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Professional Support in Teaching Mathematics through Guided Discovery: The Role of Agency on Multiple Levels

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Abstract: The paper describes a case study of mentoring support provided to a novice mathematics teacher in using Guided Discovery in her classroom. The study used qualitative methods: thematic analysis of interviews and discussions. A central theme that emerged in the results was agency, and we observed that the high level of student and teacher agency in Guided Discovery results in the importance of teacher agency within and over the mentoring framework. We conclude that when supporting teachers in using Guided Discovery in their mathematics classrooms, teacher educators need to put an increased focus on teacher agency within and over the mentoring process.

Keywords: mathematics education; guided discovery; teacher agency; teacher professional development; mentoring; task design

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1. Introduction

Guided discovery is an approach of mathematics education akin to inquiry-based learning [1], which was developed in Hungary by Tamás Varga in the 1960s and has been used and readapted since. This approach is specially suitable for the present quickly evolving and changing era, as its main aim is not to transmit fixed knowledge but to develop problem-solving and critical thinking skills that can be readapted in various future situations. It creates a democratic classroom environment using dialogue, relying equally on the students and the teacher, and thrives for equal opportunities with the principle that each student needs to be taught according to their needs [2]. Although Guided discovery influenced the mathematics curriculum and teaching methodologies and materials of Hungary, it is actively practiced only by a limited number of teachers who form a community of practice [3]. The aim of the authors is to find effective ways of involving more teachers in this community by giving them support in using guided discovery in their teaching. Professional development in guided discovery has not been extensively researched before.

As guided discovery requires creativity, adaptability, and profound mathematics content knowledge and pedagogical content knowledge from teachers [4], it is a challenge for them to learn using the method. We started a design research study [5] on finding effective ways to support teachers. The paper describes the first cycle of this research, which is a case study of a novice teacher who is mentored by two of the authors in teaching with guided discovery. As part of the mentoring process, the mentors and the teacher co-planned tasks and lessons, which will be part of a toolkit to support a wider number of teachers in teaching with guided discovery. Our research questions in this first cycle were:

1. What are the challenges and the process of growth of a novice teacher while being immersed in using guided discovery in their classroom with mentoring support in planning and reflection?
2. What aspects of the mentoring structure are effective for giving support in guided discovery, and how can it be improved and adapted to involve a larger number of teachers?

2. Background

2.1. Guided Discovery

The Hungarian guided discovery method is rooted in a heuristic approach to mathematics [6] and has been cultivated in the cooperation of mathematicians and educators as an educational approach for over a century [2,7]. The core principle of the Guided Discovery approach is to provide students scaffolding to enable them to discover mathematical concepts and knowledge with their own creative power through problem solving, rather than providing them with ready-made theorems, definitions, and strategies. This practice produced notable successes in talent nurturing programs throughout the years. From the 1970s, Tamás Varga led the attempt at extending the scope of the guided discovery approach to regular classrooms (Complex Experiment on Mathematics Education, [8]). His movement resulted in a curricular reform in grade 1-8 classrooms. Although the reform was not fully implemented, it has impacted the current curriculum, teaching methods, and materials. One of the adaptations of the guided discovery approach is the Pósa method developed for mathematics camps for gifted students and built on the use of parallel problem threads [9–11]. The method was piloted in mainstream secondary mathematics classrooms on several occasions [12]. Hungarian teachers who use the guided discovery approach in their classroom constitute a loosely organized network. Within this larger community of practice [3], there are several close-knit groups formed around schools or extracurricular student activities.

The principles of guided discovery and its adaptations were not theoretically elaborated when introduced, but there were recent attempts to derive these from the literature and documents of practice [13–15]:

- Students' creativity and responsibility in learning.
- Teachers' creativity and responsibility in teaching.
- Aptitude to utilize students' ideas in classroom dialogue.
- Problem- and activity-centered.
- The unity of mathematics; topics interlocked during instruction.
- Elongated, spiral buildup from concrete to abstract.
- Conveyance of real mathematics towards a broader student population.
- Enhancing students' competency from their individual level.

These principles place guided discovery close to inquiry-based learning. Pedaste, Mäeots et al. [1] define inquiry-based learning as a process of discovering new causal relations, with the learner formulating hypotheses and testing them by conducting experiments and/or making observations. They describe it as an approach where (1) students follow methods and practices similar to those of professional scientists in order to construct knowledge; (2) focus on problem-solving; and (3) learning emphasizes active participation and the learner's responsibility for discovering knowledge that is new to the learner. All these are important characteristics of guided discovery, but while inquiry-based learning is often self-directed and open-ended [1], in guided discovery more structured scaffolding is used to reach a specific aim.

Utilizing the guided discovery approach requires a fundamental change in one's self-image as a teacher. Guided discovery gives a central role to students as classroom actors and a central role to teachers as designers. In that, it aligns well with modern educational approaches, valuing teachers' work as design instead of implementation [16,17]. The notion of teacher design equally incorporates design for teaching and design-in-use [18].

2.2. Mentoring Framework

For teacher learning, we use the framework of situated learning, which views learning as a social phenomenon situated in a community of practice [19,20]. According to this framework, learning is most effective when taking place in the context where it is applied, which in the case of the teacher is her classroom, hence we introduced a mentoring framework. We view mentoring as a collaboration between an experienced and a novice teacher to enable the growth of the novice teacher [21].

In guided discovery, task design, lesson planning, and implementing the lesson all pose great challenges to the teacher [13,14], on top of the challenges a beginning teacher already faces [22]. Hence, in the mentoring process, we intended to alleviate the teacher's challenges by taking over the responsibility of task design, sharing the responsibility of planning, and giving her the responsibility of implementation in the classroom while supporting her with feedback on the implementation. Self-reflection is considered to be a crucial tool for teacher learning [23,24], so we also included self-reflection in our mentoring framework. Hence the mentoring framework initially included the following elements:

- Mentors designed tasks to be adapted and used by the teacher.
- Mentors and the teacher co-planned lessons.
- The teacher reflected on her lessons in writing and during mentoring sessions. Mentors gave feedback on these reflections.
- The mentors observed lessons and gave feedback to the teacher.

3. Methods

The present study was the first cycle of long-term design research study [5], the goal of which was to find ways to involve teachers in the guided discovery community of practice and support them using the methods. The long-term aim is to develop a toolkit [25] involving model activities and attached teacher support (in the form of workshops and tools for online and in-person communication between teachers). This first step was a case study with the dual aim of (1) developing and examining a mentoring structure that can be later transformed for use with a larger group of teachers and (2) developing guided discovery activities that can be used for the toolkit. The subject of the case study was Hanna, who was in her first year of teaching. Hanna experienced guided discovery as a student, and as a teacher trainee, she volunteered as an assistant in mathematics camps with guided discovery and she took part in a teacher training workshop on guided discovery. She started teaching at a high school in the school year of the study. The study was carried out in her grade 11 class (note that in Hungary mathematics is a unified subject, hence students study various topics in their mathematics class every year). The intervention started with a pilot in the second half of the school year with the unit of trigonometry, when no data was collected yet. Then it continued with the unit of coordinate geometry, where the following pieces of data were collected:

- Pre- and post-unit interviews with Hanna (semi-structured interviews about Hanna's former experience, beliefs, aims, challenges, progress).
- Video recordings of classes.
- Voice recordings of mentoring sessions.
- Emails exchanged between Hanna and the mentors.
- Written post-lesson reflections by Hanna.
- Post-unit questionnaire for students (with open-ended questions about their experiences during the intervention and Likert scale questions about their backgrounds and experiences).
- Voice recordings of post-unit interviews with three students.

Qualitative research using thematic analysis [26] was carried out on interviews with Hanna, interviews with the students, mentoring sessions, and email exchanges related to three specific classes. The three classes were selected after the unit as being representative of the following issues:

A. Lesson considered a failure by Hanna (students did not have a negative perception of any particular lesson, but Hanna perceived students to be passive during this one).
B. Lesson considered an outstanding success by Hanna and the students.
C. Lesson where Hanna piloted tasks she designed herself (which she identified as a main aim during the unit).

Video recordings of classes were used for reflection purposes by Hanna and the mentors. The student questionnaires were not formally analyzed but were used for triangulation [27] to confirm that the students interviewed represented perspectives from the whole class.

In the thematic analysis [26] recordings were transcribed and all the data used were anonymized. Then we coded the data: to each interview segment, one of the authors attached inductive codes related to the research questions. Then we recoded the data in a way that each text was coded by two different authors. We organized codes into themes and used the themes to write a narrative, which is summarized in the Results section using pseudonyms.

4. Results

The following themes emerged during thematic analysis:

- Student affect and growth: students' view.
- Student affect and growth: teacher's view.
- Teacher affect and growth.
- Teacher agency.
- What helped teacher learning.

Before we elaborate on the themes, we present sample tasks the results refer to.

4.1. Sample Tasks

As described in the Methods section, mentoring sessions and email exchanges related to three specific classes were included in the research data. Below we give a brief description and a sample task from each lesson to illustrate the results.

4.1.1. Lesson A (with "Paper-Slip Task")

The main activity of this lesson was the "paper-slip task". In the task students work in pairs, and each student receives a paper slip with a graph (partners receive different graphs and do not see each other's graph). Instructions:

Problem 1 ("paper-slip task"). *Translate the received graphs from geometry to algebra (see Figure 1). It is not completely defined what exact terms can or cannot be used, but it is something like: no terms or concepts known from geometry may be used (i.e., no straight line, no point, no segment, no circle, no circumference, no connection, no intersection, no right, no left, no up, no down, etc.). Only the coordinates of the points may be used, and of course the algebraic relationships (sum, difference, less-than, etc.).*

When you are done with the translation, pass the algebraic descriptions to each other and try to guess what the other one meant.

For three consecutive lessons, Hanna assigned some of these tasks in each lesson, besides other tasks. The lesson in question was the second lesson of these, where students worked on parts (b) and see (c) besides a problem on vectors. While students were working on the tasks, Hanna circulated in the classroom and provided help if needed. At the end of the lesson, she facilitated a whole-class discussion of the tasks. Two solutions are shown in the Figure 2.

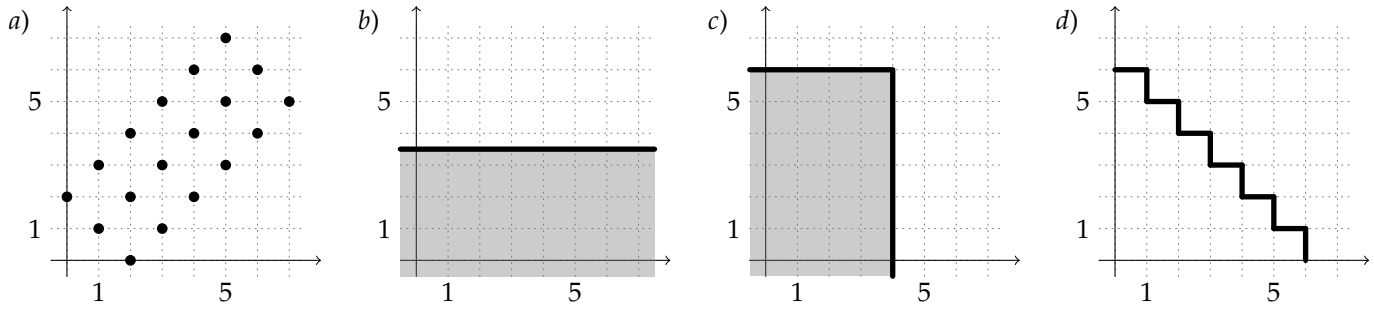


Figure 1. Four subtasks of the "paper-slip task" problem (1).

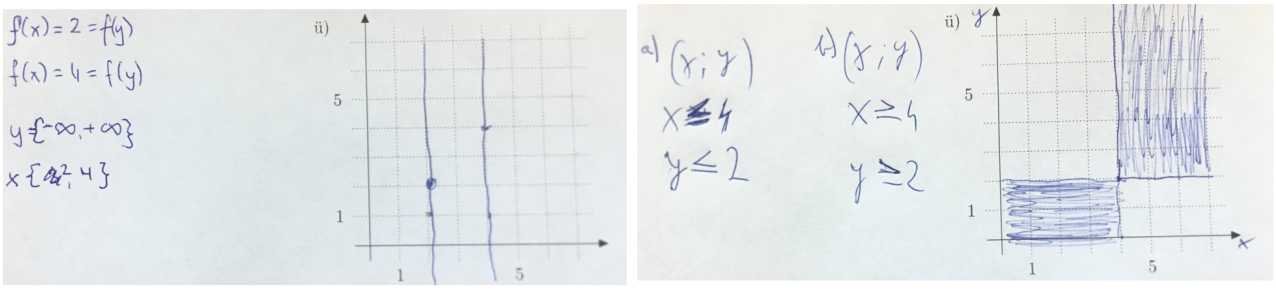


Figure 2. Student solutions for "paper-slip task".

4.1.2. Lesson B (Distance Guessing and Circle Equation)

The main activity of this lesson was the "distance guessing" task.

Problem 2 (Distance guessing). *I thought of a point, and you have to figure out which point it is. You can choose a point, and the table will tell you how far it is from the point in question. You have to guess which point I thought of. Try to find it from as few guesses as possible. Figure 3 shows the screen of the student after solving the first 5 tasks. (See https://docs.google.com/spreadsheets/d/1UCd0v4vyfKn0fYv2G8lLgfsQzmDwEuTbeGK7pKt8Y8o/copy (accessed on 25 June 2024), the online version of the "distance guessing" task at the attached link.)*

Students were prompted to proceed from task 1 to task 5 as tasks increased in difficulty. In task 1, the answer is the origin, which intends to give students the idea to guess the origin first in later tasks.

Task 1		
x	y	distance
0	0	Done!

Task 2		
x	y	distance
0	0	√2
0	1	1 (=√1)
1	1	2 (=√4)
1	-1	√8
-1	1	Done!

Task 3		
x	y	distance
0	0	√5
2	1	Done!

Task 4		
x	y	distance
0	0	3 (=√9)
0	2	√13
3	0	Done!

Task 5		
x	y	distance
0	0	5 (=√25)
0	5	√10
5	0	√80
-5	0	√20
0	-5	√90
4	3	√50
4	-3	√98
-3	-3	7 (=√49)
-3	4	Done!

Figure 3. Image of the interface with the first 5 tasks.

Figures 4 and 5 show student solutions with typical strategies on a graph, where the green dot indicates the target point, and numbers indicate the student's guesses in order.

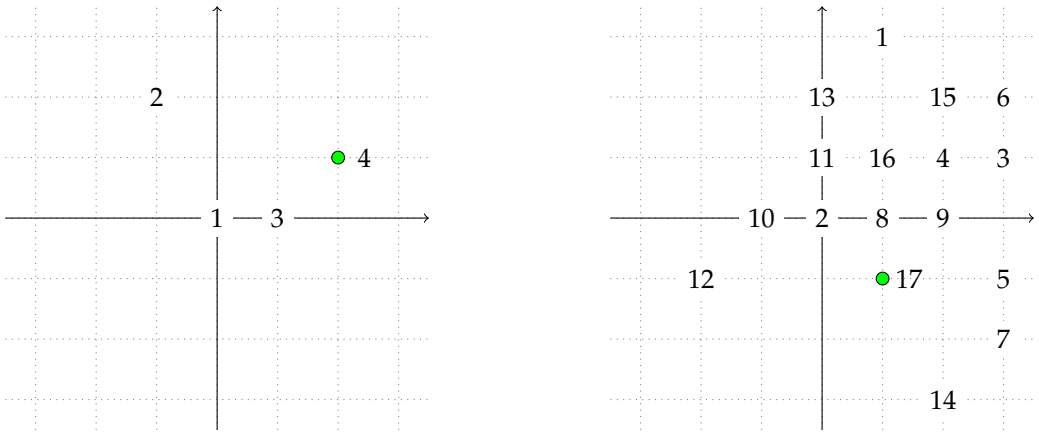


Figure 4. The figure on the **left** shows a solution where we can trace how the student made guesses informed by previous steps. In the solution on the **right**, the target point seems to have been found by chance.

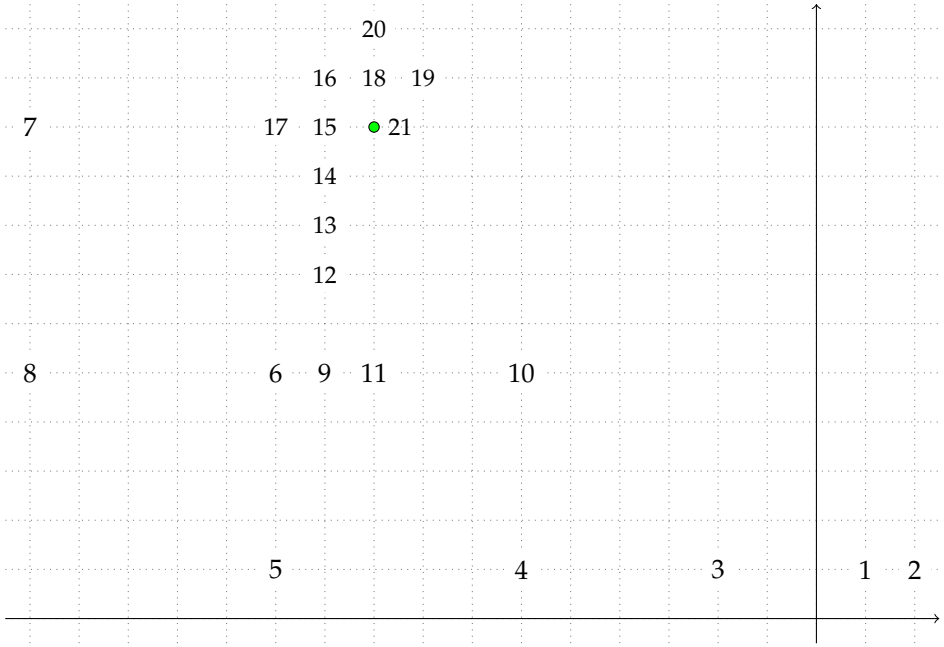


Figure 5. The figure shows a typical strategy where the student aims at decreasing the distance of their guess from the target point.

The follow-up discussion to the "distance guessing" task aimed at students discovering the circle equations. During the discussion, Hanna asked some guiding questions one by one; for each question, students had some time to consider the answer in pairs, and then they discussed the answer as a whole class. The guiding questions routed from the lesson planning discussion, where Hanna envisioned ways in which students could arrive at the circle equation based on the "distance guessing" task, but they were also adapted to the current ideas of the students. Here are the main guiding questions:

- Let us speak about what distance means. We tried $(0, 0)$, and the software said its distance from $(0, 0)$ is $\sqrt{5}$. What does this mean?
- Let us look at a specific example. How could we get $\sqrt{5}$?
- How many points are there with distance $\sqrt{5}$?
- So we have eight possibilities. What if we include points where coordinates are not integers? Brigi already said it; what shape will this be?
- Let us try looking at it in this way: we are here at $(0, 0)$, and we are looking for points at $\sqrt{5}$ distance. How could we describe them algebraically?

The last question was followed by a longer pairwork session, where several students wrote the circle equation.

4.1.3. Lesson C (with Tasks Designed by Hanna)

In the first part of the lesson, Hanna told students an "application story" about coordinate geometry. The mentors use the term application story when the teacher speaks broadly and informally about real-life applications of a mathematical topic. This is a rare instance in guided discovery when the teacher speaks and students mostly listen, with occasional brief questions for the class. This time Hanna talked about situations where coordinate geometry helps translating geometry to the language of algebra, such as with storing photos digitally, the concept of space in computer games, and GPS coordinates.

In the second part of the lesson, students worked on problems designed by Hanna. We give two examples:

Problem 3 (Lamb problem)**.** *Two lambs are tied at the points* $(2;3)$ *and* $(4;-1)$*. The rope of the first lamb is* 22 *m long, and the rope of the second lamb is* 44 *m long (Figure 6). What is the longest segment where the lambs can walk together?*

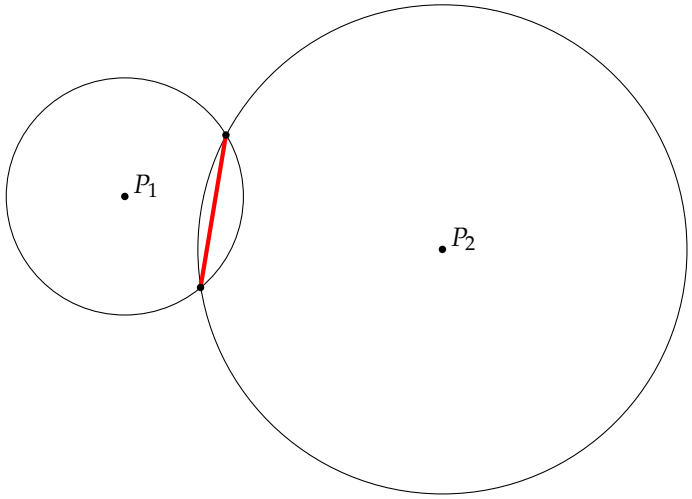


Figure 6. Note: the "longest segment where the lambs can walk together" indicates the segment between intersection points (marked in red).

Problem 4 (Cake problem)**.** *We want to slice a round cake with three cuts. The center of the cake is at the point* $K(-3;6)$*, and its diameter is 18 cm. The equations of the lines of the three cuts are:*

$$y = \frac{4}{7}x - \frac{3}{7}$$
$$y = -\frac{6}{5}x + \frac{51}{5}$$
$$y = 4x + 8$$

How many slices do we get?

Students worked on problems individually or in small groups, while Hanna circulated in the classroom and provided help if needed. At the end of the lesson, she facilitated a whole class discussion of the tasks.

4.2. Student Affect and Growth: Students' View

According to students, the teacher introduced new topics by assigning problems and not saying what the topic was. Students discovered the material themselves:

> Balázs: I could feel that the main point of the experiment was that she does not say what this is about, but we need to find it out ourselves. [...] So we basically figured out for ourselves what is written in the *Big Book*.

Students highlighted the circle equation. One student explained that she thought they were still practicing the Pythagorean theorem, when suddenly they discovered the equation of the circle. Another student said that the teacher asked them to guess what the equation of the circle was. He does not remember if they were able to say, but this definitely helped them understand why the formula worked.

Students think that with this method, they can understand the whys and the logic better. They will remember the material better, and they can solve problems more effectively than if they were told the formulas:

> Brigi: It's more interesting and more useful if we make discoveries on our own, because like that I don't need to revise the material, I will remember it well enough.

> Balázs: I think this way we can understand more what is why in math, because if we are given a formula, then maybe we know what the formula is for, and maybe we can even use it, but we don't see the logical connections behind, and what the formula expresses. And I think this is exactly why this worked, so we found out faster what we should do than if we were told what to do.

Students like pairwork, as they can help each other. They were motivated by problems with interesting wordings, game elements, the success of solving problems, and the enthusiasm of the teacher.

On the other hand, students think that this method is more time consuming. In the beginning, it was strange to go back to the basics, as they learned more complex things in the previous years. A student also mentioned that there is less structure. She is not sure how two consecutive lessons connect to each other, although she trusts they do. When she is not sure if her solution is correct, she asks the teacher to explain it to her, but it would be useful for the whole class to hear the explanation. There is less explicit knowledge written down in her notebook, so it is difficult to look up something when she needs to solve another similar problem. It would also be helpful to check homework together as a class.

Further interesting observations from the students include that with this method student's energy levels are especially important, and when the task is playful, they concentrate less.

4.3. Student Affect and Growth: Teacher's View

Hanna said that a main tool she used to help students in discovery was leaving ample time for students to think about problems. She also used scaffolding to help students discover connections, e.g., problems that build up to a certain theory, hints and guiding questions, and experimentation with physical manipulatives and interactive Google forms. She used multiple approaches to teach concepts and problems that could be solved on multiple levels. Hanna thought that wording of problems was interesting and gave students an extra challenge of interpretation, and she also told them stories about real-life applications as a means of motivation. Students usually worked in pairs or small groups, while she was going around to answer questions or give hints.

According to Hanna, students were motivated by interesting wordings and stories about real life applications, and some of them were also interested in reasoning. Hanna claimed that most students enjoyed that they did not receive a ready-made method, worked on problems actively, and some even posed new questions and tried to answer these. A few students found the challenge difficult and were less active. Overall, she reported that the gap between students in terms of readiness to think about problems has decreased. She also reported that certain students who previously had difficulty with mathematics became very active and successful in solving problems.

Hanna observed that students grew in creativity, thinking flexibly, and having courage to solve problems, and they gave beautiful solutions:

> Hanna: What specifically improved is their creativity and their courage in using things. So they gave beautiful solutions and I felt that they not only learned a formula but they tried [...] so when the task was to simply apply a formula, if a student did not know the formula, they transformed [the problem] in a way that they could solve it.

She perceived that their knowledge was stable, they understood big ideas, and they remembered connections. Students were on different levels of understanding and abstraction, but all of them were able to solve problems on the level of the final exam. Sometimes they were not sure where to start or gave random guesses instead of thinking about connections. Estimation and experimentation were challenging for students as they feared the possibility of making a mistake, but these were hands-on experiences that helped them understand concepts deeper.

4.4. Teacher Affect and Growth—Teacher's View

Hanna thinks that the main aim of mathematics education is for students to learn how to think: establish, understand, and apply connections. She is trying to reach this aim by letting students discover connections themselves with the minimum teacher help needed. During the experiment, she became braver in giving students more time for independent thinking, and she experienced that students did discover and enjoy the process.

Specific aims for teacher growth Hanna voiced before the coordinate geometry unit included awareness of the overall aims of the unit, confidence in selecting and designing tasks, making lessons engaging through games, stories about applications and exciting word problems, deciding when and how to give hints/ask helping questions, finding balance in the time given to think about problems, difficulty of problems, and amount of material taught. Homework was also an ongoing issue for her for several reasons: she wanted students to keep agency for completion; in the guided discovery tradition, homework problems are challenging, and students are expected to work on them but not necessarily to solve them; and she had difficulty with the role of being assertive about her expectations. As a result, homework completion varied among students, and she had difficulty in deciding whether to discuss answers in class. At the end of the unit, she accounted for growth in all of these areas but still felt uncertain about them.

In Lesson A Hanna felt frustrated, as she perceived that most students were passive and not cooperating. She said that when there was a specific task, they did it, but when they needed to think, talk, or cooperate, they were passive, and some of them also missed instructions, so she needed to repeat them several times. She said she was not sure why, and she gave several possible reasons. She mentioned various practical issues outside of class that probably contributed to low student energy levels and added that she was in a bad mood herself. In terms of the tasks, she said that some of the tasks may have been very easy and some of them very difficult, so students may have been stuck. Besides, students solved them at different speeds, and then had to wait for each other to be able to swap paper slips. After swapping, it was difficult for them to deal with each other's wrong answers. She also noted that when there is a playful situation, students think that they can relax.

Hanna voiced several further challenges and dilemmas. One was timing: allowing time for discovery versus making progress in the curriculum, recognizing when students are ready to move on to the next topic, and planning for timing in lessons. Often she needed to make compromises because of time: sometimes she made less progress than planned but was happy with the learning goals reached at other times she reached the goal but learning was more superficial. On the whole, she claimed that she managed to stick to the timing planned. Other challenges were detecting how confident students' knowledge is and whether they understand or simply accept connections, the amount of discovery versus whole class discussion and practice, focusing on thinking ability versus specific

curricular skills and handling differences in speed in an interactive game. Because of the complexity of the process, it was challenging for students to catch up when they were absent and for Hanna to plan for substitution when she was absent.

Hanna voiced the following further areas she grew in: being more open to student experiment activities and using several approaches in teaching.

One of her most positive experiences was when students discovered the circle equation (Lesson B). First we will summarize the co-planning phase before the lesson. Hanna started by making specific predictions about how students would be able to come up with the answer based on and extending her own experiences with engaging with the task. Then she suggested to follow up with a circle-guessing game. The mentor found this a good idea and said that this could happen after they know the circle equation. Then he suggested (tentatively) that the distance-guessing game could be used to discover the circle equation. They brainstormed jointly with Hanna about the multi-step process and possible scaffolding questions and hints given to students during class.

After the lesson, Hanna described that students were very engaged with the distance-guessing game; most of them guessed randomly, some more strategically. Then they had a whole group follow-up discussion, after which students needed to write the equation of the circle themselves. She said she was uncertain whether students would be able to find the circle equation, but it came very naturally to them. She said it was not her merit, but it was achieved by the students and previous tasks. Her strategy was not telling them the answer but asking questions:

> Hanna: it is a wonderful feeling that they discovered the circle equation [...] even after days, weeks, and now months it feels great that this is possible to achieve, possible to do, they could discover this, and I had something to do with it.

4.5. Teacher Agency

In the original mentoring framework, mentors selected and designed tasks, and Hanna and the mentors collaboratively planned lessons using and adapting these tasks. In the interview involving reflection on the first trigonometry unit and looking ahead to the coordinate geometry unit analyzed in the current paper, she voiced her desire to be more involved in designing problems in order to gain competence for the future:

> Hanna: In the future it would help me to get involved in task design on some level, how this works, how you design a task, because this is very comfortable for me and very helpful, but I don't think that in ten years they will stand by me and design tasks for me.

When observing Lesson A, one of the mentors raised the question of how much Hanna owned lesson plans. She said that sometimes yes, and sometimes not. She explained being uncertain about several issues: planning lessons herself, whether she can transmit the plans well, about the aim, the right level for students, and whether she can adapt during the lessons. She added that she said during planning that she thinks these problems would be too difficult, but she was not assertive enough.

Another issue that came up in this discussion is that sometimes Hanna slightly misunderstood or was not sure about the mentor's idea for the instructions of the task, which was also due to the fact that it was the mentors who provided the tasks.

At this point, mentors and Hanna decided to be conscious about giving more agency to Hanna in selecting and designing tasks. Based on Hanna's suggestion, they agreed that they would think about problems individually and then discuss them in meetings and via email so they could add on to each other's ideas.

During the unit, Hanna took more and more responsibility for selecting, adapting, and creating tasks. She was very determined to design tasks herself, although mentors pointed out that even experienced teachers not necessarily design task, but select and adapt existing ones. Towards the end of the unit she assigned several problems that she designed herself. She said she liked these problems and felt proud about them. She added that some tasks

she designed also worked as reference points for certain problem-solving strategies. She explained that what helped her design these problems was some of the mentor problems whose wording was far-fetched from reality, but at the same time fitting the mathematical idea in a very natural way. For some problems, she started from the mathematical idea she wanted to convey (e.g., Lamb problem described in Section 4.1.3), and sometimes she started from the real-life context and found a mathematical question (e.g., Cake problem described in Section 4.1.3).

Students highlighted several of the problems Hanna designed. They highlighted some for their sense of success when solving them and some for exciting wordings (e.g., about the Lamb problem described in Section 4.1.3):

> Panka: and there were more fun tasks like that, [...] it's nice that it's more interesting, and I don't know, it's more exciting, and it makes you feel better, because I'm not calculating two circles, but where the lambs meet.

Hanna's confidence in designing tasks grew considerably. In one of the post lesson discussions, she said: Hanna: I can't design tasks like these. So for me, I don't feel I could put together such a lesson.

However, in the final interview she said:

> Hanna: I dare to dream big, I never thought before that it was possible to come up with such tasks. Since then, I've been trying to look at the topics with an eye to how the students can be engaged, what tools could be used, what they can discover on their own, through interesting tasks.

A parallel issue of teacher agency was stories about applications. She was excited to include these in her lessons based on what she saw from the mentors, but at the beginning she found it difficult to own these instances. However, she perceived that students were interested, which helped her become more confident in the activity.

4.6. What Helped Teacher Learning

Hanna accounted for the following support in using guided discovery in her lessons:

- **Professional development workshop on guided discovery** helped her build awareness of its main components and set these components as aims for growth in her own teaching.
- **Tasks provided by mentors** gave her stability; sometimes she asked mentors to give her a task with a specific aim.
- **Joint planning with mentors** helped her think more about her lessons and have access to an external point of view. Specifically, this helped in thinking more about aims, timing, amount and difficulty of tasks, long term planning, and assessment.
- **Reflection** helped her see more clearly and adjust accordingly. Specifically, what worked and what did not, what students understood and what was difficult for them, what was left out and what can be let go, what went faster or slower than expected.
- **Mentor feedback on lessons** helped her make sense of students' understanding, and difficulties, teacher strategies concerning math content, and timing. They also strengthened her self-confidence by confirming that she was on the right track.
- **Observing mentors** teaching lessons, and telling her the story with application as if she were a student.

5. Discussion

5.1. Agency and Affect (Further Theoretical Background)

Initially, agency and affect were not in the focus of our research, but because these concepts appeared very strongly in the results, we added theoretical perspectives to aid our discussion.

Agency is defined in several different ways [28]. We will use Orland-Barak's definition, which states that agency is the capacity to make principled choices, to take action, and make that action happen. Biesta, Priestley et al. [29] claim that agency is not something

that they have; it is something that they do or achieve, so it is about the engagement of actors, not a quality of the actors themselves. In the context of our paper, agency is relevant both from the students' and the teacher's perspective.

Boaler and Greeno [30] connect student agency in the mathematics classroom to constructing meaningful understanding and capabilities to formulate questions, conjectures, and arguments that provide satisfying conceptual coherence in their practices of mathematical knowing. This aligns closely with guided discovery, where students have ample agency as they take an active part in constructing their knowledge: they take responsibility for their learning, engage actively in problems and activities, and classroom dialogues build on their ideas [13–15].

Biesta, Priestley et al. [29] claimed in 2015 that there has been little explicit research or theory development on teacher agency; however, there has been a going interest in the subject since [28,31]. Sang [28] defines teacher agency as the "capacity of teachers to act purposefully and constructively to direct their professional growth and contribute to the growth of education quality". Teachers' exercise of agency as learners is a valuable means for aligning professional development with their perceived needs, values, and epistemic beliefs. Their interactions with professional development are shaped and determined primarily by their individual goals and motivations for willing to develop professionally, which are mediated by interactions within educational contexts. Professional learning communities can improve teacher agency through involving teachers in collaborative and interactive groups. In this kind of shared communities of practice, teachers are situated and agentic human beings—they negotiate their own learning and practice by emerging learning experiences. An example of such a collaboration is described by Priestley and Biesta [32], where teachers worked with university researchers over the course of a full school year to develop the curriculum.

Hannula [33] describes affect as an umbrella term that covers attitudes, beliefs, motivation, emotions, and all other noncognitive aspects of the human mind that impact each other. They also state that in non-routine problem solving, both experienced and novice problem solvers experience positive and negative emotions and that these emotions serve an important function in a successful solution process. Negative emotion (e.g., frustration) is experienced when progress towards a goal (e.g., solving a task) is prevented, and they may suggest approaches to overcome the perceived causes, and positive emotions, on the other hand, are experienced when progress is smooth. This is relevant for guided discovery centered around non-routine problem-solving [14].

5.2. Levels of Agency

We identified four levels of agency in the results:

1. Student agency in the classroom.
2. Teacher agency in the classroom.
3. Teacher agency within the mentoring framework.
4. Teacher agency over the mentoring framework.

5.2.1. Student Agency in the Guided Discovery Classroom

When comparing the students' and Hanna's accounts of student growth (a summary of which is presented in Sections 4.1 and 4.2 above), we found that these align very closely. Hanna's account is more complex and elaborate, but students express nearly identical ideas. In terms of what happens in the classroom, they describe that students discover the materials for themselves. In terms of student growth, they think that the knowledge gained this way is retained longer and better applicable in different situations, but as the method is time consuming, it is a challenge to balance it with curricular aims (they base these ideas partly on experience and partly on beliefs). They observe that interactive and playful elements and interesting wordings motivate students and make them feel enthusiastic; however, they tend to relax and may not work to their best potential when there is a game. Challenge is often motivating for students and causes excitement and joy; however, at

times, it creates uncertainty and frustration. Less structure and teacher control and more student responsibility sometimes created uncertainty, while success created enthusiasm.

In sum, we observed that ample student agency (discovery, responsibility, less structure) can lead to heightened emotions of uncertainty, frustration, and joy, and we also observed that as success often follows challenge, positive emotions often follow negative ones. This was visible long term (there were more accounts of negative effects from the beginning of the experiment and more positive ones towards the present) and short term:

> Panka: Well, at first it was so bad, because we didn't really know what to start out from, but then we scaled it down to one degree, and then we examined how high one degree was, and then we multiplied, and then we could calculate quite close results, and then we could guess, and that felt good.

5.2.2. Teacher Agency in the Guided Discovery Classroom

In Hanna's account of her own affect and growth (summarized in Section 4.3), we found many parallels to student affect and growth (summarized in Sections 4.1 and 4.2). Hanna felt uncertain in many different situations: about her skills and tools as a teacher, specific teacher strategies connected to guided discovery, and tasks designed by mentors (aims and her efficiency facilitating them). She felt frustrated when she perceived that students were passive. She described an array of positive emotions (enthusiasm, confidence, pride, and courage), which were mostly connected to observing student success and positive affect (especially with students who were less successful or less motivated earlier), and about tasks she created and the application story she presented. Positive emotions often followed negative ones. In the long run, she accounted for much more uncertainty towards the beginning of the experiment and much more positive emotions towards the end. In the short run, she described a great amount of uncertainty before Lesson C and a great amount of enthusiasm after.

Part of these parallels can be explained by the impact of affect between Hanna and the students, which was explicitly expressed several times (students explained that Hanna's enthusiasm was motivating; Hanna felt enthusiasm when she saw that students were enthusiastic, but she was frustrated when she perceived them to be passive).

We claim, however, that most of these parallels can be explained by both parties being learners in a situation with ample agency (of mathematics and of teaching mathematics through guided discovery, respectively). Ample student agency in guided discovery requires ample teacher agency in the following ways: (1) the teacher needs to select, adapt, and possibly design tasks that provide scaffolding adequate to their specific students' needs; (2) because of ample student agency, lessons are less predictable, so during the lesson the teacher constantly needs to assess students' reactions, mathematical ideas, and affects; and react (e.g., in dialogue) or adapt the plan accordingly [2,14]. This process is illustrated by how Hanna planned for and taught the lesson with the circle equation (Section 4.3).

5.2.3. Teacher Agency within the Guided Discovery Mentoring Framework

The need for more teacher agency within the mentoring framework emerged mainly from Hanna's initiative. When deciding on the mentoring framework, mentors believed that co-planning and teaching was enough responsibility for Hanna, and they would provide tasks as means of support. However (as described in Section 4.4) Hanna suggested that she wanted to be involved in task design in order to be able to design tasks herself in the future. Besides Hanna's initiative, mentors also experienced that Hanna's use of tasks designed by mentors resulted in difficulties such as Hanna not fully owning the tasks and the tasks not necessarily fitting students' needs. Co-planning and teaching already posed many challenges for Hanna, so the reason why she even wanted to take on the challenge of task design did not result in the fact that she was not challenged enough in her growth as a teacher, but because she perceived the competencies needed to teach through guided discovery exceeding the competencies she was currently learning. Therefore, we conclude that Hanna's wish for greater agency within the mentoring structure stemmed

from her perception of the increased level of teacher agency needed in the guided discovery classroom. She even voiced that mentors providing tasks are great help, but they will not be always there for her, so she wished for adaptability of her teacher skills.

5.2.4. Teacher Agency over the Guided Discovery Mentoring Framework

Hanna's wish to have more agency within the mentoring framework resulted in changing the mentoring framework. Hanna was already exposed to guided discovery as a student and also had some experience with it as a teacher, so she may have been ready for a greater agency within the mentoring framework than other teachers. This shows that the mentoring framework itself needs to be flexible and adapt to the teacher's needs and learning goals, and even allow for teacher agency in modifying it.

We note that alongside mentoring Hanna, the mentors were already piloting the tasks that intend to serve as part of a future toolkit for teachers to teach with guided discovery, and Hanna's tasks became part of the pilot tasks, which are already available online (https://a-gondolkodas-orome.github.io/felfedezteto-matematika/docs (in Hungarian), accessed on 25 June 2024). So through increasing her agency, Hanna not only contributed to her own growth as a teacher but also had a direct impact on the growth of the guided discovery community.

Hanna gaining agency over the mentoring framework and even within the community aligns closely with Orland-Barak's description [31]: "The theme of teaching-learning environments for the development of teacher agency also invites extending the focus on teacher agency from 'outside' (i.e., conceptual and theoretical arguments as well as external research agendas tested in particular contexts) to the study of teacher agency from 'within' (i.e., creating design-based research agendas whereby teaching-learning environments geared to promoting agency are formatively developed and dynamically revised and assessed over time by the actors/participants themselves). The strength of this paradigm lies in its potential for addressing solutions to practical problems while, at the same time, empowering practitioners to become researchers into their own practices as agents of change while also implementing their research in practice".

Agency within the community could be considered a fifth level of teacher agency (the reason we have not added it as a fifth level in this paper is that it was not part of our research data and is less relevant in terms of our research question).

6. Conclusions

We observed that as the ample student agency in the guided discovery classroom requires ample teacher agency, there needs to be a focus on adapting the level of teacher agency within the mentoring structure to the teachers needs and learning goals and the structure needs to be flexible to adapt to the level of agency appropriate for the teacher, so there needs to be teacher agency in modifying the mentoring framework. So we conclude that teacher educators need to be very careful about allowing the right level of teacher agency when supporting teachers. The level of agency needs to fit the teacher's zone of proximal development [34]; too little agency may lead to less growth, and two much agency leads to failure.

Specifically in our design research setup, the results have the following implications for the future: (1) when continuing mentoring with Hanna, we will pay special attention to her agency within the mentoring structure and in structuring the mentoring process; (2) when extending support for a broader audience for teachers (e.g., providing a toolkit), we need to consider giving the right amount of flexibility to teachers' agency in making use of the support. In a way, we need to provide a guided discovery environment applied for teacher growth, where we provide the right amount of and carefully designed scaffolding for teachers towards their growth. The elements the teacher identified as helpful, e.g., sample tasks, reflection, and lesson observation, can be considered for the scaffolding structure.

The teacher involved was a beginner teacher with already some experience with guided discovery. A limitation of the study is that the process may have been different with

teachers with different backgrounds, for example, less experience with guided discovery or more experience in teaching. However, due to the fact that Hanna is a beginner teacher and that she taught using guided discovery explicitly for the first time, we think that this process caused considerable challenge for her, which represents the challenge of introducing guided discovery for other teachers. We also believe that a focus on agency within and over the type of support would be effective with any teacher exactly because it adapts to their needs. Another limitation of the study is that interviews were carried out with students on a voluntary basis, so they may not fully represent all the students in the class. To make up for this limitation, we examined the student questionnaire filled out by all students and found that answers provided in the questionnaire and during the interviews were in line with each other.

In our future research, we intend to look at further ways to support Hanna in teaching with guided discovery based on the findings of this paper and also examine ways in which support can be extended to a larger number of teachers. A further possible future area for research would be to examine parallels not only among agency between the students and the teacher but to extend the whole guided discovery framework for teachers as learners of teaching with guided discovery.

In an era where society and knowledge needed are rapidly changing, we need teachers with agency and adaptable teacher knowledge to react to these changes and provide their students with agency and adaptable knowledge in order to succeed in society.

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References

1. Pedaste, M.; Mäeots, M.; Siiman, L.A.; de Jong, T.; van Riesen, S.A.; Kamp, E.T.; Manoli, C.C.; Zacharia, Z.C.; Tsourlidaki, E. Phases of inquiry-based learning: Definitions and the inquiry cycle. *Educ. Res. Rev.* **2015**, *14*, 47–61. [CrossRef]
2. Gosztonyi, K. Mathematical Culture and Mathematics Education in Hungary in the XXth Century. In *Trends in the History of Science*; Springer International Publishing: Berlin/Heidelberg, Germany, 2016; pp. 71–89. [CrossRef]
3. Wenger, E. *Communities of Practice: Learning, Meaning, and Identity*; Cambridge University Press: Cambridge, UK, 1998. [CrossRef]
4. Loewenberg Ball, D.; Thames, M.H.; Phelps, G. Content Knowledge for Teaching: What Makes It Special? *J. Teach. Educ.* **2008**, *59*, 389–407. [CrossRef]
5. Lesh, R. Research design in mathematics education: Focusing on design experiments. In *Handbook of International Research in Mathematics Education*; Routledge: London, UK, 2002; pp. 39–62.
6. Polya, G. *How to Solve It: A New Aspect of Mathematical Method*; Number 246; Princeton University Press: Princeton, NJ, USA, 2004.
7. Gosztonyi, K.; Kosztolányi, J.; Pintér, K.; Vancsó, Ö. Varga's «complex mathematics education» reform: At the crossroad of the new math and Hungarian traditions. *ICMI Study* **2018**, *24*, 133–140.

8. Gosztonyi, K. Tamás Varga's reform movement and the Hungarian Guided Discovery approach. *Teach. Math. Comput. Sci.* **2020**, *18*, 11–28. [CrossRef]
9. Juhász, P. Talent nurturing in Hungary: The Pósa weekend camps. *Not. Am. Math. Soc.* **2019**, *66*, 898–900.
10. Pósa, L.; Juhász, P.; Matsuura, R.; Szász, R. Problem Posing in Pósa Problem Threads. In *Problem Posing and Problem Solving in Mathematics Education*; Springer Nature: Singapore, 2023; pp. 233–246. [CrossRef]
11. Győri, J.G.; Juhász, P. An extra-curricular gifted support programme in Hungary for exceptional students in mathematics. In *Teaching Gifted Learners in STEM Subjects*; Routledge: London, UK, 2017; pp. 89–106. [CrossRef]
12. Matzal, C.; Manahan, K.; Galaty, B.; Wang, H.; Barbarics, M. Guided Discovery in Hungarian Education Using Problem Threads: The Pósa Method in Secondary Mathematics Classrooms. *Teach. Math. Comput. Sci.* **2020**, *18*, 51–67. [CrossRef]
13. Gosztonyi, K. Conceiving teaching trajectories in the form of series of problems: A step for the theoretical reconstruction of the Hungarian Guided Discovery approach. In Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht, The Netherlands, 6–10 February 2019; Number 17.
14. Gosztonyi, K.; Varga, E. Teachers' practices and resources in the Hungarian "Guided Discovery" approach to teaching mathematics: Presenting and representing "series of problems". *ZDM Math. Educ.* **2023**, *55*, 641–656. [CrossRef]
15. Tamas, V. Mathematics education in Hungary today. *Educ. Stud. Math.* **1988**, *19*, 291–298. [CrossRef]
16. Gueudet, G.; Trouche, L. Teachers' Work with Resources: Documentational Geneses and Professional Geneses. In *From Text to "Lived" Resources*; Springer: Dordrecht, The Netherlands, 2011; pp. 23–41. [CrossRef]
17. Pepin, B.; Gueudet, G.; Trouche, L. Refining teacher design capacity: Mathematics teachers' interactions with digital curriculum resources. *ZDM* **2017**, *49*, 799–812. [CrossRef]
18. Brown, M.W. The teacher-tool relationship: Theorizing the design and use of curriculum materials. In *Mathematics Teachers at Work: Connecting Curriculum Materials and Classroom Instruction*; Remillard, J.T., Herbel-Eisenmann, B.A., Lloyd, G.M., Eds.; Routledge: New York, NY, USA, 2009; Chapter 2, pp. 17–36. [CrossRef]
19. Matusov, E.; Bell, N.; Rogoff, B. Situated Learning: Legitimate Peripheral Participation. JEAN Lave and ETIENNE Wenger. *Am. Ethnol.* **1994**, *21*, 918–919. [CrossRef]
20. Putnam, R.T.; Borko, H. What Do New Views of Knowledge and Thinking Have to Say About Research on Teacher Learning? *Educ. Res.* **2000**, *29*, 4–15. [CrossRef]
21. Nam, J.; Seung, E.; Go, M. The Effect of a Collaborative Mentoring Program on Beginning Science Teachers' Inquiry-based Teaching Practice. *Int. J. Sci. Educ.* **2013**, *35*, 815–836. [CrossRef]
22. Winsløw, C.; Bergsten, C.; Butlen, D.; David, M.; Gómez, P.; Grevholm, B.; Wood, T. First years of teaching. In *The Professional Education and Development of Teachers of Mathematics*; Springer: Boston, MA, USA, 2009; pp. 93–101.
23. Schön, D.A. *Educating the Reflective Practitioner*; Jossey-Bass: San Francisco, CA, USA, 1987.
24. Chapman, O. Educators reflecting on (researching) their own practice. In *The Professional Education and Development of Teachers of Mathematics: The 15th ICMI Study*; Springer: New York, NY, USA, 2009; pp. 121–126.
25. Matos, J.F.; Powell, A.; Sztajn, P.; Ejersbø, L.; Hovermill, J.; Matos, J.F. Mathematics Teachers' Professional Development: Processes of Learning in and from Practice. In *The Professional Education and Development of Teachers of Mathematics*; New ICMI Study Series; Springer: New York, NY, USA, 2009; pp. 167–183. [CrossRef]
26. Braun, V.; Clarke, V. Thematic analysis. In *APA Handbook of Research Methods in Psychology, Vol 2: Research Designs: Quantitative, Qualitative, Neuropsychological, and Biological*; American Psychological Association: Washington, DC, USA, 2012; pp. 57–71. [CrossRef]
27. Buchholtz, N. Planning and Conducting Mixed Methods Studies in Mathematics Educational Research. In *Compendium for Early Career Researchers in Mathematics Education*; Springer International Publishing: Berlin/Heidelberg, Germany, 2019; pp. 131–152. [CrossRef]
28. Sang, G. Teacher Agency. In *Encyclopedia of Teacher Education*; Springer: Singapore, 2019; pp. 1–5. [CrossRef]
29. Biesta, G.; Priestley, M.; Robinson, S. The role of beliefs in teacher agency. *Teach. Teach.* **2015**, *21*, 624–640. [CrossRef]
30. Boaler, J.; Greeno, J.G. Identity, agency, and knowing in mathematics worlds. In *Multiple Perspectives on Mathematics Teaching and Learning (International Perspectives on Mathematics Education, V. 1)*; Ablex Publishring: Westport, CT, USA, 2000; pp. 171–200.
31. Orland-barak, L. Introductory Chapter III: Learning Teacher Agency in Teacher Education. In *The SAGE Handbook of Research on Teacher Education*, 1st ed.; Jean Clandinin, D., Husu, J., Eds.; SAGE Publications Ltd.: London, UK, 2017; Volume 2, pp. 247–250.
32. Priestley, M.; Biesta, G.; Robinson, S. Teacher agency: What is it and why does it matter? In *Flip the System*; Routledge: London, UK, 2015; pp. 134–148. [CrossRef]
33. Hannula, M.S. Affect in Mathematics Education. In *Encyclopedia of Mathematics Education*; Springer International Publishing: Berlin/Heidelberg, Germany, 2020; pp. 32–36. [CrossRef]
34. Vygotsky, L.S. *Mind in Society: Development of Higher Psychological Processes*; Harvard University Press: Cambridge, MA, USA, 1980. [CrossRef]

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