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# Applications of (*h*,*q*)-Time Scale Calculus to the Solution of Partial Differential Equations †

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**Abstract:** In this article, we developed the idea of $q$-time scale calculus in quantum geometry. It includes the q-time scale integral operators and $\Delta_q$-differentials. It analyzes the fundamental principles which follow the calculus of $q$-time scales compared with the Leibnitz–Newton usual calculus and have few crucial consequences. The $\Delta_q$-differential reduced method of transformations was proposed to work out on partial $\Delta_q$-differential equations in time scale. With easily computable coefficients, the result is calculated in the version of a power series which is convergent. The performance and effectiveness of the proposed procedure are also illustrated, and Matlab software is applied for calculation with the support of some fascinating examples. It changes when $\sigma(t) = t$ and $q = 1$; then, the solution merges with usual calculus for the mentioned initial value problem. The finding of the present work is that the $\Delta_q$-differential transformation reduced method is convenient and efficient.

**Keywords:** $\Delta q$-differential; q-time scale; $q$-Integral operators; $\Delta q$-differential reduced transform method; partial differential equations

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## 1. Introduction

In the sense of mathematical objects, each and every theory of physics is articulated. Therefore, it is important to launch a number of formulas to aim any physical objects and concepts towards mathematical objectives where we study to epitomize them. As in classical mechanics, this function frequently appears in consequences of many theories, such as quantum mechanics, the mathematical objects rarely revealing. This study mainly focused on the closed quantum systems which consist of intrinsic components such as states, observables, measurements, and evolution. Quantum geometry which dates back to the early days of quantum mechanics is characterized by Heisenberg's commutation relations [1,2]

$$ih = [p, v] \tag{1}$$

These relations indicate that the classical phase space geometry is lost when position and momentum coordinates fail to commute. This leads to a non-commutative geometric space that is distinct from algebraic geometry, where the spaces are affine schemes built on a correspondence between spaces and commutative algebras. The Gelfand–Naimark theorem [3] provides a closer connection to differential geometry, associating spaces with topological spaces and commutative C*-algebras. Recent work has shown that non-commutative geometry in quantum geometry is intimately linked to delta $q$-deformed calculus, which is a generalization of quantum calculus. Our goal is to use $\Delta_q$-calculus results to study non-commutative differential equations, specifically by employing the reduced $q$ delta differential transform method to solve partial $\Delta_q$-differential equations. Furthermore, we introduce this algebraic operator [4–6] to form some non-commutative geometric spaces.

Contrary to algebraic geometry [7,8] which is established on a mapping between commutative algebras and spaces, this mapping in particular links of the space with the functions of an algebra on it, and then, in a completely algebraic structure, geometric concepts are described. This rule is the top logical initiative for general geometry such as quantum geometry. At the same time, they are affine schemes spaces for algebraic geometry. Non-commutative differential equations in time scales calculus have important applications in various areas of mathematics and physics, including quantum groups, quantum field theory, and statistical mechanics.

Regarding quantum geometry, Maliki et al. in [9] studied the concepts of deformed $q$-calculus. Here, the authors demonstrated that the $q$-calculus, an expanded version of the Leibnitz and Newton standard calculus, and the mathematical discipline of invariant geometry are intimately related. In this study, we review a few results from the $q$-time scale calculus that will aid in our observation of invariant differential equations. In particular, we will use the $q$-Delta differential transform reduced method (qDDRTM) to examine partial $q$-Delta differential equations.

## 2. Operator of the $q$-Delta Differential

For $1 < q \in< R$, we establish the delta $q$-derivative $\Delta_q$ as

$$\Delta_q f(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{2}$$

Note that $\Delta_q \to f'(t) \equiv \frac{d}{dt}$, as $q$ tends to 1.

We assume the following supporting notable points:

(a) Jackson [10] discussed the modified version of $q$-derivative and its several consequences in the twentieth century.
(b) For functions that do not have 0 in their definition domain, the $\Delta_q$-derivative can be calculated. As $q$ is at 1, it decreases to a common derivative.
(c) It can be checked easily that the $\Delta_q$-operator is a linear operator, i.e.,

$$\begin{gathered} \text{(i) } \Delta_q(f+g)\ \Delta_q f + \Delta_q g, \\ \text{(ii) } \Delta_q(\lambda f) = \lambda \Delta_q f \end{gathered} \tag{3}$$

## 3. The $\Delta_q$ Derivative of Few Transcendental Mappings and the Non-Commutative Concept

Maliki et al. [9] discussed a non-commutative differential equation in $q$ calculus is the $q$-difference equation:

$$D_q f(x) = \frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \text{where x} \neq 0, 0 < q < 1$$

$f(qx) - f(x) = \frac{\frac{d_q}{d_q x} f(x)(q-1)}{q}$, where $q$ is a deformation parameter that determines the degree of noncommutativity in the calculus.

$$f(\sigma(t)) - f(t) = \frac{\Delta_q f(t)\mu(t)}{q} \tag{4}$$

The non-commutativity of the $q$ delta derivative makes it more challenging to solve, but methods such as the $q$ delta differential transform method can be used to obtain solutions.

Following the method of calculating the $\Delta_q$-derivative (non-commutative in time scales calculus) by the first principles, we now obtain salient features for the $\Delta_q$-operator of the mappings mentioned below, such as $e^t$ and Sint.

$q$-Delta operator of the Function is

$$h(t) = sin \tag{5}$$

By definition, we have

$$\Delta_q Sin(t) = \frac{sin(\sigma(t)) - sint}{\mu(t)}, \tag{6}$$

$$\Delta_q(Sint) = \frac{\left(\sigma(t) - \frac{1}{3!}(\sigma(t))^3 + \frac{1}{5!}(\sigma(t))^5 - \frac{1}{7!}(\sigma(t))^7 + \cdots\cdots\right) - \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\cdots\right)}{\mu(t)}. \tag{7}$$

Note: by setting $\sigma(t)$ = $qt$in $\Delta_q(Sin\ t)$, we obtain the results of quantum calculus.
By using the result,
when $\sigma(t) = t$ in Equation (7), we obtain the standard derivative of the sine function, i.e.,

$$\begin{aligned} \tfrac{d}{dt}(sint) &= 1 - \tfrac{1}{3!}t^2(3) + \tfrac{1}{5!}t^4(5) - \tfrac{1}{7!}t^6(7) + \cdots\cdots \\ &= 1 - \tfrac{1}{2!}t^2 + \tfrac{1}{4!}t^4 - \tfrac{1}{6!}t^6 + \cdots\cdots = cost \end{aligned} \tag{8}$$

## 4. *q*-Time Scale Factorials and *q*-Timescale Numbers

In essence, we adopt the notations and symbols in [10]. Thus, $\mathbb{Z}^+$ denotes the set of integers which is positive. Moreover, $M$ denotes a field which has 0 characteristics throughout this research article, and *M(q)* represents the rational functions field in one parameter $q$ over *N(q)*. In the $q$-deformed setting, *N(q)* is our ground field, while $N$ is the ground field in the standard setting. We define the $q$-binomials, $q$-integers and $q$-factorials, respectively, as follows:

$$[[p]]_q = \frac{q^t - 1}{q - 1} = \sum_{j=0}^{p-1} q^j; \tag{9}$$

$$[[p]]_q! = [[p]]_q \times [[p-1]]_q \times [[p-2]]_q \times \cdots\cdots \times [[3]]_q \times [[2]]_q \times [[1]]_q \\ \text{, where } [[0]]_q! = 1; \tag{10}$$

$$\begin{Bmatrix} p \\ r \end{Bmatrix}_q = \frac{[[p]]_{q!}}{![[p-r]]_q!\ [[r]]_q}, \ \forall\ \mathrm{p,\ r} \in \mathrm{N_0},\ \mathrm{p} \geq \mathrm{r} \tag{11}$$

We have an example $g\ (t) = t^p$; then,

$$\Delta_q g(t) = \sigma(t)^{p-1} + t\sigma(t)^{p-2} + \sigma(t)^{p-3}t^2 + \cdots\cdots\cdots\cdots\ t^{p-1}$$

Setting $\sigma(t) = t$, we obtain classical derivative $\frac{d}{dt}g(t)$= p $\mathrm{t}^{\mathrm{p}-1}$.
In addition, in $q$-calculus, it becomes

$$\frac{d}{d_q t}(g(t)) = p_q\ t^{p-1}, \text{where we consider } \sigma(t) = qt. \tag{12}$$

### 4.1. Remarks

The characteristics and proofs of the $q$-factorials, $q$-integers and $q$-binomials are discussed in [9]. Now, we have the below results on the $\Delta_q$_operator. For $q \neq 1 \in \mathbb{R}$, and with $\Delta_q$ is defined here.

$$\Delta_q g(t) = \sum_{p=0}^{\infty} \frac{(\mu(t))^p}{(1+r)!} \frac{d^{p+1}}{dt^{p+1}} g(t); \tag{13}$$

$$\Delta_q^p g(t) = \frac{q^{(p-1)/2}}{\sigma(t)^p (q-1)^p} \sum_{r=0}^{p} \begin{Bmatrix} p \\ r \end{Bmatrix}_q f\left(q^{p-1-j}\sigma(t)\right)(-1)^j q^{\frac{r(r-1)}{2}}; \tag{14}$$

$$\Delta_q\{w(t)v(t)\} = v(t)\Delta_q w(t) + w(\sigma(t))\Delta_q v(t); \tag{15}$$

$$\Delta_q\left\{\frac{w(t)}{v(t)}\right\} = \frac{v(\sigma(t))\Delta_q w(t) - w(\sigma(t))\Delta_q v(t)}{v(t)v(\sigma(t))} \tag{16}$$

### *4.2. Partial $\Delta_q$ Derivative of Multivariable Function in Time Scale*

We define the continuous multivariable real valued function and partial $\Delta_q$-derivative of a function $g(t_1, t_2, \ldots, t_n)$ with respect to a variable $t_i$ by

$$\Delta_{q,\, t_j} g(t) = \frac{(\partial_{q,\, j} g)(t) - g(t)}{(1-q)t_j}, \tag{17}$$

$$\left[\Delta_{q,\, t_j} g(t)\right]_{t_j=0} = \lim_{t_j \to 0} \Delta_{q,\, t_j} g(t)\ , \tag{18}$$

where $t = (t_1,\ t_2,\ \cdots\cdots, t_p)$.

In addition, $(\partial_{q,\, j} g)(t) = g(t_1,\ t_2,\ \cdots\cdots, \sigma(t_j)\cdots\cdots\cdots t_n)$.

For the $j^{th}$ order, $\Delta_q$ order derivative, we subsequently adopt the identity with respect to $t^j$.

The solution of partial $\Delta_q$-differential equations using the innovative $q$-differential transform approach presented in [11,12] is currently our main goal.

$\Delta_q$-Differential reduced Transformation Method. Considering that all $\Delta_q$-differentials of *v(t, x)* exist in a region where $x = a$, we let

$$V_j(t) = \frac{1}{[[j]]_q!}\left[\frac{\partial_q^j}{\Delta_q x^j} v(t,x)\right]_{t=a}, \tag{19}$$

where $W_j(x)$ is the transformed spectrum function of the $x$ dimension. Consequently, the uppercase $W_j(t)$ represents the transformed mapping, while mapping of lower case $v(t,x)$ shows the original function. Now, we have the important definition below.

**Definition 1.** *The inverse transform of* $\Delta_q$*_differential of* $W_j(t)$*is defined by*

$$v(t,x) = \sum_{j=0}^{\infty} W_K(t)(x-c)^{(j)}. \tag{20}$$

*Putting Equation (19) in Equation (20), we obtain*

$$W_j(t) = \sum_{j=0}^{\infty} W_K(t) = \frac{1}{[[j]]_q!}\left[\frac{\partial_q^j}{\Delta_q x^k} v(t,x)\right]_{t=a} (x-c)^{(j)}.$$

*In the coming theorems, we let c = 0 such that*$(x-c)^{(j)} = (x-0)^{(j)} = (t)^{(j)}$.

We can construct the fact mentioned below from the linearity of the $\Delta_q$-derivative, given $z(t,x) = \beta v(t,x) \pm u(t,x)$ then $Z_j(t) = W_j(t) \pm U_j$. We have the following important theorems, $\beta$ being a constant.

**Theorem 1.** *Given*$z(t,x) = t^r x^p$, $Z_j(t) = t^r \delta(j-p)$, *where*

$$\delta(j) = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0 \end{cases}. \tag{21}$$

**Proof.** From Equation (23), we have

$$Z_j(t) = \frac{1}{[[[j]]]_q!}\left[\frac{\partial_q^j (t^r x^p)}{\Delta_q x^j} v(t,x)\right]_{x=0} = \frac{t^r}{[[[j]]]_q!}\left[\frac{\partial_q^j (t^m x^p)}{\Delta_q x^j} v(t,x)\right]_{x=0},$$

$$\begin{cases} t^r.\frac{[[p]]_{q!}}{[[p]]_q!} = t^r, \;\; j = p \\ t^r.\frac{[[p_q]].[[(p-1)]]_q...[[(p-j+1)]]_q}{[[p]]_q!} \quad x^{p-j}\Big|_{x=0,} \;\; j \neq 0 \\ t^r.0 = 0. \;\; j > p \end{cases} \tag{22}$$

$$= t\delta(j-p).$$

□

**Theorem 2.** *Given* $z(t,x)= \frac{\partial_q}{\Delta_{q}t}\mathrm{v}(t,x)$, $Z_j(t) = \frac{\Delta_q}{\partial_{q}t}\ \mathrm{V}(t)$,

**Proof.**

$$Z_j(t) = \frac{1}{[[j]]_q!}\left[\frac{\partial_q^j}{\Delta_q x^j}\left(\frac{\partial_q}{\Delta_q{}^t}v(t,x)\right)\right] = \frac{1}{[[j]]_q!}\left[\frac{\partial_q}{\Delta_q{}^x}\left(\frac{\partial_q^j}{\Delta_q x^j}v(t,x)\right)\right]_{x=0} \tag{23}$$

$$\frac{\partial_q}{\Delta_q{}^t}\frac{1}{[[j]]_q!}\left[\frac{1}{[[j]]_q!}\left(\frac{\partial_q^j}{\Delta_q x^j}v(t,x)\right)\right]_{t=0} = \frac{\partial_q}{\Delta_q{}^t}V_j(t).$$

□

**Theorem 3.** *Given* $z(t,x)= \frac{\partial_q}{\Delta_{q}t}(\mathrm{v}(t,x))$,

$$Z_j(t) = [[j+1]]_q[[j+2]]_q \cdots\cdots [[j+k]]_q V_{j+k}(t). \tag{24}$$

**Proof.**

$$\begin{aligned} Z_j(t) &= \frac{1}{[[j]]_q!}\left[\frac{\partial_q^j}{\Delta_q x^j}\left(\frac{\partial_q^k}{\Delta_q x^r}v(t,x)\right)\right] \\ &= \frac{[[j+k]]_q!}{[[j]]_q!}\frac{1}{[[j+k]]_q!}\left[\left(\frac{\partial_q^{j+k}}{\Delta_q x^{j+k}}v(t,x)\right)\right]_{x=0} \\ &= [[j+1]]_q[[j+2]]_q \cdots\cdots [[j+k]]_q V_{j+k}(t). \end{aligned}$$

□

**Example 1.**

$$\frac{\partial_q}{\Delta_q{}^x}v(t,\ x) = v^2(t,\ x) + \frac{\partial_q}{\Delta_q{}^t}v(t,\ x),\ v(t,\ 0) = 1+3t \tag{25}$$

Using the reduced $q$-differential transform method of the given partial $\Delta_q$-differential equation, we obtain

$$[[j+1]]_q V_{j+1}(t) = \sum_{p=0}^{j} V_{j-p}(t)V_j(t) + \frac{\partial_q}{\Delta_q{}^x}V_j(t) \tag{26}$$

The given initial condition forms.

$$V_O(t) = v(t,o) = 1+3t$$

Initiating with $j$ = 0, the values of the function $V_j(t)$ are calculated successively as given below :

$$[[1]]_q V_1(t) = V_o(t)V_o(t) + \frac{\partial_q}{\Delta_q{}^t} V_o(t)$$

$$= (1+3t)^2 + \frac{\partial_q}{\Delta_q{}^t} V_o(1+3t) = (1+3t)^2 + 3. \quad (27)$$

$$V_1(t) = 4 + 6t + 9t^2,$$

where $j$ = 1. We have

$$\left(\frac{q^2-1}{q-1}\right) V_2(t) = 2V_o(t)V_1(t) + \frac{\partial_q}{\Delta_q{}^t} V_1(= 2(1+3t)\left(4+6t+9t^2\right)$$

$$+ \frac{\partial_q}{\Delta_q{}^t} U_o\left(4+6t+9t^2\right)$$

$$\therefore V_2(t) = \frac{14 + 9(5t + \sigma(t)) + 54t^2 + 54t^3}{1+q} \quad (28)$$

Below is the same method. It is simple to calculate an expression for $j$ = 2.

Normally, we need solution of the partial $\Delta_q$-differential equation to be

$$v(t,\ x) = 1 + 3t + \left(4+6t+9t^2\right)x + \left(\frac{14 + 9(\sigma(t) + 5t) + 54t^2 + 54t^3}{1+q}\right)x^2 \quad (29)$$

Let us now determine the classical form of the partial $q$-Delta differential equation provided, i.e.,

$$\frac{\partial}{\Delta t} v(t,\ x) = v^2(t,\ x) + \frac{\partial}{\Delta t} v(t,\ x),$$

$$v(t,\ 0) = 1 + 3t. \quad (30)$$

By the method of characteristics, the above first-order partial differential quasilinear equation can be solved easily. The auxiliary associated equations are

$$\frac{dx}{1} = \frac{dv}{v^2} = -\frac{dt}{1}. \quad (31)$$

These provide two potential integrals that are provided by

$$x + t = a_1 \text{ and } \frac{1}{v} + x = a_2, \quad (32)$$

where $a_1$, $a_2$ are considered constants of integration which are arbitrary. Using the initial condition, we have $x = 0$, $v = 1 + 3t$. Hence, $t = a \ and \ \frac{1}{1+3t} = a_2$.

It then provides

$$\frac{1}{1+3a_1} = a_2.$$

Consequently, the necessary solution is

$$x + \frac{1}{t} = \frac{1}{1+3(x+t)} \text{ or } v(x,\ t) = \frac{1+3(x+t)}{1 - x(1+3t) - 3x^2} \quad (33)$$

Then, MatLab is used, which a software for numerical solution. Adding powers of $x$ to the formula for $v$ results in the following:

$$v(x,\ t) = (1+3t) + \left(4+6t+9t^2\right)x + \left(7 + 27t^2 + 27t^3\right)x^2 \quad (34)$$

Here, we have an interesting observation that when we put $\sigma(t) = t$ and $q = 1$ in (34), we successfully arrive at the conventional PDE solution.

## 5. Conclusions

The concept of the $q$-time scale calculus in quantum geometry is described in this research article. For this purpose, we include discussion of the principle from $q$-calculus comparing it with the usual Leibnitz and Newton calculus. Our basic aim is to obtain the consequences acquired when dealing with partial $\Delta_q$-differential equations. For this goal, we initiated the concept of the $\Delta_q$-differential reduced method of transformation that leads convergent solution of power series with easily computable sections. We are able to demonstrate the effectiveness and convenience of the proposed iteration technique using a few cases. It merges to a standard-form solution with initial value problems: when $q = 1$ and $\sigma(t) = t$. The deduced conclusion is that $q$-time scale calculus is invariant, that is the non-commutative calculus which coincides the Leibnitz–Newton standard calculus. This work introduces and generalizes the qDDTM to work on partial $q$-differential equations which represent the non-commutative forms of spaces of some dynamics.

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## References

1. Application of $q$-Calculus to the Solution of Partial $q$-Differential Equations. *Appl. Math.* **2021**, *12*, 669–678. [CrossRef]
2. Connes, A. *Noncommutative Geometry*; Academic Press: New York, NY, USA, 1994.
3. Durdevic, M. Quantum principal bundles and their characteristic classes. *arXiv* **1996**, arXiv:q-alg/9605008.
4. Connes, A. Non-Commutative Differential Geometry. *Publ. Mathématiques L'institut Hautes Études Sci.* **1986**, *62*, 41–144. [CrossRef]
5. Connes, A. Noncommutative Differential Geometry. *Publ. Mathématiques l'I.H.É.S* **1985**, *62*, 257–360.
6. Majid, S. Physics for Algebraists: Non-Commutative and Non-Commutative Hopf Algebras by a Bicrossproduct Construction. *J. Algebra* **1990**, *130*, 17–64. [CrossRef]
7. Drinfeld, G. Quantum Groups. In *Proceedings of the ICM 1986*; Gleason, A., Ed.; AMS: Providence, RI, USA, 1986; pp. 798–820.
8. Brown, L.G.; Douglas, R.G.; Filmore, P.G. Extensions of $C^*$-Algebras and $K$-Homology. *Ann. Math.* **1977**, *105*, 265–324. [CrossRef]
9. Maliki, O.S.; Ugwu, E.I. On $q$-Deformed Calculus in Quantum Geometry. *Appl. Math.* **2014**, *5*, 1586–1593. [CrossRef]
10. Jackson, H.F. $q$-Difference Equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
11. Connes, A. C*-algèbres et géométrie différentielle. *Comptes Rendus de L'académie Des Sci.* **1980**, *290*, 599–604.
12. Jain, P.; Basu, C.; Panwar, V. Reduced (P, Q)-Differential Transform Method And Applications. *J. Inequalities Spec. Funct.* **2022**, *13*, 24–40. [CrossRef]