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Short-Term Expectation Formation Versus Long-Term Equilibrium Conditions: The Danish Housing Market

Andreas Hetland * and **Simon Hetland**

Department of Economics, University of Copenhagen, 1353 Copenhagen K, Denmark; bhp240@ku.dk
* Correspondence: lxs601@ku.dk; Tel.: +45-20422606

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Abstract: The primary contribution of this paper is to establish that the long-swings behavior observed in the market price of Danish housing since the 1970s can be understood by studying the interplay between short-term expectation formation and long-run equilibrium conditions. We introduce an asset market model for housing based on uncertainty rather than risk, which under mild assumptions allows for other forms of forecasting behavior than rational expectations. We test the theory via an $I(2)$ cointegrated VAR model and find that the long-run equilibrium for the housing price corresponds closely to the predictions from the theoretical framework. Additionally, we corroborate previous findings that housing markets are well characterized by short-term momentum forecasting behavior. Our conclusions have wider relevance, since housing prices play a role in the wider Danish economy, and other developed economies, through wealth effects.

Keywords: asset pricing; cointegration; $I(2)$ analysis; housing market; imperfect knowledge; Knightian uncertainty; long swings

JEL Classification: C32; C51; D81; E22; G12

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1. Introduction

Changes in housing prices, and in turn changes in housing wealth, exert substantial effects on the economy: increased housing wealth is strongly associated with increased aggregate consumption, cf. Case et al. (2005), and vice versa, cf. Case et al. (2013). Housing wealth has (together with pension wealth) also been found to be a primary driver of the share of total wealth accruing to the middle class,[1] cf. Saez and Zucman (2016). For example, higher housing prices have historically been associated with the middle class owning a greater share of total wealth, and thereby with a lower level of inequality. Consequently, understanding the drivers and dynamics of housing prices is of material importance to economists and policy makers alike.

The market price of housing has a tendency to undergo prolonged periods of increases that outpace both incomes and other prices, see e.g., Case et al. (2003) for a study of the US housing market. We have observed similar long-swings patterns in the Danish housing market, see Figure 1.[2] The Danish national price index for housing increased by 65% between 2003:Q1 and 2007:Q4, while the general price level increased by only 8% over the same period, meaning that housing outpaced inflation by 57 percentage points. In 2008, the housing boom turned to bust as the global economy fell in to recession. Danish house prices fell by 17% between 2008:Q1 and 2012:Q4, while the general price

[1] Defined by the authors as the share of total wealth owned by the bottom 90% of the population.
[2] The data we use has been provided by the Danish Central Bank. We refer to Section 3.1 for further details.

level increased by 10%, closing most of the gap created during the boom years. Moreover, prolonged deviations are not just a phenomenon of the relatively recent past. Housing price changes have either persistently outpaced or fallen behind consumer price inflation since the late 1970s. For example, housing prices slumped during the late 1980s and early 1990s, with inflation outpacing house price increases by 43 percentage points between 1986:Q1 and 1993:Q2. The house price increases then accelerated and had completely closed the gap by 1998:Q2.

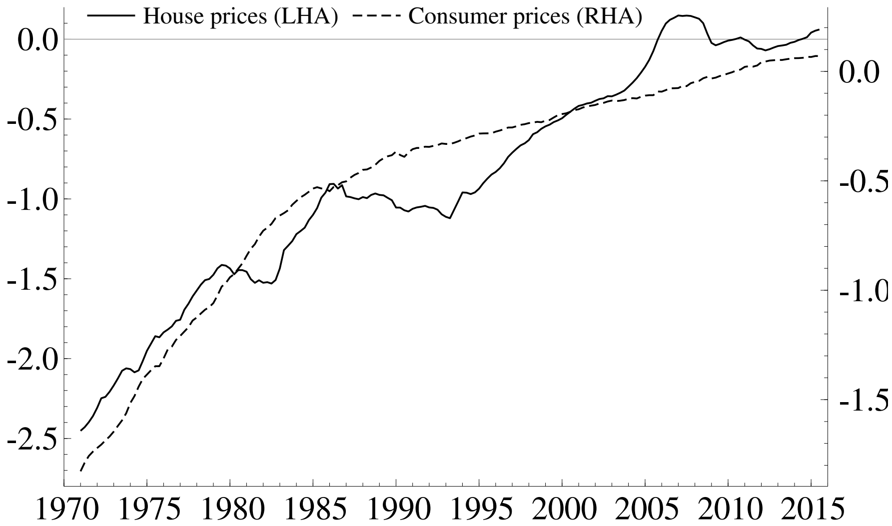

Figure 1. Danish house and consumer price indices between 1971:Q1 and 2015:Q3, log-levels.

Given that housing prices play an important role in the wider economy, the determinants of the market price of housing have been the focus of numerous studies. In this paper, we focus on the approach which treats housing as a carry-generating physical asset that can be reproduced at the cost of construction, cf. Poterba (1984). This approach falls under the general asset pricing theory, which typically represents the uncertainty associated with the future value of an asset in the form of a probability distribution, cf. e.g., Björk (2009). Investors' expectation of the future value of the asset is then assumed to follow the mathematical expectation with respect to this probability distribution. This is known as rational expectations in the wider context of economic modelling, cf. e.g., Muth (1961) and Sargent and Wallace (1975). However, in the case of housing, several studies from the US have found evidence in support of individual investors forming their expectations based on extrapolation of recent price changes, see Case and Shiller (1989), Poterba (1991), Case et al. (2003), Shiller (2008), and Piazzesi and Schneider (2009). As remarked in these studies, the observed momentum-based expectation formation appears incompatible with the assumption of rational expectations.

Some have argued that rational expectations-based approaches conflate the fundamentally different notions of risk and uncertainty, where the former may be expressed as a probability distribution while the latter cannot, see e.g., Savage (1951), Rutherford (1984), Lawson (1988) and Binmore (2009). This distinction has important implications for the appropriateness of rational expectations, and has gained traction with central bank economists in recent years, see e.g., King (2004), Carney (2016) and ECB (2016). More generally this type of thinking has led to alternative paradigms for expectation formation. For example, Akerlof and Shiller (2009) argue that investor psychology is driven by "animal spirits" such as confidence, money illusion and narratives. Frydman et al. (2007) advances the imperfect knowledge economics theory, in which individuals change their forecasting strategies in ways that need not be given by expectations with respect to a model-implied probability distribution. Elsewhere in the field of psychology, large-scale forecasting experiments have led to the finding that while "forecasting is often viewed as a statistical problem, [...] forecasts can be improved with behavioural interventions [... such as] training, teaming, and tracking.", cf. Mellers et al. (2014), which suggests that, in practice, expectation formation for macroeconomic outcomes goes beyond the problem of identifying an appropriate statistical model.

In this paper we seek to explain the long-swings dynamic in Danish housing prices (Figure 1). We do this by developing an asset market model based on uncertainty rather than risk, which allows other forms of forecasting behavior than rational expectations, such as momentum-based forecasting (cf. e.g., Case et al. (2003)). However, since we do not have survey data of individual expectations in the Danish housing market, we introduce the assumption that the expectation errors are stationary (Assumption A in Juselius (2017b)). Additionally, we introduce the notion of a gap effect as a measure of the required uncertainty premium, cf. Frydman et al. (2007), specified in terms of Tobin's q, cf. Tobin (1969). This uncertainty-based asset market approach produces a set of testable hypotheses on the long-run relationships governing the Danish housing prices, which we confront with the data via a cointegrated vector autoregressive (CVAR) model, see e.g., Juselius (2007) and Johansen (1996). The CVAR model provides a general-to-specific framework, which allows us to start the empirical analysis with a sufficiently well-specified, unrestricted VAR model of the Danish housing market, and then impose restrictions corresponding to hypotheses arising from the theoretical model, cf. e.g., Hoover et al. (2008). Importantly, this approach also allows us to infer the process by which the market adjusts when out of equilibrium; specifically, the interplay between long-run and medium-run dynamics may be able to explain long swings around the equilibrium, cf. Juselius and Assenmacher (2016, Section 5) .

The paper proceeds as follows: in Section 2, we introduce the theoretical framework for the uncertainty-based no-arbitrage condition, and derive a set of empirically testable hypotheses. Section 3 specifies an $I(2)$ CVAR model and tests the hypotheses presented in Section 2. Finally, we conclude in Section 4.

2. The Theoretical Framework

We here develop an uncertainty-based theoretical framework for housing markets with the purpose of guiding us to a set of testable hypotheses on the long-run equilibrium for housing prices. Our framework is based on uncertainty rather than risk for the reasons outlined in Section 1, and the aim is to develop a model which is simple, yet realistic enough to be empirically relevant. In the following we: introduce the classic deterministic asset-market approach to the housing market in Section 2.1; incorporate risk into the asset market model in Section 2.2; further amend the model to allow for uncertainty in Section 2.3; and finally we derive testable hypotheses from the uncertainty-based model in Section 2.4. We will confront these with historical data for the Danish housing market in Section 3.

2.1. An Asset-Market Approach to Housing

We follow the asset-market approach to modeling the price of residential housing, which was originally introduced in Poterba (1984), and rests on the premise that the price of an asset should be characterized by the absence of arbitrage opportunities. This approach centers on the equilibrium condition that individuals invest in housing until the marginal value of housing equals its cost. In line with Poterba (1984), we make several simplifying assumptions to make this condition explicit: at each point in time t, the housing stock depreciates at a rate of δ_t; housing is taxed at a rate of μ_t; all investors face a marginal income tax rate θ_t, from which they may deduct property taxes; investors may borrow or lend any amount at a nominal interest rate i_t. We assume for ease of presentation that each of these quantities are constant between periods.

The cost of a single unit of housing with nominal price, denoted $P_{h,t}$ (not in logarithmic terms), is $\omega_t P_{h,t}$, where ω_t is the sum of after-tax depreciation, property taxes, mortgage interest payments, and the opportunity cost of owning housing stock, minus the nominal capital gain

$$\omega_t = \delta_t + (1-\theta_t)\,(i_t+\mu_t) - \pi_{h,t}\,, \tag{1}$$

where $\pi_{h,t} := dP_{h,t}/P_{h,t}dt$. The benefit of owning a unit of housing is the nominal rental income, R_t, produced (or saved in the case of owner-occupied housing). In the housing market equilibrium, investors (including home owners) will price housing such that the marginal cost equals the marginal benefit of housing; formally $R_t = \omega_t P_{h,t}$, which we can rewrite as the first-order differential equation for changes in the nominal housing price

$$dP_{h,t} = c_t P_{h,t} dt\,, \tag{2}$$

where we have defined the user cost rate $c_t := \delta_t + (1-\theta_t)(i_t + \mu_t) - R_t/P_{h,t}$. We assume that the ratio $R_t/P_{h,t}$ is constant to simplify the exposition. For a given initial house price $P_{h,0}$, Equation (2) determines the nominal capital gain needed to induce investors to hold the existing housing stock.

2.2. A Risk-Based Asset-Market Approach

We next extend the no-arbitrage condition given by Equation (2) to a simple setting involving market risk. Specifically, we consider the simple case where the price of housing is given by the geometric Brownian motion

$$dP_{h,t} = (c_t + rp_t)\,P_{h,t}dt + \omega_t P_{h,t} dW_t\,, \tag{3}$$

where rp_t denotes a risk premium, dW_t is a Wiener process under the physical measure, denoted $\mathbb{P}$, and ω_t denotes the volatility of the house price changes. Investors require the risk premium rp_t to undertake the risk ω_t; the larger the risk, the larger the required premium.

Omitting here the full details, the fundamental theorem of asset pricing, see e.g., Björk (2009, Section 5.5), implies that if, and only if, there is no arbitrage in the housing market, then the current price of housing, $P_{h,t}$, must satisfy

$$P_{h,t} = E^{\mathbb{Q}}\left[P_{h,t+1} \mid \mathcal{F}_t\right] e^{-c_t}\,, \tag{4}$$

where $\mathbb{Q}$ denotes the risk-neutral measure. That is, Equation (4) states the current price of housing is equal to the discounted expected future price under the risk-neutral measure, conditional on the available information, $\mathcal{F}_t$.

Under standard regularity conditions, Girsanov's theorem tells us we can re-weigh the expectation in Equation (4) from the risk-neutral measure, $\mathbb{Q}$, to the physical measure, $\mathbb{P}$. If the investor preferences are not risk neutral then $\mathbb{P}$ will be different from $\mathbb{Q}$, such that an additional term compensating for taking on market risk enters the discount factor

$$P_{h,t} = E^{\mathbb{P}}\left[P_{h,t+1} \mid \mathcal{F}_t\right] e^{-c_t - rp_t}\,. \tag{5}$$

Next, we apply the log-transformation to Equation (5).[3] Noting that we have assumed the user cost rate and risk premium to be constant between periods, we have that

$$p^{e,\mathbb{P}}_{h,t+1|t} - p_{h,t} = c_t + rp_t\,, \tag{6}$$

where we have denoted the logarithm of the expected price by $p^{e,\mathbb{P}}_{h,t+1|t} := \log E^{\mathbb{P}}\left[P_{h,t+1} \mid \mathcal{F}_t\right]$.

Equation (6) constitutes what we will refer to as the *risk-adjusted no-arbitrage condition* from standard asset pricing theory. This equation states that for there to be no arbitrage opportunities in the housing market, the price of housing must be given by the present value of the expected price one period in the future, where the expectation is with respect to the physical measure $\mathbb{P}$, which follows

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[3] We generally let lower case letters denote logarithmic values, e.g., $p_{h,t} := \log(P_{h,t})$. The exception being rates, e.g., c_t.

from our simple model given by Equation (3). The discount factor reflects the opportunity cost under the physical measure, i.e., the user cost rate plus a premium demanded for undertaking risk.

While this risk-based asset price model is a stepping stone on our way to introduce uncertainty, we note that richer and more realistic dynamics are also possible in risk-based settings. For example, the $\mathbb{P}$-dynamics in Equation (3) may depend on the other variables such as net investments. We will not pursue these here.

2.3. An Uncertainty-Based Asset-Market Approach

If the housing market is better characterized by uncertainty than risk, then it becomes necessary to revisit the no-arbitrage condition in Equation (6). Recall that recognizing that a setting involves uncertainty implies that it is not feasible to attribute the known outcomes with unambiguous probabilities; in turn, the mathematical expectation operator in Equation (6) is not defined. We assume that investors form subjective expectations under uncertainty, but these do not necessarily follow from a model such as Equation (3). We return to the expectations formation under uncertainty in Section 2.3.1. For now, we simply denote the subjective expectations of the future house prices given the information available at time t as $p^e_{h,t+1|t}$. Based on this, we re-formulate Equation (6) as

$$p^e_{h,t+1|t} - p_{h,t} = c_t + up_t\,, \tag{7}$$

where up_t denotes an uncertainty premium. We will refer to Equation (7) as the *uncertainty-adjusted no-arbitrage* (UANA) condition. This equation states that for there to be no arbitrage in the housing market, the subjective expectation of the one-period return must equal the user cost rate plus a premium compensating for undertaking uncertainty, which is conceptually similar to the risk-adjusted no-arbitrage condition in Equation (6), except in Equation (7) the expectations are not generated by solving a stochastic model.

2.3.1. Expectation Formation under Uncertainty

Investors form subjective expectations of future house price changes, but these expectations are inherently unobservable. To relate the expectations in Equation (7) to the realized prices, we introduce Assumption A from Juselius (2017a).

Assumption A: The expectation errors of the future price levels, defined as

$$e_{t+1} := p^e_{h,t+1|t} - p_{h,t+1}\,, \tag{8}$$

are stationary; more precisely, $e_{t+1} \sim I(0)$.

This assumption implies that investors are able to assess the order of integration of house prices, $p_{h,t}$, such that the expectation errors are stationary, but not necessarily uncorrelated over time. This specification of investors' expectation formation is far less restrictive than rational expectations, and so does not preclude e.g., momentum-based forecasting.

2.3.2. The Uncertainty Premium and the Gap Effect

In a similar type of uncertainty-based asset pricing framework, Frydman et al. (2007) have introduced the notion of a gap effect to characterize the uncertainty premium. In general terms, the gap effect is defined as the difference between the current asset price and its perceived long-run fair value. In the context of housing, the Tobin's q measure, which is the ratio of the price to reproduction cost, cf. Tobin (1969), is an appropriate long-term benchmark. In the following, we will refer to Tobin's q in logarithmic terms, i.e., the difference $p_{h,t} - p_{b,t}$. Intuitively, if the price of a unit of housing is above the cost of building such a unit, then there is an incentive to construct and sell new houses until either the increased demand for construction supplies and labor pushes the cost up, the increased supply of homes pushes the price of housing down, or a combination of both. All else equal, this dynamic will tend to pull the Tobin's q ratio towards unity in the long run.

We specify the uncertainty premium as being proportional to Tobin's q (in logarithmic terms), measured using the nominal housing index, $p_{h,t}$, and the nominal price index for construction costs, $p_{b,t}$. Specifically,

$$up_t = \sigma(p_{h,t} - p_{b,t}) \,, \tag{9}$$

where σ is a positive scalar; that is, the further in excess of a Tobin's q value of one, the higher the required premium. When Tobin's q is less than unity, the premium will be negative, i.e., the required expected return will be less than the user cost rate, c_t. Substituting Equation (9) into the UANA condition in Equation (7) yields,

$$p^e_{h,t+1|t} - p_{h,t} = c_t + \sigma(p_{h,t} - p_{b,t}) \,. \tag{10}$$

In conclusion, introducing uncertainty in terms of subjective expectations and potential loss, as measured by the gap effect, results in an equilibrium condition remarkably similar to that arising from the deterministic framework originally introduced in Poterba (1984), but with an additional term accounting for the uncertainty premium as measured by Tobin's q. We now turn our attention to the testable hypotheses arising from the above framework.

2.4. Testable Hypotheses

We use Assumption A combined with with the gap effect to restate the UANA condition in terms of realized, contemporary price changes. We first rewrite the left-hand side of Equation (10) as

$$p^e_{h,t+1|t} - p_{h,t} = \Delta p_{h,t} + \Delta^2 p_{h,t+1} + e_{t+1} \,, \tag{11}$$

noting that $\Delta^2 p_{h,t+1} = \Delta p_{h,t+1} - \Delta p_{h,t}$. If the price of housing $p_{h,t}$ is non-stationary in the sense that it is integrated of order one or two, then the term $\Delta^2 p_{h,t+1}$ will be stationary. As such, under Assumption A, Equations (10) and (11) suggest the cointegration relation

$$\Delta p_{h,t} = c_t + \sigma(p_{h,t} - p_{b,t}) + w_t \,, \tag{12}$$

where $w_t := -\left(\Delta^2 p_{h,t+1} + e_{t+1}\right) \sim I(0)$ denotes stationary deviations from the long-run equilibrium.

Considering the potential for each of the variables $p_{h,t}$, $p_{b,t}$ and c_t to be either $I(1)$ or $I(2)$, there are eight different potentially relevant scenarios. We here limit our attention to scenarios where cointegration to stationarity remains a possibility, and where the price indices have the same order of integration. This leaves three different relevant scenarios.

In the first scenario, we have that $p_{h,t} \sim I(1)$, $p_{b,t} \sim I(1)$ and $c_t \sim I(1)$, such that

$$\underset{I(0)}{\Delta p_{h,t}} = \underset{I(1)}{c_t} + \underset{I(1)}{\sigma(p_{h,t} - p_{b,t})} + \underset{I(0)}{w_t} \,, \tag{13}$$

where the user cost rate, c_t, cointegrates with Tobin's q from $I(1)$ to stationarity.

In the second scenario, if $p_{h,t} \sim I(2)$, $p_{b,t} \sim I(2)$ and $c_t \sim I(1)$ then

$$\underset{I(1)}{\Delta p_{h,t}} = \underset{I(1)}{c_t} + \underset{I(1)}{\sigma(p_{h,t} - p_{b,t})} + \underset{I(0)}{w_t} \,, \tag{14}$$

where the house and construction prices, $p_{h,t}$ and $p_{b,t}$, cointegrate from $I(2)$ to $I(1)$, such that Tobin's q is $I(1)$ and cointegrates with the user cost rate, c_t, and the house price changes $\Delta p_{h,t}$ to stationarity.

Third, and last, if $p_{h,t} \sim I(2)$, $p_{b,t} \sim I(2)$ and $c_t \sim I(2)$ then

$$\underset{I(1)}{\Delta p_{h,t}} = \underset{I(2)}{c_t} + \sigma \underset{I(2)}{(p_{h,t} - p_{b,t})} + \underset{I(0)}{w_t}, \tag{15}$$

where the user cost rate, c_t, cointegrates with Tobin's q from $I(2)$ to $I(1)$, which in turn cointegrates with the changes in house prices, $\Delta p_{h,t}$ to stationarity. Only one, if any, of the three scenarios will find empirical support. We will introduce the cointegrated VAR model to investigate which one in the following section.

3. Specifying an *I*(2) CVAR Model for the Danish Housing Market

We now turn our attention to confronting the hypotheses derived from the theoretical framework presented in Section 2 with historical data for the Danish housing market. Specifically, to investigate the empirical relevance of the cointegrating relations in Section 2.4, we here introduce the $I(2)$ cointegrated vector autoregressive (CVAR) model. The method applied to arrive at a well-specified, properly identified system is outlined in Juselius (2007), while a mathematical exposition of the model, estimation and inference can be found in Johansen (1996). An appealing feature of the CVAR framework is that through testing and subsequently imposing restrictions on an unrestricted VAR, such as rank restrictions, zero parameter restrictions, and other restrictions, we arrive at a more parsimonious model with economically interpretable coefficients. As such, specifying a CVAR with an over-identified long-run structure adheres to the general-to-specific procedure outlined in e.g., Campos et al. (2005).

In the following we: introduce the information set in Section 3.1; define the $I(2)$ cointegrated VAR model in Section 3.2; develop a sufficiently well-specified unrestricted VAR model in Sections 3.3–3.5; determine the cointegration rank in Section 3.6; test the theory-derived hypotheses and interpret the over-identified long-run structure of the cointegrated VAR model in Sections 3.7–3.9; finally, we summarize the empirical findings in Section 3.10.

3.1. The Information Set

Our empirical analysis is based on variables that are part of the "MONA" database maintained by the Danish Central Bank, cf. Danmarks Nationalbank (2003). This database contains quarterly observations of variables for the Danish economy from 1971:Q1 to 2015:Q3.[4] We define the information set for our empirical analysis as: the nominal price index for goods, measured as the GDP deflator, denoted $p_{c,t}$; the nominal price index for construction costs, $p_{b,t}$; and the nominal price index for single family houses in Denmark, $p_{h,t}$. We also include the user cost rate of housing investments, c_t, which is the post-tax nominal interest rate on a highly rated bond. Since this rate represents the opportunity costs of investing in housing (i.e., the carry), depreciation and convenience yield are included and have been assumed to stay constant at 1% and 4%, respectively; an assumption similar to those made in Danmarks Nationalbank (2003, chp. 3). Finally, we also include the net investments in housing in fixed prices, Δh_t, i.e., the first-differenced real housing stock in log terms, h_t. All variables, apart from the user cost rate, c_t, are transformed with the natural logarithm. We combine these five variables in our data column vector, x_t, which we define as

$$x_t = [\ p_{c,t} \quad p_{b,t} \quad p_{h,t} \quad c_t \quad \Delta h_t \ ]'. \tag{16}$$

The levels, first-, and second differences of the data is shown in Figure A1 in Appendix A.

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[4] The dataset and variable definitions are available from the authors on request. The empirical analysis is carried out in OxMetrics v.7.1 using the Cointegration Analysis of Time Series (CATS) package v.3C.

3.2. The I(2) Cointegrated VAR Model

We formulate the $I(2)$ CVAR model in terms of acceleration rates, changes and levels (see Juselius (2007)), and use the maximum likelihood parametrization introduced by Johansen (1997). The model is shown here with $k = 2$ lags to simplify the presentation

$$\begin{aligned}\Delta^2 x_t &= \alpha\left(\beta' x_{t-1} + d'\Delta x_{t-1}\right) + \zeta\tau'\Delta x_{t-1} \\ &\quad + \mu_0 + \mu_1 t + \Phi_s D_{s,t} + \Phi_p D_{p,t} + \Phi_{tr} D_{tr,t} + \epsilon_t \\ \epsilon_t &\sim N_p(0,\Omega), \qquad t = 1,\ldots,T.\end{aligned} \tag{17}$$

Here α is a $p \times r$ matrix of adjustment coefficients, β is a $p \times r$ matrix describing long-run relationships among the variables, p is the dimension of the data vector, r is the number of multicointegration relations, s_1 is the number of cointegration relations that only become stationary by differencing, s_2 is the number of stochastic $I(2)$ trends, and $p = r + s_1 + s_2$. Moreover, $d = -((\alpha'\Omega^{-1}\alpha)^{-1}\alpha'\Omega^{-1}\Gamma)\tau_\perp(\tau_\perp'\tau_\perp)^{-1}\tau_\perp'$ is a $p \times r$ matrix of coefficients, where $\Gamma = -(\alpha d' + \zeta\tau')$. The d matrix is determined such that $(\beta' x_{t-1} + d'\Delta x_{t-1}) \sim I(0)$. Additionally, $\tau = [\beta, \beta_{\perp 1}]$ is a $(p+1) \times (r + s_1)$ matrix which describe stationary relationships among the differenced variables, where $\beta_{\perp 1}$ is the orthogonal complement of $[\beta, \beta_{\perp 2}]$. Finally, ζ is a $p \times (p - s_2)$ matrix of restricted medium-run adjustment coefficients. We follow Rahbek et al. (1999) and restrict the constant term to be in $d'\Delta x_{t-1}$ and the deterministic trend to be in $\beta' x_{t-1}$.

3.3. Lag Length Selection

Given the data vector defined in Equation (16) for the period 1971:Q1–2015:Q3 we determine the appropriate number of lags and deterministic components required to obtain a sufficiently well-specified model. First, we choose the lag length by starting with a model with $k = 4$ lags and then reduce the number of lags by removing one at a time until a LM-test is rejected and the Schwarz, Hannan-Quinn and Akaike information criteria are minimized (these are given in Table A1 in Appendix A). Based on this procedure, we choose the lag length $k = 2$. Secondly, fitting the CVAR model commonly requires a number of deterministic components to obtain a sufficiently well-specified model, e.g., shift, permanent, and/or transitory dummies. These components become necessary when the structure captured by the unrestricted VAR model falls short of explaining large movements in the data. Such a large movement could for instance be the enactment of a political reform, which changes the institutional features of the economy, or it could be a natural event affecting the economy.

3.4. Dummy Specification

We follow the method of Juselius (2007, chp. 6.6) to determine which dummies to include. This approach is based on the sequential identification of large outliers (defined as a standardized residual greater than 3.5) until a sufficiently well-specified model has been obtained. In an iterative manner, we include one dummy at a time to investigate if the specified dummy results in a well-specified model. Following this method, our final specification includes six permanent intervention dummies,[5] two transitory dummies, and three centered quarterly dummies to control for seasonality at the quarterly frequency. We specify the dummies as follows (omitting seasonal dummies)

$$D_s' = \mathbf{0}\,,\; D_p' = \left[D_{p75:1,t}\; D_{p82:1,t}\; D_{p83:1,t}\; D_{p87:1,t}\; D_{p93:1,t}\; D_{p93:3,t}\right] \text{ and } D_{tr}' = \left[D_{tr00:1,t}\; D_{tr08:1,t}\right]. \tag{18}$$

[5] In this specification there are still two large outliers remaining; 1972 Q2 and 1994 Q1, respectively. We have chosen not to include dummies for these outliers as this re-introduces residual autocorrelation. We find that including additional dummies does not produce a better model specification than the one presented here.

Reassuringly, most of the dummies coincide with economic events that we would not expect to be captured by the structure of the unrestricted VAR model. The dummies $D_{p75:1,t}$ and $D_{p83:1,t}$ are included to reduce skewness in c_t; the latter is included due to a big drop in the interest rate following the transition to a fixed exchange rate regime in 1983, whereas the former accounts for a correction from a spike in interest rates in 1975 following turbulence in the money market. The dummy $D_{p87:1,t}$ accounts for the tax reform enacted that year, which is significant in both c_t and $p_{h,t}$. The dummy $D_{p93:3,t}$ is included to correct for a large outlier in $p_{b,t}$ in the third quarter of 1993, which coincides with the abolishment of mixed loans. The dummy, $D_{tr00:1,t}$, accounts for the December storm of 1999, which caused a rise and a drop in Δh_t. The final dummy, $D_{tr08:1,t}$ accounts for the initial shocks of the financial crisis which caused a large transitory shock to Δh_t. As a robustness measure, we have also estimated the model without any dummies (included in Appendix B), where we are able to retrace the main conclusions from the analysis.

3.5. Misspecification Tests

Once the lag length and deterministic components have been chosen, we examine if the assumption on IID multivariate normality of the model innovations holds. To this end, we present a selection of common misspecification tests for the unrestricted VAR(2) in Table 1.

Table 1. Misspecification tests for the unrestricted VAR(2) model.

Multivariate Tests				
No-autocorrelation	LM(1)	$\chi^2(25) = 34.46$	$p = 0.10$	
	LM(2)	$\chi^2(25) = 24.82$	$p = 0.47$	
Normality	Doornik-Hansen	$\chi^2(10) = \mathbf{90.16}$	$p = 0.00$	
No-ARCH	LM(1)	$\chi^2(225) = \mathbf{481.45}$	$p = 0.00$	
	LM(2)	$\chi^2(450) = \mathbf{721.72}$	$p = 0.00$	

Univariate Tests					
	$\Delta^2 p_{c,t}$	$\Delta^2 p_{b,t}$	$\Delta^2 p_{h,t}$	$\Delta^2 c_t$	$\Delta^3 h_t$
No-ARCH	**17.34** [0.00]	3.06 [0.22]	3.52 [0.17]	**30.13** [0.00]	**24.77** [0.00]
Skewness	−0.25	0.22	0.01	−0.29	0.37
Kurtosis	4.85	5.55	4.19	4.33	4.68
Normality	**20.67** [0.00]	**33.95** [0.00]	**11.54** [0.00]	**27.14** [0.00]	**21.18** [0.00]
R^2	63%	54%	57%	58%	48%

Notes: Bold font indicates that the hypothesis is rejected at the 5% significance level. Graphical representations of the residual analysis can be found in Figure A2 in Appendix A. We use the multivariate tests for ARCH-effects and autocorrelation presented in Godfrey (1988) and the univariate and multivariate normality test from Doornik and Hansen (2008). The p-values for the univariate tests are shown in square brackets.

The hypotheses of no residual autocorrelation of order 1 and 2 are not rejected, which is necessary for a model to be dynamically complete. The residuals for the construction costs, $p_{b,t}$, and housing price, $p_{h,t}$, show no signs of ARCH effects. However, the residuals for the consumer price index, $p_{c,t}$, the user cost rate, c_t, and the net investments, Δh_t, do not pass the no-ARCH test. The ARCH effects in the consumer prices, $p_{c,t}$, may in part be attributed to the regime change in 1983, before which there were higher, more volatile price changes than after. The inference is robust to moderate ARCH effects, cf. Rahbek et al. (2002) and Cavaliere et al. (2010), and as such the presence ARCH effects should not invalidate inference based on our unrestricted VAR model. We note that the presence of ARCH effects is likely to contribute to excess kurtosis, and as such it is not surprising that the univariate tests of non-normality are rejected primarily due to excess kurtosis. Non-normality due to skewness is, on the other hand, a concern for inference, cf. Juselius (2007, chp. 4.3). That said, not much skewness remains in the residuals, indicating that the rejection of normality is not sufficient to invalidate inference. In sum, considering that the unrestricted VAR model is only misspecified in terms of ARCH effects and kurtosis-induced non-normality, to which the inference is robust, and given that inclusion of further dummies does not resolve these issues, we conclude that this specification constitutes an appropriate basis for further analysis.

3.6. Rank Determination

Given our sufficiently well-specified unrestricted VAR model, we proceed to determine the appropriate reduced ranks of the Π and Γ matrices. Similar to the $I(1)$ analysis, exploring whether $x_t \sim I(1)$ is facilitated by the reduced rank hypothesis $\Pi = \alpha\beta'$, implicitly assuming that Γ is full rank. Examining whether $x_t \sim I(2)$ is facilitated by the additional reduced rank hypothesis $\alpha'_{\perp}\Gamma\beta_{\perp} = \xi\eta'$, where $\alpha_{\perp}$ and $\beta_{\perp}$ are the orthogonal complements of α and β, respectively. The determination of the reduced rank indices is based on the maximum likelihood trace test procedure proposed by Bohn Nielsen and Rahbek (2007).[6] As an alternative to the analytical distribution of the rank test, one can also use critical values from a bootstrap procedure, which is outlined for the $I(1)$ model in Cavaliere et al. (2012). We refrain from using the bootstrap procedure here, as the asymptotic properties have not been shown for the $I(2)$ model yet.

Table 2 presents the $I(2)$ trace tests for rank r equal to 2 and 3, as well as the modulus of the six largest characteristic roots of the model. The trace test starts with the most restricted model $\{r = 0,\ s_1 = 0,\ s_2 = 5\}$, recalling that s_2 is the number of $I(2)$ trends, and proceeds until a reduced rank specification not rejected, and is nested in models that are also not rejected. The most restricted model we show in Table 2 is $\{r = 2,\ s_1 = 0,\ s_2 = 3\}$. The models become less and less restricted as the table is read row-wise.

Table 2. Determination of the two rank indices.

Rank Test Statistics						
$p-r$	r	$s_2=4$	$s_2=3$	$s_2=2$	$s_2=1$	$s_2=0$
3	2		134.9 [0.00]	69.8 [0.05]	55.2 [0.04]	53.4 [0.00]
2	3			44.3 [0.13]	28.6 [0.23]	23.2 [0.10]
Six Largest Roots						
Unrestricted VAR	0.98	0.94	0.92	0.92	0.65	0.42
Case 1: $r=2$						
$r=2, s_1=1, s_2=2$	1.00	1.00	1.00	1.00	1.00	0.47
$r=2, s_1=2, s_2=1$	1.00	1.00	1.00	1.00	0.79	0.51
$r=2, s_1=3, s_2=0$	1.00	1.00	1.00	0.99	0.79	0.51
Case 2: $r=3$						
$r=3, s_1=0, s_2=2$	1.00	1.00	1.00	1.00	0.66	0.53
$r=3, s_1=1, s_2=1$	1.00	1.00	1.00	0.87	0.72	0.51
$r=3, s_1=2, s_2=0$	1.00	1.00	0.98	0.88	0.65	0.52

Note: The p-values for the $I(2)$ rank test are given in square brackets below the test result.

The trace test suggests either $\{r = 2,\ s_1 = 1,\ s_2 = 2\}$ or $\{r = 3,\ s_1 = 0,\ s_2 = 2\}$. The first specification is borderline rejected (p-value of 4.6%), and it is nested in two models that are rejected (p-values of 4.0% and 0.0% respectively). The second specification is not rejected (p-value of 12.6%) and is nested in models that are also not rejected. Both specifications leave no large characteristic roots; the largest remaining root has modulus 0.47 and 0.66, respectively, indicating no residual unit roots. The first specification points to five unit roots, whereas the second specification points to four

[6] Earlier work on $I(2)$ rank tests include Rahbek et al. (1999) and Johansen (1995). Work on the distribution for the $I(2)$ rank test include Doornik (1998) and Johansen (1996).

unit roots. The modulus of the unrestricted VAR points to four unit roots, corresponding to the second specification.

Taking this into account, we proceed with the second specification, as the rank test is not rejected nor nested in models which are rejected. Moreover, there is significant error correction and stationarity in all cointegrating relations (we return to this point in Section 3.8). The chosen specification implies two stochastic $I(2)$ trends ($s_2 = 2$) and three polynomially cointegrating relations ($r = 3$), $\beta' x_t + d' \Delta x_t$, which achieve stationarity.

3.7. Hypothesis Testing

Before identifying a long-run structure, we examine if certain variables, or linear combinations of variables, relating to the UANA condition outlined in Equations (13)–(15), are found to be $I(1)$. This is done by estimating the CVAR under the reduced rank conditions, $\{r = 3,\ s_1 = 0,\ s_2 = 2\}$, also using the numerical maximum likelihood procedure outlined by Johansen (1997). Following the estimation, we impose restrictions on the $\tau = [\beta, \beta_{\perp 1}]$ vectors (see Johansen (2006, Proposition II)). This lets us examine the persistency properties of the different variables in the information set and allows us to examine how the UANA condition outlined in Section 2.4 may hold.

First, we examine if any of the variables are $I(1)$, by imposing restrictions on one τ vector. The test results are displayed in the upper half of Table 3 and we note all variables seem to be driven by one or more $I(2)$ trends, as all the hypotheses are rejected. This indicates that c_t, $p_{h,t}$, and $p_{b,t}$ are driven by $I(2)$ trends, and that they may cointegrate from $I(2)$ to $I(1)$ to form the right hand side of the UANA condition in Equation (15).

Table 3. Hypotheses.

Hypothesis	Test Statistic	p-Value
$p_{c,t} \sim I(1)$	$\chi^2(3) = \mathbf{21.83}$	$p = 0.00$
$p_{b,t} \sim I(1)$	$\chi^2(3) = \mathbf{21.17}$	$p = 0.00$
$p_{h,t} \sim I(1)$	$\chi^2(3) = \mathbf{14.86}$	$p = 0.00$
$c_t \sim I(1)$	$\chi^2(3) = \mathbf{15.31}$	$p = 0.00$
$\Delta h_t \sim I(1)$	$\chi^2(3) = \mathbf{14.47}$	$p = 0.00$
Long-run price homogeneity	$\chi^2(4) = \mathbf{25.57}$	$p = 0.00$
$p_{b,t} - p_{c,t} \sim I(1)$	$\chi^2(2) = \mathbf{9.28}$	$p = 0.01$
$p_{h,t} - p_{c,t} \sim I(1)$	$\chi^2(2) = \mathbf{11.49}$	$p = 0.00$
$p_{h,t} - p_{b,t} \sim I(1)$	$\chi^2(2) = \mathbf{10.14}$	$p = 0.01$
$c_t + \sigma(p_{h,t} - p_{b,t}) \sim I(1)$	$\chi^2(1) = 0.30$	$p = 0.58$

Note: We allow for a restricted trend in the lower half of the table. Bold font indicates that the hypothesis is rejected at the 5% significance level.

Next, by imposing restrictions on multiple τ vectors, we examine if any linear combinations of the variables in x_t are $I(1)$. The test results are shown in the bottom half of Table 3. The first test is a joint test for long-run price homogeneity; that is, we test whether $p_{b,t} - p_{c,t} \sim I(1)$ and $p_{h,t} - p_{c,t} \sim I(1)$ hold jointly. The nominal price indices would then share the same nominal $I(2)$ trend, while the real house and construction prices would be $I(1)$, which corresponds to classical dichotomy holding in the long-run, cf. Kongsted (2005). However, the joint test for long-run price homogeneity is rejected with a test statistic of $\chi^2(4) = 25.57$ and a p-value of practically zero. Testing the hypothesis for $p_{b,t} - p_{c,t} \sim I(1)$ returns a test statistic of $\chi^2(2) = 9.28$ with a p-value of $p = 0.01$, which we reject at a 5% critical level. The hypothesis for $p_{h,t} - p_{c,t} \sim I(1)$ is rejected with a test statistic of $\chi^2(2) = 11.49$ and a p-value of practically zero. The hypothesis for $p_{h,t} - p_{b,t} \sim I(1)$ corresponds to the second scenario for the UANA condition, given in Equation (14). In this scenario $p_{h,t} - p_{b,t}$ would cointegrate from $I(2)$ to $I(1)$, which could cointegrate with c_t and $\Delta p_{h,t}$ to stationarity. This scenario is based on the premise that $c_t \sim I(1)$, which we have already rejeced, and we likewise reject the test for $p_{h,t} - p_{b,t} \sim I(1)$ with a test statistic of $\chi^2(2) = 10.14$ and the p-value of $p = 0.01$. Therefore, the hypothesis for the UANA condition in Equation (14) is rejected.

The final hypothesis test corresponds to the third scenario for the UANA condition in Section 2.4, seen in Equation (15). Given the premises that $c_t \sim I(2)$, $p_{h,t} - p_{b,t} \sim I(2)$, and $\Delta p_{h,t} \sim I(1)$, we may still find that there is cointegration between the user cost rate and the relation for Tobin's q expressed in the price indices. The test for whether $c_t + \sigma(p_{h,t} - p_{b,t}) \sim I(1)$ is not rejected with a p-value of $p = 0.58$. Based on this test, it appears that there is support for the UANA condition proposed in Equation (15). However, we still have to establish that $c_t + \sigma(p_{h,t} - p_{b,t})$ cointegrates with $\Delta p_{h,t}$ from $I(1)$ to $I(0)$ for the UANA condition in Equation (15) not to be rejected by the data.

We are now ready to specify an over-identified long-run structure. This will allow us to investigate if the UANA condition in Equation (15) holds empirically, and we may examine if the equilibrium-correcting behavior of the UANA condition can explain the long-swings dynamic in the housing prices introduced in Section 1.

3.8. An Over-Identified Long-Run Structure

For the chosen rank specification, $\{r = 3,\ s_1 = 0,\ s_2 = 2\}$, there will be three polynomially cointegrating relations, $\beta_i' x_t + d_i' \Delta x_t$ for $i = 1,\ 2,\ 3$, but no stationary medium-run relations in the growth rates, $\beta_{\perp 1}' \Delta x_t$, due to no cointegration in the differences, $s_1 = 0$. We impose over-identifying restrictions on β' by testing reduced rank hypotheses on $\Pi = \alpha\beta'$ in a fashion parallel to that of an $I(1)$ analysis. We obtain an identified long-run structure by first imposing the UANA condition from Section 2.4. The UANA condition in itself is is not rejected, and we identify the second and third cointegrating relations by applying an inductive approach, in which we restrict a single variable at a time until we cannot restrict the system further.

The data lends support to a relation between the user cost rate, c_t, the price of construction, $p_{b,t}$, and the price of housing, $p_{h,t}$, in line with Equation (15); a relation between the net housing investments, Δh_t, the price of construction, $p_{b,t}$, and the price of housing, $p_{h,t}$; and a relation between the net housing investments, Δh_t, the consumer price level, $p_{c,t}$, and the user cost rate, c_t. We determine the over-identified long-run structure to be

$$\begin{bmatrix} \beta_1' \\ \beta_2' \\ \beta_3' \end{bmatrix} = \begin{bmatrix} 0 & \beta_{12} & -\beta_{12} & \beta_{14} & 0 & \beta_{16} \\ 0 & \beta_{22} & \beta_{23} & 0 & \beta_{25} & \beta_{26} \\ \beta_{31} & 0 & 0 & \beta_{34} & \beta_{35} & \beta_{36} \end{bmatrix}, \tag{19}$$

with the test statistic $\chi^2(1) = 0.30$, corresponding to $p = 0.58$. The cointegrating relations are shown graphically in Figures 2 and 3. We return to the intuition of this specification in the interpretations of the cointegrating relations in Section 3.9.

Imposing the identification scheme on the $I(2)$ model in Equation (17) results in the estimated long-run structure presented in Table 4. The asymptotic distribution of the standard errors for β are derived in Johansen (1997) and the standard errors for d are calculated using the delta method in Doornik (2016). Unfortunately, we are unable to test joint restrictions on the elements in d, which prevents us from assessing whether we can reduce the presence of first differences further.[7]

[7] An anonymous referee kindly made us aware of a recently published alternative identification scheme, which allows for more restrictions in the over-identified long-run structure, cf. Mosconi and Paruolo (2017). Unfortunately, this procedure has not yet been implemented in the software at our disposal, but it is of interest for future research.

Table 4. An identified long-run structure in β.

	$p_{c,t}$	$p_{b,t}$	$p_{h,t}$	c_t	Δh_t	Det [a)]
β_1'	0.000 [−]	0.025 [10.2]	−0.025 [−10.2]	1.000 [−]	0.000 [−]	6.28×10^{-4} [12.4]
d_1'	−0.077 [−0.5]	0.073 [0.7]	0.683 [6.8]	0.016 [3.8]	0.009 [2.3]	−0.187 [−24.3]
α_1	−0.473 [−0.6]	10.500 [4.9]	10.600 [5.0]	−0.776 [−2.6]	−0.196 [−3.6]	
β_2'	0.000 [−]	0.030 [21.0]	−0.017 [−21.0]	0.000 [−]	1.000 [−]	3.97×10^{-5} [2.1]
d_2'	0.357 [8.6]	0.260 [9.3]	0.050 [1.9]	−0.005 [−5.2]	−0.007 [−7.0]	−0.013 [−4.5]
α_2	−0.487 [−0.4]	−15.500 [−5.2]	−15.700 [−5.3]	1.160 [2.8]	0.255 [3.4]	
β_3'	0.012 [13.2]	0.000 [−]	0.000 [−]	−0.535 [−89.6]	1.000 [−]	-3.04×10^{-3} [−13.6]
d_3'	0.321 [4.6]	0.160 [3.4]	−0.353 [−8.2]	−0.013 [−7.5]	−0.011 [−6.3]	0.090 [26.2]
α_3	−1.240 [−0.9]	18.700 [4.9]	19.300 [5.1]	−1.230 [−2.3]	−0.346 [−3.6]	

Notes: t-statistics are given in brackets below the estimate. $a)$ A restricted trend is included in β' and a restricted constant in d'.

3.9. Interpreting the Long-Run Structure

We interpret a polynomially cointegrating relation as a dynamic equilibrium the same way as Juselius (2007): if $x_t \sim I(2)$, then $\beta' x_t \sim I(1)$ and we can interpret it as an equilibrium error with a high degree of persistence. This means that we can interpret α and d as two levels of equilibrium correction: α describes how the acceleration rates, $\Delta^2 x_t$, adjust to the dynamic equilibrium relations, $\beta' x_t + d' \Delta x_t$, and d describes how the growth rates, Δx_t, adjust to the long-run equilibrium error, $\beta' x_t$ (i.e., describing a medium-run adjustment, conditional on $\alpha \neq 0$). We say that a variable $x_{j,t}$, for $j = 1, 2, \ldots, 5$, is equilibrium error correcting in the long run if $\alpha_{ij}\beta_{ij} < 0$ and/or $\alpha_{ij} d_{ij} < 0$, and it is error correcting in the medium run if $d_{ij}\beta_{ij} > 0$. If we reverse the inequalities, the system is equilibrium error increasing. It is of particular interest that a variable can be error correcting in the long run ($\alpha_{ij}\beta_{ij} < 0$), while being error increasing in the medium run ($d_{ij}\beta_{ij} < 0$), or vice versa. This type of interplay between long-run and medium-run dynamics can lead to persistent swings around the long-run equilibrium, which we refer to as "long-swings dynamics" in line with Juselius and Assenmacher (2016, Section 5).

We translate the parameter estimates in Table 4 into three univariate equations, which govern the long-run error-correction mechanisms of the system, i.e., the cointegrating relations. These can be rearranged to facilitate interpretation. We do this in the following subsections.

3.9.1. The Uncertainty-Adjusted No-Arbitrage Condition

We interpret the first of the three cointegrating relations in terms of the uncertainty-adjusted no-arbitrage condition presented in Section 2.4, specifically Equation (15). The first cointegrating relation is given by

$$0.025 p_{b,t} - 0.025 p_{h,t} + c_t + 6.28 \times 10^{-4} t + 0.683 \Delta p_{h,t} + 0.016 \Delta c_t + 0.009 \Delta^2 h_t + 0.187 \sim I(0), \quad (20)$$

where we include the levels from β_1' and first differences from d_1' that are significant at the 95% critical level. We rearrange the terms in Equation (20) to relate it to the UANA condition in Equation (15):

$$\begin{aligned} \Delta p_{h,t} = & -1.464 c_t + 0.037 (p_{h,t} - p_{b,t}) \\ & - 0.013 \Delta^2 h_t - 0.023 \Delta c_t + 0.274 - 6.28 \times 10^{-4} t + u_{1,t}, \end{aligned} \quad (21)$$

where $u_{1,t} \sim I(0)$ denotes the stationary error component. As suggested by the uncertainty-based asset price approach presented in Section 2.3, the price level of housing, $p_{h,t}$, and the cost of construction, $p_{b,t}$, enter with the same coefficient and opposite sign to form Tobin's q. The user cost rate, c_t, is also present but enters with a negative coefficient. Based on asset pricing theory the user cost rate, c_t, should have entered with a positive coefficient as it represents the carry, or opportunity cost, of buying the asset. We may attempt to understand the negative coefficient on c_t by the dynamic that, in practice, a higher user cost rate would make housing less affordable, which in turn would preclude some investors from entering the housing market as they will be able to borrow less, all else equal. In addition to the levels predicted by the theory, we also find that the changes in net housing investments, $\Delta^2 h_t$, and the user cost rate Δc_t are significant.

As such, we reject the exact specification of the UANA condition as it is presented in Equation (15), noting that the primary implication from the original framework resulting in Equation (2) appears to be inconsistent with the data. However, we do find that realized changes in the price of housing, $\Delta p_{h,t}$, are positively related to Tobin's q. This is in support of the contribution made to the theory by specifying the uncertainty premium in terms of a gap effect. That said, while there are more terms present in Equation (21) than predicted, the cointegrating relation does indicate that the UANA condition in Equation (15) provides an empirically relevant characterization of the price formation in the Danish housing market over the period under consideration.

Turning our attention to the medium- and long-term dynamics, the α_1 and d_1' coefficients reveal that the price of housing, $p_{h,t}$, is strongly error correcting in the long run with respect to this cointegrating relation, but the d' coefficient suggests that the change in the price of housing, $\Delta p_{h,t}$, is error increasing in the medium run. This dynamic implies that the price of housing is prone to overshooting its long-term equilibrium level given by Equation (21). This type of overshooting is consistent with momentum-based forecasting behavior. That is, if investors base their expectations on recent price changes, then the persistence in the realized price changes will increase, cf. e.g., Shiller (2008), leading to overshooting behavior in the medium run. We return to this point in Section 3.10. We also note that the construction price index, $p_{b,t}$, is error increasing in the long run, and the change in the housing stock, Δh_t is (borderline) error correcting in the long run. Finally, the user cost rate, c_t, is (borderline) error correcting in both the medium- and long run. Moreover, graphical inspection of the cointegrating relation in the sample reveals that the relation is stationary and it exhibits relatively little persistence, see Figure 2.

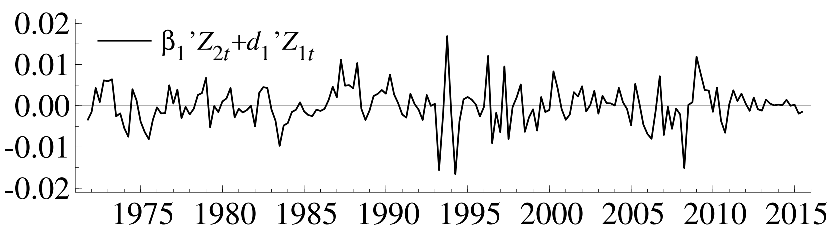

Figure 2. The first cointegration relation, see Equation (21).

3.9.2. Net Housing Investments

We interpret the second and third cointegrating relations as characterizing the long-run equilibrium of net housing investments. These relations are not implied by our theoretical framework in Section 2, instead we have identified them inductively by imposing zero restrictions on the long-run structure. In the Equations (22) and (23), we include the levels from β' and differences from d' that are significant at the 95% critical level.

The second of the three cointegrating relations is given by,

$$\begin{aligned}\Delta h_t = &+ 0.017(p_{h,t} - p_{b,t}) - 0.013 p_{b,t}\\ &- 0.357\Delta p_{c,t} - 0.260\Delta p_{b,t} + 0.005\Delta c_t + 0.007\Delta^2 h_t\\ &+ 0.010 - 3.97 \times 10^{-5} t + u_{2,t}\,,\end{aligned} \tag{22}$$

where $u_{2,t} \sim I(0)$ denotes the stationary error component. We interpret Equation (22) as characterizing the net housing investment in equilibrium as approximately proportional to Tobin's q. The levels of the variables $\{p_{h,t},\ p_{b,t},\ \Delta h_t\}$ cointegrate from $I(2)$ to $I(1)$, and in turn with the significant first-differences, $\{\Delta p_{c,t},\ \Delta p_{b,t},\ \Delta c_t,\ \Delta^2 h_t\}$, from $I(1)$ to $I(0)$ to achieve stationarity. While $p_{h,t}$ and $p_{b,t}$ do not enter with the same coefficient, they are of roughly the same magnitude and enter with opposite signs. The α_2 and d'_2 coefficients reveal that the net housing investments, Δh_t, exhibits error-increasing behavior with respect to this cointegrating relation in both the long and medium run. If there is a positive deviation from the cointegrating relation as a result of Tobin's q being above unity, then we would expect this to cause the net investments to increase. We note that the price level of housing, $p_{h,t}$, is error increasing with respect to the this cointegrating relation in the long run. On the other hand, the price index of building costs, $p_{b,t}$, is error correcting in both the long and medium run. Finally, the user cost, c_t is (borderline) error correcting in the long run. Figure 3a shows this cointegrating relation.

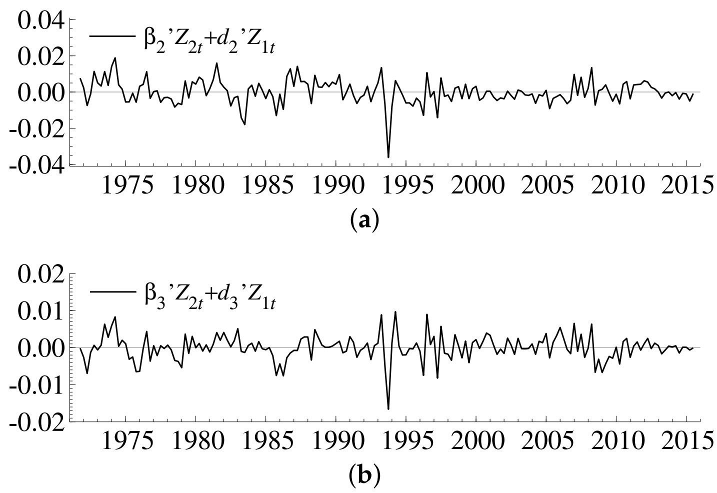

Figure 3. The second and third cointegration relations. (**a**) The second cointegration relation, see Equation (22). (**b**) The third cointegration relation, see Equation (23).

The third and final cointegration relation is a linear combination of the housing prices, $p_{c,t}$, the building costs, $p_{b,t}$, and the net investment in housing, Δh_t, which cointegrate to cancel out the $I(2)$ trends, and in turn cointegrate with the first differences to $I(0)$. The cointegrating relation is given by,

$$\begin{aligned}\Delta h_t = &- 0.012 p_{c,t} + 0.535 c_t\\ &- 0.321\Delta p_{c,t} - 0.160\Delta p_{b,t} + 0.353\Delta p_{h,t} + 0.013\Delta c_t + 0.011\Delta^2 h_t\\ &- 0.090 + 3.04 \times 10^{-3} t + u_{3,t}\,,\end{aligned} \tag{23}$$

where $u_{3,t} \sim I(0)$ denotes the stationary error component. Equation (23) is somewhat difficult to interpret in isolation. Considering the error-correcting properties, α_3 and d'_3 reveal that the housing investments, Δh_t, error correct in the long run, and error increases in the medium run with respect to this cointegrating relation. Furthermore, the user cost rate, c_t is error increasing in the long run, and error correcting in the medium run. Intuitively, if the user cost rate rises above its long-run value relative to the housing investments, we expect a negative effect on the housing investments, as a higher user cost rate will discourage investments. Finally, the construction price index, $p_{b,t}$, is error increasing in the long run, while the house price index, $p_{h,t}$, error corrects in the long run. Figure 3b shows this cointegrating relation.

3.10. Summary of Empirical Findings

In summary, we find strong evidence in support of the housing price, as well as remaining variables in our information set, defined in Equation (16), being integrated of order two, i.e., highly persistent. Moreover, we strongly reject that the $I(2)$ dynamics can be appropriately accounted for by simply transforming to real variables, i.e., we reject long-run price homogeneity. Rather, we find that the uncertainty-adjusted no-arbitrage (UANA) condition, given in Equation (15), provides an empirically relevant characterization of the long-run house price equilibrium. Furthermore, the error-correction dynamics estimated via the cointegrated VAR model can help explain the long-swings behavior in the housing price, observed in Figure 1.

On the last point, it is instructive to construct an informal example of how long-swings behavior associated with the cointegrating relation in Equation (21) relates to our theoretical framework in Section 2 in combination with momentum-based forecasting. Suppose that, at some point in time, the expected future price change is greater than the sum of the current user cost rate and the uncertainty premium; that is, there is arbitrage according to the UANA condition in Equation (12). Investors will then respond by buying housing, pushing the housing price up in the process, which increases Tobin's q (all else equal), and in turn the required uncertainty premium in the next quarter.

This pattern may continue quarter after quarter until Tobin's q has been increased to the point where the expected price change for the next quarter is less than or equal to the sum of the user cost rate and the uncertainty premium. Given the momentum in the price of housing and individual forecasts, the result can be a persistent swing upwards in the price until the required uncertainty premium has become too large relative to investors' expectations of the price change in the next quarter. At this point, further price increases will result in overshooting relative to the long-run equilibrium as well as further increases in the uncertainty premium. In this situation there is also arbitrage, but in the other direction, which will lead to investors selling housing. In turn, the momentum in housing prices will decrease, or even turn (all else equal), and the process will go into reverse. The result is in this case a persistent swing downwards in the price of housing until the required uncertainty premium has become small enough for investors to find housing attractive again relative to their expectations of future price changes.

In practice, all else is not equal, and changes in the user cost rate, building costs, and housing investments will also affect the price of housing. For example, if Tobin's q is greater than unity then the housing stock is smaller than what is demanded by investors, and so net investments will increase to generate profit from the discrepancy between the price at which housing is sold relative to its cost of construction. That is, we would expect a positive association between Tobin's q and net investments, which aligns well with our interpretation of the second cointegration relation in Equation (22).

4. Conclusions

The primary contribution of this paper was to establish that the long-swings behavior observed in the market price of Danish housing since the 1970s can be understood by studying the interplay between short-term expectation formation and long-run equilibrium conditions. We have introduced an asset market model for housing based on uncertainty rather than risk, which under mild assumptions allows for other forms of forecasting behavior than rational expectations. We have tested the theory via an $I(2)$ cointegrated VAR model and found that the long-run equilibrium for the housing price corresponds closely to the predictions from the theoretical framework. Additionally, we have corroborated previous findings that housing markets are well characterized by short-term momentum forecasting behavior. Our conclusions have wider relevance, since housing prices play a role in the wider Danish economy, and other developed economies, through wealth effects. In sum, the CVAR model and the uncertainty-based asset market approach provide a useful framework to analyzing and understanding price formation and net investments in the Danish housing market.

Supplementary Materials: The following are available online at http://www.mdpi.com/2225-1146/5/3/40/s1: the original data from the MONA database, cf. Danmarks Nationalbank (2003), and Ox code to transform the variables into the data vector in Equation (16).

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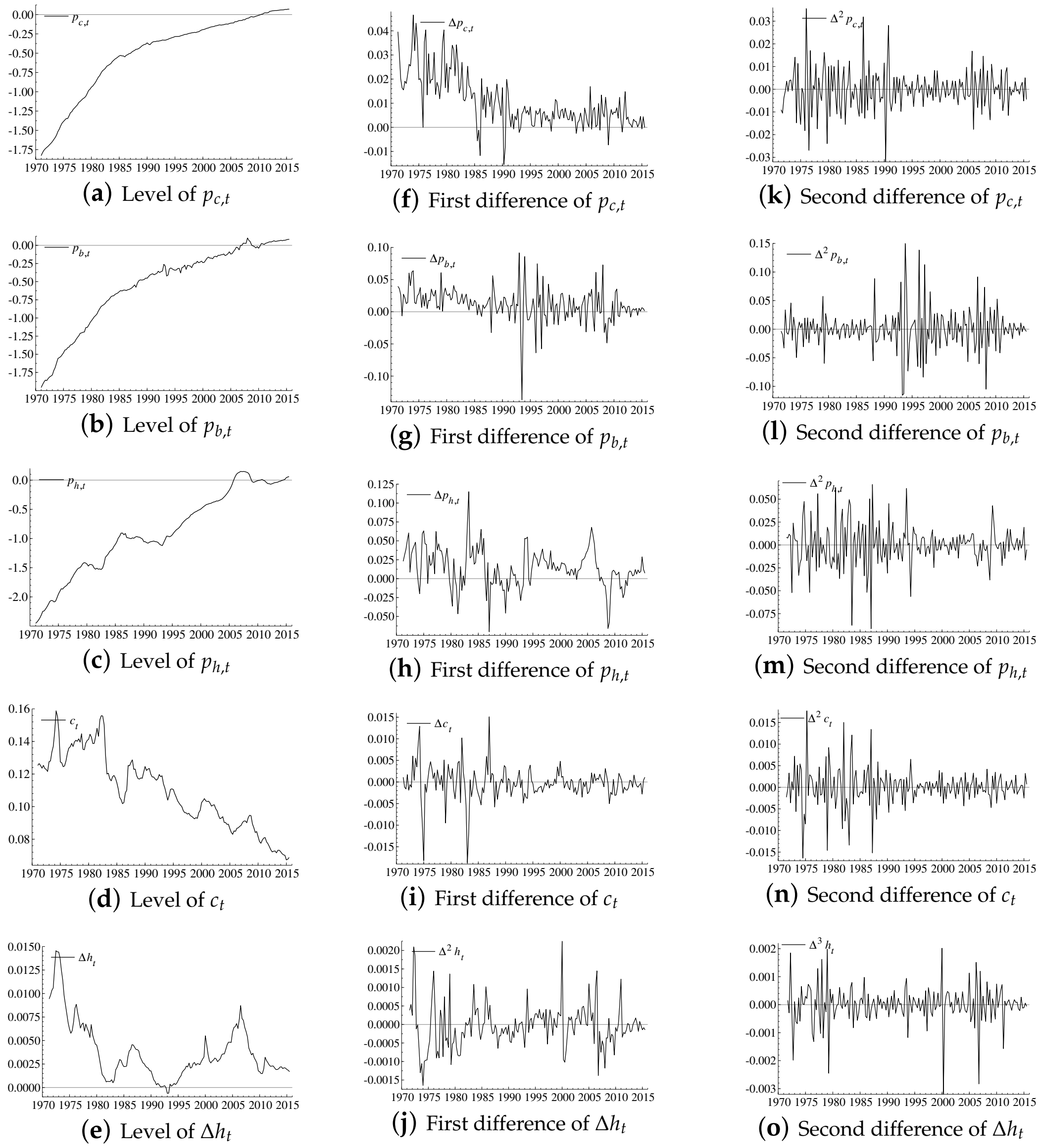
Appendix A. Supplementary Material



(**a**) Level of $p_{c,t}$ (**f**) First difference of $p_{c,t}$ (**k**) Second difference of $p_{c,t}$

(**b**) Level of $p_{b,t}$ (**g**) First difference of $p_{b,t}$ (**l**) Second difference of $p_{b,t}$

(**c**) Level of $p_{h,t}$ (**h**) First difference of $p_{h,t}$ (**m**) Second difference of $p_{h,t}$

(**d**) Level of c_t (**i**) First difference of c_t (**n**) Second difference of c_t

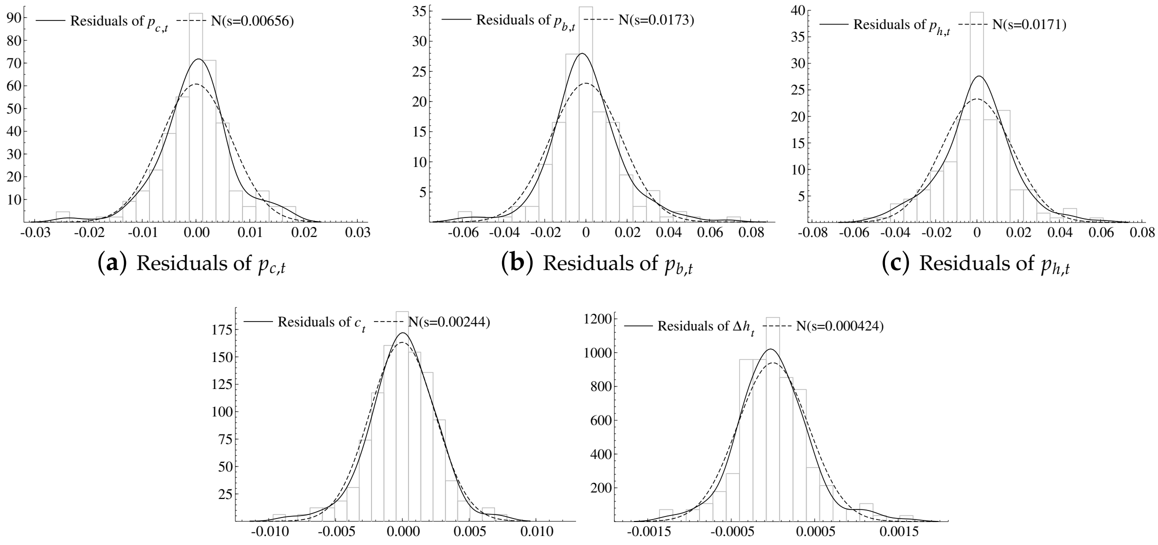
(**e**) Level of Δh_t (**j**) First difference of Δh_t (**o**) Second difference of Δh_t

Figure A1. Data in levels, first-, and second differences.

Table A1. Lag-length determination.

Lags	Log-likelihood	SC	HQ	AIC	LM-test
$k = 4$	3406.56	-35.45	-36.80	-37.72	–
$k = 3$	3395.29	-36.06	-37.14	-37.88	$F(25, 540) = 0.77$ [0.79]
$k = 2$	3375.61	-36.58	-37.39	-37.94	$F(50, 664) = 1.08$ [0.34]

Note: The LM-test is nested in $k = 4$. The preferred model minimizes the information criteria.



(**a**) Residuals of $p_{c,t}$ (**b**) Residuals of $p_{b,t}$ (**c**) Residuals of $p_{h,t}$
(**d**) Residuals of c_t (**e**) Residuals of Δh_t

Figure A2. Residuals from the unrestricted CVAR model with lag order 2.

Appendix B. Specification without Dummies

The VAR(2) model with no dummies has more traces of ARCH effects, kurtosis and skewness than the specification with dummies, as shown in Table A2. From Table A3 the rank test indicates a rank of $\{r = 4,\ s_1 = 0,\ s_2 = 1\}$. We find the same main conclusion, namely that the UANA condition exists in the over-identified long-run structure (which is not rejected, $\chi^2(2) = 0.21$, p-value of 0.90), despite the rank test pointing to a different rank. We also find that there are long swings in the house prices, i.e., the house price is error increasing in the medium run and error correcting in the long run as seen in Table A4.

Table A2. Misspecification tests for the specification with no dummies.

Multivariate tests					
No-autocorrelation	LM(1)	$\chi^2(25) = 26.4$	$p = 0.39$		
	LM(2)	$\chi^2(25) = 28.4$	$p = 0.29$		
Normality	Doornik-Hansen	$\chi^2(10) = \mathbf{238.6}$	$p = 0.00$		
No-ARCH	LM(1)	$\chi^2(225) = \mathbf{341.0}$	$p = 0.00$		
	LM(2)	$\chi^2(450) = \mathbf{630.0}$	$p = 0.00$		
Univariate tests					
	$\Delta^2 p_{c,t}$	$\Delta^2 p_{b,t}$	$\Delta^2 p_{h,t}$	$\Delta^2 c_t$	$\Delta^3 h_t$
No-ARCH	**17.57** [0.00]	**29.19** [0.00]	**9.61** [0.00]	5.75 [0.06]	**16.35** [0.00]
Skewness	−0.21	0.44	−0.26	−0.058	0.87
Kurtosis	4.59	9.15	4.75	7.72	6.05
Normality	**16.88** [0.00]	**107.61** [0.00]	**19.07** [0.00]	**84.71** [0.00]	**25.33** [0.00]
R^2	61%	32%	46%	27%	35%

Notes: Bold font indicates that the hypothesis is rejected at the 5% significance level. We use the multivariate tests for ARCH-effects and autocorrelation presented in Godfrey (1988) and the univariate and multivariate normality test from Doornik and Hansen (2008). The p-values for the univariate tests are shown in square brackets.

Table A3. $I(2)$ rank test for the specification with no dummies.

$p-r$	r	$s_2 = 4$	$s_2 = 3$	$s_2 = 2$	$s_2 = 1$	$s_2 = 0$
2	3			56.1 [0.01]	37.1 [0.03]	30.3 [0.01]
1	4				16.9 [0.13]	7.7 [0.28]

Note: The p-values for the $I(2)$ rank test are given in square brackets.

Table A4. Over-identified long-run structure for the specification with no dummies.

	$p_{c,t}$	$p_{b,t}$	$p_{h,t}$	c_t	Δh_t	$Det^{\,a)}$
β_1'	0.000	0.028 [7.2]	−0.028 [−7.2]	1.000	0.000	0.001 [15.1]
d_1'	0.393 [6.9]	0.393 [6.9]	0.697 [6.9]	0.009 [7.0]	−0.001 [−7.0]	−0.205 [−25.6]
α_1	−0.533 [−5.0]	0.238 [0.7]	0.532 [1.7]	−0.027 [−0.5]	−0.039 [−5.1]	
β_2'	0.000	0.002 [8.3]	0.000	0.000	1.000	−0.000 [−12.1]
d_2'	−0.117 [−10.5]	−0.117 [−10.5]	−0.208 [−10.5]	−0.003 [−10.5]	0.000 [10.5]	0.009 [8.4]
α_2	−7.610 [−5.7]	0.715 [0.2]	3.310 [0.8]	1.530 [2.4]	−0.377 [−4.0]	
β_3'	0.000	0.000	0.000	0.112 [17.2]	1.000	0.000
d_3'	−0.127 [−11.6]	−0.127 [−11.6]	−0.225 [−11.6]	−0.003 [−11.5]	0.000 [11.5]	−0.001 [−9.3]
α_3	6.540 [6.5]	2.220 [0.7]	0.611 [0.2]	−1.590 [−3.3]	0.220 [3.1]	
β_4'	1.000	–1.000	0.000	0.000	0.000	0.001 [8.8]
d_4'	0.340 [2.3]	0.340 [2.3]	0.603 [2.3]	0.008 [2.3]	−0.001 [−2.3]	−0.153 [−8.1]
α_4	−0.042 [−2.1]	0.385 [6.2]	0.107 [1.8]	−0.010 [−1.1]	−0.004 [−2.9]	

Notes: t-statistics are given in brackets below the estimate. $a)$ A restricted trend is included in β' and a restricted constant in d'.

References

Akerlof, George A., and Robert J. Shiller. 2009. *Animal Spirits: How Human Psychology Drives the Economy, and Why It Matters for Global Capitalism*. Princeton and Oxford: Princeton University Press, pp. 1–248.

Binmore, Ken. 2009. *Rational Decisions (The Gorman Lectures in Economics)*. Princeton and Oxford: Princeton University Press.

Björk, Tomas. 2009. *Arbitrage Theory in Continuous Time*, 3rd ed. Oxford: Oxford University Press, pp. 1–560.

Bohn Nielsen, Heino, and Anders Rahbek. 2007. The Likelihood Ratio Test for Cointegration Ranks in the I(2) Model. *Econometric Theory* 23: 615–37.

Campos, Julia, Neil R. Ericsson, and David F. Hendry. 2005. General-to-Specific Modeling: An Overview and Selected Bibliography. International Finance Discussion Papers No. 838, Board of Governors of the Federal Reserve System, Washington, USA.

Carney, Mark. 2016. Uncertainty, the Economy and Policy. Speech, Bank of England. Available online: http://www.bankofengland.co.uk/publications/Documents/speeches/2016/speech915.pdf (accessed on 31 August 2017).

Case, Karl E., John M. Quigley, and Robert J. Shiller. 2003. Home-buyers, Housing and the Macroeconomy. In *Assets Prices and Monetary Policy*. Edited by Anthony Richards and Tim Robinson. Sydney: Reserve Bank of Australia, pp. 149–88.

Case, Karl E., John M. Quigley, and Robert J. Shiller. 2005. Comparing Wealth Effects: The Stock Market versus the Housing Market. *Advances in Macroeconomics* 5: 1–32.

Case, Karl E., John M. Quigley, and Robert J. Shiller. 2013. Wealth effects revisited: 1978–2009. *Critical Finance Review* 2: 101–28.

Case, Karl E., and Robert J. Shiller. 1989. The Efficiency of the Market for Single-Family Homes. *The American Economic Review* 79: 125–37.

Cavaliere, Giuseppe Cavaliere, Anders Rahbek, and A. M. Robert Taylor. 2010. Cointegration Rank Testing Under Conditional Heteroskedasticity. *Econometric Theory* 26: 1719–60.

Cavaliere, Giuseppe Cavaliere, Anders Rahbek, and A. M. Robert Taylor. 2012. Bootstrap Determination of the Co-Integration Rank in Vector Autoregressive Models. *Econometrica* 80: 1721–40.

Danmarks Nationalbank. 2003. *MONA—En Kvartalsmodel af Dansk Økonomi*. Copenhagen: Danmarks Nationalbank.

Doornik, Jurgen A. 1998. Approximations to the asymptotic distribution of cointegration tests. *Journal of Economic Surveys* 12: 573–93.

Doornik, Jurgen A. 2016. Estimation of the I(2) Model. Institute for Economic Modelling, Oxford Martin School, and Economics Department, University of Oxford, Oxford, UK.

Doornik, Jurgen A., and Henrik Hansen. 2008. An omnibus test for univariate and multivariate normality. *Oxford Bulletin of Economics and Statistics* 70: 927–39.

ECB. 2016. The Impact of Uncertainty on Activity in the Euro Area. ECB Economic Bulletin, European Central Bank, Frankfurt am Main, Germany.

Frydman, Roman, Michael D. Goldberg, and Edmund S. Phelps. 2007. *Imperfect Knowledge Economics: Exchange Rates and Risk*. Princeton and Oxford: Princeton University Press.

Godfrey, L. G. 1988. *Misspecification Tests in Econometrics*. Cambridge: Cambridge University Press.

Hoover, Kevin D., Søren Johansen, and Katarina Juselius. 2008. Allowing the Data to Speak Freely: The Macroeconometrics of the Cointegrated Vector Autoregression. *The American Economic Review* 98: 251–55.

Johansen, Søren. 1995. A statistical analysis of cointegration for I(2) variables. *Econometric Theory* 11: 25–59.

Johansen, Søren. 1996. *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models (Advanced Texts in Econometrics)*. Oxford: Oxford University Press.

Johansen, Søren. 1997. Likelihood analysis of the I(2) model. *Scandinavian Journal of Statistics* 24: 433–62.

Johansen, Søren. 2006. Statistical analysis of hypotheses on the cointegrating relations in the I(2) model. *Journal of Econometrics* 132: 81–115.

Juselius, Katarina. 2007. *The Cointegrated VAR Model: Methodology and Applications (Advanced Texts in Econometrics)*. Oxford: Oxford University Press.

Juselius, Katarina. 2017a. A Theory-Consistent CVAR Scenario for a Standard Monetary Model using Data-Generated Expectations. Discussion Papers, Department of Economics, University of Copenhagen, Copenhagen, Denmark.

Juselius, Katarina. 2017b. Using a Theory-Consistent CVAR Scenario to Test an Exchange Rate Model Based on Imperfect Knowledge. Discussion Papers, Department of Economics, University of Copenhagen, Copenhagen, Denmark.

Juselius, Katarina, and Katrin Assenmacher. 2016. Real Exchange Rate Persistence and the Excess Return Puzzle: The Case of Switzerland Versus the US. Discussion Papers, Department of Economics, University of Copenhagen, Copenhagen, Denmark.

King, Mervyn. 2004. What Fates Impose: Facing up to Uncertainty. Available online: http://www.bankofengland.co.uk/archive/Documents/historicpubs/speeches/2004/speech234.pdf (accessed on 31 August 2017).

Kongsted, Hans Christian. 2005. Testing the nominal-to-real transformation. *Journal of Econometrics* 124: 202–25.

Lawson, Tony. 1988. Probability and Uncertainty in Economic Analysis. *Journal of Post Keynesian Economics* 11: 38–65.

Mellers, Barbara, Lyle Ungar, Jonathan Baron, Jaime Ramos, Burcu Gurcay, Katrina Fincher, Sydney E. Scott, Don Moore, Pavel Atanasov, Samuel A. Swift, and et al. 2014. Psychological strategies for winning a geopolitical forecasting tournament. *Psychological Science* 25: 1106–15.

Mosconi, Rocco, and Paolo Paruolo. 2017. Identification conditions in simultaneous systems of cointegrating equations with integrated variables of higher order. *Journal of Econometrics* 198: 271–76.

Muth, John F. 1961. Rational expectations and the theory of price movements. *Econometrica* 29: 315–35.

Piazzesi, Monika, and Martin Schneider. 2009. Momentum Traders in the Housing Market: Survey Evidence and a Search Model. *American Economic Review* 99: 406–11.

Poterba, James M. 1984. Tax Subsidies to Owner-Occupied Housing: An Asset-Market Approach. *The Quarterly Journal of Economics* 99: 729–52.

Poterba, James M. 1991. House price dynamics: The role of tax policy and demography. *Brookings Papers on Economic Activity* 1991: 143–203.

Rahbek, Anders, Ernst Hansen, and Jonathan G. Dennis. 2002. ARCH Innovations and Their Impact on Cointegration Rank Testing. Working Paper, Centre for Analytical Finance, University of Copenhagen, Copenhagen, Denmark.

Rahbek, Ander, Hans Christian Kongsted, and Clara Jørgensen. 1999. Trend Stationarity in the I(2) Cointegration Model. *Journal of Econometrics* 90: 265–89.

Rutherford, Malcolm. 1984. Rational Expectations and Kenyesian Uncertainty: A critique. *Journal of Post Keynesian Economics* VI: 377–87.

Saez, Emmanuel, and Gabriel Zucman. 2016. Wealth Inequality in the US Since 1913: Evidence from Capitalized Income Data. *The Quarterly Journal of Economics* 131: 519–78.

Sargent, Thomas J., and Neil Wallace. 1975. Rational expectations, the optimal monetary instrument, and the optimal money supply rule. *Journal of Political Economy* 83: 241–54.

Savage, Leonard J. 1951. *The Foundations of Statistics*. New York: Wiley.

Shiller, Robert J. 2008. Understanding Recent Trends in Housing Prices and Home Ownership. In Proceedings of "Housing, Housing Finance and Monetary Policy" Jackson Hole Conference Series, Federal Reserve Bank of Kansas City, Kansas City, USA, pp. 85–123.

Tobin, James. 1969. A General Equilibrium Approach to Monetary Theory. *Journal of Money, Credit and Banking* 1: 15–29.