_i \}_{1 ≤ i ≤ n} \) are the basis in the spaces \( H^2(Ω) ∩ H^1_{Γ_0}(Ω) \) and \( H^2(Γ_1) \), respectively.

Now, define for \( 1 ≤ i ≤ n \) the sequences \( φ_i(t) \) and \( ˜φ_i(t) \) as follows:

\[ \begin{aligned}
   & φ_i(t,0) = w_i, \\
   & ˜φ_i(t,0) = ˜w_i,
\end{aligned} \]

then extend \( φ_i(x,0) \) by \( φ_i(x,ρ) \) over \( L^2(Ω ∩ (0,1))), \) and \( ˜φ_i(x,0) \) by \( ˜φ_i(x,ρ) \) over \( L^2(Γ_1 ∩ (0,1))) \), and denote \( Z_n, ˜Z_n \) as the linear spaces generated by \( \{ φ_1, φ_2, ..., φ_n \} \) and \( \{ ˜φ_1, ˜φ_2, ..., ˜φ_n \} \), respectively.

Let us define the approximations \( u^n, v^n, z^n_1 \) and \( z^n_2 \) by

\[ \begin{aligned}
   & u^n(t) = \sum_{i=1}^{n} a_i^n(t)w_i, \\
   & v^n(t) = \sum_{i=1}^{n} b_i^n(t)˜w_i, \\
   & z^n_1(t) = \sum_{i=1}^{n} c_i^n(t)φ_i, \\
   & z^n_2(t) = \sum_{i=1}^{n} d_i^n(t)˜φ_i
\end{aligned} \]

where \( a_i^n, b_i^n, c_i^n \) and \( d_i^n \) are from the class \( C^2 \) and determined by the following differential equations:

\[ \begin{aligned}
   & (u^n_T, w_i) + (∇u^n, ∇w_i) + μ_1(g_1(u^n), w_i) + μ_2(g_2(z^n_1(x,1,t)), w_i) \\
   & + (v^n_T, ˜w_i)_Γ + (∇TV^n, ∇T ˜w_i)_Γ + μ_1^2(g_2(v^n_1), ˜w_i)_Γ + μ_2^2(g_2(z^n_2(x,1,t)), ˜w_i)_Γ = 0, & 1 ≤ i ≤ n,
\end{aligned} \]  

(12)
with initial data:

\[
\int_\Omega \int_0^1 (\tau z_i^n + z_{ji}^n) \phi_i d\rho dx = 0, \quad 1 \leq i \leq n
\]  

(13)

and

\[
\int_{\Gamma_1} \int_0^1 (\tau z_i^n + z_{ji}^n) \phi_i d\rho d\sigma = 0, \quad 1 \leq i \leq n,
\]  

(14)

The local existence of solutions to problems (12)–(15) is standard by the theory of ordinary differential equations. We can conclude that \( t_n > 0 \) such that in \([0, t_n]\), problems (12)–(15) have a unique local solution which can be extended to a maximum interval \([0, T]\) (with \( 0 < T \leq \infty \)) by Zorn’s lemma, since the non-linear terms in (12) are locally Lipschitz continuous.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for \((u^n, v^n, z_i^n, z_{ji}^n)\).

**The first estimate**

Since the sequences \((u_i^n)_i\), \((u_j^n)_i\), \((v_i^n)_i\), \((v_j^n)_i\), \((z_i^n)_i\) and \((z_{ji}^n)_i\) converge, the standard calculations, using (12)–(15) similar to those used to find (10), yield a number \(M_1\) independent of \(n\) such that

\[
E_\nu(t) + a_1 \int_0^t \int_\Omega u_i^n g_1(u_i^n) dx ds + a_3 \int_0^t \int_\Omega z_i^n(x, 1, s) g_1(z_i^n(x, 1, s)) dx ds
+ a_2 \int_0^t \int_{\Gamma_1} v^n g_2(v^n) d\sigma ds + a_4 \int_0^t \int_{\Gamma_1} z_{ji}^n(x, 1, s) g_2(z_{ji}^n(x, 1, s)) d\sigma ds
\leq E_\nu(0) \leq M_1,
\]  

(16)

where

\[
E_\nu(t) = \frac{1}{2} (||u^n||^2 + ||v^n||^2 + ||v_i^n||_{1,1}^2 + ||v_j^n||_{1,1}^2)
+ \xi \int_\Omega \left( \int_0^1 G_1(z_i^n(x, \rho, t) d\rho \right) dx + \xi \int_{\Gamma_1} \left( \int_0^1 G_2(z_{ji}^n(x, \rho, t) d\rho \right) d\sigma
\]

and \(a_i\), \(i = 1, ..., 4\) are defined in Lemma 1.

The estimate (16) implies that the solution \((u^n, v^n, z_i^n, z_{ji}^n)_n\) exists globally in \([0, +\infty)\).

Estimate (16) yields for any \( T > 0 \)

\[
u^n \text{ is bounded in } L^\infty\left(0, T; H^1_{00}(\Omega)\right),
\]  

(17)

\[
u^n \text{ is bounded in } L^\infty\left(0, T; H^1(\Gamma_1)\right),
\]  

(18)

\[
u_i^n \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right),
\]  

(19)

\[
u_{ji}^n \text{ is bounded in } L^\infty\left(0, T; L^2(\Gamma_1)\right).
\]  

(20)
We have the equalities

\[
\begin{align*}
\|u^n_i\|_{H^1(0, T)} & \text{ is bounded in } L^1(\Omega \times (0, T)), \\
\|v^n_i\|_{H^1(0, T)} & \text{ is bounded in } L^1(\Gamma_1 \times (0, T)), \\
G_1(z^n_i) & \text{ is bounded in } L^{\infty}\left(0, T; L^1(\Omega \times (0, 1))\right), \\
G_2(z^n_i) & \text{ is bounded in } L^{\infty}\left(0, T; L^1(\Gamma_1 \times (0, 1))\right), \\
\end{align*}
\]

\[
\begin{align*}
z^n_i(x, t)g_1(z^n_i(x, t)) & \text{ is bounded in } L^1(\Omega \times (0, T)), \\
z^n_i(x, t)g_2(z^n_i(x, t)) & \text{ is bounded in } L^1(\Gamma_1 \times (0, T)). \\
\end{align*}
\]

**The second estimate**

We need to estimate \( u^n_i(0) \) and \( v^n_i(0) \) in norms \( L^2(\Omega) \) and \( L^2(\Gamma_1) \), respectively. By taking \( t = 0 \) and considering \( \tilde{w}_i = u^n_i(0) \) and \( \tilde{w}_i = v^n_i(0) \) in (12), we obtain

\[
\begin{align*}
\|u^n_i(0)\|^2 + (\nabla u^n_i, \nabla u^n_i(0)) + \mu_1(g_1(u^n_i), u^n_i(0)) + \mu_2(g_1(z^n_i), u^n_i(0)) \\
+ \|v^n_i(0)\|^2 + (\nabla \nabla u^n_i, \nabla \nabla u^n_i(0))_\Gamma + \mu_1'(g_2(v^n_i), v^n_i(0))_\Gamma + \mu_2'(g_2(z^n_i), v^n_i(0))_\Gamma \\
= 0. \\
\end{align*}
\]

We have the equalities

\[
\begin{align*}
(\nabla u^n_i, \nabla u^n_i(0)) & = -(\Delta u^n_i, u^n_i(0)) + (\partial_t u^n_i, v^n_i(0))_\Gamma, \\
(\nabla \nabla u^n_i, \nabla \nabla u^n_i(0))_\Gamma & = - (\Delta \nabla u^n_i, v^n_i(0))_\Gamma. \\
\end{align*}
\]

Employing Young’s inequality on (28) and (29) and using the fact that if \( u^n_i \in H^1(\Omega) \cap H^2(\Gamma_1) \), then \( \partial_t u^n_i \in H^{1/2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1) \); hence, \( \partial_t u^n_i \in L^2(\Gamma_1) \); we obtain,

\[
\begin{align*}
(\nabla u^n_i, \nabla u^n_i(0)) & = -(\Delta u^n_i, u^n_i(0)) + (\partial_t u^n_i, v^n_i(0))_\Gamma \\
& \leq \frac{1}{4\epsilon} \|\Delta u^n_i\|^2 + \frac{1}{4\epsilon} \|\partial_t u^n_i\|^2_{\Gamma_1} + \epsilon \|u^n_i(0)\|^2 + \epsilon \|v^n_i(0)\|^2_{\Gamma_1}, \\
-(\Delta \nabla u^n_i, v^n_i(0))_\Gamma & \leq \frac{1}{4\epsilon} \|\Delta \nabla u^n_i\|^2_{\Gamma_1} + \epsilon \|v^n_i(0)\|^2_{\Gamma_1}, \\
\mu_1(g_1(u^n_i), u^n_i(0)) & \leq \frac{\mu_1^2}{4\epsilon} \|g_1(u^n_i)\|^2 + \epsilon \|u^n_i(0)\|^2, \\
\mu_1'(g_2(v^n_i), v^n_i(0))_\Gamma & \leq \frac{\mu_1^2}{4\epsilon} \|g_2(v^n_i)\|^2_{\Gamma_1} + \epsilon \|v^n_i(0)\|^2_{\Gamma_1}, \\
\mu_2(g_1(z^n_i), u^n_i(0)) & \leq \frac{\mu_2^2}{4\epsilon} \|g_1(z^n_i)\|^2 + \epsilon \|u^n_i(0)\|^2, \\
\mu_2'(g_2(z^n_i), v^n_i(0))_\Gamma & \leq \frac{\mu_2^2}{4\epsilon} \|g_2(z^n_i)\|^2_{\Gamma_1} + \epsilon \|v^n_i(0)\|^2_{\Gamma_1}, \\
\end{align*}
\]
by re-introducing (30)–(35) in (27), with a suitable choice of \( \epsilon \) and since \((g_1(u^n_t))_n\), \((g_1(z^n_{\Omega}))_n\)
and \((g_2(v^n_t))_n\), \((g_2(z^n_{\Omega}))_n\) are bounded in \(L^2(\Omega)\) and \(L^2(\Gamma_1)\), respectively, by \((A1)\), \((A2)\)
and initial data (15), we obtain

\[
\|u^n_t(0)\| + \|v^n_t(0)\|_{\Gamma_1} \leq M_2,
\]

where \(M_2\) is a positive constant independent of \(n\) and depends on the initial data.

Next, differentiating (12) with respect to \(t\), multiplying the resulting equation by \((a^n_t)_H(t)\) in \(\Omega\) and \((b^n_t)_H(t)\) on \(\Gamma_1\), and summing over \(i\) from 1 to \(n\), we obtain

\[
\frac{1}{2} \frac{d}{dt}\left\{ \|u^n_t\|^2 + \|\nabla u^n_t\|^2 + \|v^n_t\|^2_{\Gamma_1} + \|\nabla v^n_t\|^2_{\Gamma_1}\right\} + \mu_1 \int_\Omega |u^n_t|^2 (g_1)_i(u^n_t) dx + \mu_2 \int_\Omega u^n_t (z^n_{\Omega})_i(x,1,t)(g_1)_i(z^n_{\Omega}(x,1,t)) dx
\]

\[
+ \mu'_1 \int_{\Gamma_1} |v^n_t|^2 (g_2)_i(v^n_t) d\sigma + \mu'_2 \int_{\Gamma_1} v^n_t (z^n_{\Omega})_i(x,1,t)(g_2)_i(z^n_{\Omega}(x,1,t)) d\sigma = 0.
\]

Differentiating (13) with respect to \(t\), multiplying the resulting equation by \((c^n_t)_t(t)\) and summing over \(i\) from 1 to \(n\), it follows that

\[
\frac{\tau}{2} \frac{d}{dt}\left\|c^n_t(\rho,t)\|^2_{L^2(\Omega \times (0,1))} + \frac{1}{2} \frac{d}{d\rho}\|c^n_t(\rho,t)\|^2_{L^2(\Omega \times (0,1))} = 0.
\]

Analogously, differentiating (14) with respect to \(t\), multiplying the resulting equation by \((d^n_t)_t(t)\) and summing over \(i\) from 1 to \(n\), it follows that

\[
\frac{\tau}{2} \frac{d}{dt}\left\|d^n_t(\rho,t)\|^2_{L^2(\Gamma_1 \times (0,1))} + \frac{1}{2} \frac{d}{d\rho}\|d^n_t(\rho,t)\|^2_{L^2(\Gamma_1 \times (0,1))} = 0.
\]

Taking the sum of (37)–(39), we obtain

\[
\frac{1}{2} \frac{d}{dt}\left\{ \|u^n_t\|^2 + \|\nabla u^n_t\|^2 + \|v^n_t\|^2_{\Gamma_1} + \|\nabla v^n_t\|^2_{\Gamma_1}\right\} + \mu_1 \int_\Omega |u^n_t|^2 (g_1)_i(u^n_t) dx + \frac{1}{2} \int_\Omega |(z^n_{\Omega})_i(x,1,t)|^2 dx
\]

\[
+ \mu'_1 \int_{\Gamma_1} |v^n_t|^2 (g_2)_i(v^n_t) d\sigma + \frac{1}{2} \int_{\Gamma_1} |(z^n_{\Omega})_i(x,1,t)|^2 d\sigma = 0.
\]

Using (4) and Young’s inequality, we obtain

\[
\mu_2 \int_\Omega u^n_t (z^n_{\Omega})_i(x,1,t)(g_1)_i(z^n_{\Omega}(x,1,t)) dx \leq \epsilon \int_\Omega |(z^n_{\Omega})_i(x,1,t)|^2 dx + \left(\frac{\mu_2 \epsilon}{4}\right) \|u^n_t\|^2,
\]

\[
\mu'_2 \int_{\Gamma_1} v^n_t (z^n_{\Omega})_i(x,1,t)(g_2)_i(z^n_{\Omega}(x,1,t)) d\sigma \leq \epsilon \int_{\Gamma_1} |(z^n_{\Omega})_i(x,1,t)|^2 d\sigma + \left(\frac{\mu'_2 \epsilon}{4}\right) \|v^n_t\|^2_{\Gamma_1}.
\]
Re-introducing (41) and (42) into (40), and choosing \( \varepsilon \) small enough, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left\{ \| u^n_t \|^2 + \| \nabla u^n_t \|^2 + \| v^n_t \|^2 + \| \nabla \tau v^n_t \|^2 \right\}_H + \tau \| (z^n_{t})_1(x, \rho, t) \|^2_{L^2(\Omega \times (0,1))} + \tau \| ((z^n_{t})_1(x, \rho, t)) \|^2_{L^2(H^1_0(\Omega) \times (0,1))} \\
+ \mu_1 \int_\Omega |u^n_{tt}|^2 (g_1)_1(u^n_t) \, dx + c \int_\Omega \| (z^n_{t})_1(x, 1, t) \|^2 \, dx \\
+ \mu'_1 \int_\Gamma |v^n_{tt}|^2 (g_2)_1(v^n_t) \, d\sigma + c \int_\Gamma \| (z^n_{t})_1(x, 1, t) \|^2 \, d\sigma \\
\leq c' \left\{ \| u^n_t \|^2 + \| v^n_t \|^2 \right\}_H.
\]

Integrating the last inequality over \((0, t)\), we obtain

\[
\frac{1}{2} \left\{ \| u^n_0 \|^2 + \| v^n_0 \|^2 \right\}_H + \| \nabla u^n_t \|^2 + \| \nabla \tau v^n_t \|^2 + \tau \| (z^n_{t})_1(x, 0, t) \|^2_{L^2(\Omega \times (0,1))} \\
+ \tau \| ((z^n_{t})_1(x, \rho, t)) \|^2_{L^2(H^1_0(\Omega) \times (0,1))} \\
+ \mu_1 \int_0^t \int_\Omega |u^n_{tt}|^2 (g_1)_1(u^n_t) \, dxds + c \int_\Omega \| (z^n_{t})_1(x, 1, t) \|^2 \, dxds \\
+ \mu'_1 \int_0^t \int_\Gamma |v^n_{tt}|^2 (g_2)_1(v^n_t) \, d\sigma ds + c \int_\Gamma \| (z^n_{t})_1(x, 1, t) \|^2 \, d\sigma ds \\
\leq \frac{1}{2} \left\{ \| u^n_0 \|^2 + \| v^n_0 \|^2 \right\}_H + \| \nabla u^n_t \|^2 + \| \nabla \tau v^n_t \|^2 \\
+ \tau \| (z^n_{t})_1(x, 0, t) \|^2_{L^2(\Omega \times (0,1))} + \tau \| ((z^n_{t})_1(x, \rho, t)) \|^2_{L^2(H^1_0(\Omega) \times (0,1))} \\
+ c \int_0^t \int_\Omega \| (z^n_{t})_1(x, \rho, t) \|^2 \, dxds + \mu_1 \int_0^t \int_\Omega |u^n_{tt}|^2 (g_1)_1(u^n_t) \, dxds \\
+ \mu'_1 \int_0^t \int_\Gamma |v^n_{tt}|^2 (g_2)_1(v^n_t) \, d\sigma ds + c \int_0^t \int_\Gamma \| (z^n_{t})_1(x, 1, t) \|^2 \, d\sigma ds \\
\leq M_3,
\]

where \( M_3 \) is independent of \( n \) and for all \( t \in [0, T] \). Therefore, we conclude that

\[
u^n_t \quad \text{is bounded in} \quad L^\infty \left( 0, T; H^1_0(\Omega) \right), \quad \text{(43)}
\]

\[
u^n_t \quad \text{is bounded in} \quad L^\infty \left( 0, T; H^1(\Gamma_1) \right), \quad \text{(44)}
\]

\[
u^n_t \quad \text{is bounded in} \quad L^\infty \left( 0, T; L^2(\Omega) \right), \quad \text{(45)}
\]

\[
u^n_t \quad \text{is bounded in} \quad L^\infty \left( 0, T; L^2(\Gamma_1) \right), \quad \text{(46)}
\]

\[
u^n_t \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega \times (0,1))), \quad \text{(47)}
\]
Combining (49) and (50), we have
\[
\tau \frac{d}{dt} \| \nabla z^n_1(\rho, t) \|^2_{L^2(\Omega \times (0, 1))} + \frac{1}{2} \frac{d}{d\rho} \| \nabla z^n_1(\rho, t) \|^2_{L^2(\Omega \times (0, 1))} = 0. \tag{49}
\]

Similarly, replacing \( \tilde{\phi}_i \) with \(-\Delta \tilde{\phi}_i \) in (14), multiplying the resulting equation by \( d^n_i(t) \) and summing over \( i \) from 1 to \( n \), it follows that
\[
\tau \frac{d}{dt} \| \nabla z^n_2(\rho, t) \|^2_{L^2(\Gamma_1 \times (0, 1))} + \frac{1}{2} \frac{d}{d\rho} \| \nabla z^n_2(\rho, t) \|^2_{L^2(\Gamma_1 \times (0, 1))} = 0. \tag{50}
\]

Combining (49) and (50), we have
\[
\tau \frac{d}{dt} \| \nabla z^n_1(\rho, t) \|^2_{L^2(\Omega \times (0, 1))} + \frac{1}{2} \frac{d}{d\rho} \| \nabla z^n_1(\rho, t) \|^2_{L^2(\Gamma_1 \times (0, 1))} + \frac{1}{2} \left\{ \| \nabla u^n \|^2 + \| \nabla T v^n \|^2_{\Gamma_1} \right\} = 0.
\]

Integrating the last inequality over \((0, t)\) and using Gronwall’s lemma, we have
\[
\tau \frac{1}{2} \| \nabla z^n_1(\rho, t) \|^2_{L^2(\Omega \times (0, 1))} + \frac{1}{2} \frac{\tau}{2} \| \nabla z^n_2(\rho, t) \|^2_{L^2(\Gamma_1 \times (0, 1))} \leq e^{\tau T} \left\{ \frac{1}{2} \| \nabla z^n_1(x, \rho, 0) \|^2_{L^2(\Omega \times (0, 1))} + \frac{1}{2} \left\| \nabla T z^n_2(x, \rho, 0) \right\|^2_{L^2(\Gamma_1 \times (0, 1))} \right\},
\]
for all \( t \in [0, T] \). Therefore, we conclude that
\[
z^n_1 \text{ is bounded in } L^\infty \left(0, T; H^1_{10} \left( \Omega; L^2(0, 1) \right) \right), \tag{51}
\]
\[
z^n_2 \text{ is bounded in } L^\infty \left(0, T; H^1 \left( \Gamma_1; L^2(0, 1) \right) \right). \tag{52}
\]

**Passing the limit**

Applying Dunford–Petti’s theorem, we conclude that there exists subsequences of \((u^n)_n\), \((v^n)_n\), \((z^n_1)_n\) and \((z^n_2)_n\) which we still denote by \((u^n)_n\), \((v^n)_n\), \((z^n_1)_n\) and \((z^n_2)_n\), respectively, such that from (17), (43) and (45), we obtain
\[
(u^n, u^n, u^n_H) \rightharpoonup (u, u_t, u_H) \text{ weakly star in } L^\infty \left(0, T; \left[ H^1_{10}(\Omega) \right]^2 \times L^2(\Omega) \right), \tag{53}
\]
from (18), (44) and (46), we obtain
\[
(v^n, v^n_t, v^n_H) \rightharpoonup (v, v_t, v_H) \text{ weakly star in } L^\infty \left(0, T; \left[ H^1(\Gamma_1) \right]^2 \times L^2(\Gamma_1) \right), \tag{54}
\]
from (51) and (52), we find
\[
z^n_1 \rightharpoonup z_1 \text{ weakly star in } L^\infty \left(0, T; H^1_{10} \left( \Omega; L^2(0, 1) \right) \right), \tag{55}
\]
\[
z^n_2 \rightharpoonup z_2 \text{ weakly star in } L^\infty \left(0, T; H^1 \left( \Gamma_1; L^2(0, 1) \right) \right), \tag{56}
\]
We have the following two lemmas, (for the proof, see [4]):

Thanks to Aubin–Lions’s theorem, (see [5]), we deduce that there exists subsequences from (19)–(26), we have

and from (19)–(26), we have

We can deduce from (60) and (62) that

Analysis of the non-linear terms

Denoted by \( Q = \Omega \times (0, T) \) and \( \Sigma = \Gamma_1 \times (0, T) \).

We can deduce from (60) and (62) that

and

We have the following two lemmas, (for the proof, see [4]):

**Lemma 2.** For each \( T > 0 \), \( g_1(u^n_t) \), \( g_1(z_1^n, 1, t) \) \( \in L^1(Q) \) and \( g_2(v^n_t) \), \( g_2(z_2^n, 1, t) \) \( \in L^1(\Sigma) \), we have

where \( A_1 \) and \( A_2 \) are constants independent of \( t \).
Lemma 3. We have the following convergences

\[
\begin{aligned}
\left\{ \begin{array}{l}
g_1(u^n_t) \to g_1(u_t) \text{ in } L^1(\Omega \times (0, T)), \\
g_2(v^n_t) \to g_2(v_t) \text{ in } L^1(\Gamma_1 \times (0, T)).
\end{array} \right.
\end{aligned}
\]

Hence, from Lemma 3, we deduce that

\[
\begin{aligned}
\left\{ \begin{array}{l}
g_1(u^n_t) \to \chi_1 = g_1(u_t) \text{ weakly in } L^2(\Omega \times (0, T)), \\
g_2(v^n_t) \to \chi_2 = g_2(v_t) \text{ weakly in } L^2(\Gamma_1 \times (0, T)),
\end{array} \right. \tag{67}
\end{aligned}
\]

and

\[
\begin{aligned}
\left\{ \begin{array}{l}
g_1(z^n_1(x, 1, t)) \to \Psi_1 = g_1(z_1(x, 1, t)) \text{ weakly in } L^2(\Omega \times (0, T)), \\
g_2(z^n_2(x, 1, t)) \to \Psi_2 = g_2(z_2(x, 1, t)) \text{ weakly in } L^2(\Gamma_1 \times (0, T)).
\end{array} \right. \tag{68}
\end{aligned}
\]

Now, returning to (12) and using standard arguments, we can show from the above estimates that

\[
u_{tt} - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_1(z_1(., 1, .)) = 0, \quad \text{in } \mathcal{D}'(\Omega \times (0, T)). \tag{69}\]

Since \( u_{tt}, g_1(u_t) \) and \( g_1(z_1(., 1, .)) \in L^2(0, T; L^2(\Omega)) \), we obtain from identity (69)

\[
\Delta u \in L^2(0, T; L^2(\Omega)),
\]

and therefore identity (69) yields

\[
u_{tt} - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_1(z_1(., 1, .)) = 0, \quad \text{in } L^2(0, T; L^2(\Omega)). \tag{70}\]

Taking (70) into account and making use of the generalized Green’s formula, we deduce that

\[
\partial_t \nu - \Delta \nabla v = -\mu'_1 g_2(v_t) - \mu'_2 g_2(z_2(., 1, .)) - v_{tt}, \quad \text{in } \mathcal{D}'(0, T; H^{-1}(\Gamma_1)),
\]

and since \( v_{tt}, g_2(v_t) \) and \( g_2(z_2(., 1, .)) \in L^2(0, T; L^2(\Gamma_1)) \), we infer that

\[
\partial_t \nu - \Delta \nabla v = -\mu'_1 g_2(v_t) - \mu'_2 g_2(z_2(., 1, .)) - v_{tt}, \quad \text{in } L^2(0, T; L^2(\Gamma_1)),
\]

then

\[
v_{tt} + \partial_t \nu - \Delta \nabla v + \mu'_1 g_2(v_t) + \mu'_2 g_2(z_2(., 1, .)) = 0, \quad \text{in } L^2(0, T; L^2(\Gamma_1)).
\]

Exploiting convergences (55)–(58), (63) and (64), we pass to the limit in (13) and (14) to obtain

\[
\int_0^T \int_0^1 \int_\Omega (\tau z_{1_t} + z_{1_p}) \theta_1 dx dp dt = 0, \quad \forall \theta_1 \in L^2\left(0, T; H^1_0(\Omega \times (0, 1))\right),
\]

\[
\int_0^T \int_0^1 \int_{\Gamma_1} (\tau z_{2_t} + z_{2_p}) \theta_2 dx dp dt = 0, \quad \forall \theta_2 \in L^2\left(0, T; H^1(\Gamma_1 \times (0, 1))\right).
\]

Uniqueness
Let \((u, v, z_1, z_2)\) and \((\tilde{u}, \tilde{v}, \tilde{z}_1, \tilde{z}_2)\) be two solutions to problem (6). Then \((U, V, Z_1, Z_2) = (u, v, z_1, z_2) - (\tilde{u}, \tilde{v}, \tilde{z}_1, \tilde{z}_2)\) verifies the following system of equations:

\[
\begin{align*}
U_t - \Delta U + \mu_1 g_1(u_t) - g_1(u_t) \\
+ \mu_2 g_1(z_1(x, 1, t)) - g_1(z_1(x, 1, t)) = 0, \quad &\text{in} \quad \Omega \times (0, \infty), \\
V_t + \partial_v U - \Delta T V + \mu'_1 g_2(v_t) - \mu'_{12}(\tilde{v}_t) \\
+ \mu'_{12} g_2(z_2(x, 1, t)) - g_2(z_2(x, 1, t)) = 0, \quad &\text{on} \quad \Gamma_1 \times (0, \infty), \\
\tau Z_1(x, \rho, t) + Z_{1t}(x, \rho, t) = 0, \quad &\text{in} \quad \Omega \times (0, 1) \times (0, \infty), \\
\tau Z_2(x, \rho, t) + Z_{2t}(x, \rho, t) = 0, \quad &\text{on} \quad \Gamma_1 \times (0, 1) \times (0, \infty), \\
U = V, \quad &\text{on} \quad \Gamma_0 \times (0, \infty), \\
U = 0, \quad &\text{on} \quad \Omega \times (0, \infty), \\
Z_1(x, 0, t) = U_t(x, t), \quad &\text{in} \quad \Omega \times (0, \infty), \\
Z_2(x, 0, t) = V_t(x, t), \quad &\text{on} \quad \Gamma_1 \times (0, \infty), \\
(U(0), V(0)) = (0, 0), \quad &\text{in} \quad \Omega \times \Gamma, \\
(U_t(0), V_t(0)) = (0, 0), \quad &\text{in} \quad \Omega \times \Gamma, \\
Z_1(x, \rho, 0) = 0, \quad &\text{in} \quad \Omega \times (0, 1), \\
Z_2(x, \rho, 0) = 0, \quad &\text{on} \quad \Gamma_1 \times (0, 1).
\end{align*}
\tag{71}
\]

Multiplying the first equation of (71) by \(U_t\), we have

\[
(U_{tt}, U_t) - (\Delta U, U_t) + \mu_1 (g_1(u_t) - g_1(\tilde{u}_t), U_t) + \mu_2 (g_1(z_1(x, 1, t)) - g_1(\tilde{z}_1(x, 1, t)), U_t) = 0,
\tag{72}
\]

Next, integrating over \(\Omega\), we obtain

\[-(\Delta U, U_t) = (\nabla U, \nabla U_t) - (\partial_v U, V_t)_{\Gamma_1},\]

then, (72) becomes

\[
(U_{tt}, U_t) + (\nabla U, \nabla U_t) - (\partial_v U, V_t)_{\Gamma_1} + \mu_1 (g_1(u_t) - g_1(\tilde{u}_t), U_t) + \mu_2 (g_1(z_1(x, 1, t)) - g_1(\tilde{z}_1(x, 1, t)), U_t) = 0,
\]

which is equivalent to

\[
\frac{1}{2} \frac{d}{dt} \|U_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla U\|^2 - (\partial_v U, V_t)_{\Gamma_1} + \mu_1 (g_1(u_t) - g_1(\tilde{u}_t), U_t) + \mu_2 (g_1(z_1(x, 1, t)) - g_1(\tilde{z}_1(x, 1, t)), U_t) = 0.
\tag{73}
\]

Multiplying the second equation of (71) by \(V_t\), we have

\[
(V_{tt}, V_t)_{\Gamma_1} + (\partial_v U, V_t)_{\Gamma_1} - (\Delta T V, V_t)_{\Gamma_1} + \mu'_{1} (g_2(v_t) - g_2(\tilde{v}_t), V_t)_{\Gamma_1} + \mu'_{12} (g_2(z_2(x, 1, t)) - g_2(\tilde{z}_2(x, 1, t)), V_t)_{\Gamma_1} = 0,
\tag{74}
\]

Next, integrating over \(\Gamma_1\), we obtain

\[-(\Delta T V, V_t)_{\Gamma_1} = (\nabla T V, \nabla V_t)_{\Gamma_1},\]
then (74) becomes
\[
\begin{align*}
(V_t, V_t)_{\Gamma_1} + (\partial_t U, V_t)_{\Gamma_1} + (\nabla V, \nabla V)_{\Gamma_1} + \mu_1'(g_2(u) - g_2(\tilde{u}), V_t)_{\Gamma_1} \\
+ \mu_2'(g_2(z_2(x, 1, t)) - g_2(\tilde{z}_2(x, 1, t)), V_t)_{\Gamma_1}
\end{align*}
\]

\[= 0,
\]
which is equivalent to
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|V_t\|^2 + 1 \frac{d}{dt} \|\nabla V\|^2 + 1 \frac{d}{dt} \|V_t\|^2 + 1 \frac{d}{dt} \|\nabla V\|^2_{\Gamma_1} \\
+ \mu_1(g_1(u) - g_1(\tilde{u}), U_t) + \mu_1'(g_2(v) - g_2(\tilde{v}), V_t)_{\Gamma_1} \\
+ \mu_2(g_1(z_1(x, 1, t)) - g_1(\tilde{z}_1(x, 1, t)), U_t) \\
+ \mu_2'(g_2(z_2(x, 1, t)) - g_2(\tilde{z}_2(x, 1, t)), V_t)_{\Gamma_1}
\end{align*}
\]

\[= 0. \tag{75}
\]

Now, summing (73) and (75), we obtain
\[
\frac{\tau}{2} \frac{d}{dt} \int_0^1 \|Z_1(x, \rho, t)\|^2 d\rho + \frac{1}{2} \left\{ \|Z_1(x, 1, t)\|^2 - \|U_t(x, t)\|^2 \right\} = 0, \tag{77}
\]
\[
\frac{\tau}{2} \frac{d}{dt} \int_0^1 \|Z_2(x, \rho, t)\|^2 d\rho + \frac{1}{2} \left\{ \|Z_2(x, 1, t)\|^2 - \|V_t(x, t)\|^2 \right\} = 0. \tag{78}
\]

Similarly, multiplying the third and fourth equations of (71) by $Z_1$ and $Z_2$, respectively, and integrating over $\Omega \times (0, 1)$ and $\Gamma_1 \times (0, 1)$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|U_t\|^2 + \|\nabla U\|^2 + \|V_t\|^2_{\Gamma_1} + \|\nabla V\|^2_{\Gamma_1} \\
+ \tau \int_0^1 \|Z_1(x, \rho, t)\|^2 d\rho + \tau \int_0^1 \|Z_2(x, \rho, t)\|^2_{\Gamma_1} d\rho \right\}
\]
\[= -\mu_2(g_1(z_1(x, 1, t)) - g_1(\tilde{z}_1(x, 1, t)), U_t) \\
- \mu_2'(g_2(z_2(x, 1, t)) - g_2(\tilde{z}_2(x, 1, t)), V_t)_{\Gamma_1}
\]
\[+ \frac{1}{2} \|U_t(x, t)\|^2 + \frac{1}{2} \|V_t(x, t)\|^2_{\Gamma_1}
\]
\[\leq \frac{1}{2} \|U_t(x, t)\|^2 + \|g_1(z_1(x, 1, t)) - g_1(\tilde{z}_1(x, 1, t))\|_2^2 \|U_t(x, t)\|^2 \\
+ \frac{1}{2} \|V_t(x, t)\|^2_{\Gamma_1} + \|g_2(z_2(x, 1, t)) - g_2(\tilde{z}_2(x, 1, t))\|_2^2 \|V_t(x, t)\|^2_{\Gamma_1},
\]

Next, using condition (4) and Young’s inequality, we find
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|U_t\|^2 + \|\nabla U\|^2 + \|V_t\|^2_{\Gamma_1} + \|\nabla V\|^2_{\Gamma_1} \\
+ \tau \int_0^1 \|Z_1(x, \rho, t)\|^2 d\rho + \tau \int_0^1 \|Z_2(x, \rho, t)\|^2_{\Gamma_1} d\rho \right\}
\leq c \left\{ \|U_t(x, t)\|^2 + \|V_t(x, t)\|^2_{\Gamma_1} + \|Z_1(x, 1, t)\|^2 + \|Z_2(x, 1, t)\|^2_{\Gamma_1} \right\}
\]
where \( c > 0 \). Then, integrating over \((0, t)\) and using Gronwall’s lemma, we conclude that
\[
\|U_t\|^2 + \|\nabla U\|^2 + \|V_t\|_{\Gamma_1}^2 + \|\nabla T V\|_{\Gamma_1}^2 + \tau \int_0^1 \|Z_1(x, \rho, t)\|^2 d\rho + \tau \int_0^1 \|Z_2(x, \rho, t)\|^2_{\Gamma_1} d\rho
\]
\[
= 0,
\]
which implies \((U, V, Z_1, Z_2) = 0\).
This finishes the proof of Theorem 1. \( \square \)

3. Conclusions

In this article, we have proven the existence of a unique solution to a wave equation with dynamic Wentzell-type boundary conditions on a part of the boundary \( \Gamma_1 \) of the domain \( \Omega \) with non-linear delays in non-linear dampings in \( \Omega \) and on \( \Gamma_1 \), using the Faedo–Galerkin method.

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