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Data-Based Parametrization for Affine GARCH Models Across Multiple Time Scales—Roughness Implications

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Abstract: This paper revisits the topic of time-scale parameterizations of the Heston–Nandi GARCH (1,1) model to create a new, theoretically valid setting compatible with real financial data. We first estimate parameters using three US market indices and six frequencies to let data reveal the correct, data-implied, time-scale parameterizations. We compared the data-implied parametrization to two popular candidates in the literature, demonstrating structurally different continuous-time limits, i.e., the data favor fractional Brownian motion (fBM)—instead of the standard Brownian motion (BM)-based parametrization. We then propose a theoretically flexible time-scale parameterization compatible with this fBM behavior. In this context, a fractional derivative analysis of our empirically based parametrization is performed, confirming an anomalous diffusion in the continuous-time limit. Such a finding is yet another endorsement of the recent and popular stylized fact known as rough volatility.

Keywords: Affine GARCH; maximum likelihood estimation; time-scale parameterization; rough volatility



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1. Introduction

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is a discrete-time model widely used in financial analysis and modeling because of its ability to capture the dynamic nature of volatility in economic time series data [Bollerslev \(1986\)](#). The GARCH family of models effectively addresses fundamental properties of financial data, including time-dependent volatility, volatility clustering (where periods of high or low volatility tend to cluster together), and leverage effect (this is the correlation between returns and volatility). Extending the original GARCH model from [Bollerslev \(1986\)](#), [Heston and Nandi \(2000\)](#) proposed a new structure known as Affine GARCH that allows for closed-form moment-generating functions. Given that existing GARCHs depend on simulations for financial applications, these results opened the door to analytical solutions in many financial applications like derivative pricing and portfolio optimization (see [Christoffersen et al., 2013](#) and [Escobar-Anel et al., 2022a](#)).

Accurate parameter estimation is essential for investors and traders to understand financial asset behavior better. In this spirit, most market participants use daily frequency to analyze and estimate asset return models, even when working with continuous or discrete-time models of a single frequency like GARCHs. However, data do not come in

continuous time, and are available on many different time scales (i.e., Δ , also referred to as frequency or time scale) of interest to practitioners, e.g., Δ representing every minute, every five minutes, hourly, daily, weekly, etc.

The Affine GARCH stream of models is the first advanced discrete-time class of models that allows for analytical solutions to financial problems, becoming a solid alternative to continuous-time stochastic volatility models like [Heston \(1993\)](#). Discrete-time models offer a hidden level of flexibility, as they allow users to entertain different parameter values for various time scales even while keeping the modeling within the same family, e.g., each frequency generating the same type of GARCH, with parameters adapting to the frequency, e.g., $\beta(\Delta)$. The flexibility of keeping the same model structure for different time scales permits the study of continuous-time models as limits of discrete-time models, with the potential of revealing continuous-time structures that are not available in raw data due to the physical impossibility of observing in continuous time.

It is important to understand that discrete-time models for different time scales may not be nested; in particular, they could fail the temporal aggregation property; see [Drost and Nijman \(1993\)](#). For instance, the weekly model implied by a pre-specified daily GARCH model might not belong to the same type of GARCH, or it might not be a GARCH model at all; consequently, it could differ from a pre-specified weekly GARCH model. Therefore, we do not claim that the GARCH model in this paper fulfills temporal aggregation; we will treat the model for every frequency as a stand-alone GARCH of the same family. Another relevant aspect we are not tackling here is the documented intra-day behavior of volatilities (see [Andersen & Bollerslev, 1997](#)) starting and ending the trading day high, while being lower mid-day. Our approach is akin to how continuous time models are studied on data by fitting them to the available frequency. The two aspects described above could be attacked in extensions and refinements of this work.

The objective of our paper is to study how the parameters of a specific Affine GARCH model, the proposal of [Heston and Nandi \(2000\)](#) (HN-GARCH from now on), change depending on the frequency of observation, the time scale, and the time series. Special attention is mainly dedicated to limiting high-frequency cases ($\Delta \rightarrow 0$) and low-frequency cases ($\Delta \rightarrow \infty$), as they are strategically important. First, the high-frequency case helps connect our model to the popular literature on continuous time. By contrast, the low-frequency case is of vital importance to practitioners as a way to bypass the lack of data and poor estimation properties of such a scenario.

Our analysis relies on the empirical study of several indexes at different frequencies: the S&P500, Dow Jones Industrial Average, and NASDAQ, all from 2007 to 2023 for the following frequencies: one hour, two hours, four hours, one day, two days, and one week. These analyses reveal several patterns for this parametrization across various assets. In the second step, we formalize the finding by proposing an analytical representation of the frequency parameters. As by-products of our study, we confirm that the other two proposals in the literature, the proposal of the original authors in 2000 [Heston and Nandi \(2000\)](#), and the more recent 2019 proposal in [Badescu et al. \(2019\)](#), are incompatible with our empirical findings. Our study suggests the need for fractional Brownian motions as continuous-time limits when studying the parameter-implied fractional derivatives concerning time scale. This confirms the roughness of volatility (see [Gatheral et al., 2018](#)), albeit not as a by-product of long memory. Furthermore, unlike the recent work of [Augustyniak et al. \(2024\)](#), our assumptions regarding the scaling of the parameters are not pre-specified; instead, they represent a degree of freedom that can be calibrated to data.

In summary, our contributions are as follows:

- We propose a new, theoretically sound, empirically-based time-scale parametrization for the HN-GARCH model, adding to the existing literature on the topic as per [Heston](#)

and Nandi (2000), Badescu et al. (2019), and Augustyniak et al. (2024). The main novelty of this parametrization is its convergence to fBM instead of BM.

- We demonstrate the compatibility of our proposal with financial data via the estimation of the HN-GARCH model on three indexes for six frequencies: minute, five minutes, hourly, daily, and weekly.
- Using fractional derivatives instead of long-memory analysis to tackle fBM as motivated by the work Jumarie (2005), real financial data reveal a preference for an fBM in the continuous-time limit with a Hurst coefficient of less than 0.5, together with a non-linear drift. This confirms the roughness of volatility, reported in Gatheral et al. (2018), from the perspective of time-scale limits instead of using implied volatility surfaces from option prices.

This paper is structured as follows: Section 2 presents the HN-GARCH model, the existing parameterization in the literature, and the new proposed parameterization. Section 3 describes the indexes used for estimation, the estimation results, and plots illustrating the relationships between frequencies and estimated parameters, focusing on the S&P500. Section 4 summarizes key findings and suggests areas for improvement and future study.

2. Model and Parameterization

We assume that the asset price follows Heston Nandi's GARCH (1,1) model for a given frequency Δ ; this can be specified as:

$$y_t = \log\left(\frac{S_t}{S_{t-\Delta}}\right) = r(\Delta) + \lambda(\Delta)h_t + \sqrt{h_t}z_{t,\Delta}, \quad (1)$$

$$h_t = \omega(\Delta) + \beta(\Delta)h_{t-\Delta} + \alpha(\Delta)\left(z_{t-\Delta,\Delta} - \gamma(\Delta)\sqrt{h_{t-\Delta}}\right)^2, \quad (2)$$

where $\{z_{t,\Delta}\}$ is a sequence of independent standard normal random variables for a given frequency Δ , in easy notation; we will drop the second subindex Δ , working with $\{z_t\}$. The parameters are: $\lambda(\Delta)$, $\omega(\Delta)$, $\beta(\Delta)$, $\alpha(\Delta)$ and $\gamma(\Delta)$, where the dependence on Δ is explicit in the representation. $r(\Delta)$ is the short-term risk-free rate for the time interval Δ . The process h_t represents the variance in the time interval $[t - \Delta, t]$.

We aim to analyze how the GARCH (1,1) parameters depend on Δ using real data in a discrete-time setting (weekly, every second day, daily, half-day, every two hours, hourly). Given the importance and popularity of daily data, we use $\Delta = 1$ to represent the daily frequency. Since most estimations use daily data as the benchmark frequency, we denote the parameters of this estimation with the subscript 1:

$$y_t = r_1 + \lambda_1 h_t + \sqrt{h_t} z_t, \quad (3)$$

$$h_t = \omega_1 + \beta_1 h_{t-1} + \alpha_1 \left(z_{t-1} - \gamma_1 \sqrt{h_{t-1}}\right)^2, \quad (4)$$

In order to transition into a continuous-time setting, we define the variance per unit of time as $v_t = \frac{h_t}{\Delta}$; this leads to:

$$\begin{aligned} y_t &= r(\Delta) + \lambda(\Delta)v_t\Delta + \sqrt{v_t}z_t\sqrt{\Delta}, \\ v_t &= \frac{\omega(\Delta)}{\Delta} + \beta(\Delta)v_{t-\Delta} + \frac{\alpha(\Delta)}{\Delta}\left(z_{t-\Delta} - \gamma(\Delta)\sqrt{\Delta}\sqrt{v_{t-\Delta}}\right)^2 \\ &= \omega_v + \beta_v v_{t-\Delta} + \alpha_v (z_{t-\Delta} - \gamma_v \sqrt{v_{t-\Delta}})^2, \end{aligned} \quad (5)$$

where:

$$\omega_v = \frac{\omega(\Delta)}{\Delta}, \quad \beta_v = \beta(\Delta), \quad \alpha_v = \frac{\alpha(\Delta)}{\Delta}, \quad \gamma_v = \gamma(\Delta)\sqrt{\Delta}.$$

The representation v_t facilitates the analysis of the model in the limits, as Δ goes to 0 or to ∞ .

2.1. Existing Parameterizations

In this section, we would like to remind the reader of two existing parameterizations designed to achieve a specified continuous-time limit and introduce an empirically based approach to achieve the limiting process. The first parameterization, derived from [Heston and Nandi \(2000\)](#), is given in Equation (9). The second, presented in Equation (13), is from [Badescu et al. \(2019\)](#). Both are consistent with the continuous-time model of [Heston \(1993\)](#). We then review these parameterizations and discuss their limitations.

2.1.1. Heston and Nandi (2000)'s Parameterization

The authors proposed the following parameterization:

$$\begin{aligned}\gamma(\Delta) &= \frac{2}{\sigma\Delta} - \frac{\kappa}{\sigma}, \\ \omega(\Delta) &= \left(\kappa\theta - \frac{1}{4}\sigma^2\right)\Delta^2, \\ \beta(\Delta) &= 0, \\ \alpha(\Delta) &= \frac{1}{4}\sigma^2\Delta^2, \\ \lambda(\Delta) &= \lambda.\end{aligned}\tag{6}$$

It is easy to check that such a structure ensures weak convergence, as per [Francq and Zakoian \(2019\)](#), to the Heston model in [Heston \(1993\)](#), e.g., Cox–Ingersoll–Ross variance as per [Cox et al. \(2005\)](#), as $\Delta \rightarrow 0$, producing:

$$\begin{aligned}d \log S_t &= (r + \lambda v_t)dt + \sqrt{v_t}dW_t, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t.\end{aligned}\tag{7}$$

The parameterization is expressed in terms of the continuous-time parameters, which are unknown. More conveniently, we can rewrite it in terms of the one-day parameters, as they are the most commonly estimated. First, note that:

$$\gamma_1 = \frac{2}{\sigma} - \frac{\kappa}{\sigma}, \quad \omega_1 = \left(\kappa\theta - \frac{1}{4}\sigma^2\right), \quad \alpha_1 = \frac{1}{4}\sigma^2.\tag{8}$$

This leads to the following relationships in terms of daily estimates:

$$\begin{aligned}\gamma_H(\Delta) &= \frac{2}{\sigma\Delta} - \frac{\kappa}{\sigma} = \frac{1}{\sqrt{\alpha_1}}\left(\frac{1-\Delta}{\Delta}\right) + \gamma_1, \\ \omega_H(\Delta) &= \omega_1\Delta^2, \\ \beta_H(\Delta) &= 0, \\ \alpha_H(\Delta) &= \alpha_1\Delta^2, \\ \lambda_H(\Delta) &= \lambda.\end{aligned}\tag{9}$$

There are several drawbacks to the HN-GARCH parameterization. For example, when $\Delta = 1$, it fails to recover all the corresponding parameters, such as $\beta(1) \neq \beta_1$. Additionally, $\beta(\Delta)$ is always zero, which is unrealistic and does not align with the estimated value at $\Delta = 1$. For $\Delta > 1$, $\gamma(\Delta)$ may become negative. Lastly, the parameterization does not address the behavior for $\Delta > 1$ or ensure proper fitting to real data.

2.1.2. Badescu et al. (2019)'s Parameterization

The notation in Badescu et al. (2019) is different; the authors work with the following representation (taking $t = k\Delta$ and $\epsilon_{k\Delta} = z_t$):

$$\begin{aligned} y_t &= (r + \lambda h_t^B) \Delta + \sqrt{\Delta} \sqrt{h_t^B} z_t, \\ h_t^B &= \omega_B(\Delta) + \beta_B(\Delta) h_{t-\Delta}^B + \alpha_B(\Delta) \left(z_t - \gamma_B(\Delta) \sqrt{h_{t-\Delta}^B} \right)^2. \end{aligned} \quad (10)$$

Note, that their h_t^B is the v_t in Heston and Nandi's notation; hence, when comparing both parameterizations, we need to be careful; this is why the parameters have the superscript B . In other words, Badescu et al. (2019)'s representation expressed in Heston and Nandi's notation means replacing their $h_t^B = v_t$ with $\frac{h_t}{\Delta}$:

$$\begin{aligned} y_t &= \left(r + \lambda \frac{h_t}{\Delta} \right) \Delta + \sqrt{\Delta} \sqrt{\frac{h_t}{\Delta}} z_t \\ &= r\Delta + \lambda h_t + \sqrt{h_t} z_t, \\ \frac{h_t}{\Delta} &= \omega_B(\Delta) + \beta_B(\Delta) \frac{h_{t-\Delta}}{\Delta} + \alpha_B(\Delta) \left(z_t - \gamma_B(\Delta) \sqrt{\frac{h_{t-\Delta}}{\Delta}} \right)^2, \\ &\Leftrightarrow \\ h_t &= \omega_B(\Delta) \Delta + \beta_B(\Delta) h_{t-\Delta} + \alpha_B(\Delta) \Delta \left(z_t - \frac{\gamma_B(\Delta)}{\sqrt{\Delta}} \sqrt{h_{t-\Delta}} \right)^2. \end{aligned}$$

We then have the following mapping between Badescu et al. (2019)'s parameters and our Equation (1):

$$\begin{aligned} \omega(\Delta) &= \omega_B(\Delta) \Delta, \quad \beta(\Delta) = \beta_B(\Delta), \quad \alpha(\Delta) = \alpha_B(\Delta) \Delta, \\ \gamma(\Delta) &= \frac{\gamma_B(\Delta)}{\sqrt{\Delta}}, \quad \lambda(\Delta) = \lambda, \quad r(\Delta) = r\Delta. \end{aligned}$$

The authors propose the following time-scale parameterization based on capturing the continuous-time CIR process:

$$\begin{aligned} \gamma_B(\Delta) &= \frac{\gamma_1}{\sqrt{\Delta}}, \\ \omega_B(\Delta) &= \omega_1 \Delta, \\ \alpha_B(\Delta) &= \alpha_1 \Delta, \\ \beta_B(\Delta) &= 1 - \left(1 - \beta_1 - \alpha_1 \gamma_1^2 \right) \Delta - \alpha_B(\Delta) \gamma_B^2(\Delta), \\ &= 1 - \left(1 - \beta_1 - \alpha_1 \gamma_1^2 \right) \Delta - \alpha_1 \gamma_1^2. \end{aligned} \quad (11)$$

In particular, the authors show the following connection between continuous time parameters and one-day parameters:

$$\theta = \frac{\omega_1 + \alpha_1}{1 - \beta_1 - \alpha_1 \gamma_1^2}, \quad \sigma = 2\alpha_1 \gamma_1, \quad \kappa = 1 - \beta_1 - \alpha_1 \gamma_1^2. \quad (12)$$

Rewriting [Badescu et al. \(2019\)](#)'s parameterization formulas in terms of our notation, we obtain:

$$\begin{aligned}\gamma(\Delta) &= \frac{\gamma_B(\Delta)}{\sqrt{\Delta}} = \frac{\gamma_1}{\Delta}, \\ \omega(\Delta) &= \omega_B(\Delta)\Delta = \omega_1\Delta^2, \\ \beta(\Delta) &= \beta_B(\Delta) = 1 - \left(1 - \beta_1 - \alpha_1\gamma_1^2\right)\Delta - \alpha_1\gamma_1^2, \\ \alpha(\Delta) &= \alpha_B(\Delta)\Delta = \alpha_1\Delta^2, \\ \lambda(\Delta) &= \lambda.\end{aligned}\tag{13}$$

The main limitation of this parameterization is that if $\Delta > 1$, then $\beta(\Delta)$ could become negative. Moreover, there is no attention to the behavior when $\Delta > 1$ with respect to observed market data.

2.2. New Parameterization for High and Low Frequency

The parameter ω is very unstable for estimation; some authors report nonsignificant values (see, for instance, [Christoffersen et al., 2006](#)); others report negative values, which would make the model ill-defined (e.g., [Christoffersen et al., 2012](#); [Ornthanalai, 2014](#); [Babaoğlu et al., 2018](#)). By contrast, others set it to zero (see [Christoffersen et al., 2013](#)). Therefore, we will assume $\omega(\Delta) = 0$ here as well as in the empirical section of the paper.

We can also observe the following patterns for β and λ : $\beta(\Delta) = \beta_1$ and $\lambda(\Delta) = \lambda_1$. Lastly, we also find clear patterns for persistence approaching 1, denoted as $\pi(\Delta)$, and finite (annualized) long-term variance¹, denoted $\bar{v}(\Delta)$, these are:

$$\pi(\Delta) = \beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) \xrightarrow{\Delta \rightarrow 0} \pi(0) = 1,\tag{14}$$

$$\bar{v}(\Delta) = \frac{\alpha(\Delta)}{1 - \beta(\Delta) - \alpha(\Delta)\gamma^2(\Delta)} \frac{1}{\Delta} \xrightarrow{\Delta \rightarrow 0} \bar{v}(0) < \infty,\tag{15}$$

where $\bar{v}(0)$ ($\bar{v}$ for short) is a non-zero finite value, $\pi(0)$ (π in easy notation) seems to be 1, and we have $\bar{v}(\Delta) = \lim_{t \rightarrow \infty} E[h_t]$, $\pi(\Delta) = \frac{E_{t-2\Delta}[h_t]}{h_{t-\Delta}}$. We will use all these observations to place constraints on the new parametrization.

In addition to the conditions above, we want to parameterize $\alpha(\Delta)$ and $\gamma(\Delta)$, in a way that ensures that daily estimates are matched, i.e., $\alpha(1) = \alpha_1$ and $\gamma(1) = \gamma_1$. Given the good fit of a power function on real data, we assume $\gamma(\Delta)$ as follows:

$$\gamma(\Delta) = \frac{\gamma_1}{\Delta^b},$$

with $b > 0$. We also choose $\alpha(\Delta)$ as a power function in the form:

$$\alpha(\Delta) = \frac{1 - \beta_1}{\gamma_1^2} \Delta^{2b} + \left(\alpha_1 - \frac{1 - \beta_1}{\gamma_1^2}\right) \Delta^a,$$

where we assume $a > 2b$ to guarantee a limiting persistence equal to 1:

$$\begin{aligned}\pi(\Delta) &= \beta_1 + \left(\frac{1 - \beta_1}{\gamma_1^2} \Delta^{2b} + \left(\alpha_1 - \frac{1 - \beta_1}{\gamma_1^2}\right) \Delta^a\right) \frac{\gamma_1^2}{\Delta^{2b}} \\ &= \beta_1 + \left(1 - \beta_1 + \left(\alpha_1\gamma_1^2 - 1 + \beta_1\right) \Delta^{a-2b}\right) \\ &= 1 - \left(1 - \alpha_1\gamma_1^2 - \beta_1\right) \Delta^{a-2b}.\end{aligned}$$

Note, that if $a = 2b$, then $\pi(0) = \alpha_1 \gamma_1^2 + \beta_1 = \pi(1)$, which is not compatible with the observation from data where $\pi(0) > \pi(1)$.

Moreover, to guarantee that the long-term annualized variance converges to a finite term, we require either $a = 4b - 1$ and $b > \frac{1}{2}$, or $b = \frac{1}{2}$ with $a < 4b - 1$:

$$\begin{aligned}\bar{v}(\Delta) &= \frac{\alpha(\Delta)}{1 - \beta(\Delta) - \alpha(\Delta)\gamma^2(\Delta)} \frac{1}{\Delta} = \frac{\frac{1-\beta_1}{\gamma_1^2}\Delta^{2b} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2}\right)\Delta^a}{1 - \beta_1 - \left(\frac{1-\beta_1}{\gamma_1^2}\Delta^{2b} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2}\right)\Delta^a\right) \frac{\gamma_1^2}{\Delta^{2b}}} \frac{1}{\Delta} \\ &= \frac{(1 - \beta_1)\Delta^{2b} - (1 - \gamma_1^2\alpha_1 - \beta_1)\Delta^a}{\gamma_1^2(1 - \alpha_1\gamma_1^2 - \beta_1)\Delta^{a-2b}} \frac{1}{\Delta} = \frac{(1 - \beta_1)\Delta^{4b-a-1}}{\gamma_1^2(1 - \alpha_1\gamma_1^2 - \beta_1)} - \frac{\Delta^{2b-1}}{\gamma_1^2}.\end{aligned}$$

We first focus on a parametrization for high frequency, $\Delta \leq 1$. Given that the real data favor $b > \frac{1}{2}$, we assume $a = 4b - 1$, leading to the following proposal in terms of the daily estimates:

$$(\lambda, \omega, \alpha, \beta, \gamma)(\Delta) = \left(\lambda_1, 0, \frac{1-\beta_1}{\gamma_1^2}\Delta^{2b} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2}\right)\Delta^{4b-1}, \beta_1, \frac{\gamma_1}{\Delta^b} \right), \quad (16)$$

where b is a free (hyper-) parameter that shall be calibrated to estimates using different frequencies.

As for low frequency, the case when $\Delta \rightarrow \infty$, we should think in terms of the limiting distributions of y_Δ given y_0 , and therefore we need v_Δ given (z_0, v_0) . We first write:

$$v_\Delta = \frac{\omega(\Delta)}{\Delta} + \beta(\Delta)v_0 + \frac{\alpha(\Delta)}{\Delta} \left(z_0 - \gamma(\Delta)\sqrt{\Delta}\sqrt{v_0} \right)^2,$$

which reminds us that v_Δ is not random. We also obtain:

$$y_\Delta = r(\Delta) + \lambda(\Delta)v_\Delta\Delta + \sqrt{v_\Delta}\sqrt{\Delta}z_\Delta. \quad (17)$$

Hence, $y_\Delta \mid y_0$ distributes normally with the mean $r(\Delta) + \lambda(\Delta)v_\Delta\Delta$, and variance $v_\Delta\Delta$.

It is reasonable and empirically supported that the mean and variance of the log returns are unbounded for infinitely large horizons and independent of the initial value of the log return. Therefore, we assume $v_\Delta \xrightarrow{\Delta \rightarrow \infty} v_\infty < \infty$ and $v_0 = v_\infty$ with $z_0 = 0$:

$$\frac{\omega(\Delta)}{\Delta} + \beta(\Delta)v_0 + \frac{\alpha(\Delta)}{\Delta} \left(z_0 - \gamma(\Delta)\sqrt{\Delta}\sqrt{v_0} \right)^2 = \frac{\omega(\Delta)}{\Delta} + \left(\beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) \right) v_\infty \xrightarrow{\Delta \rightarrow \infty} v_\infty.$$

As we have assumed $\omega(\Delta) = 0$, the expression above leads to one condition:

$$\left(\beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) \right) \xrightarrow{\Delta \rightarrow \infty} \pi(\infty) < \infty, \quad (18)$$

where $\pi(\infty)$ is constant. Assuming $\beta(\Delta) = \beta_1$, $\gamma(\Delta) = \frac{\gamma_1}{\Delta^c}$, $\alpha(\Delta) = \alpha_1\Delta^d$ with $d \leq 2c$, we obtain:

$$\beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) = \beta_1 + \alpha_1\gamma_1^2\Delta^{d-2c} \xrightarrow{\Delta \rightarrow \infty} \pi(\infty) < \infty.$$

In particular, we can take $d = 2c$ for any free (hyper) parameter c leading to $\pi(\infty) = \beta_1 + \alpha_1\gamma_1^2$.

Together, we obtain the final proposal:

$$(\lambda, \omega, \alpha, \beta, \gamma)(\Delta) = \begin{cases} \left(\lambda_1, 0, \frac{1-\beta_1}{\gamma_1^2}\Delta^{2b} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2}\right)\Delta^{4b-1}, \beta_1, \frac{\gamma_1}{\Delta^b} \right) & \text{if } \Delta \leq 1 \\ \left(\lambda_1, 0, \alpha_1\Delta^{2c}, \beta_1, \frac{\gamma_1}{\Delta^c} \right) & \text{if } \Delta \geq 1 \end{cases}, \quad (19)$$

where $b > \frac{1}{2}$ and c shall be calibrated to estimates from different frequencies.

2.3. Continuous-Time Limit Implications

Analog to [Francq and Zakoian \(2019\)](#)'s Theorem 12.1, we first derive the continuous-time limit of our proposed parameterization by using the weak convergence of a Markov process to diffusion.

We use the generic notation $X_{t+\Delta} - X_t = \mu_\Delta \Delta + \sigma_\Delta (W_{t+\Delta} - W_t)$, with $dX_t = \mu_0 dt + \sigma_0 dW_t$ for $\Delta \rightarrow 0$, where W_t is a Brownian motion.

We want to find the drift, μ_Δ , and diffusion term, σ_Δ^2 , of the increase in variance per unit of time, $\frac{v_{t+\Delta} - v_t}{\Delta}$

$$\begin{aligned} E_{t-\Delta} \left[\frac{v_{t+\Delta} - v_t}{\Delta} \right] &= \mu_\Delta \\ \text{Var}_{t-\Delta} \left[\frac{v_{t+\Delta} - v_t}{\Delta} \right] &= \sigma_\Delta^2 \frac{1}{\Delta} \end{aligned}$$

Using the parametrization in (19), we obtain the following for the drift:

$$\begin{aligned} E_{t-\Delta} \left[\frac{v_{t+\Delta} - v_t}{\Delta} \right] &= \frac{(\beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) - 1)}{\Delta} v_t + \frac{\alpha(\Delta)}{\Delta^2} \\ &= \left(\alpha_1 \gamma_1^2 - 1 + \beta_1 \right) \Delta^{2b-2} v_t + \frac{1 - \beta_1}{\gamma_1^2} \Delta^{2b-2} + \left(\alpha_1 - \frac{1 - \beta_1}{\gamma_1^2} \right) \Delta^{4b-3}. \end{aligned}$$

It is easy to see that we need $b = 1$ for the limit to be finite and non-zero; otherwise, if $b < 1$, the limit is unbounded, and if $b > 1$, the limit is 0.

We can also obtain the diffusion term of the instantaneous variance:

$$\text{Var}_{t-\Delta} \left[\frac{v_{t+\Delta} - v_t}{\Delta} \right] \Delta = \frac{\text{Var}_{t-\Delta}[v_{t+\Delta}]}{\Delta} = \sigma_\Delta^2$$

$$\begin{aligned} \frac{\text{Var}_{t-\Delta}[v_{t+\Delta}]}{\Delta} &= 2 \frac{\alpha^2(\Delta)}{\Delta^3} + 4 \frac{\alpha^2(\Delta)}{\Delta^2} \gamma^2(\Delta) v_t \\ &= 2 \left(\left(\frac{1 - \beta_1}{\gamma_1^2} \right)^2 \Delta^{4b-3} + \left(\alpha_1 - \frac{1 - \beta_1}{\gamma_1^2} \right)^2 \Delta^{8b-4} + 2 \frac{1 - \beta_1}{\gamma_1^2} \left(\alpha_1 - \frac{1 - \beta_1}{\gamma_1^2} \right) \Delta^{6b-4} \right) \\ &\quad + 4 \left(\left(\frac{1 - \beta_1}{\gamma_1^2} \right)^2 \frac{1}{\gamma_1^2} \Delta^{2b-2} + \left(\alpha_1 - \frac{1 - \beta_1}{\gamma_1^2} \right)^2 \frac{1}{\gamma_1^2} \Delta^{6b-3} + 2(1 - \beta_1) \left(\alpha_1 - \frac{1 - \beta_1}{\gamma_1^2} \right) \Delta^{4b-3} \right) v_t. \end{aligned}$$

Similarly, we need $b = 1$ for the limit to be finite and non-zero. Otherwise, if $b < 1$, the limit is unbounded, and if $b > 1$, the limit is 0.

Lastly, for any $b > \frac{1}{2}$, we obtain:

$$\begin{aligned} \text{Corr}_{t-\Delta}[v_{t+\Delta}, y_t] &= -\text{sign}(\gamma(\Delta)) \frac{\sqrt{2\gamma^2(\Delta)\Delta v_t}}{\sqrt{1 + 2\gamma^2(\Delta)\Delta v_t}} \\ &= -\text{sign}(\gamma_1) \frac{\sqrt{2\gamma_1^2\Delta^{1-2b}v_t}}{\sqrt{1 + 2\gamma_1^2\Delta^{1-2b}v_t}} \xrightarrow{\Delta \rightarrow 0} -\text{sign}(\gamma_1). \end{aligned}$$

Therefore, for the new parametrization in Equation (19), the model converges to the Heston model ([Heston, 1993](#)) in continuous time, i.e., as $\Delta \rightarrow 0$, only if $b = 1$. In such a case, Equation (12.10) in [Francq and Zakoian \(2019\)](#) is also satisfied for $\delta = 1$, producing:

$$\begin{aligned}d \log S_t &= (r + \lambda v_t)dt + \sqrt{v_t}dW_t, \\dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t.\end{aligned}\quad (20)$$

with

$$\begin{aligned}\sigma &= 2\left(\frac{1-\beta_1}{\gamma_1^2}\right)\frac{1}{\gamma_1}, \\ \kappa &= (1 - \alpha_1\gamma_1^2 - \beta_1), \\ \theta &= \frac{1 - \beta_1}{\gamma_1^2(1 - \alpha_1\gamma_1^2 - \beta_1)}.\end{aligned}$$

The interesting and empirically observed case is when $b \neq 1$; here we would obtain:

$$(\mu_\Delta, \sigma_\Delta^2) \xrightarrow{\Delta \rightarrow 0} \begin{cases} (0, 0) & b > 1 \\ (\infty, \infty) & b < 1 \end{cases}$$

Hence, values of $b \neq 1$ can be interpreted as a limitation of classical continuous-time diffusion models to explain the behavior of some financial time series.

Anomalous Diffusion Limit

Alternatively, we can think in terms of weak limits to anomalous diffusion, i.e., fractional Brownian motions (fBM) instead of Brownian motions. As per [Jumarie \(2005\)](#), an fBM can be expressed as a fractional derivative. In other words, the author shows that the increments of an fBM with a Hurst coefficient $H \in (0, 1)$ can be written in the Maruyama notation as $dW_t^{(H+1/2)} = Z_t(dt)^H$ with Z_t being white noise.

In this notation, a BM can be represented as $dW_t^{(1)} = dW_t = Z_t(dt)^{1/2}$. In particular, we have $\text{Var}_t[dW_t] = dt$ and $\text{Var}_t[dW_t^{(H+1/2)}] = (dt)^{2H}$. Note, that for high frequencies, the previous property implies $\text{Var}_t\left[\frac{W_{t+\Delta}^{(H+1/2)} - W_t^{(H+1/2)}}{\Delta^{H+1/2}}\right] = \frac{1}{\Delta}$, and in particular, $\text{Var}_t\left[\frac{\Delta W_t}{\Delta^{1/2+1/2}}\right] = \frac{1}{\Delta}$ for a BM.

As acknowledged by the author, the notation has the limitation of failing to reflect the dependency of increments. Still, it is helpful to us as we are only concerned with the conditional distribution²; needless to say, the HN-GARCH model is not meant to capture such dependencies either (see [Augustyniak et al. \(2022\)](#) for an Affine GARCH with fractional behavior).

Let us use this notation to write the variance per unit of time, $v_t = \frac{h_t}{\Delta}$, process as follows:

$$\begin{aligned}v_{t+\Delta} - v_t &= \Delta v_t = \mu_\Delta \Delta^B + \sigma_\Delta \Delta W_t^{(H+1/2)} \\ \frac{v_{t+\Delta} - v_t}{\Delta^{H+1/2}} &= \mu_\Delta \Delta^{B-1/2-H} + \sigma_\Delta \frac{\Delta W_t^{(H+1/2)}}{\Delta^{H+1/2}}\end{aligned}$$

where B is a scalar and H is the Hurst coefficient. This implies:

$$\begin{aligned}E_{t-\Delta}\left[\frac{v_{t+\Delta} - v_t}{\Delta^{H+1/2}}\right] &= \mu_\Delta \Delta^{B-1/2-H} \\ \text{Var}_{t-\Delta}\left[\frac{v_{t+\Delta} - v_t}{\Delta^{H+1/2}}\right] &= \sigma_\Delta^2 \frac{1}{\Delta}\end{aligned}$$

We are now ready to find out what are the values of B and H such that μ_Δ and σ_Δ^2 are finite and non-zero on real data. In other words, for our parametrization in (19), with an

empirically estimated value of b , what should be the value of B and H so that μ_Δ and σ_Δ^2 are finite and non-zero.

We obtain for the drift μ_Δ :

$$\begin{aligned} E_{t-\Delta} \left[\frac{v_{t+\Delta} - v_t}{\Delta^{H+1/2}} \right] &= \frac{(\beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) - 1)}{\Delta^{H+1/2}} v_t + \frac{\alpha(\Delta)}{\Delta^{H+3/2}} \\ &= \left(\alpha_1 \gamma_1^2 - 1 + \beta_1 \right) \Delta^{2b-H-3/2} v_t + \frac{1-\beta_1}{\gamma_1^2} \Delta^{2b-H-3/2} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2} \right) \Delta^{4b-H-5/2}. \end{aligned}$$

This implies:

$$\mu_\Delta = \left(\alpha_1 \gamma_1^2 - 1 + \beta_1 \right) \Delta^{2b-1-B} v_t + \frac{1-\beta_1}{\gamma_1^2} \Delta^{2b-1-B} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2} \right) \Delta^{4b-2-B}$$

It is easy to see that we need $2b - B - 1 = 0$, and thus $B = 2b - 1$ for the limit to be finite and non-zero.

We can also obtain the term σ_Δ^2 of the instantaneous variance:

$$Var_{t-\Delta} \left[\frac{v_{t+\Delta} - v_t}{\Delta^{H+1/2}} \right] \Delta = \frac{Var_{t-\Delta}[v_{t+\Delta}]}{\Delta^{2H}} = \sigma_\Delta^2$$

$$\begin{aligned} \frac{Var_{t-\Delta}[v_{t+\Delta}]}{\Delta^{2H}} &= 2 \frac{\alpha^2(\Delta)}{\Delta^{2+2H}} + 4 \frac{\alpha^2(\Delta)}{\Delta^{1+2H}} \gamma^2(\Delta) v_t \\ &= 2 \left(\left(\frac{1-\beta_1}{\gamma_1^2} \right)^2 \Delta^{4b-2-2H} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2} \right)^2 \Delta^{8b-3-2H} + 2 \frac{1-\beta_1}{\gamma_1^2} \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2} \right) \Delta^{6b-3-2H} \right) \\ &\quad + 4 \left(\left(\frac{1-\beta_1}{\gamma_1^2} \right)^2 \frac{1}{\gamma_1^2} \Delta^{2b-1-2H} + \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2} \right)^2 \frac{1}{\gamma_1^2} \Delta^{6b-2-2H} + 2(1-\beta_1) \left(\alpha_1 - \frac{1-\beta_1}{\gamma_1^2} \right) \Delta^{4b-2-2H} \right) v_t. \end{aligned}$$

Similarly, we need $2b - 1 - 2H = 0$ ($H = b - 1/2$) for the limit to be finite and non-zero. Lastly, for any $b > \frac{1}{2}$, we obtain:

$$\begin{aligned} Corr_{t-\Delta}[v_{t+\Delta}, y_t] &= -sign(\gamma(\Delta)) \frac{\sqrt{2\gamma^2(\Delta)\Delta v_t}}{\sqrt{1+2\gamma^2(\Delta)\Delta v_t}} \\ &= -sign(\gamma_1) \frac{\sqrt{2\gamma_1^2\Delta^{1-2b}v_t}}{\sqrt{1+2\gamma_1^2\Delta^{1-2b}v_t}} \xrightarrow{\Delta \rightarrow 0} -sign(\gamma_1). \end{aligned}$$

Summarizing, the implied continuous-time model requires $B = 2H = 2b - 1$ to produce well-defined, non-trivial drifts and volatilities.

This has several implications. First, only when $b = 1$, we obtain a standard diffusion process. On the other hand, if $1/2 < b < 1$, then $0 < H < 1/2$, which corresponds to a rough volatility process as documented in [Gatheral et al. \(2018\)](#). With $B \in (0, 1)$, the drift term is nonlinear as explicitly entertained by [Arutkin et al. \(2020\)](#), while implicitly implied by [Gatheral et al. \(2018\)](#) thanks to their exponential of fBM construction.

As documented by [Corsi \(2009\)](#) and also [Comte and Renault \(1998\)](#), volatilities display long memory, while the HN-GARCH is a pure short-term memory for volatility. Therefore, it is very important to understand that the analysis above does not convey that an HN-GARCH can capture long-term memory in the continuous time limit. The analysis only demonstrates that if the parameters of the GARCH behave in a certain way with respect to frequency, then it converges, in terms of fractional derivatives, to an fBM. However, as

explained by [Jumarie \(2005\)](#), this analysis excludes other aspects of the behavior of the fBM, particularly the dependence on increments (long memory).

3. Estimation

We selected three index funds for HN-GARCH (1,1) parameter estimation: the S&P 500, NASDAQ, and Dow Jones Industrial Average from 2007 to 2023. Our selection aims to cover a broad spectrum of US stocks, providing a comprehensive representation of the US stock market.

The S&P 500 comprises the 500 largest publicly traded companies in the US stock exchanges and serves as a benchmark for the overall performance of the US economy. On the other hand, NASDAQ is one of the three most-followed stock market indices in the United States, heavily weighted towards companies in the information technology sector. Finally, the Dow Jones is one of the oldest indices, comprising the 30 largest US companies, focusing on the industrial sector that significantly influences US economic growth.

In our analysis, we aimed to ensure a sufficiently large number of observations, N , particularly for the lowest frequency, which is weekly. A sufficiently large sample, N , is crucial for achieving accuracy thanks to the consistency of the MLE estimator.

We employed one data source for our estimation: hourly data from Firstratedata, calculating the log asset returns, denoted as y_t , using the standard formula:

$$y_t = \ln\left(\frac{S_t}{S_{t-\Delta}}\right).$$

3.1. Application of Maximum Likelihood Estimation (MLE)

Maximum likelihood estimation is widely used to determine the optimal parameter values for a statistical model. The approach described in this paper follows the same method used in the seminal work of [Heston and Nandi \(2000\)](#) and more recently in [Escobar-Anel et al. \(2022b\)](#) for the HN-GARCH model. To simplify the notation, we explain this section using $\Delta = 1$, though the same approach applies to any time scale. Just so you know, the estimation will be performed for various values of Δ , producing corresponding estimates. The parameter set with the highest likelihood, denoted as $\hat{\theta}(\lambda, \omega, \beta, \gamma)$, will be considered the optimal parameter set.

In a discrete-time setting, we observe the asset price for N periods $S_0, S_1, \dots, S_N$; we can derive the log-likelihood function using conditional probabilities as:

$$\log L(z | \theta) = -\frac{1}{2} \sum_{t=1}^N \left(\log(2\pi) + \log(h_t) + z_t^2 \right), \quad (21)$$

where $z_t = \frac{\log \frac{S_t}{S_{t-1}} - r - \lambda h_t}{\sqrt{h_t}}$ is the implied daily noise for log asset price returns. Now, we can obtain $\hat{\theta}$, which maximizes the joint density function for the log asset price through:

$$\hat{\theta} = \arg \max \log L(\theta). \quad (22)$$

For a random sample of N independent and identically distributed (i.i.d.) random variables, denoted as $z_1, z_2, \dots, z_N$, we can construct Fisher's information matrix to quantify how the log-likelihood function varies with changes in our parametric space θ :

$$I(\theta) = -E \left[\frac{\partial^2 \log L(z | \theta)}{\partial \theta \partial \theta^T} \right]. \quad (23)$$

Under mild conditions, the estimator $\hat{\theta}$ obtained in Equation (22) has important statistical properties such as consistency and asymptotic normality; see [Ling and McAleer \(2003\)](#) or [Francq and Zakoian \(2019\)](#), Chapter 6. We can estimate the Fisher information matrix as follows:

$$\hat{I}(\theta) = \frac{1}{N} \sum_{t=1}^N \left[\frac{\partial^2 \log L(z_t | \theta)}{\partial \theta \partial \theta^T} \right]. \quad (24)$$

The inverse of the Fisher information corresponds to the standard error of the estimators.

3.2. Methodology and Results

Now, we are ready to estimate the HN-GARCH model's parameters. Maximum likelihood methodologies on real data have the challenge of needing a good initial point to start up the maximization algorithm. To address this challenge, we aimed to use a wide range of initial parameter sets; for instance, we combined the following values for daily data, e.g., $\Delta = 1$:

$$\begin{cases} \lambda = 1, 4, 9, \\ \omega_1 = 0 \\ \alpha_1 = 3 \times 10^{-6}, 5 \times 10^{-6}, 8 \times 10^{-6}, \\ \beta_1 = 0.6, 0.75, 0.9, \\ \gamma_1 = 150, 250, 350. \end{cases} \quad (25)$$

After combining the values in Equation (25), we have a total of $3 \times 3 \times 3 \times 3 = 81$ initial sets to start with.

To meet the stationary condition, we need to impose:

$$\beta_1 + \alpha_1 \gamma_1^2 < 1 \quad (26)$$

Following the exclusion of sets that do not meet the criteria outlined in Equation (26), we conducted maximum likelihood estimation on the chosen index data. We employed several MATLAB, 2024b optimizer functions sequentially to ensure success in finding the global optimal rather than a local: `fminunc`, `fminsearchbnd`, `MultiStart`, `GlobalSearch` and `simulannealbnd`.

Table 1 reports the estimates and standard errors for all four parameters, $(\alpha, \beta, \gamma, \lambda)$, and for all six frequencies under consideration, namely hourly, two hours, four hours, daily, two days and weekly. The table also shows the persistence, computed as $\pi = \beta + \alpha \gamma^2$ and the long-term volatility annualized: $\frac{\alpha}{1-\pi} \frac{1}{\Delta}$.

Table 1. S&P500 estimation results.

	1 h	2 h	4 h	1 Day	2 Days	1 Week
lambda	2.24	2.23	2.1	2.092	2.127	1.74
(SE lambda)	(1.336)	(1.335)	(1.364)	(1.349)	(1.468)	(1.461)
alpha	0.105	0.47	1.86	5.91	19.3	100.7
(SE alpha)	(0.0015)	(0.0116)	(0.057)	(0.356)	(1.62)	(9.83)
beta	0.784	0.791	0.796	0.762	0.797	0.746
(SE beta)	(0.0021)	(0.0029)	(0.0054)	(0.08)	(0.01)	(0.0146)
gamma	1408.6	638.4	304.4	179.1	79.59	34.23
(SE gamma)	(13.05)	(8.07)	(7.44)	(7.34)	(5.75)	(2.92)
Persistence	0.99234	0.98255	0.96835	0.95157	0.91926	0.86399
Long Run Vol	10.4693	10.3798	10.8408	11.0473	10.9323	12.1687

In Figure 1, the x-axis $\Delta = 1/8, 1/4, 1, 2, 5$ corresponds to data frequencies of hourly, every two hours, half-day, daily, every second day, and weekly. The y-axis shows the estimates for each of the four parameters ($\alpha, \beta, \gamma, \lambda$) distributed in subfigures (a), (b), (c), and (d), respectively.

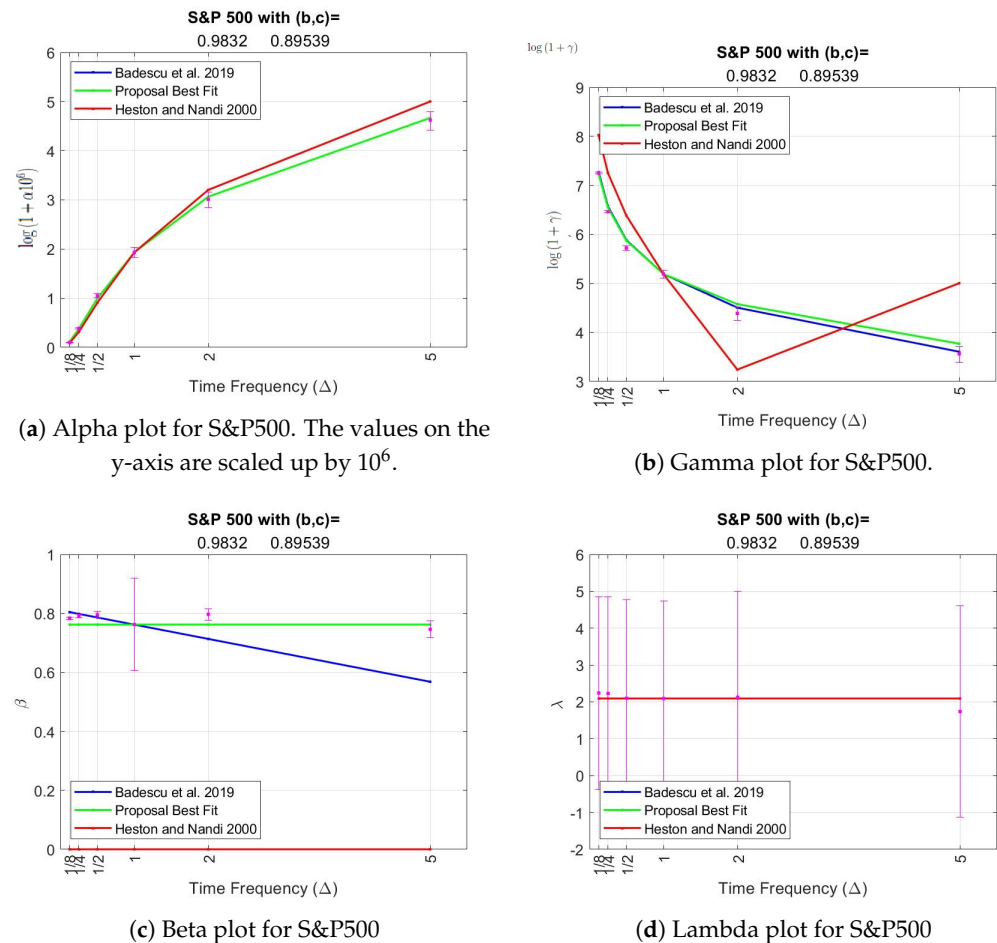


Figure 1. For the S&P500, Estimates (y-axis) for all four parameters: $\log(1 + \alpha 10^6)$ in (a), $\log(1 + \gamma)$ in (b), β in (c), and λ in (d), per frequency (x-axis), as well as 95% confidence intervals. Each plot shows the parameterization by [Heston and Nandi \(2000\)](#) (red), [Badescu et al. \(2019\)](#) (blue) and our proposal in green. Note, in (a) the blue coincide with red, while blue and green coincide with red in (d).

These tables and figures, together with similar analyses carried out for Dow Jones and NASDAQ in Appendix A, reveal various important and clear features in terms of time-scale parameterization, which are all compatible with our proposed parameterization in Equation (19). These are:

- The parameters α and γ require $0.7 < b \leq 1$ for $\Delta < 1$, and $0.8 < c < 1.2$ for $\Delta > 1$ to match the estimates from frequencies;
- The parameters λ and β are independent of the frequency;
- Long-run volatility is also independent of frequency;
- Persistence goes to 1 as $\Delta \rightarrow 0$.

Notably, none of the existing parameterizations coincide with the empirical behavior revealed in the figures.

4. Conclusions and Further Research Directions

This paper proposes a theoretically viable, time-scale parameterization of the HN-GARCH model compatible with real data. The parameterization is new in the literature as compared to the existing cases of [Heston and Nandi \(2000\)](#), [Badescu et al. \(2019\)](#), and [Augustyniak et al. \(2024\)](#). For this, the plots and tables in Section 3 and the Appendix A, based on various indexes from 2007 to 2023, demonstrate the empirical relationships between estimated parameters and multiple frequencies. In particular, our findings reveal that the variance converges to an fBM in terms of fractional derivatives with respect to time scale, confirming the well-established roughness of volatility, albeit not in the sense of long memory.

The need for fractional derivatives to explain the empirical behavior of the estimates is what set the parametrization apart from those in the literature in the realm of Affine GARCH models, which are known to lead to Brownian motions in the continuous-time limit. Knowledge of a correct time-scale parametrization, together with the closed-form nature of the Affine GARCH models, shall continue raising the usefulness and relevance of GARCH models in finance and economics.

The work could be extended to explore European and Asian indexes and individual stocks. Finer frequencies (e.g., every 5 min or minutely) or other GARCH models could also be considered in this area of time-scale parameterizations of financial models.

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Appendix A. Additional Plots

Appendix A.1. Plots for Estimated Parameters Against Frequency Using Dow Jones Hourly Data

Table A1. Dow Jones estimation results.

	1 h	2 h	4 h	1 Day	2 Days	1 Week
lambda	1.348	1.366	1.378	1.378	1.365	1.74
(SE lambda)	1.324	1.33	1.355	1.387	1.453	1.123
alpha	0.159	0.544	1.63	4.38	17.8	135.1
(SE alpha)	0.00347	0.0221	0.0993	0.333	1.72	14.91
beta	0.9478	0.9372	0.9248	0.9116	0.8875	0.7452
(SE beta)	0.00138	0.00249	0.00418	0.0053	0.00762	0.0202
gamma	507.45	283.4	169.56	106.6	44.59	22.15
(SE gamma)	13.02	12.71	12.5	10.39	7.55	2.98
Persistence	0.98874	0.98089	0.97166	0.96137	0.92289	0.81148
Long Run Vol	10.6302	10.6713	10.7259	10.6485	10.7434	11.972

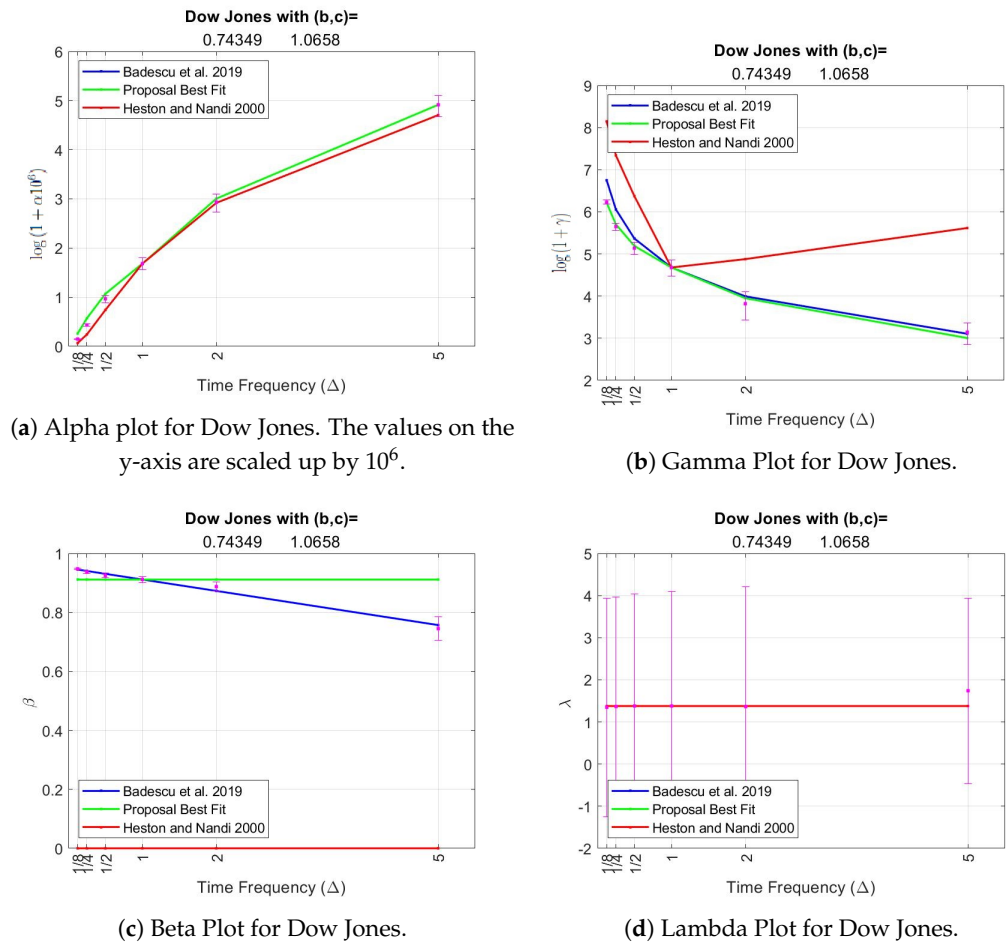


Figure A1. For Dow Jones, Estimates (y-axis) for all four parameters: $\log(1 + \alpha 10^6)$ in (a), $\log(1 + \gamma)$ in (b), β in (c), and λ in (d), per frequency (x-axis), as well as 95% confidence intervals. Each plot shows the parameterization by [Heston and Nandi \(2000\)](#) (red), [Badescu et al. \(2019\)](#) (blue) and our proposal in green. Note, in (a) the blue coincide with red, while blue and green coincide with red in (d).

Appendix A.2. Plots for Estimated Parameters Against Frequency Using Nasdaq Hourly Data

Table A2. Nasdaq estimation results.

	1 h	2 h	4 h	1 Day	2 Days	1 Week
lambda	3.331	3.292	3.303	3.208	3.083	2.971
(SE lambda)	(1.158)	(1.167)	(1.185)	(1.169)	(1.172)	(1.265)
alpha	0.177	0.665	2.5	7.19	20	100.3
(SE alpha)	(0.00297)	(0.0175)	(0.119)	(0.422)	(1.62)	(13.91)
beta	0.8682	0.8763	0.8802	0.8562	0.828	0.821
(SE beta)	(0.00161)	(0.00168)	(0.00305)	(0.00512)	(0.0105)	(0.0196)
gamma	830.9	399.4	186.49	111.24	74.33	24.5
(SE gamma)	(9.52)	(5.35)	(4.894)	(3.843)	(4.309)	(2.936)
Persistence	0.9904	0.98238	0.96715	0.94517	0.9385	0.88121
Long Run Vol	12.1449	12.2871	12.3365	11.4515	12.7514	12.9947

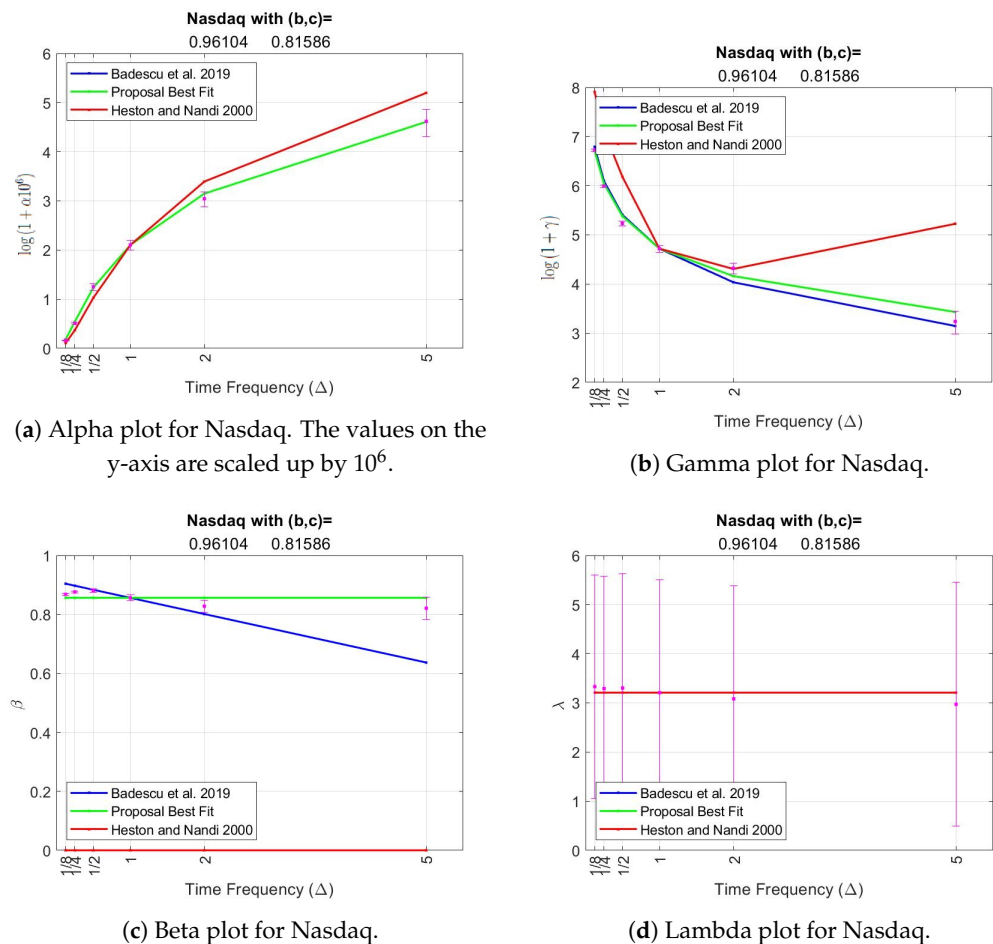


Figure A2. For Nasdaq, Estimates (y-axis) for all four parameters: $\log(1 + \alpha 10^6)$ in (a), $\log(1 + \gamma)$ in (b), β in (c), and λ in (d), per frequency (x-axis), as well as 95% confidence intervals. Each plot shows the parameterization by Heston and Nandi (2000) (red), Badescu et al. (2019) (blue) and our proposal in green. Note, in (a) the blue coincide with red, while blue and green coincide with red in (d).

Notes

- The annualized is computed as the long-term variance for the frequency Δ , e.g., daily $\Delta = \frac{1}{252}$, times $\frac{1}{\Delta}$.
- A complete analysis of how discrete time converges to anomalous diffusion would require more than weak convergence and conditional distributions.

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