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Application of Fuzzy Discount Factors in Behavioural Decision-Making for Financial Market Modelling

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Abstract: This paper presents an innovative approach to financial market modelling by integrating fuzzy discount factors into the decision-making process, thereby reflecting the complexities of human behaviour. Traditional financial models often fail to account for market dynamics' psychological factors. The proposed method utilizes fuzzy logic to encapsulate the uncertainty and subjective judgment inherent in financial decisions. By representing financial variables as fuzzy numbers, the model better simulates the way humans assess information and make decisions under uncertainty. The incorporation of fuzzy discount factors marks a significant shift from deterministic to a more realistic representation of financial markets, suitable for practical application. This methodology offers a nuanced investment strategy that balances theoretical rigour with real-world applicability, appealing to a broad spectrum of investors. The aim of the following paper is to introduce an alternative to price modelling with the use of fuzzy return rates, which results in some errors in the mathematical model. The solution has the form of introducing fuzzy discount factors (FDFs) that retain the advantages of the fuzzy approach (e.g., encompassing subjectivity and imprecision) while preserving the shape of the fuzzy number modelling a price.

Keywords: discount factor; fuzzy number; portfolio management

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1. Introduction

The financial market, prone to human behaviours, presents a complex environment for modelling and prediction. Traditional financial models, rooted in precise mathematical frameworks, often fail to capture the nuanced realities of market dynamics influenced by human behaviour. These models typically overlook the psychological underpinnings that significantly affect investment decisions, such as fear, greed, overconfidence, and herding behaviour. Behavioural finance attempts to fill this gap by integrating insights from psychology into financial modelling. However, a significant challenge remains in quantifying and incorporating the inherently subjective and imprecise nature of human behaviour into these models (Suresh, 2024).

The need for a more holistic approach is evident, one that considers not just the mathematical probabilities but also the human elements of decision-making (Stasiak & Staszak, 2024). Fuzzy logic, with its ability to handle imprecision and subjective judgment, offers a promising solution. By representing financial variables as fuzzy numbers, these models can better mimic the way humans process information and make decisions under uncertainty. This approach acknowledges that financial decision-making is rarely black and white but often involves shades of grey, impacted by both quantifiable factors and unquantifiable human sentiments (Rutkowska & Szyszko, 2024).

Fuzzy logic's incorporation into financial models marks a paradigm shift, moving from the traditional deterministic approach towards a more realistic representation of the financial market's complexity. This integration is critical for developing models that are theoretically sound and practically applicable in the real-world environment, where human behaviour plays a pivotal role. Implementing fuzzy discount factors in portfolio construction is a prime example of this new approach, offering a way to model investment decisions that align more closely with human reasoning (Piasecki & Łyczkowska-Hanćkowiak, 2021).

This paper contributes to the market modelling problem by highlighting the error of using directly fuzzified return rates and presenting an alternative in the form of fuzzy discounting factors, resulting logically from investor's preferences and retaining the advantages of the fuzzy approach.

2. Research Background

The Markowitz model of portfolio theory (Markowitz, 1991) laid the foundation for modern financial modelling, emphasizing the importance of diversification and the trade-off between risk and return. However, its reliance on crisp data and the assumption of rational behaviour limit its effectiveness in capturing the nuances of real-world investing. Real-life investment decisions are rarely made based on purely rational calculations; they are influenced by a myriad of factors, including investors' personal experiences, cognitive biases, and emotional responses to market events (Pfiffelmann et al., 2016).

The introduction of the direct fuzzification of crisp variables in financial modelling has been a significant step forward. This process involves converting precise numerical data into fuzzy numbers that better represent the inherent uncertainty and subjectivity of the financial markets. Such an approach aligns more closely with the real-world scenario where information is often incomplete, ambiguous, or subjective (Kacprzak & Kosiński, 2014; Łyczkowska Hanćkowiak, 2021; Piasecki, 2018).

Despite these advancements, a gap remains between the complexity of these models and their usability for the average investor. Many investors, especially those without a strong background in finance or mathematics, find these models intimidating and challenging to understand. The use of linguistic variables, akin to natural human language, addresses this issue (Piasecki & Siwek, 2018; Siwek, 2017). Terms like "high risk", "moderate return", or "likely profitable" make the models more relatable and easier to interpret.

The evaluation of financial portfolios has long been focused on minimizing the risks associated with uncertain returns (Rusu & Bolos, 2024). However, another significant challenge lies in addressing the risk of imprecision inherent in the information used for decision-making. Imprecision, often associated with vagueness or ambiguity in financial data, affects the accuracy of present value assessments for financial instruments. Traditional approaches tend to overlook this behavioural and subjective dimension, wherein investors rely on approximate estimations influenced by their experience, market conditions, and inherent risk aversion. These approximations can result in imprecise decision-making processes that influence portfolio outcomes (Aggarwal & Mohanty, 2022).

The foundational work on imprecision in finance has evolved significantly, leveraging mathematical tools such as fuzzy sets, first introduced by (Zadeh, 1965), and further developed by (Dubois & Prade, 1986), and (Zimmermann, 2011). Fuzzy arithmetic has become a cornerstone for addressing imprecision in financial contexts, with applications ranging from the valuation of instruments to portfolio optimization. Subsequent advancements extended fuzzy concepts to model present values and return rates using triangular and trapezoidal and many different types of fuzzy numbers. This approach has highlighted the limitations of purely probabilistic methods in capturing the behavioural aspects of financial decision-making under imprecise conditions.

Despite advancements, the practical application of fuzzy systems to portfolio theory reveals persistent challenges. Notably, imprecision in estimating present values of assets, especially with algorithmic and high-frequency trading in mind, stems from delays arising from technical constraints, decision lags, and market fluctuations. These factors make the current value inherently subjective and resistant to future validation. Furthermore, while ambiguity risk can be quantified and incorporated into portfolio models as a linear combination of individual risks, the risk of vagueness, measured by entropy, remains unaffected by diversification. This underscores the need for models that better address the interplay between imprecision and traditional measures of uncertainty (Kaczmarek et al., 2020).

Additionally, the common approach of directly fuzzyfying the rates of return is not logically sound—the imprecision influences the present values of assets, so it is the present value that is the token of behavioural aspects and imprecision risk (Wang et al., 2023). Subsequently, fuzzy returns calculated based on the fuzzy present value have some interpretation issues, like not retaining the shape of the base fuzzy number. The following paper shows how to amend this problem in a model of fuzzy portfolio optimization.

Recent research has explored the feasibility of constructing portfolio optimization tasks under fuzzy frameworks, demonstrating the potential for integrating imprecision into decision-making. However, limitations persist, particularly when dealing with discrete fuzzy representations, which challenge the efficiency of optimization models. As the field progresses, there is a growing need to refine mathematical tools and incorporate imprecision more effectively into financial analysis, providing a pathway for both theoretical and practical advancements in portfolio management.

3. Materials and Methods

Triangular fuzzy numbers offer a solution to the challenge of balancing model complexity with user-friendliness. Their simplicity and ease of interpretation make them an excellent tool for representing uncertainty in financial variables (Alfonso et al., 2017; Piasecki & Siwek, 2015). The three parameters of a triangular fuzzy number—the lower limit, the peak, and the upper limit—provide a straightforward way to encapsulate the range of possibilities that a financial variable can take, reflecting the uncertainty and subjectivity inherent in financial decision-making.

Another problem is that not all parameters of financial instruments can be fuzzified. For example, in the case of fuzzy return rates, the assumption of simple return rates and attempt of fuzzification lead to bimodal distributions and the fact that, a return rate that is calculated, for instance, based on triangular fuzzy present values, is also a fuzzy set, but does not retain the triangular shape. This leads to problems in creating and managing fuzzy portfolios.

3.1. Portfolio with Fuzzy Present Value

Incorporating fuzzy logic into portfolio construction, particularly in calculating present values, has provided a significant improvement in aligning financial models with the realities of the market. The use of triangular fuzzy numbers to express present values acknowledges the inherent ambiguities and subjective judgments in financial evaluations (Siwek, 2017). This approach contrasts sharply with traditional models that often assign a single, crisp value to financial instruments, overlooking the complex reality that these valuations are not absolute but are influenced by various subjective factors.

Let $PV_i = T(\hat{C}_{i\,\min}; \hat{C}_i; \hat{C}_{i\,\max}), i = 1,2$ be triangular fuzzy numbers representing the present values of two assets. The membership function of each asset is then given by

$$\mu_{PVi}(x) = \begin{cases} \frac{x-\hat{C}_{\min}}{\hat{C}-\hat{C}_{\min}} & \text{for } \hat{C}_{\min} \leq x < \hat{C}, \\ 1 & \text{for } x = \hat{C}, \\ \frac{x-\hat{C}_{\max}}{\hat{C}-\hat{C}_{\max}} & \text{for } \hat{C} < x \leq \hat{C}_{\max}, \\ 0 & \text{for } \hat{C}_{\min} < x < \hat{C}_{\max}, \end{cases} \tag{1}$$

where the following are denoted:

- $\hat{C}$ is the market price of the instrument observed at the moment of determining its current value;
- $\hat{C}_{\min} \in [0; \hat{C}]$ is the maximum lower estimate of the possible market price;
- $\hat{C}_{\max} \in [\hat{C}; +\infty)$ is the minimum upper estimate of the possible market price.

However, modelling return rates as fuzzy numbers presents challenges. While it is possible to calculate fuzzy return rates, these often do not adhere to the simplicity and clarity of the triangular fuzzy number due to their bimodal distribution. This deviation from the triangular structure introduces complexity into the model, potentially making it less intuitive for users.

The membership function of a simple fuzzy return rate calculated based on the portfolio π consisting of the two assets above would have the following form:

$$\rho\left(r \middle| \hat{C}^{\pi}_{\min}, \hat{C}^{\pi}_{\max}\right) = \begin{cases} \frac{\hat{C}^{\pi}_{\min}\cdot(1+\hat{r})-\hat{C}^{\pi}_{\min}}{1+r} \cdot \frac{1}{\hat{C}^{\pi}-\hat{C}^{\pi}_{\min}}, & \text{for } \frac{\hat{C}^{\pi}_{\min}}{\hat{C}^{\pi}} \leq \frac{1+\hat{r}}{1+r} < 1, \\ 1, & \text{for } \frac{1+\hat{r}}{1+r} = 1, \\ \frac{\hat{C}^{\pi}\cdot(1+\hat{r})-\hat{C}^{\pi}_{\max}}{1+r} \cdot \frac{1}{\hat{C}^{\pi}_{\max}-\hat{C}^{\pi}}, & \text{for } 1 < \frac{1+\hat{r}}{1+r} \leq \frac{\hat{C}^{\pi}_{\max}}{\hat{C}^{\pi}}, \\ 0, & \text{for } \frac{1+\hat{r}}{1+r} \notin \left[\frac{\hat{C}^{\pi}_{\min}}{\hat{C}^{\pi}}, \frac{\hat{C}^{\pi}_{\max}}{\hat{C}^{\pi}}\right]. \end{cases} \tag{2}$$

To address these challenges, the concept of fuzzy discounting factors is introduced. These factors, derived from triangular fuzzy present values, maintain the triangular structure, thereby simplifying the modelling process. This simplification is crucial in making the model more accessible and user-friendly. By employing fuzzy discount factors, portfolio construction can be made more aligned with the uncertain and subjective nature of financial markets, reflecting real-world scenarios more accurately:

$$\eta_{D_1}(v) = \begin{cases} \frac{\hat{C}_i\cdot v - v\cdot\hat{C}_{\min}}{\hat{C}_i\cdot\hat{v}_i - \hat{v}_i\cdot\hat{C}_{\min}}, & \text{for } \frac{\hat{C}_{\min}}{\hat{C}_i} \leq v \leq \hat{v}_i, \\ 1, & \text{for } v = \hat{v}_i, \\ \frac{\hat{C}_i\cdot v - v\cdot\hat{C}_{\max}}{\hat{C}_i\cdot\hat{v}_i - \hat{v}_i\cdot\hat{C}_{\max}}, & \text{for } \hat{v}_i \leq v \leq \frac{\hat{C}_{\max}}{\hat{C}_i}, \\ 0, & \text{for } \frac{\hat{C}_{\max}}{\hat{C}_i} < v, \hat{v}_i < \frac{\hat{C}_{\min}}{\hat{C}_i}, \end{cases} \tag{3}$$

where v_i is a discount factor calculated for a given asset, based on a simple return rate r, and $\hat{v}_i$ is an expected discount factor.

In constructing portfolios using fuzzy present values, investors can better account for the ambiguities and subjective assessments that are a natural part of financial decision-making. This approach also allows for more flexible and nuanced investment strategies, as it enables the modelling of a broader range of scenarios and outcomes, taking into account the fuzziness and uncertainties of financial market data.

3.2. Validation of the Portfolio

Validating a portfolio constructed using fuzzy discount factors involves the application of the energy measure, which assesses the variability and risk associated with the portfolio. The energy measure, defined mathematically, quantifies the spread or divergence within a fuzzy number, providing insights into the risk profile of the portfolio:

$$d(D^{\pi}) = \left(\frac{p_1}{\hat{v}_1} + \frac{p_2}{\hat{v}_2}\right)^{-1}\left(\frac{p_1}{\hat{v}_1} \cdot d(D^1) + \frac{p_2}{\hat{v}_2} \cdot d(D^2)\right), \tag{4}$$

where p_i are the shares of the assets in the portfolio, D^{π} is the discount factor of the portfolio, and D^i are the discounting factors of assets.

By incorporating this measure, a risk minimization problem can be formulated for portfolio creation. This enables the development of more sophisticated and nuanced investment strategies that consider the fuzzy nature of financial market data. The ability to define and solve a risk minimization problem is a significant advancement in portfolio management, offering a more realistic and practical approach to investment strategy development.

3.3. Simulations

This study investigates multi-asset portfolios where the present values of financial instruments are modelled using trapezoidal fuzzy numbers. The process involves several structured steps.

First, the foundational assumptions for multi-asset portfolios were established. The current values of financial instruments, denoted as $PV_i = Tr(\underline{m}_i; \underline{c}_i; \overline{c}_i; \overline{m}_i)$, were modelled as trapezoidal fuzzy numbers. These parameters allowed for the representation of both the imprecision in valuation and the associated membership functions. The overall current value of the portfolio, PV_{π}, was also derived as a trapezoidal fuzzy number:

$$PV_{\pi} = Tr\left(\sum_{i=1}^{n} \underline{m}_i; \sum_{i=1}^{n} \underline{c}_i; \sum_{i=1}^{n} \overline{c}_i; \sum_{i=1}^{n} \overline{m}_i\right). \tag{5}$$

Next, the expected returns of individual assets and the portfolio were computed. Assuming a multivariate normal distribution for the simple rates of return $\tilde{r}_t = (\tilde{r}_{t1}, \tilde{r}_{t2}, \ldots, \tilde{r}_{tn})^{\top}$, with mean vector $\bar{r}$ and covariance matrix Σ, this study derived the fuzzy expected return for the portfolio. The fuzzy arithmetic approach facilitated the incorporation of imprecision into the return estimates, expressed through energy and entropy measures:

$$d(R_{\pi}) = \int_{x \in \mathbb{S}(R)^{\pi}} \rho(x)\, dx \tag{6}$$

and

$$e(R_{\pi}) = \frac{\sum_{r \in \mathbb{S}(R_{\pi})} \min\{\rho_{R_{\pi}}(r), 1 - \rho_{R_{\pi}}(r)\}}{\sum_{r \in \mathbb{S}(R_{\pi})} \max\{\rho_{R_{\pi}}(r), 1 - \rho_{R_{\pi}}(r)\}}. \tag{7}$$

where ρ is the membership function of the return rate and π indicates the portfolio.

The third step involved calculating the discount factors for the portfolio and its components. The fuzzy discount factor D_{π} was modelled using the fuzzy expected return R_{π}:

$$D_{\pi} = \left(\sum_{i=1}^{n} \frac{p_i}{\bar{c}_i}\right)^{-1} \otimes \bigoplus_{i=1}^{n}\left(\frac{p_i}{\bar{c}_i} \otimes D_i\right), \tag{8}$$

where p_i represents the proportion of the i-th instrument in the portfolio. Measures of energy and entropy for D_π were also derived:

$$d(D_\pi) = \sum_{v \in \mathcal{S}(D_\pi)} \eta_{D_\pi}(v), \quad e(D_\pi) = \frac{\sum_{v \in \mathcal{S}(D_\pi)} \min\{\eta_{D_\pi}(v), 1 - \eta_{D_\pi}(v)\}}{\sum_{v \in \mathcal{S}(D_\pi)} \max\{\eta_{D_\pi}(v), 1 - \eta_{D_\pi}(v)\}}. \tag{9}$$

where η denotes the membership function of the discount factor D_π.

Following this, this study formulated a portfolio optimization problem to minimize the fuzzy discount factor under constraints for energy, entropy, and variance. This involved solving the following:

$$\min\, d(D_\pi), \quad \text{subject to}\, e(D_\pi) \leq e_{\max}, \, \sigma_\pi^2 \leq \sigma_{\max}^2, \tag{10}$$

where σ_π^2 is the variance of the portfolio.

Finally, simulations using synthetic data validated the models. The results can be found in the next section.

3.4. Numeric Example

Assume that there is an instrument A_1 with a present value PV_1, defined by a triangular fuzzy number $T(60; 90; 120)$, and an instrument A_2 with a present value PV_2 equal to $T(30; 35; 36)$. The membership function plots for these numbers are shown in Figure 1.

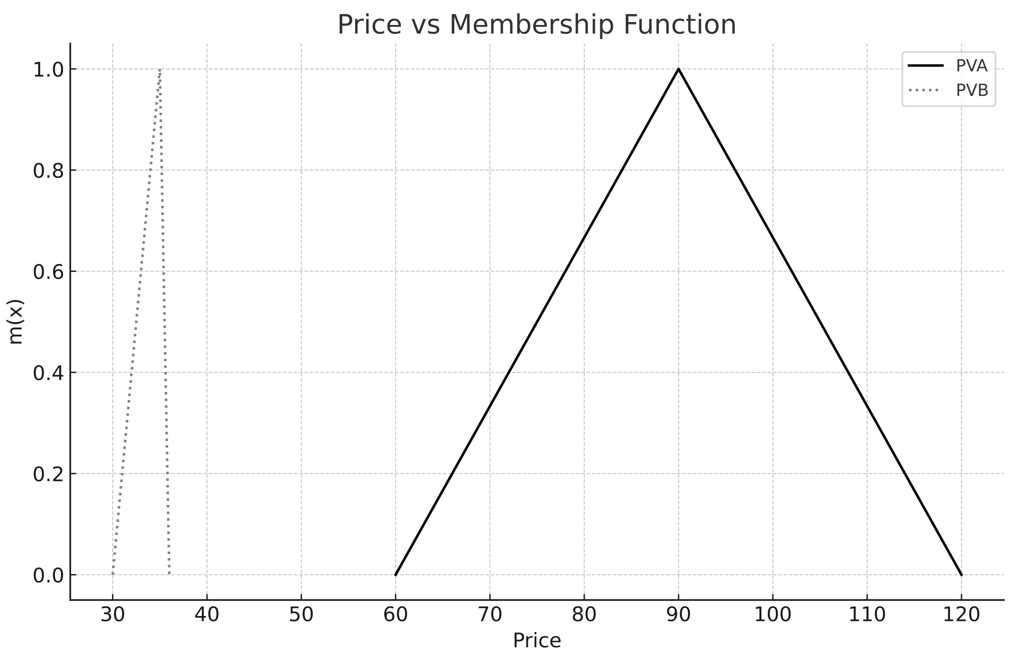

Figure 1. Membership function plots for the proposed present values of instruments A_1 and A_2.

We created a portfolio composed of instruments A_1 and A_2. Based on the market price of the instruments, we calculated their shares in the portfolio.

The present value of the portfolio composed of both these instruments, presented in Figure 2, is

$$PV_\pi = T(60 + 30; 90 + 35; 120 + 36) = T(90; 125; 156).$$

Assume also that there is a two-dimensional random variable with a joint distribution, where Σ is the covariance matrix of this variable.

For the given instruments and the portfolio constructed from them, we determined the expected return rates. They are fuzzy numbers R_1, R_2, and R_π defined by their membership functions, and are shown in Figures 3 and 4.

Figure 2. Membership function plot of the PV of the portfolio composed of instruments A_1 and A_2.

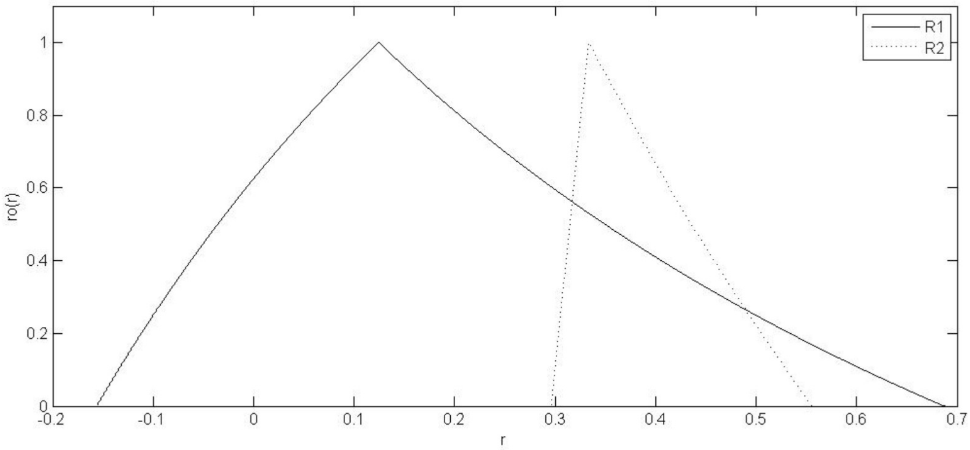

Figure 3. Membership function plot for the expected returns from instruments A_1 and A_2.

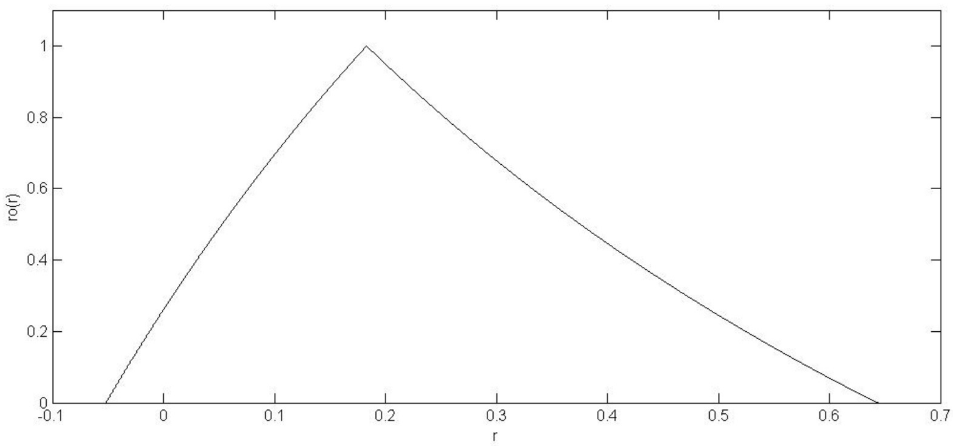

Figure 4. Membership function plot for the expected returns for the portfolio.

We can now determine the energy measures for the expected return from each instrument individually and from the portfolio:

$$d(R_1) = 0.0442,$$

$$d(R_2) = 0.0310,$$

$$d(R_\pi) = 0.0513.$$

Next, we obtain entropy measures for the expected return from each instrument and the portfolio:

$$e(R_1) = 0.0549,$$

$$e(R_2) = 0.0440,$$

$$e(R_\pi) = 0.0628.$$

The variance for the components and the portfolio is equal to, respectively:

$${\sigma_1}^2 = 0.0900,$$

$${\sigma_1}^2 = 0.1000,$$

$$\sigma^2 = 0.0545.$$

Thus, for the considered case, we have

$$d(R^\pi) > d(R^1) > d(R^2),$$

$$e(R^\pi) \geq e(R^1) \geq e(R^2),$$

$$\sigma_2^2 \geq \sigma_1^2 > \sigma^2.$$

In this numerical example, the energy and entropy measures of the expected return rate of the portfolio are higher than the analogous measures determined for the individual components of the portfolio. This means that diversification of the portfolio may increase the risk of ambiguity and fuzziness measured by the energy and entropy of the expected return rate, and it may also reduce the risk of uncertainty measured by variance.

For the instruments and the constructed portfolio, we determined the expected discount factors. The fuzzy sets corresponding to the expected rates of return from the individual instruments and the expected discount factors are presented in Figures 5 and 6.

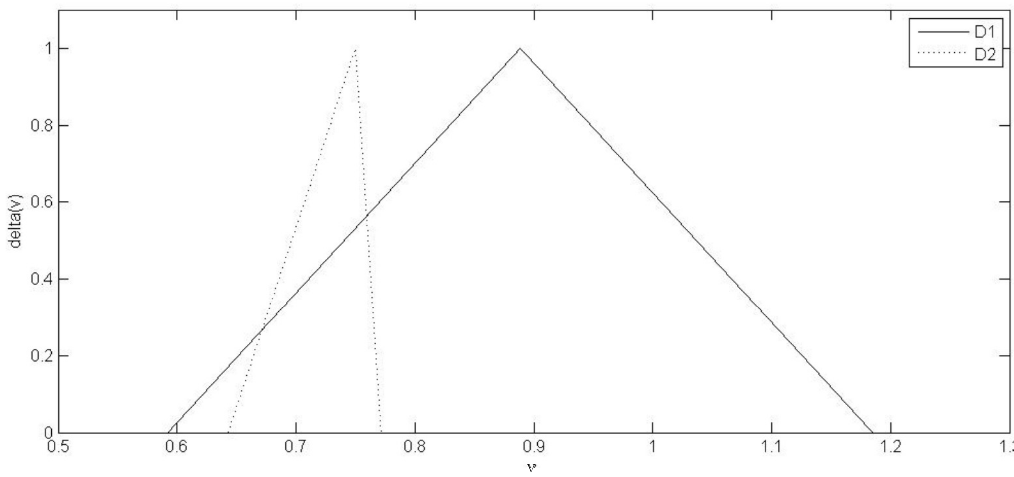

Figure 5. Plots of the membership functions for the expected discount factors of the instruments.

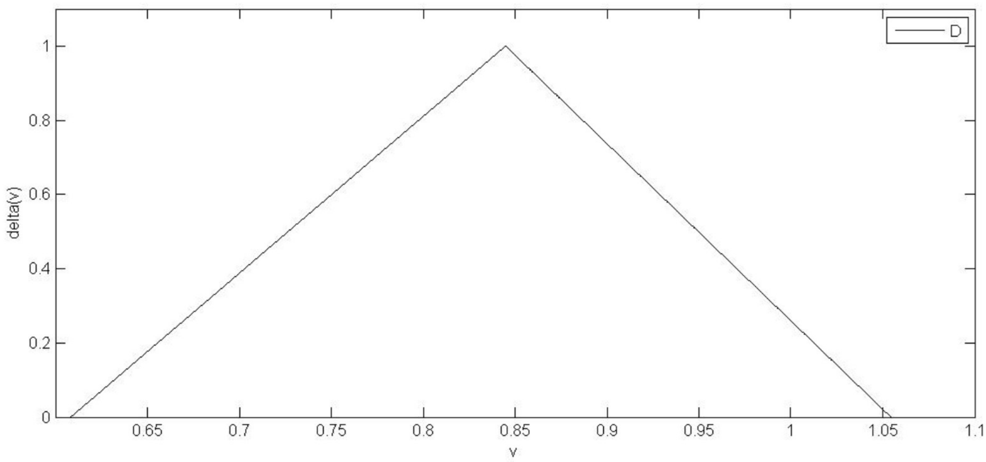

Figure 6. Plots of the membership functions for the expected discount factor of the portfolio.

Using Equation (5), we determine the energy measures for the expected discount factor of each instrument individually and for the portfolio:

$$d(D_1) = 0.2963,$$

$$d(D_2) = 0.0643,$$

$$d(D_\pi) = 0.2231.$$

The entropy measures for the expected discount factor of each instrument and for the portfolio are constant and equal to 1/3. Thus, for the case of the discount factor, we have $d(D_1) > d(D) > d(D_2)$:

$$e(D_1) = e(D_\pi) = e(D_2),$$

$$\sigma_2^2 > \sigma_1^2 > \sigma^2.$$

In the cited example, the energy measure for the portfolio is an intermediate value between the energy measures for the instruments, and the entropy is a constant value.

From the relation above, it follows that in portfolio creation, the risk of ambiguity measured by the energy of the discount factor is averaged. On the other hand, this relation suggests that the risk of fuzziness does not change in this case. This indicates that portfolio management using the discount factor may reduce the risk of imprecision.

4. Results

The conducted research highlights several key findings regarding the role of imprecision in financial portfolio analysis. First, it is evident that constructing a portfolio of financial assets does not mitigate the risk associated with imprecision. This insight underscores the inherent limitations of portfolio diversification in addressing uncertainty arising from vague or ambiguous information about the current value of financial instruments. Consequently, focusing on the expected fuzzy rate of return may not be as effective as shifting the emphasis toward the expected discount factor derived from the portfolio, which provides a more robust foundation for managing imprecision in financial decision-making.

Second, within portfolios that account for imprecision, reducing the uncertainty of returns does not necessarily lower the risk associated with the imprecision of the discount factor. This distinction emphasizes that the imprecision in valuation, influenced by subjective and behavioural factors, persists despite efforts to optimize other portfolio characteristics. It has been demonstrated that portfolio optimization tasks can indeed be formulated for portfolios involving fuzzy current values modelled as triangular or trapezoidal fuzzy numbers. However, it is crucial to note that treating portfolio risk as a homogeneous property is inappropriate, as different types of risk interact in complex and non-linear ways.

Finally, the ambiguity risk, expressed as the energy of the discount factor for triangular and trapezoidal fuzzy numbers, can be represented as a linear combination of the corresponding measures for individual portfolio components. This property facilitates the development of portfolio optimization problems that incorporate ambiguity risk as a quantifiable element. However, the imprecision risk represented by the entropy of the discount factor is not reduced by portfolio diversification. Moreover, for present values given by triangular fuzzy numbers, this risk remains insensitive to the structure of the portfolio, presenting a unique challenge to traditional diversification strategies.

A more detailed case study can be found in (Piasecki & Siwek, 2018) and (Siwek, 2017). In future research, the model will be tested on real market data, with a subjective investor input.

5. Discussion

The application of fuzzy discount factors in financial modelling represents a significant leap forward in the field. This approach is not only theoretically sound but also practical and accessible to a wide range of investors, including those without extensive financial training. By integrating behavioural aspects into financial modelling, this approach offers a more realistic portrayal of market dynamics, acknowledging the role of human behaviour in financial decision-making.

The presented method is not in line with the efficient market hypothesis since it assumes that the simple return rates from assets in the portfolio (and resulting discounting factors) have a distribution that can be calculated based on historical data and projected on future prices in the form of forecasts. Similarly, hypotheses like the fractal market hypothesis or coherent market hypothesis do not apply here—the first one connects the risk strictly with the time horizon and the other one requires the investor's rationality. The proposed approach allows for any subjective preference, not necessarily rational, and connects the risk with imprecision rather than changing the probability scope with time.

Actually, the described method suggests a slightly different approach to market modelling, which assumes that the market risk consists not only of the uncertain part but also of imprecision. The presented research proves that minimizing uncertainty does not have to reduce imprecision. Risk connected with imprecision has to be modelled separately to predict future prices more accurately. Yet, this does not mean that the whole notion of "risk" is encompassed by the model—the authors are aware that other factors that influence future prices may exist.

This model's intuitive nature and the use of linguistic variables make it particularly appealing to novice investors and those who prefer a more straightforward approach to financial analysis. The ability to define and solve risk minimization problems in portfolio construction provides a practical tool for effective risk management, accommodating the often imprecise nature of market information.

The model, to be even more intuitive, can be reconstructed with discrete fuzzy numbers. Yet, the first tentative approaches to the problem show that energy measures for current values modelled as discrete triangular and discrete trapezoidal fuzzy numbers do not provide a viable basis for formulating efficient portfolio optimization tasks. This limitation highlights the need for further refinement of methodologies and models when dealing with discrete fuzzy representations of financial instrument values. Overall, these findings contribute to a more nuanced understanding of the interplay between uncertainty and imprecision in financial portfolios, offering valuable insights for both theoretical advancements and practical applications in the field.

During the research, we detected the following limitations that need to be addressed in future works. The first limitation is that the model requires high user input in the form of deciding on the shape of the fuzzy number defining the prices. This means that the model is prone to "overfitting"—due to the subjectivity, the optimal selection of parameters for one investor can be not optimal for another. A possible solution to this problem is constructing decision-support systems that allow for preference management. Another problem at hand is the discrete case—discrete fuzzy numbers are more natural to model the market, yet defining fuzzy operations on them is not intuitive for the investors due to the change in the support domain. The answer to this is the introduction of a discretization unit and the proper definition of operations, retaining the required market characteristics. Further limitations that have to be addressed are the selection of the return rate form (e.g., simple or logarithmic), the return rate distribution, and the fuzzy number shape. The first two can be appointed using statistical methods on historical data, while the third one requires interpretative justification.

6. Future Research Direction

Beyond portfolio management, this approach has potential applications in various areas of finance, including financial planning, risk assessment, algorithmic trading, and more. The development of automated decision-making systems in finance that can mimic human-like reasoning and preferences is another exciting possibility opened up by this approach. The use of fuzzy discount factors lays the groundwork for more advanced, automated financial decision-making systems, reflecting the nuanced and often subjective nature of human investment decisions.

The proposed approach can be incorporated into other methods of financial modelling as a tool to incorporate the imprecision of return rates in a mathematically proper way, while still allowing for modelling behavioural aspects of the investors' decision-making. The future research direction in this topic is focused on discrete fuzzy numbers, which are closer to the representation of financial value in the human psychological model. Both the researched and future research plans can be incorporated into high-frequency trading to increase the algorithmic trading probability estimators by adding imprecision, besides uncertainty, to predict transitions of market states.

For example, the introduced method can be used in high-frequency trading state modelling by substituting transition probabilities with the reaching of a particular membership level in the fuzzy number representing a discount factor. In algorithmic trading, the decision about a transaction can be made via fuzzy controller (the approach is already implemented with fuzzy return rates (Abidin et al., 2023)), but with fuzzy discount factors as an input, since the first approach can lead to false conclusions. In automated trading, the fuzzy case allows for the fine-tuning of the model to a particular investor's input but needs the user's subjective assessment of the market prices.

In high-frequency trading, using fuzzy discount factors in the decision-making process allows for substituting the simple statistical dominance over a price by incorporating other preference aspects (e.g., making transactions that result in lower revenue due to choosing transactions that are not optimal in terms of the price, but instead satisfying, for instance, the requirement of oscillating around a set price, dictated by the investor's experience).

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J.S.; project administration, J.S.; funding acquisition, J.S and P.Ż. All authors have read and agreed to the published version of the manuscript.

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