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Investigation of Equilibrium in Oligopoly Markets with the Help of Tripled Fixed Points in Banach Spaces

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Abstract: In the study we explore an oligopoly market for equilibrium and stability based on statistical data with the help of response functions rather than payoff maximization. To achieve this, we extend the concept of coupled fixed points to tripled fixed points. We propose a new model that leads to generalized triple fixed points. We present a possible application of the generalized tripled fixed point model to the study of market equilibrium in an oligopolistic market dominated by three major competitors. The task of maximizing the payout functions of the three players is modified by the concept of generalized tripled fixed points of response functions. The presented model for generalized tripled fixed points of response functions is equivalent to Cournot payoff maximization, provided that the market price function and the three players’ cost functions are differentiable. Furthermore, we demonstrate that the contractive condition corresponds to the second-order constraints in payoff maximization. Moreover, the model under consideration is stable in the sense that it ensures the stability of the consecutive production process, as opposed to the payoff maximization model with which the market equilibrium may not be stable. A possible gap in the applications of the classical technique for maximization of the payoff functions is that the price function in the market may not be known, and any approximation of it may lead to the solution of a task different from the one generated by the market. We use empirical data from Bulgaria’s beer market to illustrate the created model. The statistical data gives fair information on how the players react without knowing the price function, their cost function, or their aims towards a specific market. We present two models based on the real data and their approximations, respectively. The two models, although different, show similar behavior in terms of time and the stability of the market equilibrium. Thus, the notion of response functions and tripled fixed points seems to present a justified way of modeling market processes in oligopoly markets when searching whether the market has reached equilibrium and if this equilibrium is unique and stable in time.

Keywords: fixed point; tripled fixed point; market equilibrium; response functions; oligopoly market

MSC: 46B07; 46B20; 46B25; 55M20; 65D15

JEL Classification: C02; D43; C62

1. Introduction

Modern markets are characterized by the consolidation of producers, which naturally leads to the replacement of the free market with a noncompetitive one. Since Cournot announced the theory of oligopolistic markets Cournot (1897) more than a century ago, there has been a growing interest in studying the dynamics in these markets. The classical payoff maximization technique has its drawbacks due to the possible irrational behavior
of the participants. This behavior can be due both to their technological limitations or the presence of long-term contracts that do not allow them to make quick changes in the quantities of goods, as well as to the lack of reliable information about the demand function and, from there, the function of the price. Incorrect information about the price function can lead to the solution to the payoff maximization problem being different from the one generated by the market. Another disadvantage is the assumption of differentiability for the payoff functions, which in most cases can be even discontinuous ones. An approach proposed in (Dzhabarova et al. 2020; Kabaivanov et al. 2022) avoids these drawbacks by reducing the task of finding market equilibrium to the task of finding fixed points for maps that are the natural response functions of producers participating in the market under consideration. The results in (Dzhabarova and Zlatanov 2022; Dzhabarova et al. 2020; Kabaivanov et al. 2022) are theoretical. The first application of these results to the study of equilibrium, its uniqueness, and stability is proposed in Badev et al. (2024). We will try to demonstrate that this approach can be applied to other markets as well.

The Banach fixed point theorem Banach (1922), although a century old, has an enormous number of generalizations and applications. The generalizations may be classified in several directions: changing the contractive type condition, the underlying space, the fixed point notion, or all of them simultaneously. We will concentrate on only two of them, related to our investigation. The first is to change the contractive condition. Contractive assumptions on the self-map \( T : X \to X \) have been considered in Reich (1971), which cover several generalizations. The presented condition in Reich (1971) has been later simplified in Hardy and Rogers (1973) by exploiting the symmetry of the metric function. Another direction in the generalizations is the alteration of the fixed point notion. Coupled fixed points, introduced in Guo and Lakshmikantham (1987), are relevant from this perspective. Unfortunately, the system of equations \( x = F(x, y) \) and \( y = F(y, x) \) that define the coupled fixed points (Bhaskar and Lakshmikantham 2006; Guo and Lakshmikantham 1987) usually leads to an ordered pair \((x, y)\) such that \( x = y \). This drawback has been overcome in Zlatanov (2021), where a generalized coupled fixed point for an ordered pair of maps \((F, G)\) has been introduced. Following Guo and Lakshmikantham (1987), it has been proposed to search for an ordered triple \((x, y, z)\) satisfying \( x = F(x, y, z), y = F(y, x, y), \) and \( z = F(z, y, x) \), Borcut and Berinde (2011). This idea has been further developed in Samet and Vetro (2010) by suggesting the notion of \( N \)-tuple \((x_1, x_2, \ldots, x_N)\) being a fixed point of \( N \)-order for the map \( F : X \times \cdots \times X \to X \). This idea has been exploited in Dzhabarova and Zlatanov (2022). Combining all mentioned above, we will attempt to generalize the notion of a tripled fixed point to generalized tripled points for ordered triplets of maps \((F, G, H)\), satisfying the contractive condition of the Hardy-Rogers type Hardy and Rogers (1973).

We will apply the obtained results to the study of market equilibrium in noncompetitive markets where three manufacturers hold a dominant position.

An investigation of oligopoly markets, i.e., markets dominated by a few big producers that control the price and supply quantity of the goods being traded is presented in Cournot (1897). While the research in Cournot (1897) employed a highly simplified model, it has served as a fundamental framework for numerous investigations over a century.

The classical oligopoly model assumes that the firms sequentially respond to each other’s output in a rational way, i.e., pursuing to maximize their payoff function. The maximization problem leads to the solution of the first-order equations, which define the response function for each of the producers. The concept of deviating from rational player behavior has been explored in prior studies, such as (Rubinstein 1998; Ueda 2019). In another enhancement of the approach by Cellini and Lamberti (2004), each company endeavors to anticipate the production changes of other players at time \( t \), while still relying on data up to period \( t \).

Let us recall the Cournot model, with three companies producing perfectly substitute goods and competing for the same consumers. Let the overall production in the market be \( X = x_1 + x_2 + x_3 \), where \( x_i \), \( i = 1, 2, 3 \) is the quantity produced by the \( i \)th player. The
Assuming that all players are acting rationally, i.e., they are interested only in maximizing their profit, the payoff functions of the three players are represented by

\[ \Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i), \quad i = 1, 2, 3, \]

respectively. Let us put \( x = (x_1, x_2, x_3) \), just to simplify some of the notations. Each company, if with a rational behavior, wants to maximize its profit, which is

\[ \max \{ \Pi_i(x) : x_i, \text{ presuming that } x_i, k_i, j, k \neq i \text{ are fixed} \} \quad i = 1, 2, 3. \]  

We obtain the system of equations

\[
\begin{align*}
\frac{\partial \Pi_1(x_1, x_2, x_3)}{\partial x_1} &= P(x_1 + x_2 + x_3) + x_1 P'(x_1 + x_2 + x_3) - c'_1(x_1) = 0 \\
\frac{\partial \Pi_2(x_1, x_2, x_3)}{\partial x_2} &= P(x_1 + x_2 + x_3) + x_2 P'(x_1 + x_2 + x_3) - c'_2(x_2) = 0 \\
\frac{\partial \Pi_3(x_1, x_2, x_3)}{\partial x_3} &= P(x_1 + x_2 + x_3) + x_3 P'(x_1 + x_2 + x_3) - c'_3(x_3) = 0,
\end{align*}
\]

assuming that functions \( P \) and \( c_i, i = 1, 2, 3, \) are differentiable.

There are various approaches to pursuing a solution to (2). The direct approach of obtaining \( x_1 = b_1(x_2, x_3), \) \( x_2 = b_2(x_1, x_3) \) and \( x_3 = b_2(x_1, x_2), \) called response functions Friedman (1983), may encounter significant technical limitations. For example, it may be difficult or impossible to obtain an exact solution to (2). Even if approximate solutions are found, if the model is not stable, this approximation is far from the equilibrium of the market. The solutions of (2) are only necessary conditions for a triple \((y_1, y_2, y_3)\) to be an equilibrium point that maximizes the payoff functions \( \Pi_i. \) A sufficient condition for a solution \((y_1, y_2, y_3)\) of (2) to also be a solution of (1) involves satisfying either that the payoff functions \( \Pi_i \) are concave or ensuring the second order conditions, i.e., \( \frac{\partial^2 \Pi_i(x_1, x_2, x_3)}{\partial x_j^2}(y_1, y_2, y_3) < 0 \) for \( i = 1, 2, 3 \) (Bischi et al. 2010; Okuguchi and Szidarovszky 1999). Finally, if the payoff functions \( \Pi_i \) are not differentiable or even continuous, different optimization techniques should be applied.

An alternative approach is pioneered in Dzhabarova et al. (2020) by finding an implicit formula for the response functions.

\[ x_i = \frac{c'_i(x_i) - P(x_1 + x_2 + x_3)}{P'(x_1 + x_2 + x_3)} = F_i(x_1, x_2, x_3), \quad i = 1, 2, 3. \]

The general framework proposed in Dzhabarova et al. (2020) is as follows: for each ordered pair of sold production \((x_1, x_2, x_3)\), where \( x_i \) is the quantity sold by the \( i \)th, \( i \)th producer chooses a new level of production \( y_i, \) depending on their technological options and their policy on the market. Thus, in the next time interval, the \( i \)th player will produce \( y_i = F_i(x_1, x_2, x_3). \) We are searching for a triple \( x = (x_1, x_2, x_3) \) of productions that will not lead to a change in the productions of any of the players, i.e., \( x_i = F(x_1, x_2, x_3). \)

Importantly, using the response function enables the transformation of the maximization problem from (1) into a tripled fixed point one, thus all assumptions of concavity, differentiability, or even continuity can be bypassed (Dzhabarova et al. 2020; Kabaivanov et al. 2022). Specifically, Kabaivanov et al. (2022) show that in the settings of Andaluz et al. (2020) their criteria for guaranteeing the uniqueness and existence of the market equilibrium can be expanded outside the settings of payoff function maximization.

One can find a comprehensive analysis of oligopolistic markets in (Bischi et al. 2010; Matsumoto and Szidarovszky 2018; Okuguchi 1976). Specifically in duopoly markets (Baik and Lee 2020; Liu and Sun 2020; Wang et al. 2020), some recent results on oligopoly markets are found (Alavifard et al. 2020; Geraskin 2020; Siegent and Ulbricht 2020; Strand-
holm 2020; Xiao and Wang 2020). Response function-based equilibrium analysis in duopoly markets was first presented in Dzhabarova et al. (2020) and has recently been further explored in Kabaivanov et al. (2022).

2. Materials and Methods

2.1. Fixed Points

Let $A$ be a nonempty set. A point $x \in A$ is said to be a fixed point for a map $T : A \to A$, if $x = Tx$.

The Banach fixed point theorem Banach (1922), although being a century old, has many applications and generalizations. One such generalization is altering the contractive condition. We will introduce a version from Reich (1971), which will be necessary for our subsequent discussion.

**Definition 1** (Reich (1971)). Let $(X, \rho)$ be a metric space and the map $T : X \to X$ is called Hardy–Rogers, provided that the inequality

$$\rho(Tx, Ty) \leq a_1 \rho(x, y) + a_2 \rho(x, Tx) + a_3 \rho(y, Ty) + a_4 \rho(x, Ty) + a_5 \rho(y, Tx)$$

(3)

holds for all $x, y \in X$ and for some constants $a_i \in [0, 1)$, $i = 1, 2, 3, 4, 5$, verifying $\sum_{i=1}^{5} a_i < 1$.

As pointed out in Hardy and Rogers (1973), the symmetry of the metric function $\rho(\cdot, \cdot)$ implies that if (3) holds then the following inequality will also hold:

$$\rho(Tx, Ty) \leq a_1 \rho(x, y) + \frac{a_2 + a_3}{2} (\rho(x, Tx) + \rho(y, Ty)) + \frac{a_4 + a_5}{2} (\rho(x, Ty) + \rho(y, Tx)).$$

Thus, we may take into consideration maps that meet the inequality

$$\rho(Tx, Ty) \leq k_1 \rho(x, y) + k_2 (\rho(x, Tx) + \rho(y, Ty)) + k_3 (\rho(x, Ty) + \rho(y, Tx)),$$

(4)

so that $k_1 + 2k_2 + 2k_3 < 1$, without sacrificing generality.

When $k_2 = k_3 = 0$ a Banach contraction, if $k_1 = k_3 = 0$ a Kannan contraction Kannan (1968), and if $k_1 = k_2 = 0$ a Chatterjea contraction Chatterjea (1972) are obtained.

We shall assume in the remainder of our study that (4) is satisfied by a Hardy–Rogers map.

**Theorem 1** (Hardy and Rogers (1973)). Let $(X, \rho)$ be a complete metric space and $T : X \to X$ be a Hardy–Rogers map, then:

1. a unique fixed point $\xi \in X$ of $T$ exists and, moreover, for any first estimate $x_0 \in X$, the iterated sequence $x_n = Tx_{n-1}$ for $n = 1, 2, \ldots$ converges to the fixed point $\xi$.
2. there comes a priori error estimate $\rho(\xi, x_n) \leq \frac{k^n}{1-k} \rho(x_0, x_1)$
3. there comes a posteriori error estimate $\rho(\xi, x_n) \leq \frac{k^n}{1-k} \rho(x_{n-1}, x_n)$
4. the rate of convergence is $\rho(\xi, x_n) \leq k \rho(\xi, x_{n-1})$.

where $k = \frac{k_1 + k_2 + k_3}{1-k_2 - k_3}$ and $k_i = 1, 2, 3$ are the constants from (4).

2.2. Coupled Fixed Points

A different approach in the generalization of fixed points has been proposed in Guo and Lakshmikantham (1987).

**Definition 2** (Bhaskar and Lakshmikantham 2006; Guo and Lakshmikantham 1987). Let $A \neq \emptyset$ be a set in a metric space $(X, \rho)$, $F : A \times A \to A$. An ordered pair $(x, y) \in A \times A$ is named a coupled fixed point of $F$ in $A$, provided that $x = F(x, y)$ and $y = F(y, x)$.

The initial results regarding the existence and uniqueness of coupled fixed points were obtained in Bhaskar and Lakshmikantham (2006), assuming that the underlying space is a
We say that \((x, y) \in A \times A \) holds for any \(x, y \in X\). \(F\) and \(G\) are called coupled fixed points if \(x = F(x, y)\) and \(y = G(y, x)\).

**Theorem 2:** Let \(A, B \neq \emptyset\) be closed sets in a complete metric space \((X, \rho)\). The ordered pair \((F, G)\) has a unique common coupled fixed point \((x_0, y_0) \in A \times A \cup B \times B\), i.e., \(x_0 = F(x_0, y_0) = G(x_0, y_0)\) and \(y_0 = F(y_0, x_0) = G(y_0, x_0)\), provided that \((F, G)\) is a cyclic contraction.

Moreover, Zlatanov (2021) proved that \(x_0 = y_0\).

Zlatanov (2021) proposed considering two maps \(F : A \times A \rightarrow A, f : A \times A \rightarrow A\) and define an ordered pair \((x, y)\) as a coupled fixed point for \((F, f)\) if \(x = F(x, y)\) and \(y = f(x, y)\). If \(f(x, y) = f(y, x)\) we get the definition from (Bhaskar and Lakshmikantham 2006; Guo and Lakshmikantham 1987).

In order to apply the coupled fixed points approach to the study of market equilibrium, an extension of the previously discussed concepts was given Dzhabarova et al. (2020). Each participant in a duopoly market will inevitably respond differently depending on how their competition performs and how well they do on the market. Therefore, two response functions \(F_i\) for \(i = 1, 2\) were taken into consideration in Dzhabarova et al. (2020), assuming \(F_i : X_1 \times X_2 \rightarrow X_i, i = 1, 2\), wherein \(X_i\) is the player \(i = 1, 2\)'s production set, and \(x = F_1(x, y)\) and \(y = F_2(x, y)\) define the coupled fixed points. Thus we reach maps that are not self-maps and are not cyclic ones from Definition 3. The authors have termed these new types of maps cyclic once again in Dzhabarova et al. (2020). A more natural name a semi-cyclic map is introduced in Ajeti et al. (2022).

We obtain the concept of coupled fixed points from Definition 2 anytime \(X_1 = X_2 = A\) and \(F_2(x, y) = F_1(y, x)\).

**Definition 6:** Let \(A, B \neq \emptyset\) be sets in a metric space \((X, \rho)\) and \(F : A \times B \rightarrow A, G : A \times B \rightarrow B\). An ordered pair \((F, G)\) is said to be an semi-cyclic ordered pair of maps.
1795. The method is usually associated with linear (univariate and multivariate) statistical models where its efficacy is most evident. Determining the dependency of a single indicator variable, on a set of other \( x_1, x_2, \ldots, x_k \)—independent variables is the fundamental task, i.e., a model of the form

\[
y = f(x_1, x_2, \ldots, x_k) + \epsilon,
\]

Tests involving lagged changes are called augmented Dickey-Fuller tests (Dickey and Fuller 1979). The Dickey-Fuller test evaluates the hypothesis of a unit root

\[
\Delta Y_t = \gamma Y_{t-1} + \epsilon_t,
\]

where \( Y_t \) denotes the value of a data point at period \( t \), \( \Delta Y_t = Y_t - Y_{t-1} \), \( \epsilon_t \) a pure random disturbance in \( t \). Tests involving lagged changes are called augmented Dickey-Fuller tests and are also used for hypothesis testing \( \gamma = 0 \). The null hypothesis \( H_0 \) means that the time series is non-stationary and indicates the presence of a trend; the alternative hypothesis \( H_1 \) means stationarity. Put otherwise, the small p-value indicates that it is improbable that a unit root exists.

The attempt to generalize the coupled fixed point notion for ordered triples seems to lack uniqueness as different ideas have been proposed. Let \( F : X \times X \times X \to X \). Following (Amini-Harandi 2013; Samet and Vetro 2010) a triple \((x, y, z)\) is said to be a tripled fixed point, provided that \( x = F(x, y, z) \), \( y = F(y, z, x) \) and \( z = F(z, x, y) \). Following Borcut and Berinde (2011) a triple \((x, y, z)\) is said to be a tripled fixed point, provided that \( x = F(x, y, z) \), \( y = F(y, x, y) \) and \( z = F(z, y, x) \).

An idea to consider \( n \)-tupled fixed points is presented in Samet and Vetro (2010).

It is proposed in Zlatanov (2021) to consider three maps \( F_i : X \times X \times X \to X \) and to define a tripled fixed point by the equalities \( x_i = F_i(x_i, x_{i+1}, x_{i+2}) \) for \( i = 1, 2, 3 \). This approach gives the possibility to consider models where the tripled fixed points \((x_1, x_2, x_3)\) do not need to satisfy \( x_1 = x_2 = x_3 \). By a particular choice of the functions \( F_i, i = 2, 3 \) Zlatanov (2021) it is possible to get the investigated maps in (Amini-Harandi 2013; Borcut and Berinde 2011; Samet and Vetro 2010).

We obtain the concept of coupled fixed points from Definition 2 anytime \( A = B \) and \( G(x, y) = F(y, x) \).

The existence and uniqueness of an equilibrium in duopoly markets are confirmed by a series of studies about coupled fixed points for semi-cyclic maps (Ajeti et al. 2022; Dzhabarova et al. 2020; Kabaivanov et al. 2022).

2.3. Tripled and n-Tupled Fixed Points

Investigating the stationarity of time series is an important factor for our study. A stationary time series is a series whose statistical measures, such as mean, variance, covariance, and standard deviation, are not a function of time. In other words, stationarity in a time series means a time series without trends or seasonal components. The two most popular statistical tests to ascertain whether or not a time series is stationary are the Phillips-Perron (PP) and augmented Dickey-Fuller (ADF) tests (Dickey and Fuller 1979). The Dickey-Fuller test evaluates the hypothesis of a unit root

\[
\Delta Y_t = \gamma Y_{t-1} + \epsilon_t,
\]

where \( Y_t \) denotes the value of a data point at period \( t \), \( \Delta Y_t = Y_t - Y_{t-1} \), \( \epsilon_t \) a pure random disturbance in \( t \). Tests involving lagged changes are called augmented Dickey-Fuller tests and are also used for hypothesis testing \( \gamma = 0 \). The null hypothesis \( H_0 \) means that the time series is non-stationary and indicates the presence of a trend; the alternative hypothesis \( H_1 \) means stationarity. Put otherwise, the small p-value indicates that it is improbable that a unit root exists.

2.4. Testing the Stationarity of Time Series Data

The Least Squares Method (LSM) is one of the most widely used methods in statistics and holds a special place in the development of science. Adrien-Marie Legendre Bretscher (1997) initially published the LSM, which was established by Carl Friedrich Gauss in 1795. The method is usually associated with linear (univariate and multivariate) statistical models where its efficacy is most evident. Determining the dependency of a single indicator \( y \), the dependent variable, on a set of other \( x_1, x_2, \ldots, x_k \)—independent variables is the fundamental task, i.e., a model of the form

\[
y = f(x_1, x_2, \ldots, x_k) + \epsilon,
\]
where $\epsilon$ denotes the summed effect of the factors, which are not reported in the explicit form $f(x_1, x_2, \ldots, x_k)$. The residual component $\epsilon$ in the model (6) is required to be a relatively small random variable.

Depending on the type of function $f$, regression models can be linear or nonlinear. In much of the study of multivariate dependencies, the form of $f$ is assumed to be linear. This can be justified in part by the fact that the Taylor expansion of a function $f$ can separate the linear term and assign all subsequent terms, quadratic, cubic, etc., to the residual component $\epsilon$. Assuming a linear form of dependence, model (6) can be written as:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \epsilon,$$

where the coefficients $\beta_0, \beta_1, \beta_2, \ldots, \beta_k$ are the ones we are searching for.

For $n$ joint observations of the dependent and independent variable $(y_i; x_{1i}, x_{2i}, \ldots, x_{ki})$, $i = 1, 2, \ldots, n$ (7) is presented in the following matrix form:

$$
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  \vdots \\
  y_n
\end{bmatrix}
= \begin{bmatrix}
  1 & x_{11} & x_{21} & \cdots & x_{k1} \\
  1 & x_{12} & x_{22} & \cdots & x_{k2} \\
  1 & x_{13} & x_{23} & \cdots & x_{k3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & x_{1n} & x_{2n} & \cdots & x_{kn}
\end{bmatrix}
\begin{bmatrix}
  \beta_0 \\
  \beta_1 \\
  \beta_2 \\
  \vdots \\
  \beta_k
\end{bmatrix}
+ \begin{bmatrix}
  \epsilon_1 \\
  \epsilon_2 \\
  \epsilon_3 \\
  \vdots \\
  \epsilon_n
\end{bmatrix}
$$

or in an equivalent form

$$Y = X\beta + \epsilon,$$

$Y = (y_1, y_2, \ldots, y_n)^T$ is the variable observation vector of $n \times 1$, $X$ is the known matrix of $n \times (k + 1)$ with the first column consisting of ones, $\beta = (\beta_0, \beta_1, \ldots, \beta_k)^T$ is the unknown parameter vector of $(k + 1) \times 1$, and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T$ of $n \times 1$ is the error vector. Finding the parameters $\beta$ so that the sum of the squares of the errors is as little as possible is the basic goal of the least squares approach, i.e.,

$$(Y - X\beta)^T(Y - X\beta) \rightarrow \min.$$

To apply the LSM in regression analysis, the dependent variable must be normally distributed. We verify the latter with Shapirio and Wilk (1965) and Kolmogorov-Smirnov tests.

2.6. Statistical Measures for Model Estimation

We employ two basic statistical metrics to assess the effectiveness of the built models: the mean absolute percentage error (MAPE) and the coefficient of determination $R^2$, which can be computed using the following equations

$$R^2 = 1 - \frac{\sum_{i=1}^{n} (P_i - Y_i)^2}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}, \quad MAPE = \frac{1}{n} \sum_{i=1}^{n} \left| \frac{P_i - Y_i}{Y_i} \right|,$$

where the sample size is $n$, the predicted values are $P_i$, the mean is $\bar{Y}$, and the values of the dependent variable are $Y_i$. Our objective is to develop a model with the lowest MAPE and the maximum potential value of $R^2$.

We aim to generalize the proposed concepts of tripled fixed points so that the obtained results can be applied to the study of market equilibrium based on response functions for markets dominated by three producers. Using the statistical techniques discussed, we will construct response function models that are based on real data and examine these real markets for the presence of equilibrium and stability.
3. Main Result

The concepts from Dzhabarova et al. (2020) will be expanded upon by taking into account three distinct metric spaces \((Z_i, d_i)\), where \(i = 1, 2, 3\).

**Definition 8.** Let \(X_1, X_2\) and \(X_3\) be nonempty sets. We will call the ordered triple of maps \((F_1, F_2, F_3)\), \(F_i : X_1 \times X_2 \times X_3 \rightarrow X_i\) for \(i = 1, 2, 3\) a semi-cyclic map.

Let us emphasize that even if we confine ourselves to a single set \(X\), it is not possible to assume that \(F_i : X \times X \times X \rightarrow X\), where \(X = X_1 \cup X_2 \cup X_3\) in the application section. Here, we will investigate market equilibrium in oligopoly market, where the percentage shares of the three biggest players are \(X_1 = [25, 45], X_2 = [20, 35]\) and \(X_3 = [20, 30]\) and the response functions satisfy \(F_i : X_1 \times X_2 \times X_3 \rightarrow X_i\) for \(i = 1, 2, 3\).

**Definition 9.** Let \(X_1, X_2\) and \(X_3\) be nonempty sets and the ordered triple \((F_1, F_2, F_3)\), \(F_i : X_1 \times X_2 \times X_3 \rightarrow X_i\) for \(i = 1, 2, 3\) be a semi-cyclic map. An ordered pair \((\xi, \eta, \zeta)\) \(X_1 \times X_2 \times X_3\) is called a tripled fixed point of \((F_1, F_2, F_3)\) if \(\xi = F_1(\xi, \eta, \zeta), \eta = F_2(\xi, \eta, \zeta)\) and \(\zeta = F_3(\xi, \eta, \zeta)\).

**Definition 10.** Let \(X_1, X_2, X_3\) be nonempty sets and the ordered triple \((F_1, F_2, F_3)\), \(F_i : X_1 \times X_2 \times X_3 \rightarrow X_i\) for \(i = 1, 2, 3\) be a semi-cyclic map. We construct the sequences \(\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty\) and \(\{z_n\}_{n=0}^\infty\) for any triple \((x, y, z)\) in \(X_1 \times X_2 \times X_3\) by \(x_0 = x, y_0 = y, z_0 = z\) and \(x_{n+1} = F_1(x_n, y_n, z_n), y_{n+1} = F_2(x_n, y_n, z_n), z_{n+1} = F_3(x_n, y_n, z_n)\) for all \(n \geq 0\).

We will assume that the sequences \(\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty\) and \(\{z_n\}_{n=0}^\infty\) are those specified in Definition 10 everywhere.

**Theorem 3.** Let \((X_1, d_1), (X_2, d_2),\) and \((X_3, d_3)\) be three complete metric spaces and the ordered triple \((F_1, F_2, F_3)\), \(F_i : X_1 \times X_2 \times X_3 \rightarrow X_i\) for \(i = 1, 2, 3\) be a semi-cyclic map. To fit some of the formulas within the text box, let us assign \(X = (x_1, x_2, x_3), U = (u_1, u_2, u_3) \in X_1 \times X_2 \times X_3\).

Let there be constants \(k_1 \in [0, 1]\) for \(i = 1, 2, 3\), such that \(k_1 + 2k_2 + 2k_3 < 1\) and the ordered triple of maps \((F_1, F_2, F_3)\) satisfy the inequality:

\[
\sum_{i=1}^{3} d_i(F_i(X), F_i(U)) \leq k_1 \sum_{i=1}^{3} d_i(x_i, u_i) + k_2 \sum_{i=1}^{3} d_i(x_i, F_i(X)) + k_3 \sum_{i=1}^{3} d_i(F_i(U)) + d_i(u_i, F_i(X))
\]

(12)

for any \(X, U \in X_1 \times X_2 \times X_3\). Then:

1. a unique tripled fixed point \((\xi, \eta, \zeta)\) in \(X_1 \times X_2 \times X_3\) of \((F_1, F_2, F_3)\) exists and, moreover, for any first estimate \((x_0, y_0, z_0)\) in \(X_1 \times X_2 \times X_3\) the iterated sequences \(x_{n+1} = F_1(x_n, y_n, z_n), y_{n+1} = F_2(x_n, y_n, z_n), z_{n+1} = F_3(x_n, y_n, z_n)\) converge to the tripled fixed point \((\xi, \eta, \zeta)\).

2. there comes an a priori error estimate

\[
d_1(\xi, x_n) + d_2(\eta, y_n) + d_3(\zeta, z_n) \leq \frac{k^n}{1-k}(d_1(x_0, x_1) + d_2(y_0, y_1) + d_3(z_0, z_1))
\]

3. there comes a posteriori error estimate

\[
d_1(\xi, x_n) + d_2(\eta, y_n) + d_3(\zeta, z_n) \leq \frac{k}{1-k}(d_1(x_n, x_{n-1}) + d_2(y_n, y_{n-1}) + d_3(z_n, z_{n-1}))
\]

4. the rate of convergence is

\[
d_1(\xi, x_n) + d_2(\eta, y_n) + d_3(\zeta, z_n) \leq k(d_1(\xi, x_{n-1}) + d_2(\eta, y_{n-1}) + d_3(\zeta, z_{n-1})),
\]

where \(k = \frac{k_1 + k_2 + k_3}{1-k_2-k_3}\).
If \( X_1 \equiv X_2 \equiv X_3 \subseteq X \), wherein \((X, d)\) be a complete metric space and \( F_3(x, y, z) = F_1(y, z, x) \), \( F_3(x, y, z) = F_1(z, x, y) \) then the tripled fixed point \((\xi, \eta, \zeta)\) verifies \( \zeta = \eta = \xi \).

If \( X_1 \equiv X_2 \equiv X_3 \subseteq X \), wherein \((X, d)\) be a complete metric space, and \( F_3(x, y, z) = F_2(x, z, y) \), then the tripled fixed point \((\xi, \eta, \zeta)\) verifies \( \eta = \zeta \).

**Proof.** Consider the product space \((X_1 \times X_2 \times X_3, \rho)\), supplied with the following metric:

\[
\rho(\cdot, \cdot) = d_1(\cdot, \cdot) + d_2(\cdot, \cdot) + d_3(\cdot, \cdot).
\]

The assumption that \((X_i, d_i)\) be complete metric spaces implies that \((X_1 \times X_2 \times X_3, \rho)\) be a complete metric space, too.

Following a smart idea from Petrușel (2018), let us define a map \( G : X_1 \times X_2 \times X_3 \rightarrow X_1 \times X_2 \times X_3 \) by \( G(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \). Then inequality (12) is equivalent to:

\[
\rho(G(X), G(U)) \leq k_1\rho(X, U) + k_2(\rho(X, G(X)) + \rho(U, G(U)) + k_3(\rho(X, G(U)) + \rho(U, G(X)))
\]

and consequently \( G : X_1 \times X_2 \times X_3 \rightarrow X_1 \times X_2 \times X_3 \) is a Hardy–Rogers map in the complete metric space \((X_1 \times X_2 \times X_3, \rho)\). As a result, we can use Theorem 1, and we will conclude that there exists a unique \((\xi, \eta, \zeta) \in X_1 \times X_2 \times X_3\), such that \((\xi, \eta, \zeta) = G(\xi, \eta, \zeta), F_1(\xi, \eta, \zeta), F_2(\xi, \eta, \zeta), F_3(\xi, \eta, \zeta))\), i.e., \( \xi = F_1(\xi, \eta, \zeta), \eta = F_2(\xi, \eta, \zeta) \) and \( \zeta = F_3(\xi, \eta, \zeta) \). The error estimates were derived right away from the definition of the metric \( \rho \) and Theorem 1.

If \( X_1 \equiv X_2 \equiv X_3, F_2(x, y, z) = F_1(y, z, x), F_3(x, y, z) = F_1(z, x, y) = F_2(y, z, x) \), with \( d_1 = d_2 = d_3 = d \), subsequently for the fixed point \( X = (x, y, z) \) of \( G \), employing (12) for \( u = y, v = x, \) and \( w = z \) we establish:

\[
\begin{align*}
S_1 &= d(x, y) + d(y, z) + d(z, x) \\
&= d(F_1(x, y, z), F_2(x, y, z)) + d(F_2(x, y, z), F_3(x, y, z)) + d(F_3(x, y, z), F_1(x, y, z)) \\
&= d(F_1(x, y, z), F_1(x, z, y)) + d(F_2(x, y, z), F_2(y, z, x)) + d(F_3(x, y, z), F_3(y, z, x)) \\
&\leq k_1(d(x, y) + d(y, z) + d(z, x)) \\
&\quad + k_2(d(y, z, x) + d(z, F_3(x, y, z)) + d(z, F_3(x, y, z)) + d(x, F_3(y, z, x)) \\
&\quad + k_3(d(x, F_3(y, z, x)) + d(y, F_3(y, z, x)) + d(z, F_3(x, y, z)) + d(x, F_3(y, z, x)) \\
&= k_1S + k_2(d(x, y) + d(y, z) + d(z, x) + d(x, z) + d(y, z) + d(z, x)) \\
&\quad + k_3(d(x, y) + d(y, z) + d(z, x) + d(x, y) + d(y, z) + d(z, x)) \\
&= (k_1 + k_3)S_1 < S_1,
\end{align*}
\]

which is a contradiction and therefore \( d(x, y) + d(y, z) + d(z, x) = 0 \), i.e., \( x = y = z \).

If \( X_1 \equiv X_2 \equiv X_3, F_2(x, y, z) = F_3(x, y, z), \) and \( d_1 = d_2 = d_3 = d \), subsequently for the fixed point \( X = (x, y, z) \) of \( G \), using (12) for \( u = y, v = x, \) and \( w = z \), we obtain

\[
\begin{align*}
S_2 &= d(x, y) + d(y, z) + d(z, y) \\
&= d(F_1(x, y, z), F_1(x, y, z)) + d(F_2(x, y, z), F_3(x, y, z)) + d(F_3(x, y, z), F_2(x, y, z)) \\
&= d(F_1(x, y, z), F_2(x, y, z)) + d(F_2(x, y, z), F_3(x, y, z)) + d(F_3(x, y, z), F_2(x, y, z)) \\
&\leq k_1(d(x, y) + d(y, z) + d(z, x)) \\
&\quad + k_2(d(y, z, x) + d(z, F_3(x, y, z)) + d(z, F_3(x, y, z)) + d(x, F_3(y, z, x)) \\
&\quad + k_3(d(x, F_3(y, z, x)) + d(y, F_3(y, z, x)) + d(z, F_3(x, y, z)) + d(x, F_3(y, z, x)) \\
&= k_1S + k_2(d(x, y) + d(y, z) + d(z, x) + d(x, z) + d(y, z) + d(z, x)) \\
&\quad + k_3(d(x, y) + d(y, z) + d(z, x) + d(x, y) + d(y, z) + d(z, x)) \\
&= (k_1 + k_3)S_2 < S_2,
\end{align*}
\]
which is a contradiction and therefore \( d(x, y) + d(y, z) + d(z, x) = 0 \), i.e., \( y = z \). □

3.1. Extension of Already Known Results on Tripled Fixed Points and Corollaries

Let us recall the main result from Dzhabarova and Zlatanov (2022).

**Theorem 4.** Let \( X_1 \subseteq X \) be nonempty, closed sets in the complete metric space \((X, d)\). Let the ordered triple of maps \( F_i : X_1 \times X_2 \times X_3 \rightarrow X_1 \) for \( i = 1, 2, 3 \) be a semi-cyclic one. Let there be \( a_i, b_i, \gamma_i \in [0, 1) \), such that \( \max\{\sum_{i=1}^{\nu} a_i, \sum_{i=1}^{\nu} b_i, \sum_{i=1}^{\nu} \gamma_i\} < 1 \) and let the ordered triple \((F_1, F_2, F_3)\) satisfy the inequality

\[
\sum_{i=1}^{3} \rho(F_i(x_i, y_i, z_i), F_i(u_i, v_i, w_i)) \leq \sum_{i=1}^{3} (a_i \rho(x_i, u_i) + b_i \rho(y_i, v_i) + \gamma_i \rho(z_i, w_i)) \tag{14}
\]

for every \((x_i, y_i, z_i), (u_i, v_i, w_i) \in X_1 \times X_2 \times X_3\). Then there exists a unique tripled fixed point \((x, y, z) \in X_1 \times X_2 \times X_3\). Moreover, for any first estimate \((x_n, y_n, z_n) \in A_1 \times A_2 \times A_3\) the iterated sequences \(x_{n+1} = F_1(x_n, y_n, z_n), y_{n+1} = F_2(x_n, y_n, z_n), z_{n+1} = F_3(x_n, y_n, z_n)\) converge to the tripled fixed point \((x, y, z)\).

Let us consider Theorem 3 for \( X_1, X_2, X_3 \) as closed and nonempty sets in a complete metric space \((X, d)\), rather than being subsets of three distinct metric spaces and for constants \(k_2 = k_3 = 0\). If we put \((x_i, y_i, z_i) = (x, y, z), (u_i, v_i, w_i) = (u, v, w)\) for \(i = 1, 2, 3\) in (14), we get:

\[
\sum_{i=1}^{3} d(F_i(x, y, z), F_i(u, v, w)) \leq \sum_{i=1}^{3} (a_i \rho(x, u) + b_i \rho(y, v) + \gamma_i \rho(z, w)) \leq s(d(x, u) + d(y, v) + d(z, w)) \tag{15}
\]

where \(s = \max\{\sum_{i=1}^{\nu} a_i, \sum_{i=1}^{\nu} b_i, \sum_{i=1}^{\nu} \gamma_i\} < 1\). Thus, Theorem 4 follows from Theorem 3.

A weaker contractive condition for the ordered triple of semi-cyclic maps \((F_1, F_2, F_3)\) than that in Theorem 4 can be obtained if we treat \(X_1, X_2, X_3\) as closed and nonempty sets in a complete metric space \((X, d)\), as opposed to subsets of three distinct metric spaces and constants \(k_2 = k_3 = 0\).

3.2. Application in the Study of Market Equilibrium in Tripodal Markets

The main task in most economic studies is to determine the factors affecting competitiveness, economic success, predicting consumer behavior, forecasting demand and sales. For this purpose, high-performance statistical models are built using classical or machine learning methods. Classical methods include panel regression, multiple linear regression, principal component analysis, Box-Jenkins ARIMA methodology (Török et al. 2020; Van Trang et al. 2022). The authors Janytik et al. (2020) used most effective with high predictive power machine learning methods like as convolutional neural networks, long short-term memory network and ensemble methods. The present study is not concerned with building a high-performance statistical model of real data for prediction purposes. We are seeking a reasonably adequate statistical model that meets the conditions of Theorem... to examine the equilibrium point.

3.2.1. The Basic Model

Assume three enterprises compete for the same customers Friedman (1983) and aim to meet request with a total production of \(Z = x_1 + x_2 + x_3\). The market price, \(P(Z) = P(x_1 + x_2 + x_3)\), is the inverse of the demand function. The cost functions of the three players are \(c_i(x_i)\) for \(i = 1, 2, 3\), respectively. The payoff functions are \(\Pi_i(Z) = x_i P(Z) - c_i(x_i)\)
for \( i = 1,2,3 \). Assuming that every participant is rational, their purpose is to maximize profits, i.e.,

\[
\max \{ \Pi_i(Z) : x_i, \text{ assuming that } x_j, x_k, j \neq i \text{ are fixed} \}, i = 1,2,3.
\]

We get the equations:

\[
\begin{align*}
\frac{\partial \Pi_1(x_1, x_2, x_3)}{\partial y_1} &= P(x_1 + x_2 + x_3) + x_1 P'(x_1 + x_2 + x_3) - c_1'(x_1) = 0 \\
\frac{\partial \Pi_2(x_1, x_2, x_3)}{\partial y_2} &= P(x_1 + x_2 + x_3) + x_2 P'(x_1 + x_2 + x_3) - c_2'(x_2) = 0 \\
\frac{\partial \Pi_3(x_1, x_2, x_3)}{\partial y_3} &= P(x_1 + x_2 + x_3) + x_3 P'(x_1 + x_2 + x_3) - c_3'(x_3) = 0,
\end{align*}
\]

provided the functions \( P \) and \( c_i, i = 1,2 \) are differentiable.

The solution of (16) represents the equilibrium triple of production Friedman (1983). Equation (16) frequently contains solutions in the form of \( x_1 = b_1(x_2, x_3), x_2 = b_2(x_1, x_3), \) and \( x_3 = b_3(x_1, x_2) \), known as response functions. Friedman (1983).

It may be challenging or even impossible to solve (16), hence it is commonly advised to look for an approximate solution. One big downside is that it may be unstable. Luckily, an implicit formula for the response function can be obtained through (16), i.e.,

\[
x_i = \frac{\partial \Pi_i(x_1, x_2, x_3)}{\partial y_i} + x_i = F_i(x_1, x_2, x_3).
\]

We could eventually end up with response functions that do not maximize the payoff \( \Pi \). It is usually considered that every participant’s reaction is determined by both their own and others’ outputs. For instance, if the resulting quantities are \( (x_n, y_n, z_n) \) at time \( n \), and the first player adjusts their output to levels \( x_{n+1} = F_1(x_n, y_n, z_n) \), the second and third players will also alter their outputs to amounts \( y_{n+1} = F_2(x_n, y_n, z_n) \) and \( z_{n+1} = F_3(x_n, y_n, z_n) \), respectively. If there are \( x_0, y_0, z_0 \) satisfying \( x = F_1(x, y, z), y = F_2(x, y, z) \) and \( z = F_3(x, y, z) \), we get an equilibrium. To guarantee that the solutions of (16) maximize the payoff functions, either \( \Pi_i \) is concave or (17) is satisfied. Matsumoto and Szidarovszky (2018):

\[
\begin{align*}
\frac{\partial^2 \Pi_1(x, y, z)}{\partial y^2}(x_0, y_0, z_0) &\leq 0 \\
\frac{\partial^2 \Pi_2(x, y, z)}{\partial y^2}(x_0, y_0, z_0) &\leq 0 \\
\frac{\partial^2 \Pi_3(x, y, z)}{\partial z^2}(x_0, y_0, z_0) &\leq 0.
\end{align*}
\]

The inequalities (17) are referred to in the literature as second-order conditions.

The inclusion of response functions changes the maximizing problem into a tripled fixed-point problem, removing all concavity and differentiability requirements. The problem of finding triple fixed points for an ordered pair of mappings \( (F_1, F_2, F_3) \) is the challenge of solving the equations \( x = F_1(x, y, z), y = F_2(x, y, z), \) and \( z = F_3(x, y, z) \) Samet and Vetro (2010). One important restriction, though, might be that participants aren’t allowed to alter output too quickly, which could prevent them from making the most money possible.

3.2.2. Relationship between the Contraction-Type Conditions and the Second-Order Ones

For an instance where \( k_2 = k_3 = 0 \) and \( X_1, X_2, \) and \( X_3 \) are subsets of a metric space \( (X, d) \), we shall rephrase Theorem 3 in the language of economics. We employ Corollary 1 to demonstrate that, in the event that the response functions are differentiable, they fulfill the second-order criterion. Furthermore, Corollary 1 ensures the iterated process’s stability.

**Corollary 1.** Let us consider an oligopoly market with three players, satisfying:

1. the three players are producing perfect substitutes of homogeneous
2. the first player can produce quantities from the set \( X_1 \), and the second and the third ones can produce quantities from the sets \( X_2 \) and \( X_3 \), respectively, where \( X_1, X_2 \) and \( X_3 \) are closed, nonempty subsets of a complete metric space \( (\mathbb{R}, | \cdot |) \)
3. Let the ordered tripled \((F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))\), \(F_i : X_1 \times X_2 \times X_3 \to X_i\), \(i = 1, 2, 3\) be a semi-cyclic map that presents the response functions for players one, two, and three, respectively.

4. Let there be a \(\alpha < 1\), so that the inequality:

\[
\sum_{i=1}^{3} |F_i(x, y, z) - F_i(u, v, w)| \leq \alpha (|x - u| + |y - v| + |z - w|)
\]

holds for all \((x, y, z), (u, v, w) \in X_1 \times X_2 \times X_3\).

Then there is a unique market equilibrium triple \((\xi, \eta, \zeta)\) in \(X_1 \times X_2 \times X_3\), i.e., \(\xi = F_1(\xi, \eta, \zeta), \eta = F_2(\xi, \eta, \zeta)\) and \(\zeta = F_3(\xi, \eta, \zeta)\).

Additionally, if \(F_2(x, y, z) = F_1(y, z, x)\) and \(F_3(x, y, z) = F_1(z, x, y)\) are fulfilled, then \(\xi = \eta = \zeta\) is satisfied by the market equilibrium triple \((\xi, \eta, \zeta)\).

In the event that \(F_2(x, y, z) = F_2(x, z, y)\) is also true, \(\eta = \zeta\) is satisfied by the market equilibrium triple \((\xi, \eta, \zeta)\).

**Example 1.** Let us obtain the response functions \(F_i\) by maximizing \(\Pi_i\), where \(i = 1, 2, 3\). Assume all partial derivatives involved in (16) and (17) exist. Let \((x_0, y_0, z_0)\) be a unique solution of (16). It is widely known that the optimization of the payoff functions in the Cournot model is ensured if (17) is met.

We will show that from (18) follows (17).

Following Kabaivanov et al. (2022), the response functions are defined as:

\[
\begin{align*}
F_1(x, y, z) &= \frac{\partial \Pi_1(x, y, z)}{\partial x} + x \\
F_2(x, y, z) &= \frac{\partial \Pi_2(x, y, z)}{\partial y} + y \\
F_3(x, y, z) &= \frac{\partial \Pi_3(x, y, z)}{\partial z} + z.
\end{align*}
\]

From (18) we get:

\[
\begin{align*}
0 &\leq \lim_{\Delta x \to 0} \frac{|F_1(x + \Delta x, y, z) - F_1(x, y, z)|}{\Delta x} \leq \lim_{\Delta x \to 0} \sum_{i=1}^{3} \frac{|F_i(x + \Delta x, y, z) - F_i(x, y, z)|}{\Delta x} \leq \alpha \\
0 &\leq \lim_{\Delta y \to 0} \frac{|F_2(x, y + \Delta y, z) - F_2(x, y, z)|}{\Delta y} \leq \lim_{\Delta y \to 0} \sum_{i=1}^{3} \frac{|F_i(x, y + \Delta y, z) - F_i(x, y, z)|}{\Delta y} \leq \alpha \\
0 &\leq \lim_{\Delta z \to 0} \frac{|F_3(x, y, z + \Delta z) - F_3(x, y, z)|}{\Delta z} \leq \lim_{\Delta z \to 0} \sum_{i=1}^{3} \frac{|F_i(x, y, z + \Delta z) - F_i(x, y, z)|}{\Delta z} \leq \alpha,
\end{align*}
\]

and therefore it follows that \(\left|\frac{\partial F_1}{\partial x}(x_0, y_0, z_0)\right| \leq \alpha < 1\), \(\left|\frac{\partial F_2}{\partial y}(x_0, y_0, z_0)\right| \leq \alpha < 1\) and \(\left|\frac{\partial F_3}{\partial y}(x_0, y_0, z_0)\right| \leq \alpha < 1\). Then from (19) we get:

\[
\begin{align*}
\frac{\partial^2 \Pi_1(x, y, z)}{\partial x^2}(x_0, y_0, z_0) &= \frac{\partial F_1}{\partial x}(x_0, y_0, z_0) - 1 < \alpha - 1 < 0, \\
\frac{\partial^2 \Pi_2(x, y, z)}{\partial y^2}(x_0, y_0, z_0) &= \frac{\partial F_2}{\partial y}(x_0, y_0, z_0) - 1 < \alpha - 1 < 0 \\
\end{align*}
\]

and

\[
\frac{\partial^2 \Pi_3(x, y, z)}{\partial z^2}(x_0, y_0, z_0) = \frac{\partial F_3}{\partial z}(x_0, y_0, z_0) - 1 < \alpha - 1 < 0.
\]

Corollary 1 not only provides sufficient requirements for the presence of a market equilibrium, but it also provides sufficient conditions for the process of the players’ successive reactions to be stable in the event that their behavior remains unchanged.
Example 2. A model with the price function $P(x, y, z) = 100 - x - y - z$ will be examined and, for $i = 1, 2, 3$, the cost functions $c_i(x) = \frac{x^2}{z}$.

By (16) we get:

\begin{align*}
\frac{\partial \Pi_1}{\partial x} &= 100 - 3x - y - z = 0 \\
\frac{\partial \Pi_1}{\partial y} &= 100 - x - 3y - z = 0 \\
\frac{\partial \Pi_1}{\partial z} &= 100 - x - y - 3z = 0.
\end{align*}

The equilibrium points are the solution of the system of Equation (21) because it satisfies the second-order conditions, which are

\begin{align*}
\frac{\partial^2 \Pi_1}{\partial x^2} &= -3 < 0, \\
\frac{\partial^2 \Pi_2}{\partial y^2} &= -3 < 0,
\end{align*}

and \( \frac{\partial^2 \Pi_3}{\partial z^2} = -3 < 0 \). Sadly, condition (18) will not be satisfied by the response functions in the model, which are

\begin{align*}
F_1(x, y, z) &= 100 - 2x - y - z, \\
F_2(x, y, z) &= 100 - x - 2y - z, \\
F_3(x, y, z) &= 100 - x - y - 2z.
\end{align*}

Let us consider three starting states in the market from Example 2. The iterations of the successive changes of the output of the three players is presented in Tables 1 and 2.

Table 1. The iterated sequence’s values \((x_n, y_n, z_n)\) with initial conditions of \((21, 20, 19)\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>2k−1</th>
<th>2k</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_n)</td>
<td>21</td>
<td>19</td>
<td>21</td>
<td>\ldots</td>
<td>19</td>
<td>21</td>
</tr>
<tr>
<td>(y_n)</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>\ldots</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>(z_n)</td>
<td>19</td>
<td>21</td>
<td>19</td>
<td>\ldots</td>
<td>21</td>
<td>19</td>
</tr>
</tbody>
</table>

In the event that the starting point differs, Table 2 results.

Table 2. The iterated sequence’s values \((x_n, y_n, z_n)\) with initial conditions of \((25, 18, 17)\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_n)</td>
<td>25</td>
<td>15</td>
<td>25</td>
<td>15</td>
<td>25</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>(y_n)</td>
<td>18</td>
<td>22</td>
<td>18</td>
<td>22</td>
<td>18</td>
<td>22</td>
<td>18</td>
</tr>
<tr>
<td>(z_n)</td>
<td>17</td>
<td>23</td>
<td>17</td>
<td>23</td>
<td>17</td>
<td>23</td>
<td>17</td>
</tr>
</tbody>
</table>

In both Tables 1 and 2, we see that the process is not converging.

Note that the system (16) may have several solutions \((x, y, z)\) that meet the second-order criteria (17). In this scenario, further inquiry will be required to determine which is the answer to the Cournot model’s optimization problem. Even though (18) imposes a stronger limitation than (17), the model from Corollary 1 differs from the well-known Cournot optimization problem.

The obtained results convince us that the study of market equilibrium using response functions is a generalization of Cournot’s theory for obtaining equilibrium using payoff maximization. As we have shown, if the response functions are obtained from the first-order conditions, then the tripled fixed points will necessarily be solutions to the optimization problem, and even more, we will have market stability. On the other hand, it is possible to have a market with irrational behavior among the participants, but if it is possible to find their response functions, then it is possible to give an answer: what are the market equilibrium levels, are they unique, and are they stable over time?

4. Empirical Data

In our analysis, we have monthly data on beer sales (1000 HLTRs) by five significant corporations that cover the whole Bulgarian beer industry. The percentage share of each company in the market distribution is calculated. To determine the market equilibrium, we consider the three companies with the largest percentage share of sales.
Monthly data on the percentage participation of each company (Company 1, Company 2, Company 3) contains a total of $N = 48$ data from January 2017 until December 2020. Over the period, the three companies covered an average of 88.73%.

Table 3 presents the descriptive statistics on the values of the volume of beer sold for each of the three company brands. The close values of the mean and the median, as well as the values close to zero of the skewness and kurtosis for the three variables, indicate that we may assume the data is normally distributed. This is confirmed by the Kolmogorov-Smirnov and Shapiro-Wilk tests, which are insignificant with $p$-value $> 0.05$ for the variables under consideration. Also, Table 3 shows the first firm has the largest average proportion of sales.

Table 3. Descriptive statistics of the initial variables used.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Company 1, %</td>
<td>35.931</td>
<td>35.83</td>
<td>4.18</td>
<td>17.47</td>
<td>−0.064</td>
<td>0.343</td>
<td>−0.369</td>
<td>0.674</td>
</tr>
<tr>
<td>Company 2, %</td>
<td>27.593</td>
<td>27.46</td>
<td>2.26</td>
<td>5.09</td>
<td>−0.152</td>
<td>0.343</td>
<td>−0.766</td>
<td>0.674</td>
</tr>
<tr>
<td>Company 3, %</td>
<td>25.202</td>
<td>24.95</td>
<td>1.83</td>
<td>3.36</td>
<td>0.632</td>
<td>0.343</td>
<td>0.102</td>
<td>0.674</td>
</tr>
</tbody>
</table>

Figure 1 shows a sequence plot of the three companies’ percentage participation. The leading company’s trend is increasing, while the other two are somewhat decreasing.

Figure 1. Percentage shares in the market over time for Company 1 in blue, Company 2 in red, and Company 3 in green color.

4.1. Results of the Unit Root Test for the Original and Differenced Time Series

The outcomes of the unit root tests for the original time series are shown in Table 4. Since all $p$-values for the initial variables are negligible, as Table 4 demonstrates, $γ = 0$ is the outcome of the unit root tests and the null hypothesis $H_0$ cannot be rejected. As a result, every initial time series exhibits a first-order trend and is non-stationary.

Table 4. Result from the unit root tests of the initial time series.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Company 1</th>
<th>Company 2</th>
<th>Company 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dickey-Fuller F test</td>
<td>statistic</td>
<td>0.167359</td>
<td>−0.237856</td>
</tr>
<tr>
<td></td>
<td>$p$-value</td>
<td>0.723573</td>
<td>0.624571</td>
</tr>
<tr>
<td>Philips-Perron F test</td>
<td>statistic</td>
<td>0.183871</td>
<td>−0.225644</td>
</tr>
<tr>
<td></td>
<td>$p$-value</td>
<td>0.727902</td>
<td>0.627348</td>
</tr>
</tbody>
</table>
For the differenced series DCompany1, DCompany2, and DCompany3, calculated by
\[ DY_t = Y_t - Y_{t-1}, \] \hfill (22)
and Table 5 provides the corresponding indices of the unit root tests. Since all of the 
P-values in this table are essentially zero, the unit root problem is resolved. Consequently, it is possible to reject the null hypothesis \( H_0 \) in this situation, and it is presumed that these variables are stationary.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>DCompany1</th>
<th>DCompany2</th>
<th>DCompany3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dickey-Fuller F test</td>
<td>statistic</td>
<td>−55.6533</td>
<td>−56.0131</td>
</tr>
<tr>
<td></td>
<td>p-value</td>
<td>( 5.120 \times 10^{-9} )</td>
<td>( 4.824 \times 10^{-8} )</td>
</tr>
<tr>
<td>Philips-Perron F test</td>
<td>statistic</td>
<td>−55.599</td>
<td>−57.4133</td>
</tr>
<tr>
<td></td>
<td>p-value</td>
<td>( 5.166 \times 10^{-8} )</td>
<td>( 3.840 \times 10^{-8} )</td>
</tr>
</tbody>
</table>

The unit root tests corroborate the first-order trends in all variables. This effectively indicates that the time series values for each month can be expressed in terms of the preceding month’s data.

4.2. Modeling the Response Functions of the Three Producers

We have genuine statistics on the quantity sold for 48 consecutive months. As long as beer consumption varies by season, it is not suitable to include the quantities in the model. As a result, we converted the quantities into percentages. Figure 1 demonstrates substantial swings over 48 months. This is the basis for considering two models. The first will be based on true empirical data, while the second will be based on weighted moving average-smoothed data. We will suppose in the second instance that the huge oscillations in the real data (Figure 1) could be related to the fact that the producers do not sell their product.

Distributors may order a large quantity of items while their warehouses are empty, and a small quantity of goods when their warehouses are full. For the procedure to average over time, the percentage shares appear to be a better representation of producer-consumer interaction, as we will see later.

All of the reasons we mentioned for using moving averages are merely assumptions; therefore, we will present the two models, because working with real data is always the preferred option.

In the genuine data green the three players follow a normal distribution. An issue was that none of the algebra computer software could solve the least squares problem for arbitrary constants. Thus, we made some assumptions regarding three of the coefficients that represent each producer’s conduct in relation to their own results from the previous month. These coefficients were chosen to be similar to the correlation coefficients that we calculated for real data. As a result of these assumptions, we searched for the coefficient values that represent each producer’s conduct in relation to their competitors’ performance.

We choose to convert the real data using weighted moving averages. Although this strategy is mostly employed in financial markets Lorenzo (2013) to discover the exact time of trend change, it appears that this approach can be applied in the inquiry we are conducting. Using the classical average yields new data that is not normally distributed to the two smaller market players. When we employ a moving average using the first 5 Fibonacci numbers, the converted data follows a normal distribution for all producers. It is interesting to point out that in this example the algebra computer software was able to perform the least squares assignment with no additional assumptions.
4.3. Application of Lsm under Additional Conditions for Building Interactive Models

The presence of a first-order trend indicates that we can reasonably look for a dependence that describes the current state of each producer relative to the state one month ago. For this reason, we form three new lagged variables with one lag back, namely LagC1, LagC2, and LagC3, based on the main variables. For the three lagged variables, the Shapiro-Wilk and Kolmogorov-Smirnov tests are negligible for (p-values > 0.05). Therefore, the variables have a normal distribution. This allows us (Table 6) to determine the Pearson’s correlation coefficients. The table demonstrates that all correlation coefficients of the variables Company1, Company2, and Company3 and their lagged variables are statistically significant at a significance level $\alpha = 0.05$.

Table 6. Pearson Correlation Coefficients. We denote by Ci Company i.

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>Lag C1</th>
<th>Lag C2</th>
<th>Lag 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>Pearson Cor</td>
<td>−0.827</td>
<td>−0.712</td>
<td>0.891</td>
<td>−0.790</td>
<td>−0.551</td>
</tr>
<tr>
<td></td>
<td>Sig (2-tailed)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>C2</td>
<td>Pearson Cor</td>
<td>1</td>
<td>0.256</td>
<td>−0.777</td>
<td>0.790</td>
<td>0.422</td>
</tr>
<tr>
<td></td>
<td>Sig (2-tailed)</td>
<td></td>
<td>0.079</td>
<td>0.000</td>
<td>0.000</td>
<td>0.003</td>
</tr>
<tr>
<td>C3</td>
<td>Pearson Cor</td>
<td>1</td>
<td>0.537</td>
<td>0.369</td>
<td>0.416</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Sig (2-tailed)</td>
<td></td>
<td>0.000</td>
<td>0.010</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>Lag C1</td>
<td>Pearson Cor</td>
<td>1</td>
<td>−0.834</td>
<td>−0.754</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Sig (2-tailed)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lag C2</td>
<td>Pearson Cor</td>
<td>1</td>
<td>0.323</td>
<td></td>
<td>0.025</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sig (2-tailed)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let $x_n$, $y_n$, and $z_n$, $n = 1, \ldots, 48$, be the data for the three companies: Company 1, Company 2, and Company 3. Following Dzhabarova et al. (2020) we are looking for a model of the form (23)

$$
\begin{align*}
&x'_n = a_1 x_{n-1} + b_1 y_{n-1} + c_1 z_{n-1} + d_1, \\
y'_n = a_2 x_{n-1} + b_2 y_{n-1} + c_2 z_{n-1} + d_2, \\
z'_n = a_3 x_{n-1} + b_3 y_{n-1} + c_3 z_{n-1} + d_3
\end{align*}
$$

$n = 2, 3, \ldots, 48, a_i, b_i, c_i, d_i \in \mathbb{R}, i = 1, 2, 3$ subject to conditions

$$
|a_1| + |a_2| + |a_3| < 1, \ |b_1| + |b_2| + |b_3| < 1, \ |c_1| + |c_2| + |c_3| < 1.
$$

From Table 6 we can estimate $a_1 = 0.89$, $b_2 = 0.79$ and $c_3 = 0.416$. The coefficients $a_i, b_i, c_i, d_i \in \mathbb{R}, i = 1, 2, 3$ are found by minimizing the sum of the squares of the differences

$$
\sum_{i=2}^{n} ((x_i - x'_i)^2 + (y_i - y'_i)^2 + (z_i - z'_i)^2) \rightarrow \text{min},
$$

subject to the constraints (24) and the additional restriction on $a_1, b_2,$ and $c_3$. The optimization problem has the following solution:

$$
\begin{align*}
x'_n &= 0.89 x_{n-1} - 0.062 y_{n-1} + 0.419 z_{n-1} - 4.66, \\
y'_n &= -0.009 x_{n-1} + 0.79 y_{n-1} + 0.141 z_{n-1} + 2.46, \\
z'_n &= -0.005 x_{n-1} + 0.128 y_{n-1} + 0.42 z_{n-1} + 11.19
\end{align*}
$$

where $n = 2, 3, \ldots, 48$. 
Therefore, we get the ordered tripled \((F_1, F_2, F_3)\) defined by
\[
\begin{aligned}
F_1(x, y, z) &= 0.89x - 0.062y + 0.419z - 4.66, \\
F_2(x, y, z) &= -0.009x + 0.79y + 0.141z + 2.469, \\
F_3(x, y, z) &= 0.005x + 0.128y + 0.42z + 11.19
\end{aligned}
\] (27)
that represent the response functions of each of the three players in the market and \(\max\{ |a_1| + |a_2| + |a_3|, |b_1| + |b_2| + |b_3|, |c_1| + |c_2| + |c_3| \} = \max\{0.904, 0.98, 0.98\} \leq 0.98\)

Figure 2 presents the observed and approximated values from the built models for each of the companies.

In Table 7, for all companies, we see a high percentage: 97%, 98%, and 87% respectively matching the data to the model, with an average absolute percentage error of less than 3%.

Table 7. Statistical performance of the created models.

<table>
<thead>
<tr>
<th>Model Company 1</th>
<th>Model Company 2</th>
<th>Model Company 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R^2)</td>
<td>0.966</td>
<td>0.981</td>
</tr>
<tr>
<td>MAPE</td>
<td>0.025</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Figure 3 presents histograms of the residuals, which are seen to be normally distributed. The resulting models are statistically adequate. 

The Kolmogorov-Smirnov tests for the match between the distribution of the data and the values predicted by the model in all cases are insignificant with \(p\)-value 0.96, 0.85 and 0.25, respectively. This means that the null hypothesis that the datasets of real and predicted data have the same distribution is not rejected and the models adequately describe the data.

Applying the iterated process using the response functions (27) with the initial start of the market \((x_0, y_0, z_0) = (25.5745, 30.7458, 29.6924)\), we obtain how percentage shares will
change over time. Figure 4a represents Company 1, Figure 4b represents Company 2, and Figure 4c represents Company 3.

![Graphs showing data for different companies](image)

Figure 4. Real data vs iterated one. (a) Simulation of the sequence of successive percentage shares for Company 1, blue color for the real data, red color for $x_n = F_1(x_{n-1}, y_{n-1}, z_{n-1})$, and green color for $x_n = F_1(x_0, y_0, z_0)$. (b) Simulation of the sequence of successive percentage shares for Company 2, blue color for the real data, red color for $y_n = F_2(x_{n-1}, y_{n-1}, z_{n-1})$, and green color for $y_n = F_2(x_0, y_0, z_0)$. (c) Simulation of the sequence of successive percentage shares for Company 3, blue color for the real data, red color for $z_n = F_3(x_{n-1}, y_{n-1}, z_{n-1})$, and green color for $z_n = F_3(x_0, y_0, z_0)$.

In Figure 5, we plot the predicted iterations of the three companies.

![Graph showing predicted iterations](image)

Figure 5. Blue, red, and green colors for Companies 1, 2, and 3, respectively, for the sequences of successive iterations $x_n = F_1(x_0, y_0, z_0)$, $y_n = F_2(x_0, y_0, z_0)$, $z_n = F_3(x_0, y_0, z_0)$.

### 4.4. Modeling the Response Functions after Smoothing the Empirical Data with Moving Averages

By replacing the data with its weighted moving averages at period 5 values we get Figure 6.

![Graphs showing smoothed data for different companies](image)

Figure 6. (a) Company 1’s real data in blue vs the moving average data in red. (b) Company 2’s real data in blue vs the moving average data in red. (c) Company 3’s real data in blue vs the moving average data in red.
With the least squares method without imposing additional constraints we get the following response functions \((G_1, G_2, G_3)\) defined by

\[
\begin{align*}
G_1(x, y, z) &= 0.97x - 0.019y - 0.135z - 2.67, \\
G_2(x, y, z) &= -0.003x + 0.882y + 0.186z - 1.38, \\
G_3(x, y, z) &= -0.005x + 0.061y + 0.659z + 6.98
\end{align*}
\] (28)

and it is easy to observe that \(\max\{|a_1| + |a_2| + |a_3|, |b_1| + |b_2| + |b_3|, |c_1| + |c_2| + |c_3|\} \leq 0.98\) and the ordered triple \((G_1, G_2, G_3)\) satisfies all the conditions in Corollary 1.

The Kolmogorov-Smirnov tests for the match between the distribution of the data and the values predicted by the model in all cases are insignificant with \(p\)-value 0.9, 0.55 and 0.25, respectively. This means that the null hypothesis that the datasets of real and predicted data have the same distribution is not rejected and the models adequately describe the data.

Figures 7a, 8a, and 9a present the real data in blue, the predicted values by iterating \(F_n^1(x_0, y_0, z_0)\) (the response function constructed by the real data) in red, predicted values by iterating \(G_n^1(x_0, y_0, z_0)\) (the response function constructed by the moving average data) in green.

Figures 7b, 8b, and 9b present the real data in blue, predicted values by iterating \(F_1(x_n, y_n, z_n)\) (the response function constructed by the real data) in red, predicted values by iterating \(G_1(x_n, y_n, z_n)\) (the response function constructed by the moving average data) in green.

![Figure 7](image1)

**Figure 7.** (a) Company 1’s real data in blue, predicted values by iterating \(F_n^1(x_0, y_0, z_0)\) in green, predicted values by iterating \(G_n^1(x_0, y_0, z_0)\) in red (the response functions constructed by the real data and by the moving average data, respectively). (b) Company 1’s real data in blue, predicted values by \(F_1(x_n, y_n, z_n)\) (the response function constructed by the real data) in green, predicted values by \(G_1(x_n, y_n, z_n)\) (the response function constructed by the moving average data) in red.

![Figure 8](image2)

**Figure 8.** (a) Company 2’s real data in blue, predicted values by iterating \(F_n^2(x_0, y_0, z_0)\) in green, predicted values by iterating \(G_n^2(x_0, y_0, z_0)\) in red (the response functions constructed by the real data and by the moving average data, respectively). (b) Company 2’s real data in blue, predicted values by \(F_2(x_n, y_n, z_n)\) (the response function constructed by the real data) in green, predicted values by \(G_2(x_n, y_n, z_n)\) (the response function constructed by the moving average data) in red.
After summing (29), (30), and (31), we get
\[
\sum_{n=1}^{3} |F_n(x, y, z) - F_n(u, v, w)| \leq \alpha (|x - u| + |y - v| + |z - w|),
\]
where \( \alpha = \max\{0.89 + 0.009 + 0.05, 0.062 + 0.79 + 0.128, 0.419 + 0.141 + 0.42\} = 0.98.\)

Using the empirical data, we see that the sets of the percentage shares of the three major players in the market are \( X_1 = [25, 44], \ X_2 = [20, 35], \) and \( X_3 = [20, 30]. \) It is easy to calculate
\[
24 \leq \{F_1(x, y, z): x + y + z = 109, 25 \leq x \leq 44, 20 \leq y \leq 35, 20 \leq z \leq 30\} \leq 44,
\]
\[
23 \leq \{F_2(x, y, z): x + y + z = 109, 25 \leq x \leq 44, 20 \leq y \leq 35, 20 \leq z \leq 30\} \leq 32
\]
and
\[
22 \leq \{F_3(x, y, z): x + y + z = 109, 25 \leq x \leq 44, 20 \leq y \leq 35, 20 \leq z \leq 30\} \leq 28.
\]

Thus, the assumptions of Corollary 1 are satisfied, and there exists a unique market equilibrium \((37.4, 26.9, 24.9),\) which is the market shares in percentage. The total percentage \(37.41 + 26.95 + 25.09 = 89.18\) is smaller than 100, because we have removed several small producers that represent approximately 10% of the market.

Using comparable computations, we determine that the ordered triple of response functions \( (G_1, G_2, G_3) \) meets Corollary 1’s assumption, and therefore there is a unique market equilibrium. Solving the system of equations \( x = G_1(x, y, z), y = G_2(x, y, z), \) and \( z = G_3(x, y, z), \) we get that the market equilibrium, predicted by the smoothing data model, will be \((39.81, 26.076, 24.5).\)

This model has an advantage compared to the first one. From (28) we see that the increase in sales of any of the companies leads to an increase in their production. We also
see that any increase in the market shares of the second and third companies leads to a very small decrease in the performance of the first company (the largest one). Any increase in the percentage shares of the first player (the largest one) leads to a small decrease in the other two, and finally, any increase in the performance of any of the two small producers leads to an increment in the production of the other small one, i.e., there is a kind of symmetry between the responses of the big producer and the two smaller ones. Let us point out that in the optimization with least squares, there was no need for additional assumptions on the coefficients $a_i, b_i, c_i$, $i = 1, 2, 3$.

6. Conclusions

Rather than solving a payoff maximization issue, we use an equilibrium theory based on reaction functions to experimentally examine the beer manufacturers in Bulgaria. Overall, this serves as an example of the paradigm that (Dzhabarova et al. 2020; Kabaivanov et al. 2022) presented to represent market equilibrium via response functions.

As demonstrated in Badev et al. (2024) during the examination of equilibrium in the US market for wireless telecommunications, the assumptions in the main theorem are only sufficient ones. This is demonstrated by presenting the market behavior, provided that the starting point of the market lies outside the domain of the response functions.

Data analysis shows that the idea proposed can be applied to real markets and determine whether the market is stable, has reached equilibrium, and whether this equilibrium is unique. This analysis is not specifically related to the beer industry. The obtained results demonstrate the possibility that the theory of equilibrium based on response functions has the potential to study the dynamics in oligopolistic markets. Such an example is proposed for mobile services in the US. It is possible to apply this technique in other markets as well, such as aircraft manufacturing, operating systems, and mobile phone manufacturers, where there are also a small number of manufacturers holding a large market share. As we mentioned in the text, the results can also be used by government bodies, statistical institutes, and companies. Firms may assume that the results show that to change their market share, it is necessary to make a significant change in their policy and technology.

If we try to simulate such a state for the beer market with the obtained response functions and an initial start of the market arbitrary chosen, for example $(10, 15, 70), (10, 50, 25) \notin X_1 \times X_2 \times X_3$, we obtain Figure 10.

![Figure 10](image-url)

**Figure 10.** (a) Simulation, with the two models, of the market with an initial start at levels $(10, 15, 70)$ Company 1, 2, and 3 in blue, red, and green colors respectively. (b) Simulation, with the two models, of the market with an initial start at levels $(10, 15, 70)$ Company 1, 2, and 3 in blue, red, and green colors respectively.

This simulation suggests that if $x_0, y_0, z_0 \notin X_1 \times X_2 \times X_3$, then it is possible for a sequence of successive productions to converge to the same market equilibrium too, which coincides with the observations made in Badev et al. (2024). Therefore, we can conclude that the imposed conditions in Corollary 1 and Theorem 3 are only sufficient ones.
Let us mention that when looking to model discrete data with continuous functions, there are infinite number models that can describe the available data statistically reliably. Therefore, the imposition of additional constraints on the modeling of real processes is adopted. Once the model is constructed, it is checked to see how much the continuous model differs statistically from the actual discrete data (Carr and Lykkesfeldt 2023; Eid et al. 2023; Gobin et al. 2023; Krishnasamy et al. 2023). What is interesting in the research is that both considered models give similar results for the levels at which market equilibrium is reached, that it is unique, and that it is stable. Also, simulations with an arbitrarily chosen initial state of the economy show that both models quickly stabilize around equilibrium levels.

An intriguing endeavor is to investigate other oligopolistic markets. Another open problem that we find interesting is the following: Let us have available data on the quantity sales and cost functions for the participants and have constructed the response functions. If we assume that the participants behave rationally, then let’s try to recover the price function and, from there, the demand function. Or if we have the price function, try to recover the cost function for the participants.

Given enough data, for example, light beer, non-alcoholic beer, and dark beer, a response function can be constructed for the entire basket of products offered. In addition, a Cournot-Bertrand model where producers compete simultaneously for quantities and prices is considered in Dzhabarova et al. (2020). If for the mentioned types of beer there are also their prices on the market, an equilibrium can be sought in terms of price and quantity, as proposed in Dzhabarova et al. (2020). In the orderly triad of quantity and price, one can add the code of marketing performance and seek balance in terms of quantities, prices, and marketing. This can be done for different types of industries, for example, petrol producers.
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