

Article

“Precision”, “Approximation”, and Experimental Mathematics at School in the Age of Digital Educational Technology

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Abstract

This paper explores the interrelations between “precision”, “approximation”, and experimental mathematics supported by digital educational technology in the classroom. It examines the epistemic and pedagogical challenges and opportunities these links create in school mathematics and what is required of teachers to address them. Instances of “approximation” mathematics, emerging from experiments performed with digital tools, are analyzed and evaluated from the standpoint of “precise,” deductive mathematics. The study presents examples where “approximate” computations—or inferences based upon them—are incorrect from the standpoint of “precision” mathematics. It also shows how to correct or further investigate such discrepancies and how school mathematical practices would profit from such approaches. In addition, it discusses instances where experimental mathematics reveals issues typically outside the school curriculum but still conceptually accessible, yielding unexpected or unfamiliar results. The paper also considers recurrent sequences and their hands-on exploration in the context of the interrelation between the three realms of mathematics. The analysis of the examples through the prism of three realms of mathematics linked by digital technologies outlines the ensuing epistemic shift in the mathematics classroom. The teachers’ expectations in view of this shift are analyzed and presented.

Keywords: epistemic shift at school; error as a learning opportunity; weighing digital outputs; informal introduction of advanced mathematical concepts; guided inquiry based on experimental mathematics

1. Introduction

This paper analyzes the interplay between “precision”, “approximation” (Klein, 2016), and experimental mathematics within high-school education in the age of digital educational technology. It investigates the epistemic and pedagogical challenges and opportunities this interplay creates and what is required of teachers to address them effectively.

The terms “precision mathematics” (PM) and “approximation mathematics” (AM) come from the English translation of the third volume of Felix Klein’s *Elementary Mathematics from a Higher Standpoint* (Klein, 2016). Experimental mathematics (EM) refers to gaining mathematical insight through experimentation. It has always been one of mathematicians’ ways of arriving at conjectures prior to proving them.

Although these three facets of mathematics—precision, approximation, and experimentation—are deeply interconnected across the discipline, they are rarely addressed to-

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gether in school education (Devlin, 2019). This paper introduces a “mathematical triangle” (Figure 1) in which PM, AM, and EM are interconnected by digital technology (DT), arguing that this framework can expand the epistemic and conceptual boundaries of school mathematics and promote students’ epistemic development (Abramovich, 2025).

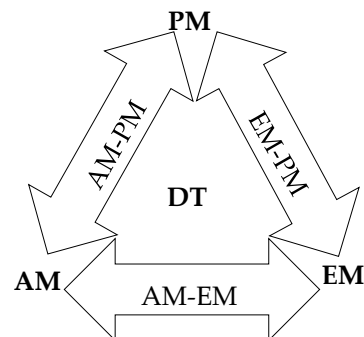


Figure 1. PM–AM–EM triangle with DT as a central link.

1.1. Digital Technology

Digital technology’s relations with mathematics has deep historical and philosophical roots (Hansson, 2020). Here, digital technology refers to mathematically robust interactive software enabling symbolic computation, dynamic visualization, and geometric construction, such as Desmos, GeoGebra, Wolfram Mathematica¹ (WM), and more.

Two key qualities define the software used for EM: **dynamicity** and **interactivity**. Dynamicity allows configurations to change while maintaining user-defined invariants. Observing variations under parameter changes enables a deeper analysis. For example, if two segments are defined as perpendicular, every triangle built on them as sides will be a right triangle. By allowing the lengths and orientations to vary dynamically, one obtains an infinite “example space” (Watson & Mason, 2005) of right triangles to explore. The distinction between dynamic and static platforms mirrors the difference between algebra and arithmetic—representing relations and structures rather than isolated instances. Dynamicity also extends the ideas of independent and dependent variables to independent and dependent *objects*. Apropos right-angle triangles thus constructed, their elements, i.e., angles, medians, length of the hypotenuse, etc.—are dependent objects. Interactivity implies an immediate response of the platform to the user’s actions, so that the user may learn from it and take further action according to the response (see, e.g., Naftaliev, 2018).

However, integrating digital technology into school mathematics also introduces challenges. Teachers must select appropriately among diverse affordances—deciding when technology should support open-ended exploration rather than rote “drill and practice” based on a right-or-wrong pedagogy (Abramovich, 2014). To draw meaningful conclusions from computer-generated results, one must discriminate between relevant and irrelevant, invariant and variable, and systematic and random patterns.

While end users can work effectively without knowing every underlying mechanism (few phone users understand how their phones operate), teachers benefit from understanding the logic behind the mathematical software—even if not its programming code. Dilling et al. (2024) propose the MPC-model (=Media, Pedagogy, and Content). In the present paper, the focus is on “math-specific software”, and on its opportunities and advantages for pedagogy and content compared to static teaching media.

GeoGebra serves as the principal platform discussed in this paper, occasionally supplemented by Wolfram Mathematica. Designed for education, GeoGebra has been widely embraced by teachers for its accessible syntax, dual algebraic–geometric representation,

intuitive interface, dynamic construction, multilingual adaptability, and retrievable construction steps (Sangwin, 2007; Hohenwarter et al., 2008).

Despite their capabilities, computers remain neither omniscient nor infallible. Borwein and Devlin (2008) warn against “misleading cases” and provide an assortment of examples convincing one that, however surprising, promising, and precise the outputs may seem, a pinch of doubt is desired in appraising their meaning and credibility. Several examples in this paper were intentionally or accidentally constructed to demonstrate such pitfalls—each analyzed for its mathematical and didactic implications. These examples will be analyzed from the point of view of credibility, counterposing conviction to formal reasoning and argumentation.

1.2. Experimental Mathematics (EM) and Digital Technology (DT)

EM includes identifying patterns, exploring the possible results to determine whether they merit formal proof, and gathering evidence for or against conjectures (Borwein & Devlin, 2008; De Villiers, 2010; Borwein, 2012, 2016; Devlin, 2019). Modern computers have greatly expanded this mode of mathematical work. As early as 1995, Epstein and Levy observed that “the introduction of the computer into mathematics ... has been quite revolutionary, profoundly affecting the basic philosophy of professional mathematicians” (Epstein & Levy, 1995, p. 671).

DT-supported EM makes it possible to collect computational evidence for or against mathematical assertions. De Villiers (2010) emphasizes that conviction often precedes proof: “If uncertain about a result, one would rather look for a counterexample than for a proof.” Devlin (2019) asserts that modern EM, in contrast to the classical conception and practice of mathematics, may appear “as a significant part of the mathematical enterprise in its own right” (cf. Hansson, 2020).

The inductive side of mathematical discovery is usually hidden from the learner’s eye in books and papers that present only the final appearances of theorems and proofs (Devlin, 2019). EM restores the connection between inductive exploration and deductive verification, creating powerful opportunities for developing mathematical understanding through digital technology.

1.3. Precision Mathematics and Approximation Mathematics

Klein (2016) delineated two realms of mathematics, “precision” (axiomatic, deductive, and abstract), and “approximation” (empirical and applied) mathematics, as complementary parts of a unified whole: “We get the whole science only when we embrace the two parts” (p. 7). The line separating them is “a threshold of precision”, which distinguishes the abstract mathematical objects, such as number, line, and function, from their physical embodiments, inevitable in all practical contexts, including teaching.

An abstract point is infinitely precise; it has no dimensions. Even viewed under magnification, it does not get magnified. But, as with any ideal, it is physically unattainable. Unlike an abstract point, a “practical” point, however precisely generated, always has an extent. This equally applies to a point on the number line representing a real number, or to a geometric point and, therefore, line, which entails issues with the precision of geometric drawing and constructions, drawing graphs of functions, etc.

Writing in the early 20th century, Klein could not foresee the computational precision of modern digital tools. Nevertheless, different applications and digital platforms choose different default thresholds of precision. For instance, GeoGebra defaults to two decimal places ($\sqrt{2} \approx 1.41$), which is often sufficient for basic tasks but can lead to epistemic contradictions if not managed by educated use, though precision can be extended to fifteen places, while WM allows one to preset the precision levels. To improve or not to improve

is a matter of educated choice, when a user discerns that, for some reason, the default threshold is insufficient.

Though the title of Klein's book entails a discussion on elementary mathematics, most topics and examples are hardly of school mathematics level. Herein, we will consider examples that are closer to school mathematics, and analyze their mathematical validity, didactic relevance, and epistemic value. We will be interested in looking more deeply into situations when "approximate" computation brings about dubious but correctable results; when the results are unacceptable from the PM point of view; and when the results are unexpected and different from the precise setting.

1.4. Teachers Facing DT in the Mathematics Classroom—Enter the "Mathematical Triangle"

The increasing appearance of technological tools in mathematics education demands a redefined view of teacher competence needed to cope with the teaching and learning modes based on these tools. Building on Shulman's (1986) Pedagogical Content Knowledge and Loewenberg Ball et al.'s (2008) Mathematical Knowledge for Teaching, Niess (2005) suggested the technology-enhanced PCK (TPCK) model of integrating technology with mathematics (and sciences) teaching and learning for pre-service teachers, which later evolved into the Technological Pedagogical Content Knowledge (TPACK) model (see, e.g., Mishra & Koehler, 2006). Geraniou et al. (2024) later defined *mathematical digital competence* for teaching, emphasizing the reflective, context-aware use of digital tools (cf. Dilling et al., 2024; Pedersen et al., 2021 for the citation of the Danish KOM "flower" of competencies). Barabash (2025) extended these characteristics to incorporate EM in "doing mathematics" and added the component of critical thinking (CT) necessary in engaging with EM.

Referring to DT-supported EM at school, Kortenkamp (2004) indicates the requirement of teachers' deep (mathematical) knowledge in "the a priori work (designing the experiment) and the a posteriori work (deducing the results)" (p. 62). Weighing the AM–EM results by PM considerations extends "doing mathematics" to encompass the whole "mathematical triangle" and, therefore, extends the expectations of teachers' digital mathematical competences.

1.5. Epistemic Development of Students

Devlin (2019) announced a "revolution in what it means doing mathematics today" as a shift from execution to understanding made possible by DT use (cf. also De Villiers, 2010). This has implications for what should be learned at school, and how, aiming at the epistemic development of students and at the emergence of their new mathematical knowledge (Abramovich, 2025). When appropriately guided, DT-based experiments may cultivate students' critical thinking in appreciating the results and their ability and willingness to learn from experiences. "Exercising in mathematics" thus evolves from routine calculation into a process of reasoning and inference, supporting students' conceptual and intellectual growth.

2. Materials and Methods

The paper presents eight examples that illustrate interactions among the three mathematical domains (PM, AM, and EM) mediated by digital technology. Each episode—whether arising from real classroom experience or constructed intentionally for analysis—demonstrates the interplay of precision, approximation, and experimentation. Together, they form the empirical foundation for a proposed model of balanced mathematics teaching: the PM–AM–EM triangle centered on DT.

Example 1. Are the lines parallel?²

The configuration shown in Figure 2 was produced by an applet designed to demonstrate that equality of slopes indicates parallelism of lines. The situation arose unintentionally during experimentation:

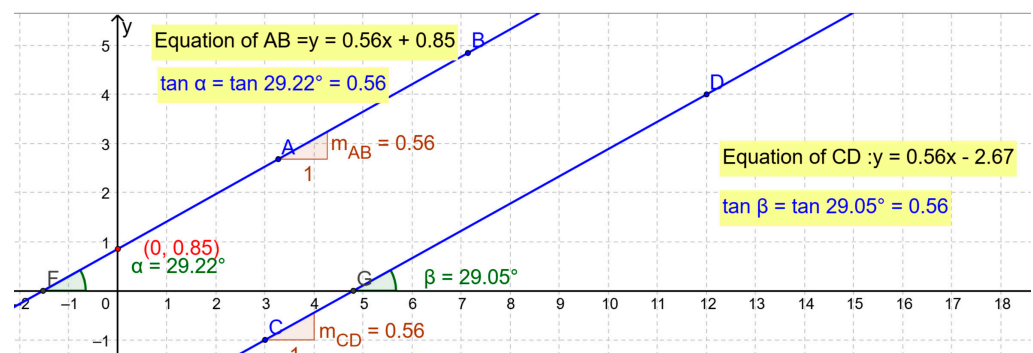


Figure 2. GeoGebra applet calling for improvement in precision, originally intended to demonstrate that parallel lines have equal slopes, or “gradients”.

Approaching this situation through PM lens, it does not make sense; the default two-decimal places precision is insufficient for angles or their tangents or both. In Figure 2, $\tan 29.22^\circ \approx 0.559$; $\tan 29.05^\circ \approx 0.555$. Rounded to two decimals, both become 0.56, creating the illusion of equality.

From the AM viewpoint, this is a good opportunity to look more deeply into the meaning of precision. Questions like “Are the lines *practically* parallel?”, or “Does an imperceptible angle difference matter?” should be attached to the problem being solved. The difference between the angles of the lines with the x-axis in the print-screen is 0.17° . In a bounded static sketch, the difference is imperceptible. But, if, for example, the context is astronomical calculations, it is known that the unit of 1° appeared to be too crude for astronomic distances already in ancient astronomy, which caused introduction of a smaller unit of $1' = \frac{1}{60} 1^\circ \approx 0.17^\circ$. This was only the first (“prima”) subdivision of degree; apparently, $1'$ was still too crude, leading to the second (“secunda”) subdivision, $1'' = \frac{1}{60} 1'$. In the long run, this too appeared insufficiently fine a unit. On down-to-earth scale, if the angle between rails on railway track were only $0.17^\circ \approx 0.003$ rad, along 1000 m of the track, its width would alter by about 3 m—a catastrophic result. Thus, whether 0.17° is significant depends entirely on situation and scale. Another appearance of scaling refers to units of angle measurements by the platform: should the angles be measured in radians, within two-digits precision, both would render 0.51—thus, no difference between them would be perceptible, whether the angles are equal or not. The angle of 0.17° on the GeoGebra screen is visually perceptible due to dynamic properties of the screen enabling zooming out to watch it; zooming out of only one axis deforms the angle and provides unreliable image, highlighting how scale and precision interact (<https://www.geogebra.org/m/yt25bbtw> (accessed on 14 May 2026)). The differences in discerning the angle of 0.17° in print-screen vs. railways or astronomy bring into consideration the idea of scales and units of measurement and, thus, connects precision mathematics with applied reasoning.

Example 2. Visual vs. numeric.

The following case emerged unexpectedly during a lesson on parametric representation of a straight line. The goal was to link the analytic and graphical representations of Point E, allegedly belonging to line f:

$$x = (0, 4) + \lambda(2, 5).$$

Point E was visually placed on f , to inspect its coordinates in the Algebra window (Figure 3a). The expected value for E was (1, 6.5). Surprisingly, the GeoGebra output was (1, 6.51) (Figure 3b). Thus, either GeoGebra located Point E as a point on f with $x(E) \approx 1.004$, which equals 1 within two-digits precision, or it identified its coordinates as (1, 6.51), indicating that E does not belong to f .

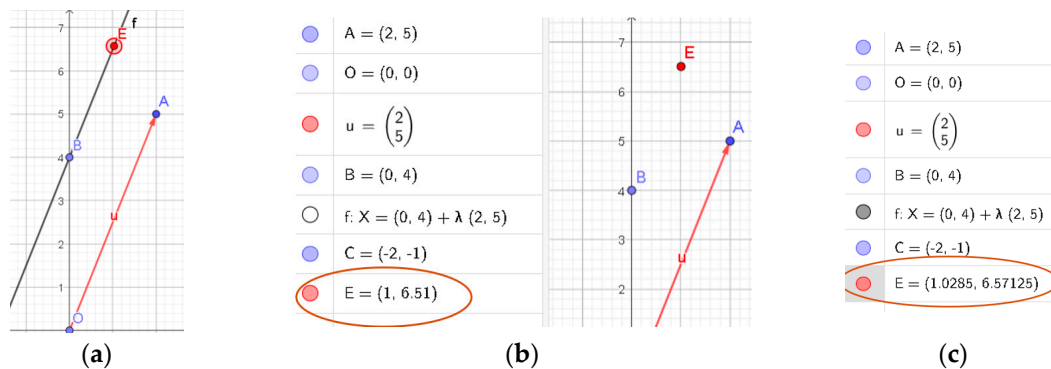


Figure 3. Does Point E belong to line f or does it not? (a) Point E selected on line $f: x = (0, 4) + \lambda(2, 5)$. (b) Algebra window with coordinates of E. The line f is hidden. (c) Values for E with Rounding of 5 decimal places.

The re-defined Rounding from 2 to 5 decimal places yielded (1.0285, 6.57125) (Figure 3c). Thus, within this precision, GeoGebra discerns that Point E belongs to the line. The 5-place rounding does not fit the initial coordinates of E (Figure 3b): rounding them back to 2 decimal places renders (1.03, 6.57). This discrepancy arises because visual placement of a point and its numeric computation both depend on the chosen precision. GeoGebra's default point size roughly matches its grid cell, so a slight misplacement can shift results at the third or fourth decimal place. Consequently, a higher precision alters both coordinates and output values, effectively "updating" the data.

Example 3. *Graphic solutions and explorations: is "what you see" really "what you have"?*

Zoom in: graphic methods for solving equations and systems are a powerful mathematical tool—especially when symbolic or numeric solutions are unavailable or difficult to obtain. In all these cases, the precision of an approximate solution is defined by the needs of the user and affordances of a platform.

To teach this method, a simple didactic example can be used where the intersection has integer coordinates easily spotted from grid lines—for instance: $x + 2y = 5$, $3x - 4y = 5$. But, if the students are actually to use this method, they are prone to facing configurations in which coordinates of intersection points are not integers, but finite decimal values, as shown by the intersection of lines i and h in Figure 4:

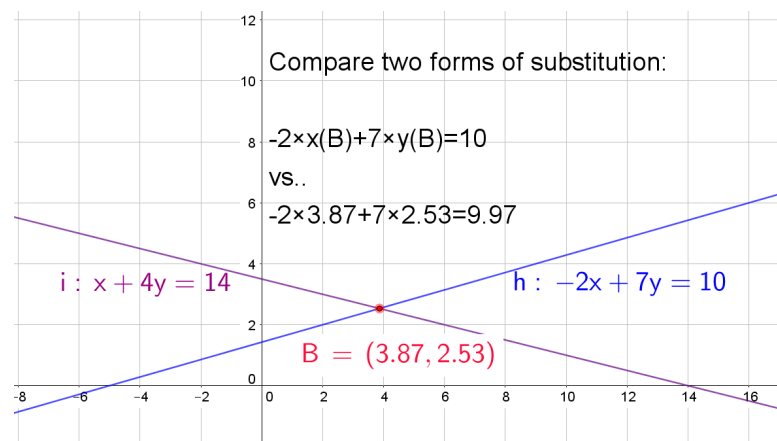


Figure 4. Can a graphical solution be exact?

The exact solution $(\frac{58}{15}, \frac{38}{15})$ in decimal form renders decimals³ of infinite length. That the solutions are rational in this case follows from the equations, supposing that the coefficients are exact integers and not approximations, but, visually, only a decimal approximation can be read from the plot.

Substituting it back into equation is expected to satisfy it within precision afforded by the tool or chosen by the user; see Figure 4.

If the solution is sought not as a mere exercise but as a step in a more complicated procedure, the user is to account for error propagation and consider improving the approximation prior to applying the result.

An important feature of the platform is the difference in substitution of “exact” value of coordinates of solution point B symbolically, $(x(B), y(B))$, vs. numerically, (3.87, 2.53); see Figure 4. The first computation, symbolic, yields equality; the second, numeric, reflects finite precision. This provides a glimpse of the way the platform operates with data, indicating the essential difference between symbolic and numerical computations afforded by the platform. The “exactness” means that the platform can compute the coordinates in real time with any precision dictated by the procedure.

A complementary side of digital graphs is to explore the existence of solutions, their number, and (at least approximate) location. Global and local behaviors of functions can be dynamically investigated through zooming and rescaling (<https://www.geogebra.org/m/hadrd26z> (accessed on 14 May 2026)). In such investigations, the interlink between EM and AM-PM is inseparable. Nevertheless, “what you see” on screen may be misleading: visual appearances of roots, inflection points, asymptotes, etc. in the same coordinate plane are prone to “concealing” valuable information. Therefore, they are to be cross-checked with analytic investigation or numeric evaluation of higher precision. Special challenge is behavior “at infinity”, i.e., for large values of variables and local behavior around closely located roots. GeoGebra’s ability to stretch or shrink axes independently allows closer inspection of such cases, for example, distinguishing between two nearby roots and one double root (Figure 5b). However, manipulating scales distorts slopes and relative shapes, and portions of graphs can disappear altogether. Therefore, graphical appearances must always be checked against analytic or higher-precision numeric methods.

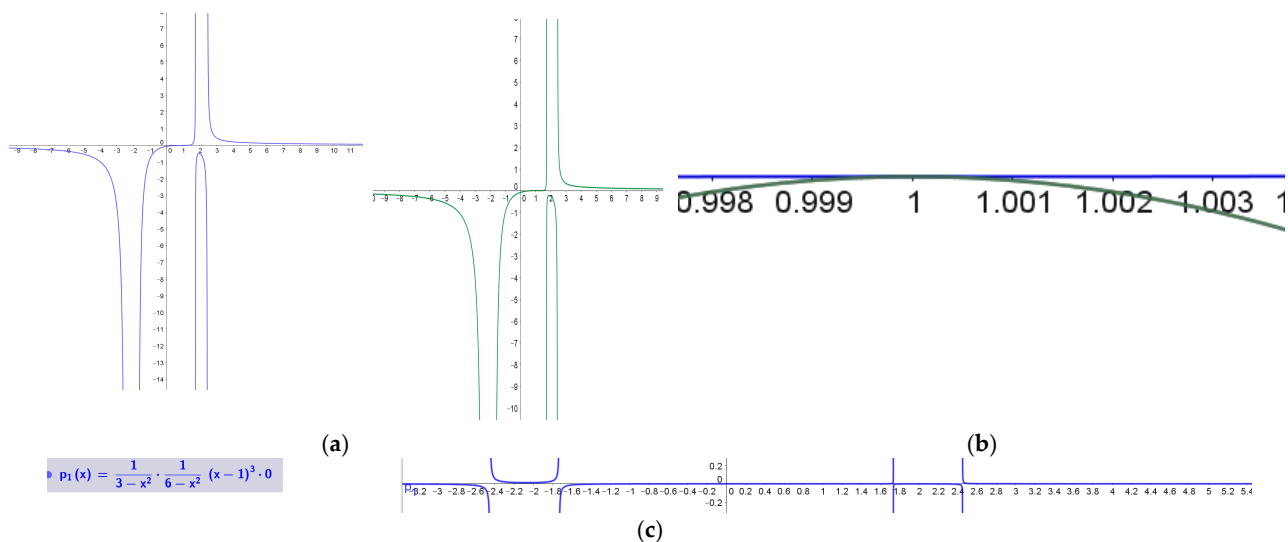


Figure 5. Observing local and global behavior of functions depends on scales on axes (<https://www.geogebra.org/m/hadrd26z> (accessed on 14 May 2026)). (a) At first glance, green and blue graphs of functions seem similar—are they? How many roots appear between 0 and 2? What happens near -4 to -1 ? What happens between -1 and -4 ? (b) Zooming reveals differences in local behavior around $x = 1$; note the changed x-axis scale. (c) Algebra window vs. the graph of function $p_1(x) = 0.001 \frac{1}{3-x^2} \frac{1}{6-x^2} (x-1)^3$.

Figure 5a presents graphs of two functions as they look on the GeoGebra screen. It takes PM-considerations to understand that something is amiss and to look for the missing parts between $(-4, -1)$. Furthermore, the “Zoom-axis” tool (Figure 5b) reveals that digital graphs can “conceal” information. Two closely located roots of one of the functions look like a single double root of the other until the x-axis is stretched. This teaches students that “seeing the whole picture” on a screen is an illusion; investigation must be supported by further AM or PM considerations. As suggested by one of the reviewers of this paper, “it could be accompanied by posing the problem of the actual approximate computation of a root or roots, which is related to algorithms for replacing one function (polynomial) with another that has the same roots but without multiplicities. This would lead to the elimination of flattening, and scaling could then refine the value of the root(s).”

A notable phenomenon occurs if one multiplies a function by a small number to reduce its values to inspect the graph for large values of y , e.g., by entering “ $h(x) \times 0.001$ ” and keeping the axes scales unchanged. Within the default precision, the result in Algebra window is $\frac{1}{3-x^2} \frac{1}{6-x^2} (x-1)^3 \times 0$, while the plotted graph of function differs from zero (Figure 5c).

Such contradictory appearances of outputs rooted in default precision are in the basis of the following examples, as well.

Zooming out may create an illusion that one sees the whole picture, which is not necessarily true. Notwithstanding the fact that DT enables close examination, more local or more global, it can never exhaust the whole range if it is infinite. Thus, a part of graph “too high” between -1 and -4 is missing in both functions.

The implicit function $s(\alpha)$ presented by the equation $\frac{\sin(\alpha+s)}{\sin \alpha} = \frac{\cos \frac{\alpha+s}{2}}{\cos \frac{3\alpha+s}{2}}$ (to be considered further on, see Example 6) is another valuable example of partial information provided by limited DT presentation. The “real” graph of function is totally different from its local picture near $\alpha = 0$, though the local information was relevant for the purpose of the problem considered.

Visual information and numeric data must be interpreted critically. DT tools expand perception but also expose users to subtle computational artifacts—valuable material for classroom discussion on *scale, rounding, and error tolerance*.

2.1. Geometric Constructions: The Ideal vs. The Digital

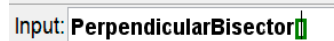
This section examines how digital geometry environments embody “precise” mathematical logic, yet remain subject to computational and visual limitations. Awareness of this dual nature is vital for teachers who use such tools in the classroom, creating a fertile ground for critical thinking.

“... abstract geometry starts by putting up front, in the form of axioms, those statements that are given in an approximate formulation in practical geometry. ... Relations, which are, in practice, only approximately correct, are postulated as exact...” (Klein, 2016, p. 8). This encapsulates the long-term shift from empirical observation to deductive abstraction. Not every “approximately correct” construction, however, leads to a valid relation in “precise” geometry.

Dynamic geometric tools serve both practical and “precise” geometric purposes. Understanding the interplay between them is no trivial matter. Compass and straightedge, with which it is impossible to trisect an angle, are physically non-existent ideal tools belonging to the realm of PM. Their earthly embodiments, even if sharp and straight, are crude enough to encourage one to try the “impossible” and produce seemingly credible results.

Geometric tools in GeoGebra enable implementation of Euclidean constructions with straightedge and compass at two levels:

- Procedures “mimicking” geometric construction, for example, drawing two congruent intersecting circles to find a perpendicular to a line passing through their centers;
- Command-level operations or input-line commands, e.g.,  Perpendicular Bisector



, as a ready-to-use Euclidean tool, in full conformity with the logic of mathematics: once a problem is solved, one may use the solution.

Thus considered, these tools enact PM in a computerized platform. But its environment also includes non-PM behaviors. For instance, while one cannot solve PM-unsolvable problems using Euclidean steps, a user can compute $\sqrt[3]{2}$ to any number of digits, effectively “solving” the classic *doubling the cube* problem. Therefore, awareness of the contrast between what is geometrically possible and what is computationally feasible calls for critical reflection—an ideal context for cultivating students’ analytical thinking.

The following examples analyze situations where geometric constructions performed with insufficient precision lead to seemingly correct, yet mathematically invalid results. While improving precision might seem to be the obvious resolution, analyzing these failures is valuable for three reasons:

- Skepticism: As Borwein and Devlin (2008) warn, we must not blindly accept seemingly precise outputs.
- Intellectual Honesty: As De Villiers (2010) notes, it is “intellectually dishonest to pretend in the classroom that conviction only comes from deductive reasoning; ... even “adult mathematicians” experimentally investigate already-proved results”. Conviction often comes from experimental investigation rather than just deductive reasoning.
- Discovery: Erroneous results, when carefully analyzed, may uncover deeper conceptual insights.

Example 4. Misleading “example spaces” (Watson & Mason, 2005).

A dynamic applet (<https://www.geogebra.org/m/gxhmn8bp> (accessed on 14 May 2026)) was created to demonstrate a triangle in which angle bisector and median coincide, while the height from the same vertex lies outside the triangle (Figure 6a).

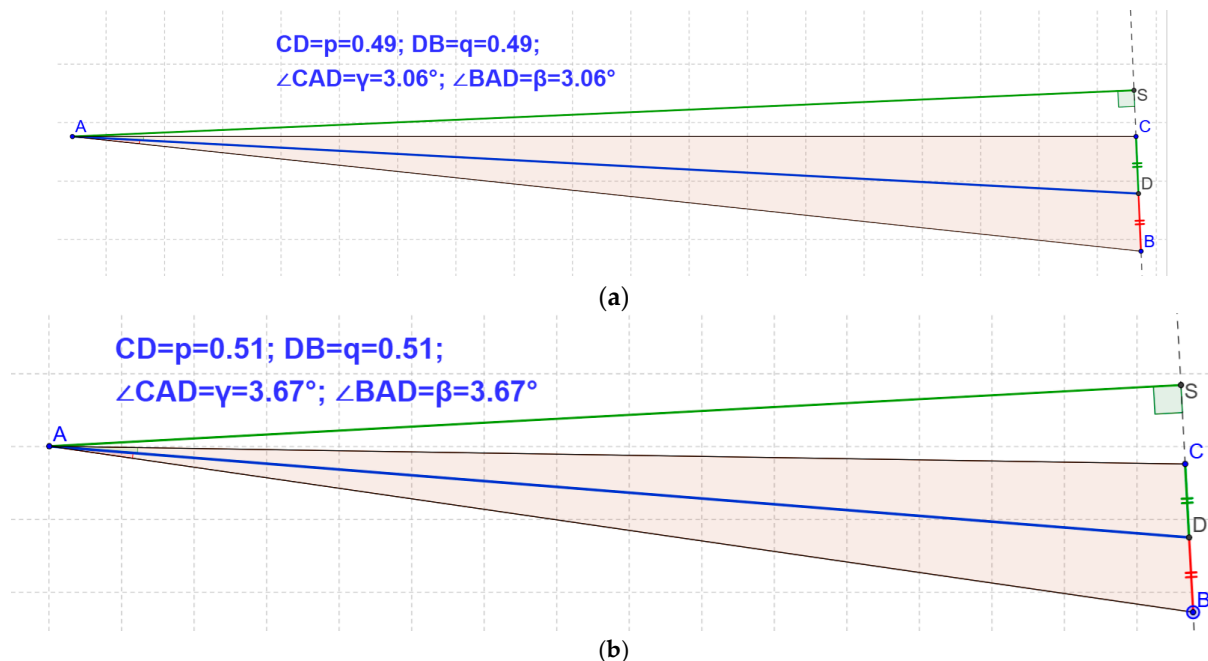


Figure 6. (a) The median and angle bisector from vertex A coincide, while the height AS is external to the triangle. (b) Dragging vertices within certain limits produces more “triangles” with the same apparent properties; here is one more.

This applet was constructed to highlight the challenges that the visual DT-supported outputs present to our perception, judgement, and critical thinking. GeoGebra tools enabled visual “support” to an explicitly deceptive configuration: it is possible to denote any two segments as equal by applying LineDecorations tool from the Graphics menu. This is how CD and DB segments were denoted as “equal”. The angles were plotted as equal—otherwise, it is impossible to denote them as such, unlike segments. The dynamic texts above the drawing employ the platform feature and provide the angle and segment magnitudes within defined precision, with two-digit precision being the default applied here.

The applet is dynamic; dragging the vertices of the triangle within a certain range of values produces more triangles with the same property (Figure 6b). Thus, an utterly misleading “example space” (Watson & Mason, 2005) of deceptive artifacts is formed, contradicting formal geometric inferences. According to formal geometry, such triangles cannot exist. The configuration contradicts the theorem that concurrence of the median and bisector implies the triangle is isosceles and the height from the same vertex coincides with them.

This applet looks weird and implausible to a mathematically trained observer, but it may look entirely reasonable to uncritical users—illustrating how digital precision can lend false conviction that triangles exist for which such location of three special segments is possible.

Didactically, those “impossible triangles” provide perfect material for discussing proof by contradiction, logical consistency, and the limits of visualization. Teachers can guide students to question even high-quality digital diagrams, showing that correctness ultimately lies with PM reasoning, not visual presentation, persuading as it may seem.

2.1.1. Angle Trisection—A Never-Ending Story

Angle trisection epitomizes the tension between precise and approximate reasoning. The “exact” geometry of the problem posing is associated with abstract algebra of the proof of impossibility of solution. The solution which states that there is no solution, is being tested once and again by various people and methods; as De Corte (2013) put it, “we felt the urge to try this”. Some solutions by straightedge and compass are declared as approximate and are supplemented by error evaluation; see, e.g., De Corte (2013) and Ben-Ari (2022) for discussion on approximate straightedge and compass trisections. Other, exact solutions apply different tools, such as origami or Archimedean spiral, which renders the problem to be a different one from the original, for the problem is formulated tool-wise: it is straightedge and compass trisection that is impossible. Thus, Huzita–Hatori axioms for origami geometry are different from the Euclidean axiomatics (see, e.g., Huzita, 1989). Angle trisection by paper folding is compatible with this axiomatic system. See also Shima, 2013 for algebraic corroboration of this method using basic trigonometry and solution of the cubic equation. As for the trisection using the Archimedean spiral—see, e.g., the applet in <https://www.geogebra.org/m/brGBZ44Q> (accessed on 19 July 2026). which also contains the proof of the construction accessible to anyone familiar with polar coordinates in plane.

Example 5. *Angle trisection—an urge to try.*

The trisection method shown in a YouTube video (<https://www.youtube.com/watch?v=OkzA-do1PV8>, accessed on 19 July 2026) was reconstructed in GeoGebra to estimate its relative error (Figures 7 and 8).

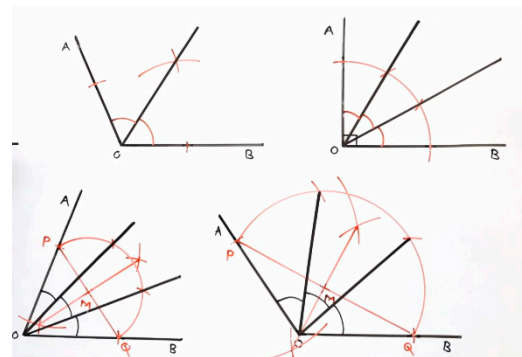


Figure 7. The trisection proposed in <https://www.youtube.com/watch?v=OkzA-do1PV8> (accessed on 19 July 2026).

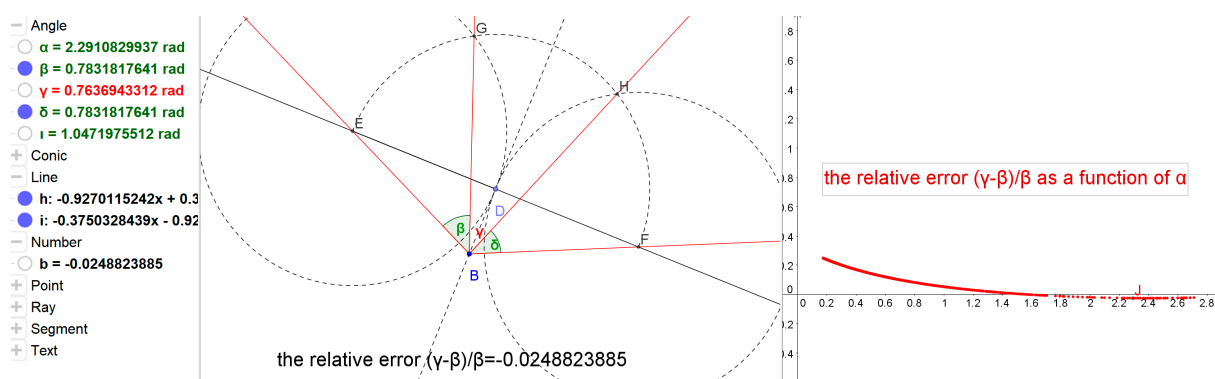


Figure 8. “YouTube”-inspired trisection implemented in GeoGebra, showing relative error as a function of α —the angle being trisected.

The resulting error graph (Figure 8) shows that the difference between the middle angle (i.e., the angle between the trisecting segments) and a side angle (an angle formed by a trisecting segment and the adjacent side of the triangle) decreases as α increases, becoming nearly negligible for obtuse angles. For small angles, however, the error is visibly larger. Around $\alpha \approx 1.5$ rad, the mid-angle becomes slightly smaller than the side angles, implying a crossing point where equality holds exactly. This does not contradict the general impossibility theorem—approximate or even exact equality can always occur for specific magnitudes.

Visually, for large α , the differences between the three “equal” parts become imperceptible, giving users a false impression of precision. Nevertheless, analyzing the relative error helps transform this illusion into a learning opportunity. Students encounter an *approximate but correct approach*: solutions acceptable within tolerance that remain fully consistent with theoretical impossibility.

This analysis is valid for this construction and may vary for other methods leading to approximate solutions. Construction like this and other ones (e.g., <https://www.youtube.com/watch?v=FuRWd6a5RIU> (accessed on 19 July 2026) “very close to ideal solution”), as long as they are declared as approximate solutions, should be considered “approximate and correct”: they do not contradict any PM-originating impossibility, and actually belong to the PM realm, notwithstanding their approximate character.

Example 6. *Angle trisection through the Einstellung prism.*

The following construction created as a GeoGebra experiment represents an attempt to perform the angle trisection by trisecting the opposite base of equilateral triangle, in the spirit of the Einstellung effect—the tendency to apply familiar strategies to new problems even when inappropriate (Abramovich, 2025). Here, the familiar idea of “similar bisecting” was applied incorrectly, yet appeared to “work.”

Though, by “precise” mathematics, this idea is decidedly wrong, an experiment performed by dynamic application shows “successful” results (Figure 9, <https://www.geogebra.org/m/chzbp9rc> (accessed on 14 May 2026)). The applet is dynamic, and, like Example 4, dragging the vertices within certain limits does not violate the result, thus demonstrating another occurrence of misleading example space. However, extending the movement beyond certain limits invalidates the construction, hinting at underlying inconsistency. When angles are measured in degrees, two-digit rounding masks the inequality among the three angles (Figure 9a). Using radians with three-digit rounding conceals the same issue (Figure 9b). Further rounding—with two digits in radians—extends the apparent validity range even more (Figure 9c).

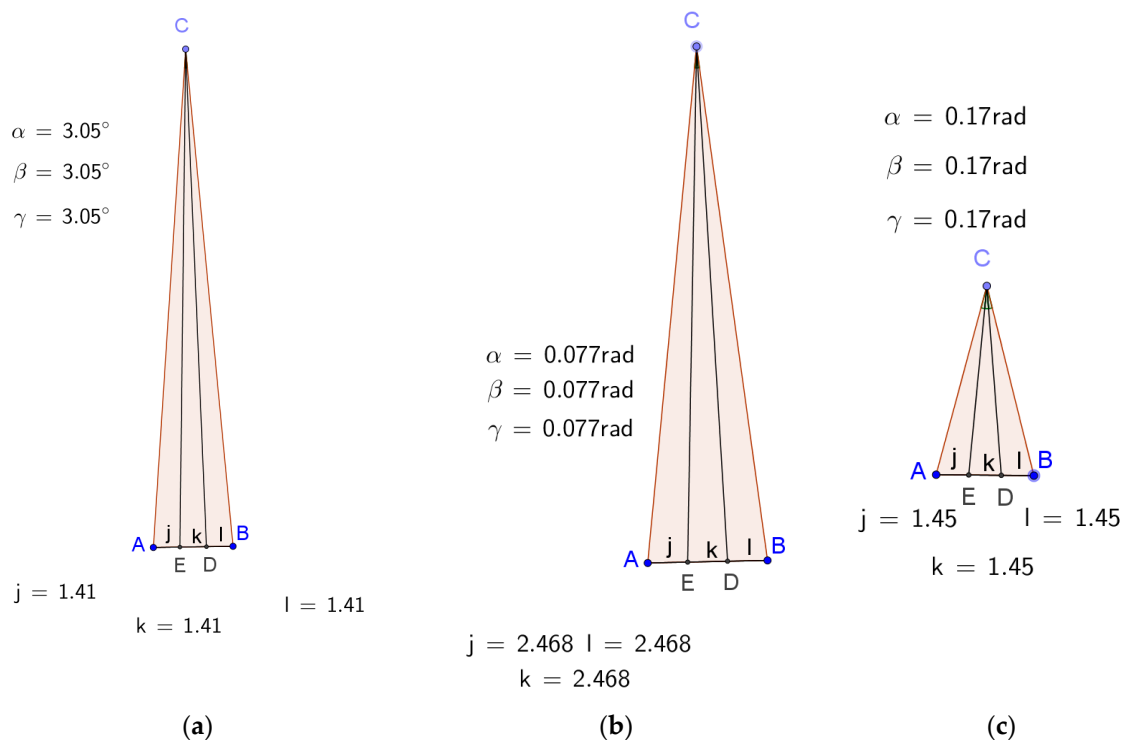


Figure 9. Einstellung-effect-motivated “trisection”. $\alpha = \angle ACE$, $\beta = \angle ECD$, $\gamma = \angle DCB$. (a) Angles in degrees, 2-digit rounding—differences invisible. (b) Angles in radians, 3-digit rounding—still indistinguishable. (c) Angles in radians, 2-digit rounding—appears valid for a wider range.

A way to confront the results of experiments is to improve the precision right away. More precise angle measurement would indicate that not all three angles equal each other, though three-digits improvement (Figure 9b) appears to be insufficient. To act this way, one should probably know in advance that such construction contradicts a theorem from PM⁴. A valuable alternative is to analyze why the apparent trisection works under coarse measurement but fails under scrutiny.

The side triangles ACE and DCB are congruent by the Side-Angle-Side theorem, readily available at the secondary-school level. To assess the difference s between the middle angle and a side angle α (Figure 10), we assume the base is perfectly trisected. Using basic trigonometric identities, one obtains the relation (see Appendix A.1 for the detailed derivation):

$$\frac{\sin(\alpha+s)}{\sin \alpha} = \frac{\cos \frac{\alpha+s}{2}}{\cos \frac{3\alpha+s}{2}} \quad (1)$$

This equation is unsolvable explicitly, but the GeoGebra Implicit Curve tool plots $s(\alpha)$ as a solution to it and enables exploring its values:

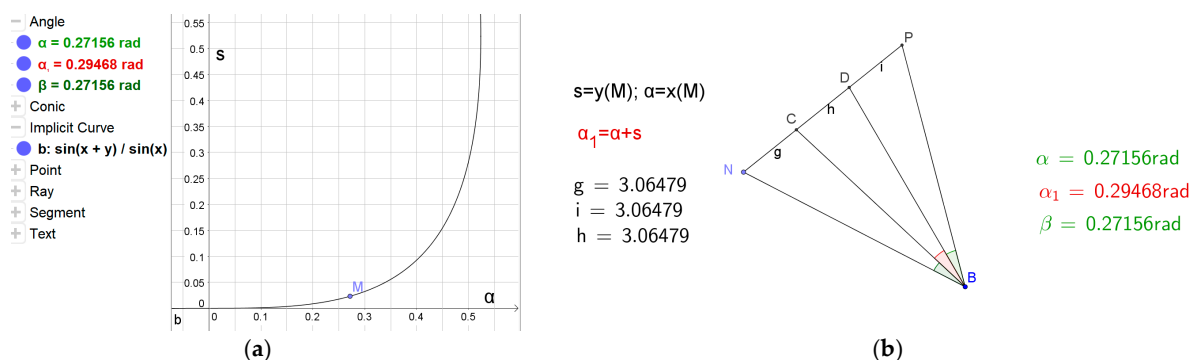


Figure 10. s is the difference between α_1 (the middle angle) and α (the side angle) for construction by base trisection. (a) Graph of s as a function of α (point M represents (α, s)). (b) Base segments and respective opposite angles in an equilateral triangle ($PB = BN$).

This construction has several implications:

- Unlike the construction in Example 5, this construction is more precise for small angles being trisected: s ascends steeply to the right of $\alpha \approx 0.1$ rad, that is, for “trisected” angle bigger than 0.3 rad (Angle B, Figure 10b). This is consistent with values of the “trisected” angle in Figure 9c (Angle C); thus, it can be extended to approximately 0.51 rad without “spoiling” the “example”.
- The values of s are positive for all values of α ; hence, “If an equilateral triangle’s base is trisected, the middle angle between trisecting segments is always bigger than the side angles”. This illustrates how experimental explorations of invalid construction can suggest valid theorems. The proof is in Appendix A.2. It uses basic school geometry and trigonometry.
- Both this and Example 4 show that extremely “long and narrow” triangles produce deceptively convincing visuals: relative errors of about 1% fall below visual and round-off detection. Example 4 employs the angle bisector theorem for triangle: $CA:BA = CD:CB$ (Figure 6). The values taken from the applet are⁵ as follows: $CA = 9.12$, $BA = 9.22$, $BC = 0.98$. The relative difference between CA and BA is roughly 1%. BC is divided in the ratio of $CA:BA$; thus, $CD = 0.4873 \approx 0.49$; $CB = 0.4926 \approx 0.49$. The difference of 1% between these sub-segments is indistinguishable at two-digit precision, none the more visually, especially when they are viciously denoted as equal. Similarly, the error graph in Figure 10a indicates low deviations for small α , explaining why “long” triangles are more deceptive and serve well for such examples.

Example 7. *This multifaceted deceptive parabola.*

Klein (2016) differentiated between “precise” mathematical functions and *empirical curves*—the latter “drawn” by not ideal dimensionless points but rather by “physical” points of finite size. This means that the value of such function can be defined within a certain threshold of precision. Klein defines an empirical curve as “a function stripe” (p. 17). A visual example of such stripe is a Confidence Band around a regression curve obtained in polynomial regression (green curve, Figure 11); the red curves form a stripe, the Confidence Band, while the “true” curve (the blue one) and its estimate lie within the strip.

Klein’s stripe around a curve can be treated as its ε -neighborhood⁶. We will focus on the ε -neighborhood of the parabola $y = x^2$, using it to investigate the differences between this *empirical stripe* and the ideal parabola itself. Before doing this, we revisit the parabola as a *locus* and as an *envelope* as a gateway to further experimental explorations by DT.

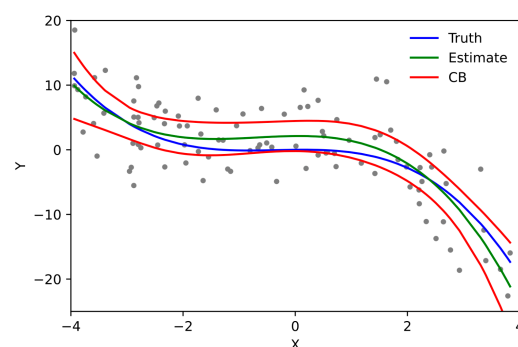


Figure 11. From https://en.wikipedia.org/wiki/Ordinary_least_squares#/media/File:Polyreg_scheffe.svg (accessed on 19 July 2026).

By geometric definition, a parabola is the locus of points equidistant from a focus (F) and a directrix (AB) (Figure 12). A GeoGebra applet in <https://www.geogebra.org/m/bCtCtGCM9> (accessed on 19 July 2026) employs this definition using Envelope GeoGebra tool. Here, the *Envelope* tool in GeoGebra traces the set of all perpendicular bisectors of FP as P moves along AB—so that each bisector is tangent to the parabola at one point. While “envelope” is an advanced concept, its intuitive link with tangency is accessible at the secondary level.

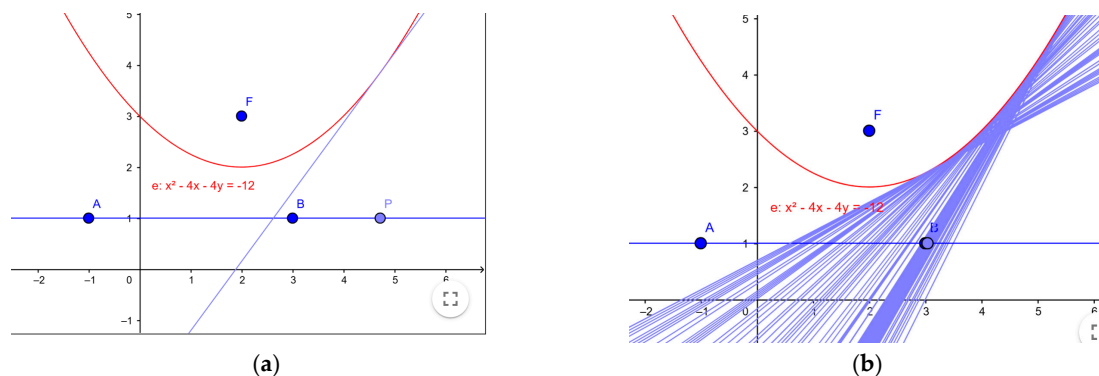


Figure 12. From <https://www.geogebra.org/m/bCtCtGCM9> (accessed on 19 July 2026). (a) Construction of a parabola, given its Focus F and directrix AB; P is a moving point on AB. (b) Parabola as an envelope of perpendicular bisectors of FP, appearing as blue lines.

The formal proof that “a parabola is tangent to all such perpendicular bisectors” requires only high-school geometry (Appendix A.3), providing an instance of an activity leading to new knowledge.

Investigating the ε -Neighborhood of $y = x^2$

To visualize an ε -neighborhood of parabola $y = x^2$ and to investigate it, we turn to EM, drawing a circle of radius ε around an arbitrary point A(x, x^2) on $y = x^2$, and dragging the center along the parabola. The ε -neighborhood is bounded by traces of points B and C on the circle farthest from the parabola—these are endpoints of diameter perpendicular to the tangent of the parabola (Figure 13a). By changing the magnitude of ε , one obtains stripes of various width around the parabola.

An investigation of an envelope of the parabola is included in Dana-Picard and Kovács, 2023 as an example of a wider issue of merged application of PM with CAS (Computer Algebra System) and DGS (Dynamic Geometry System), in particular, with GeoGebra, to the study of envelopes. This paper focuses on systematic mathematical corroboration of tools employed by both kinds of DT, and on fruitful mutual support of these types of platforms. However, in the context of the present paper, the emphasis is on investigating the implications of Klein’s idea of $f(x) \pm \varepsilon$ stripe at a more intuitive basis, and on geometric inferences based on such EM-DT investigation which lead to high-school level argumentation. Similarly to (Dana-Picard & Kovács, 2023), different DT platforms are employed when needed, as is seen in what follows.

After acquaintance (albeit informal) with notion of envelope, it is plausible to suggest that points B and C (Figure 13a) draw envelopes of the circles (Figure 13b).

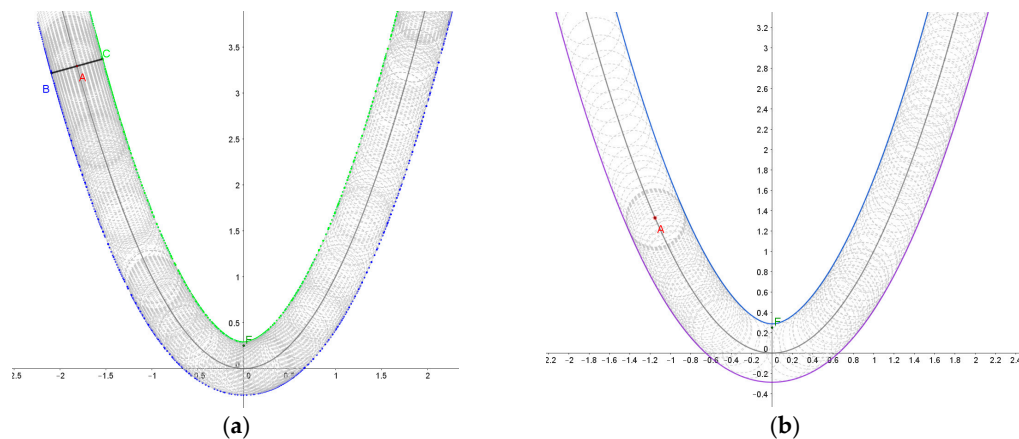


Figure 13. The stripe: ϵ -neighborhood around $y = x^2$ formed by dragging Center A of circle of radius ϵ along the parabola. Slider for ϵ changes the width of the stripe. <https://www.geogebra.org/m/vkbhkq3c> (accessed on 14 May 2026). (a) Traces of endpoints B and C of the diameter draw the boundaries of the ϵ -neighborhood of the parabola. (b) The envelopes and the circles tangent to them.

Visually, at least for small values of ϵ , the boundaries appear to be two parabolas. This conjecture based on visual image must be tested. We look for ways of experimentally testing if it is compatible with a high-school curriculum. If confirmed, the conjecture will necessitate further PM proof; if refuted, students will face another case when visual information is deceitful.

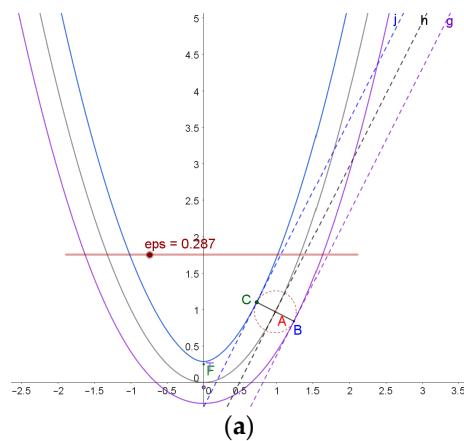
Parabola is historically known to suggest “impostures”; for example, the chain curve was long thought to be parabola. Richmond and Richmond (2009) proposed several PM-warranted criteria for recognizing true parabolas. We apply one of them, the focal property test (Richmond & Richmond, 2009, p. 919; see also Rousseau & Saint-Aubin, 2008):

Suppose $f(x)$ is a differentiable function. Then, $f(x)$ is a concave-up parabola if and only if it has the property that all vertical rays coming downward from above $f(x)$ are reflected to a single point F .

A school-level proof of the “if” direction of this property appears in Appendix A.57. Unlike the proof in Richmond and Richmond (2009), it does not use any analytic apparatus. The “only if” direction, its proof being beyond the school curriculum, is accepted here as given.

Since the focal property involves tangent, to apply EM, it is not enough that the curves are traced by points B and C; they must be defined so that their tangents can be plotted. To calculate these envelopes, we turn to Wolfram Mathematica (WM) for a PM investigation (see, e.g., <https://mathworld.wolfram.com/Envelope.html> (accessed on 19 July 2026)). WM symbolically computes the family of curves $U(x, y; x_A) = 0$ that are tangent to all circles centered on $A(x_A, x_A^2)$ with radius ϵ (Appendix A.4). Admittedly, this investigation is far beyond the school curriculum; therefore, the PM investigation will stop here. We will discuss classroom appearances of PM tools beyond the school curriculum in Section 4.

WM discovered two lines (blue and violet) tangent to the circles (Figure 13b). To apply the focal property to the envelopes, we first plot their tangents at points B and C. These tangents are also tangent to the circle whose diameter is BC. Figure 14a suggests that these tangents are parallel to each other and to the tangent to the parabola at point A, though their slopes appearing in the Algebra window coincide only up to 6 digital places (Figure 14b). In this case, despite differences in numeric values, the visual information is PM-supported, since all the tangents are perpendicular to Diameter BC, by definition of tangent to a line (including tangent to a circle).



$$h : y = 1.9699494623x - 0.970175221$$

$$j : y = 1.96995143x - 0.3361274058$$

$$g : y = 1.9699504841x - 1.6042257387$$

(b)

Figure 14. Tangents to the original parabola at Point A (line h) and to envelopes at points B (line g) and C (line j) and their slopes. (a) Tangents to the original parabola at Point A and to envelopes at points of tangency B and C look parallel. (b) The slopes of the tangents coincide up to six digital places.

2.1.2. Testing the Envelopes by the Focal Property

If the outer curves were true parabolas, all vertical rays reflected by them would pass through single, fixed foci, similar to rays reflected by the true parabola, that all meet in its Focus F.

Therefore, we follow the rays parallel to the y-axis after they are reflected by the three curves (Figure 15a). The vertical rays hitting the curves are obviously parallel to each other. The reflected rays form equal angles with CB which belongs to common normal to the three parallel tangents; therefore, they are parallel to each other, as well. Hence, SCBT is a trapezoid, and AF is its midsegment (Figure 15b).

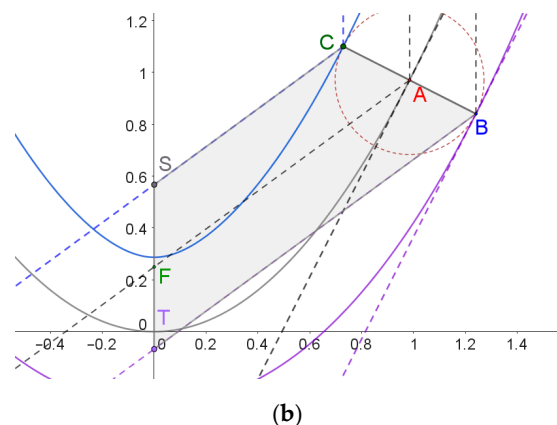
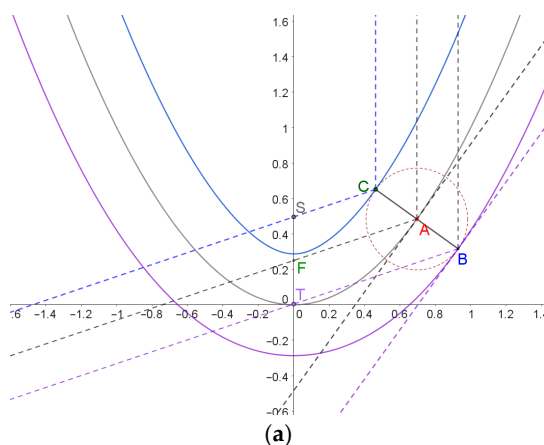


Figure 15. Testing the envelopes by the focal property (<https://www.geogebra.org/m/smey8zzn> (accessed on 14 May 2026)). (a) Vertical rays before and after the reflection in the three curves. (b) Trapezoid SCBT, whose bases (segments of the reflected rays) shift as A moves.

As Point A glides along the parabola, the rays reflected at A keep passing through F, by the focal property of parabola. The angle between BC and y-axis changes with the change of position of A. Since the length of BC is constant, this implies that the lengths of FT and FS change, so that points S and T change their locations. Thus, the rays reflected by envelopes do not pass through a constant point, which implies that the envelopes are not parabolas.

Figure 16 illustrates these varying intersections for several ε values, providing an EM-produced illustration to the PM-stated fact that the envelopes are not parabolas.

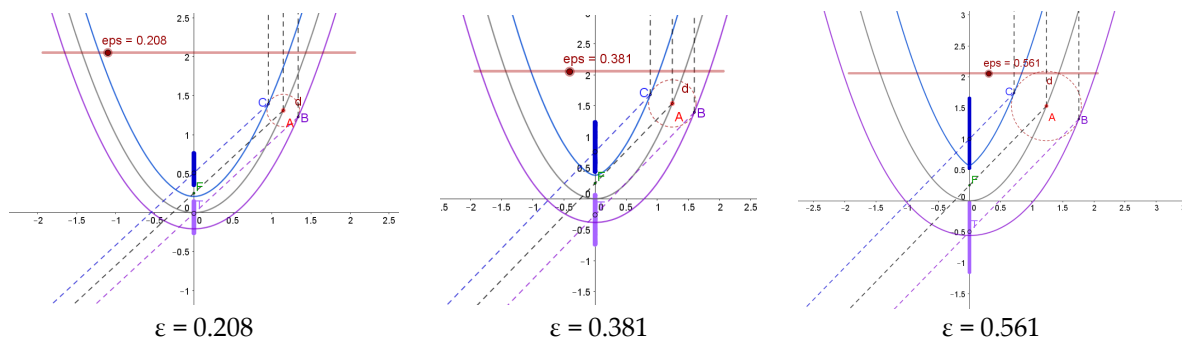


Figure 16. Testing the envelopes by focal property. As the Point A is dragged along the original parabola, the rays reflected by it (grey) all pass through F—the focus, while intersection points of rays reflected by envelopes, change their location on y-axis (blue and lilac).

Conclusion: No matter how small ε is, the bounding curves of the ε -stripe around $y = x^2$ are not parabolas. As ε increases, distortions become more pronounced, producing additional unexpected properties—several of which are proposed for further research.

Example 8. Recursive sequences step-by-step.

A mixed arithmetic–geometric sequence provides a versatile model for dynamic phenomena such as drug concentration in the bloodstream, pollution levels in a lake, or repayment plans for bank loans (Kalman, 2023). Each context combines constant additive and multiplicative changes: an intake or deposit repeated regularly and a proportional reduction or growth at each step.

Let

$$a_{n+1} = qa_n + d, 0 < q < 1,$$

where (a_n) is the quantity at the beginning of the (n) -th time unit, (q) is the retention (or decay) rate, and (d) is the constant inflow. For a drug, (d) represents dosage at regular intervals; for debt, the installment; and, for pollution, the constant inflow rate.

Though each situation differs in purpose—stabilizing medication levels, repaying debt efficiently, or limiting contamination—the same mathematical structure applies. This universality becomes an educational advantage: once students recognize the shared recursion, they can model multiple phenomena by adjusting parameters.

Geometric and arithmetic sequences are familiar at secondary level, but this *mixed model* rarely is. The recursive formula $a_{n+1} = qa_n + d$ guiding the process is not very eloquent. It is not clear from it that the process stabilizes. It does not explicitly instruct how to choose the values of guiding parameters to achieve the goals. However, it enables step-by-step following of the process by performing the addition and multiplication at each step, which entails trial-and-error method performed by DT (Figure 17) (<https://www.geogebra.org/m/qwtqxufx> (accessed on 14 May 2026)). Dynamic control of one guiding parameter at a time enables the user to take in the dependence of the process on each parameter.

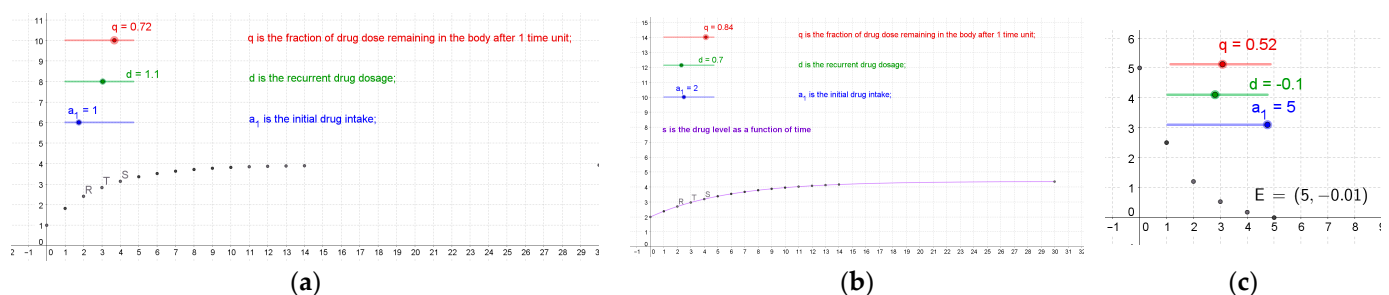


Figure 17. Mixed-sequence graphs representing compound additive and multiplicative phenomena. (a) Discrete representation of drug concentration levels over time. (b) Discrete and continuous representations of the same process. (c) Discrete model of bank loan payments (here, d is negative). The computation halts after zero-level is attained.

These dynamic trials yield several insights:

- For various parameter sets, the process stabilizes—the sequence approaches a *fixed point*.
- Graphical sliders facilitating parameter changes allow exploration of sensitivity and mutual dependence between the system parameters: increasing or decreasing q or d alters the final equilibrium or the speed of convergence.
- Certain phenomena require axis scale adjustments to reveal asymptotic behavior precisely.

In the pharmacological case, the goal is to attain and maintain a target level efficiently; in pollution models, to ensure concentrations stay below limits; and, in finance, to balance installments and total interest paid. Each context defines “acceptable approximation” differently—an authentic AM consideration.

The recursive process can be expressed in closed form as

$$a_n = a_1 q^n + d \frac{1 - q^n}{1 - q},$$

or as a continuous analogue:

$$s(x) = a_1 q^x + d \frac{1 - q^x}{1 - q}.$$

The first suits discrete phenomena like loan payments, while the second models continuous flows like medicine absorption. Seeing both together helps students connect sequences with functions, and discrete change with continuity.

The experiment suggests that the continuous function $s(x)$ has a horizontal asymptote—see Kalman (2023) for the detailed analysis from the PM–AM point of view. Eventually, he sums it up:

“It is a very good approximation. For all practical purposes, the model can be assumed to reach a state of equilibrium. But the sequence values cannot ever reach the fixed point with perfect, mathematical, equality” (p. 266).

In Figure 18, the sequence approaches equilibrium at zero, so that the asymptote is the x -axis. The values of y -coordinates indicate that the points approach it but are not located on it.

Such “step-by-step” acquaintance with a model is applicable to other discrete models, e.g., logistic models of population growth (see, e.g., Kalman, 2023, pp. 293–295). Like mixed models, logistic models of population growth link together two mathematical objects included in a school curriculum: multiplicative algorithm reflecting geometric growth, and linear dependence of the growth factor on the population size.

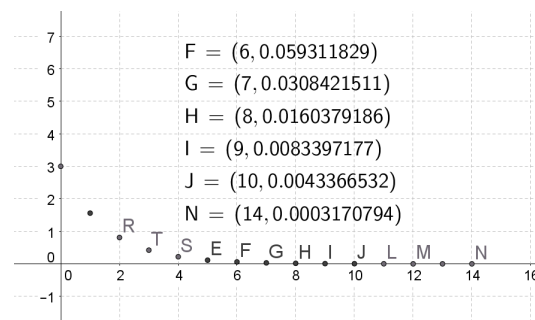


Figure 18. Final values approaching equilibrium—the apparent zero level on screen hides residual quantities depending on precision chosen.

The same experimental approach may be applied to mathematical recursive algorithms, for example, to the Newton–Raphson algorithm for approximating $\sqrt{N}$ (Figure 19):

$$x_{n+1} = 0.5 \left(x_n + \frac{N}{x_n} \right).$$

True, GeoGebra environment is not planned for programming, and step-by-step typing in of expressions like $(2, 0.5(y(B) + 7/y(B)))$ may seem tiresome, but, since the method converges fast and the platform enables changing only three characters at a time—the step number as x-coordinate and the name of the previous point in y-coordinate—the effort seem worth it. Implementing this step by step makes tangible the idea of *convergence through successive refinement*. Students can observe how additional iterations increase decimal accuracy until rounding ceases to change results, illustrating both the concept of limit and the dependence of computation on chosen precision.

Linking to the notion of a fixed point, $\sqrt{N}$ is the fixed point of $0.5 \left(x_n + \frac{N}{x_n} \right)$.

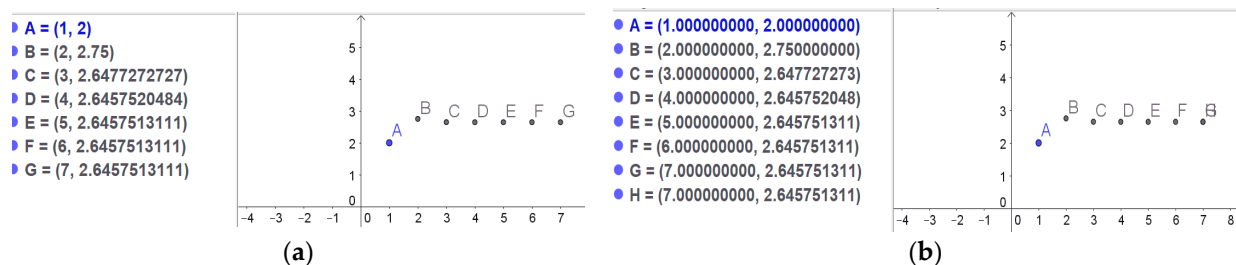


Figure 19. Step-by-step Newton–Raphson computation of $\sqrt{7}$. (a) Ten-decimal rounding. (b) 10-significant-figure rounding.

This hands-on process demonstrates that the sequence continually improves the approximation but never *reaches* the exact irrational value—an excellent introduction to ideas of infinite improvement and limit within reachable school mathematics.

3. Results

The results are integrated in Table 1 which considers each example through the prism of the EM–AM–PM triangle and summarizes each example’s mathematical, didactic, and epistemic implications.

Across the eight examples, several recurring insights emerge:

Mathematically, an appropriately guided digital experimentation enacting the “mathematical triangle” doctrine exposes the limits of numerical representation, precision control, and exact symbolism. It illustrates that “precision” in digital environments is a

parameter to be managed—not an absolute property. It also facilitates the emergence of new mathematical knowledge by virtue of a guided inquiry.

Didactically, it transforms “errors” and inconsistencies into learning opportunities, and “hands-on” experiences into an inductive foundation for further deductive inference. Digital tools thus applied have a potential of reframing mathematics education as an inquiry rather than routine execution.

Epistemically, it encourages a critical evaluation of visual and digital outputs and helps students navigate the gap between intuitive, approximate, and rigorous knowledge.

Table 1. EM–AM–PM DT-linked triangle and implications for school mathematics (Examples 1–8).

Examples	PM–AM–EM Interrelation	Implications for School Mathematics: Mathematical; Didactic; Epistemic
Example 1.	Parallel lines belong to abstract geometry, that is, to PM. EM investigation of a didactic applet reveals that two-decimal precision is insufficient for representing equality of slopes measured in degrees, while may be misleading if performed in radians. PM reasoning identifies the contradiction; AM interpretation contextualizes relevance of small angular differences.	Mathematical: Illustrates the limits of digital precision and the impossibility of representing irrational numbers exactly. Didactic: Promotes discussion of error, scaling, and real-world implications (e.g., railway or astronomical contexts). Epistemic: Strengthens critical appraisal of visual output.
Example 2.	The PM statement “Point A lies on f ” is approximated by DT analytically and visually. Ultimately, the platform’s answer to the question “Does E belong to f ?” depends on graphic and computational precision.	Mathematical: Highlights the differences between PM and EM–DT representations of objects. In PM, the ways of expressing the fact that a point belongs to a line are by verbally stating it or by producing precise analytic information. AM–EM–DT realizations of point location involve the precision threshold. Didactic: Encourages students to investigate inconsistencies rather than dismiss them. Epistemic: Enhances awareness of precision in AM vs. PM information.
Example 3.	Graphic solutions of equations in general are neither exact nor exhaustive; the method belongs totally to the AM realm. An important PM aspect is the existence of solutions, and their number and location; EM dynamic tool helps investigate it, subject to DT limitations. The situation where the Algebra window suggests that the function equals 0, while the graphic window depicts its graph pours light on what 0 means in PM and AM contexts and how such expression should be understood in the AM setting.	Mathematical: Highlights the meaning of exact vs. approximate solution. An exact number in mathematics is either a rational number represented as a fraction of two integers (including finite decimals), or an irrational number written as a result of appropriate operations, such as $\sqrt[3]{5}$ or $\sqrt{2} + 3$, or using special symbols, like e or π . Whether a solution is finite rational, or irrational or rational decimal of infinite length is imperceptible from the graphic representation, and irrelevant for an AM quest. In particular, integers are, in effect, absent from a digital environment such as GeoGebra, and so are rational numbers. In DT, “exact” is embodied by symbolic representation, e.g., $(x(B), y(B))$, in contrast to numerical values. Integrating PM considerations with DT-experimenting provides information that is as exhaustive and comprehensive as possible about the solutions, their number, locations, etc. Didactic: Suggests teaching towards educated usage of graphic exploration of equations and functions; elucidates real numbers and their rational estimates in the context of mathematical exploration; imparts ideas of order of magnitude, significant decimal figures, and global behavior of function. Epistemic: Places graphic solution as an important and legitimate method and as a mathematical “tool of trade”, whose place is between PM and EM–AM; DT enhances the link between them.
Example 4		

Dynamic geometry enables generating plausible but impossible figures through insufficient precision. PM logic exposes the contradictions.	Mathematical: Cast doubt at supportive examples, even if they form infinite example space; they should be validated by deductive reasoning. Didactic: Affords and triggers a classroom discussion on deductive vs. experimental mathematics and on appraising DT outputs. Computer-produced diagrams are not necessarily trustworthy: the same diagram (e.g., Figure 6) may include both reliable and totally arbitrary information. Epistemic: Calls for establishing PM as an eventual “arbitrator” of correctness of a visual representation.
Example 5	Mathematical: Refines the understanding of a construction problem as tool-dependent—different tools used for the same construction comprise different problems with different solutions—and of error estimation as method-dependent—the error analysis is per method, not per problem. Didactic: Highlights the importance of purpose-based precision: constructing an angle of 20° for applied work may be entirely sufficient, though geometrically impossible. This leads to important classroom activity focusing on compatibility of DT geometric tools with Euclidean geometry. Ben-Ari (2022) refers to never-ending attempts to trisect angles claiming that “... a solution would have no practical use”. Usefulness and practicality depend on the aim. For didactic purposes, such attempts, both successful and erroneous, are incentives for hands-on learning. Epistemic: Addresses important epistemic features not to be avoided at school, of tension between abstract deductive claims and intuitive assertions and experiences, and of existence of special cases vs. impossibility of the general solution.
Example 6	Mathematical: Analyzes the flaw in construction and reasoning discovered by a contradiction to theorems of elementary geometry. Didactic: Reveals the cognitive value of erroneous Einstellung-originating solutions: rather than rejecting them outright, students can analyze by available mathematical tools where and why they fail. Epistemic: Discerns “correct–incorrect” vs. “precise–imprecise” considerations.
Example 7.	Mathematical: Exposes the role of “small changes” in mathematics, inherent, e.g., to the analysis of infinitesimals, catastrophe theory, theory of perturbations, and more, far beyond the school limits. A structured series of exercises in school geometry reveals conceptual power of “small change leads to new object.” Didactic: Poses exercises adapted to inquiry-based activities as source of new knowledge. The volume of curriculum is limited; such approach may extend it and upgrade the learning experience. Thus, the focal property of parabola can be studied through a series of school-level geometry exercises involving EM and PM components.

envelope is a more advanced one and presented eventually at an intuitive level.

Investigation of parabola as an AM object, i.e., empirical curve (stripe), leads to unexpected conclusions. On the way to experimental (EM) investigation of the stripe around parabola, several PM facts emerge through a series of short pieces of geometric argumentation within school level.

Epistemic: Includes and applies advanced mathematical objects at intuitive level, thus extending the students' perception of mathematics and the curriculum repertoire. In this example, the envelope appears twice, exemplifying application of recently acquired new knowledge.

Element of surprise in view of unexpected results or of refuted allegations is a valuable experience, familiar to mathematicians in their research. In the present case, discovering that “parabola-like” curves are not parabolas has such epistemic value. This discovery may be enhanced by further expansion of the neighborhood of the parabola to watch visible divergence of the envelopes from the parabola-like curve and from similarity between them (<https://www.geogebra.org/m/vkbhkq3c> (accessed on 14 May 2026))

Example 8.

The example presents DT-afforded trial-and-error experiments leading to the discovery of fixed point (PM) of the recursive sequence, and to understanding the mutual dependence between parameters.

Though the process tends to a limiting value, whether and at which step the desired values are attained and the algorithm should stop depends on the context; thus, AM, in this case, is the eventual arbitrator.

Mathematical: Exemplifies how one PM object serving as a mathematical model reflects various real-life or mathematical phenomena. Different models involve different units and quantities; here, in most cases, the x-axis variable was time, but on different scales; in Newton–Raphson approximations, it was a number of steps. Units on the y-axis are context-dependent. In mathematics, applicability of a model to different physical phenomena is resolved by non-dimensional units on both axes resulting from dimensional analysis, thus paving way to PM to apply abstract tools to physical phenomena. This transition is not sufficiently exposed in school mathematics.

Didactic: Epitomizes the educational value of DT-supported experimental recursion: students gain intuition about convergence and approximation—bridging the purely formal definition of recurring sequences with tangible visual experience.

Epistemic: Connects mathematical notions of algorithm; fixed point; limit; and steady state. The limiting value of the dependent variable is presented both by asymptote and as a “fixed point”, thus linking discrete and continuous representations.

4. Discussion

The eight examples analyzed in this paper—each rooted in elementary or high-school mathematics—demonstrate the deep interdependence of the three sides of the PM–AM–EM triangle linked through digital technology (Figure 1). Collectively, they reveal significant implications for epistemology, didactic practice, and teacher preparation in mathematics education.

The discussion focuses on three intersecting domains:

1. Facing the mathematical triangle as an integral approach to school mathematics.
2. The epistemic shift in teaching and learning that promotes students' epistemic development.
3. Teacher expectations arising from these shifts enhanced by technological realities.

4.1. Facing the “Mathematical Triangle”

School mathematics offers many opportunities to interconnect precision, approximation, and experimentation; however, these opportunities are rarely utilized systematically. Precision and error—central to modern digital environments—are often treated as minor technical details rather than conceptual issues.

For instance, when Example 4 was presented to an experienced teacher and two students, they initially perceived the paradoxical configuration as mysterious. When it became clear that the issue stemmed from insufficient numerical precision, their reaction—“Oh, it's only precision”—revealed how undervalued this aspect remains. Similarly, teacher–students encountering Example 2 for the first time confessed they had never previously discussed rounding discrepancies as part of mathematics learning.

In digital contexts, however, precision must be intentionally addressed. Visual accuracy is constrained by screen resolution and human perception, while computational precision depends on rounding, representation type (numeric or symbolic), and software defaults. Teachers and students alike must learn to interpret precision as a mathematical factor to be studied and accounted for.

To be sure, “approximate” reasoning may belong to the PM. Thus, for example, equalities like $\sin x \approx x \approx \tan x$ (for small values of x in radians), are of this nature. Conversely, PM problems may emerge from AM settings, as has occurred in several examples. Echoing Klein's maxim, “We get the whole science only when we embrace the two parts”. Thus, school mathematics should fully incorporate both precise and approximate reasoning through inquiry-based, experimentally supported learning activities.

One topic revealed by the examples—mostly missing in many curricula—is the structure of the real number system at a conceptual level: the relationship among integers, rationals, and irrationals, the nature of decimals of infinite length, and the logic of rounding and error propagation. At the school level, real numbers are often reduced to truncated decimals—e.g., “ $\pi = 3.14$ ”—without emphasizing that this is only an approximation.

Digital technology offers an accessible pathway to address this. Using a zoomable digital number line, teachers can dynamically illustrate the process of approximation: each additional decimal brings the representation closer to its true position on the number line. Students thus experience how “improving precision” refines approximations indefinitely without ever reaching an exact symbolic expression—building an intuitive understanding of real numbers and limits.

4.2. Epistemic Shift in School Mathematics—Epistemic Development of Students

Learning mathematics balances intuition and formal knowledge. Experimental mathematics, enhanced by digital interactivity, nurtures intuition, while deductive mathematics refines it into structured reasoning.

Devlin (2019) asserts that the goal of mathematics teaching today is *understanding*, not *execution*. De Villiers (2010) and Engelbrecht and Borba (2024) further note that digital technologies reshape both how and where mathematical thought develops. According to Abramovich (2025), the pragmatic use of technology in classrooms stimulates the *epistemic development* of learners by enabling genuine mathematical exploration.

Within classroom practice, EM encourages students to test conjectures, interpret anomalies, and relate computation to conceptual reasoning, thus balancing mathematical methods (Engelbrecht & Borba, 2024), and reshapes the notion of mathematical knowledge and its development (Arzarello et al., 2012; Kortenkamp, 2004; Abramovich, 2025). As De Villiers (2010, p. 208) observed, many mathematicians “are often convinced of the truth of their results (on the basis of experimental evidence) long before they have proofs.”

Devlin claims that there may be no need in precise definitions at a certain level—present examples when there is need in it. This perspective invites educators to transform mechanical exercises into a creative inquiry—where routine computation acts as a stepping-stone to new insight. Examples in this paper allow secondary-level students to encounter advanced mathematical notions (*implicit curves*, *envelopes*, and *limits*) intuitively through guided digital experimentation, without needing university-level formalism.

Such an epistemic shift in school mathematics is not an objective per se, but a means to impact the epistemic development of students. In systems dominated by high-stakes examinations (e.g., Wells, 2016), learning often emphasizes procedural mastery. Nevertheless, inquiry-based learning can coexist with assessment if the evaluation also measures reasoning, interpretation, and the ability to evaluate digital evidence critically.

In the analysis of the examples, key epistemic proficiencies emerge repeatedly:

- The critical interpretation of digital and visual information;
- The identification and investigation of inconsistencies;
- The application of PM and AM reasoning to graphical exploration, including scale and unit considerations;
- Distinguishing between “correct/incorrect” and “precise/imprecise”;
- Questioning supportive examples, even if they seem to form an example space;
- Conjecturing, testing, and revising through EM processes;
- Learning by trial-and-error through DT-afforded experiments;
- Intuitive engagement with advanced constructs such as limits, fixed points, or convergence.

The perception of visual information mentioned above appeared in several contexts in this paper, probably expectedly, accounting for its ever-emerging source—the digital graphic output. Visualization in mathematics is a tool of the trade. On the one hand, it is a part of the mathematical apparatus at any level, and, on the other hand, one needs a mathematical background to use it productively. Thus, for example, Arcavi (2003) addresses the difference in the perception of visual representations of mathematical objects by experts and by novices, noting, however, that, “in different contexts, the ‘same’ visual objects may have different meanings even for experts”, as we have observed, e.g., in Example 1. As a tool of the trade, visualization realizes the power of multiple representations for those who know how to use it, in his example—the way the students may use it to support and illustrate symbolic results, to resolve possible conflicts between rigor and intuition, and to “recover conceptual underpinnings which may be easily bypassed by

formal solutions” (p. 224). Illustrations of these occurrences also appear in the examples above. Another example appearing in Kidron and Tall (2015) considers the application of DT (WM, in this case) to illustrate the process of tending to the limit of a sequence of graphically represented functions as “‘getting close’ replaced by the practical notion of being ‘good enough’ for the graphs to look the same on a computer screen (p. 186), so that the visual representation considered simultaneously with the corresponding numerical calculations leads to the formal definition. This example reflects the multifaceted challenge and benefits of applying visualization in an educated way, implementing the PM–EM–AM-interlock by DT, at a higher level of PM.

4.3. Expectations from Teachers

Modifications in school mathematics implied by the discussion above entail teachers’ preparedness and competence to bear a significant load on their practice, characterized “by the deep structure of teaching ... and taking an intellectual risk by going into an uncharted territory brought to light through an open-ended classroom discourse”, indicating “the need in strong preparation in technology, pedagogy and mathematics” (Abramovich, 2014, p. 12). An analysis of the examples suggests three essential areas of competence: seizing and creating opportunities; teachers’ know-how in interactive DT for teaching; and guiding students’ inquiry learning of topics, integrating all three mathematical realms by the use of interactive DT tools.

Seizing and creating opportunities. Opportunities for inquiry can arise unexpectedly or through deliberate design. Teachers must recognize such moments and decide whether to explore them immediately, postpone them, or connect them later. This professional judgment depends on identifying the mathematical value, the didactic potential, and the confidence to deviate temporarily from the planned curriculum. Examples 1 and 2 emerged spontaneously during lessons, whereas Examples 4–7 were intentionally constructed. Teachers’ motives may also be of a purely didactic nature, such as appearances of the Einstellung effect or turning to the experimental inductive reinforcement of deductive results motivated by “intellectual honesty” (De Villiers, 2010; see also Hansson, 2020 on the experimental–deductive interplay), i.e., by recognizing the fact that a purely deductive presentation of mathematics is not sufficiently persuasive.

Teachers’ know-how in interactive DT for teaching. As Naftaliev (2018) notes, interactive tools challenge teachers “to figure out how to promote and guide exploration.” Teachers must understand how digital tools operate, not merely how to use them. Competence includes the following:

- Distinguishing symbolic from numeric representations;
- Adjusting rounding and precision settings;
- Discerning when a dragging-point image of a curve is sufficient, and when its mathematical representation is needed (e.g., for plotting tangent);
- Managing axis scaling and visual distortion;
- Interpreting discrepancies logically rather than attributing them to “bugs.”

Perhaps the most important ability of a teacher to face the challenge is the ability and preparedness to learn “on the go”, having acquired a basic familiarity with a tool: most features needed for the use of DT are acquired when needed. Moreover, digital technology is in a state of permanent evolution—see, e.g., Devlin (2019), who claims that this places a requirement of teachers’ understanding of basic mathematics behind these platforms. Continuous professional development is needed to keep pace with evolving technologies (Devlin, 2019; Engelbrecht & Borba, 2024). The choice whether to construct an applet or to use an existing one implies that a teacher has time to do so on their own, which is far from being the case in usual teachers’ practice.

Albaladejo and García López (2024) address the teachers' use of GeoGebra to cope with precision issues, focusing on students' "careless calculations or geometrical constructions". They apply GeoGebra as a tool to advance the students' awareness of precision in calculations and geometric representations and question "approximate solutions' righteousness or adequacy" as opposed to precision and rigor (p. 1082). Discerning the cases in which the results "are not accurate, possibly because they lean on a mistaken reasoning" from those that are not accurate for deeper mathematical reasons, is also one of expectations of teachers. Error is sometimes opposed to rigor, as in Albaladejo and García López (2024). Nevertheless, in AM, it is a part of the rigor considerations. Considering DT merely as a tool for precise geometric representations or calculations may be missing the bigger picture of it as a tool for versatile mathematical investigations. Following Dilling et al.'s (2024) MPC-model, the interactive DT advantage, compared to static media that also enable high-quality drawings, is not in its ability to produce "exact" results (sometimes questionable, as we have seen), but in its compatibility with the desired pedagogy mode of inquiry. Hence, the "scant tolerance of errors in geometric constructions and arithmetic operations" (p. 1082) should probably be extended to a deeper analysis of errors as a way of learning "for reasoning and understanding".

Guiding students' inquiry learning. Enabling students' open inquiry as DT-enhanced learning, a teacher is expected to react to unexpected outputs, resolve DT jams and bugs, weigh the pros and cons of deviation from the planned course of the lesson, and decide on the spot on how to go on, which questions to deal with, and which surprise phenomena are worth discussing on the spot or later. In such decisions, curriculum recommendations and the epistemic development of students are not always in harmony and the dilemma resolution is not a simple decision, requiring teachers to rely on their professional knowledge for teaching.

Speaking of teachers' professional knowledge for teaching in the spirit of embodying the "mathematical triangle" and supporting hands-on learning from experiments, there are (at least) two issues to consider. The first one is upgrading this key concept to support teachers' ability to master ever-evolving mathematical DTs and to weigh their use in the mathematics classroom more or less "on the go". The TPACK model mentioned above (Mishra & Koehler, 2006), which expands Loewenberg Ball et al.'s (2008) Mathematical Knowledge for Teaching to the demands of DT-integration with mathematics in the classroom, is itself evolving to reflect the DTs evolution. Petko et al. (2025) state that TPACK includes knowledge about educational environments (contextual knowledge). Therefore, the concept of "content" is being revised in this framework. The pedagogical content knowledge necessitates an educated choice of tools appropriate for a mathematical situation and teachers' intentions within guided learning (see, e.g., Geraniou et al., 2024; Barabash, 2025), which presupposes their acquaintance, albeit superficial, with more than one tool and their features. Another issue to consider is related to advanced mathematical notions in the classroom. "Advanced" is challenging. Mathematical challenges at the school level are a challenge in itself, and it is not mere wordplay; challenges in the mathematical classroom are, first and foremost, challenges for teachers. Thus, the 16th ICMI year study (Taylor & Barbeau, 2009) refers, to a great extent, albeit not always, to extra-curricular, "beyond the classroom" challenging mathematics, in particular, challenging problems. On the other hand, Zachariades et al. (2025) consider teachers' advanced knowledge in mathematics needed to cope with students' misunderstandings of concepts close to those discussed in this paper, namely, on decimal representations of rational numbers. It is this aspect of "advanced" that is applied in this paper. As one can trace the mathematical problems that emerged here in the course of experiments and investigations, they are not necessarily "advanced" in the sense of "challenging Olympiad problems". However, these problems stemmed from the application of advanced mathematical ob-

jects at the intuitive level, which demands that the teachers possess and master mathematical knowledge to face this expectation.

Teachers must be ready to discern and introduce advanced notions intuitively when they surface naturally during inquiry—such as “envelope,” “parametric curve,” or “limit”. This does not mean formalizing these concepts beyond the curriculum but positioning them as *windows onto mathematics*—stimulating curiosity and modelling how mathematicians build knowledge. An example of the parametric curve actually appears in high-school curricula which include the trigonometric unit circle, so the gap, in this case, is not too big; it is just for a teacher to seize the opportunity to mention it and bridge the gap when needed. This expectation does not contradict school mathematical epistemology, but, rather, calls for expanding it. Behind the scenes of the school curriculum stand advanced mathematical notions. Inquiry learning may pull some of them forward in the framework of “an open-ended classroom discourse”.

Finally, lapses, contradictions, and accidental discoveries—so common in digital exploration—should be viewed as pedagogical gifts; see, e.g., Abramovich (2023) for an example of channeling an error in programming into an entirely new conceptual domain “when recognized and attended through an epistemic lens either by a mindful student or by a teacher as a ‘more knowledgeable other’” (p. 11). They humanize mathematics as a process of inquiry, trial, and refinement, rather than a collection of immutable rules.

5. Conclusions

The examples presented in the paper balance on the brink between traditional school mathematics and issues outside the school curriculum that have the potential of enriching it mathematically, didactically, and conceptually.

Mathematically, these cases reveal new perspectives on familiar topics and uncover connections between everyday classroom content and higher-level mathematical structures. They show that even simple tasks—drawing lines, measuring angles, and iterating formulae—can generate substantive discussions about precision, proof, and approximation.

Didactically, the examples advocate active exploration through experimentation and conjecture rather than mere technical procedure. They show how digital tools, when used critically, transform “errors,” rounding issues, and apparent contradictions into spaces for investigation and understanding. Instead of transmitting ready-made knowledge, teachers can foster an environment where students *discover why mathematics behaves as it does*.

Conceptually, intuitive reasoning derived from digital experimentation provides a viable gateway to abstract ideas such as limits, envelopes, infinitesimal change, or convergence—concepts that underpin higher mathematics. Digital experimentation’s *dynamism* allows learners to move gradually from visual intuition to deductive reasoning, effectively bridging AM, EM, and PM.

Allowing open digital inquiry in classrooms also demands flexibility from teachers. They must balance spontaneity with curricular goals, respond constructively to unexpected tool behavior, and make informed decisions about whether emerging questions deserve immediate exploration. Such decisions rely on teachers’ professional judgment—mathematical, pedagogical, and technological.

In practice, challenges and opportunities rarely align perfectly. Curricular objectives, examination pressures, and students’ epistemic development often pull in different directions. It is precisely within these tensions that teachers’ creativity and mathematical professionalism become essential. Through the deliberate integration of digital experimentation, precision analysis, and approximate reasoning, school mathematics can evolve from a discipline of procedures into a discipline of inquiry.

Ultimately, embracing the PM–AM–EM triangle in the age of digital technology re-defines mathematics education itself:

- *Precision* is no longer the opposite of error but a tool to understand its nature;
- *Approximation* is not a flaw but a bridge between concrete and abstract;
- *Experimentation* is not an end but a method for uncovering truth.

Together, they invite teachers and students to see mathematics not as a static certainty but as an evolving way of thinking—dynamic, approximate, and profoundly human.

Limitations and Suggestions for Further Research

Some closely related topics—such as STEM-oriented interdisciplinarity, curiosity and motivation in mathematics teaching, or the role of artificial intelligence in mathematics education—fall outside this discussion. Each deserves separate treatment, given their philosophical and pedagogical scope

This study revealed but “the tip of the iceberg” of issues related to the relevance of the “mathematical triangle” to school mathematics and to mathematics education. To be sure, there is much more to work out and to study about the mathematical, didactic, epistemic, and theoretical-educational aspects. Suggested herein are some possible further lines of research.

Some of these lines of research were suggested by the reviewers of this paper, and I include them here with their kind permission, and appreciate their constructive input.

One of the reviewers suggested the following open problem (which I cite verbatim) “of the advisability of solving particular tasks in different environments. For instance, the problem of parallelism of the lines $2 \times x + 3$ and $2 \times x + 5$ is natural, but the problem of parallelism of $2.12 \dots \times x + 3$ and $2.12 \dots \times x + 5$, where the slopes are unknown in advance, appears artificial. This question follows directly from Klein’s work, which describes “approximate” mathematics as that which is naturally formed by a person through interaction with the environment and serves as an object of abstracting individual properties of surrounding objects to advance toward “exact” mathematics. By introducing digital tools into the mathematics learning environment, we introduce new representations, which also constitute “approximate” mathematics. Do we want this “approximate” mathematics to be further removed from “precise” mathematics than the natural, “non-digital” representations are?”

Another reviewer suggested considering the importance of the semiotic analysis of representations appearing aplenty in this approach, following Duval’s registers of representations (see Pedersen et al., 2021 for a concise reference). Multiple representations appearing in DT manifest themselves in the differences in numeric vs. symbolic representations rendering different kinds of outputs, synchronized dynamic multiple representations (for example, multiple co-varying representations as applied in examples of trisection), representations embodying PM vs. AM, inter alia. This is far from an exhaustive reference to the semiotic issues stemming from DT use reviewed by Pedersen et al. (2021), though it does not include the teachers’ knowledge and needs, which seem to be an integral part of such further study.

An additional suggestion of this reviewer which I totally share is to consider the issue of didactic transposition indispensable when content involving mathematics in all its embodiments is to be transformed into a source of learning for school students. It is particularly relevant when some of the mathematical topics are at the brink of the school curriculum, integrating several objects, some of them of a more advanced nature; therefore, design and research projects in this stance will hopefully contribute to developments stemming from the present research.

Example 7 of the neighborhood of the parabola suggests a further experimental investigation whose results must be substantiated by PM; as has been indicated above, the expansion of the neighborhood as the radius grows exposes the pair of curve families that no more resemble either parabolas or each other. In addition to the purely mathematical essence of the problem which may be distant from the school curriculum, some of which is covered in Dana-Picard and Kovács (2023), its epistemic value at the school level is yet to be studied and employed.

Mathematical modelling in view of EM–PM–DT–AM links is another strand of design and research that seems to be naturally adjacent to these links and undoubtedly suggests rich and useful topics to engage in.

The role of the mathematical triangle in STEM-wise-oriented curricula is also a strand not to be overlooked; it seems but natural to investigate and design STEM-related teaching and learning while enacting all four components of EM–PM–DT–AM links.

The role of artificial intelligence in the further design of learning sequences employing the approach suggested here, and analyzing their output, is an inexhaustible source of possibilities and challenges, including challenges for critical thinking.

Last, but far from the least, is the matter of creativity, motivation, and curiosity necessarily arising in relation to DT-afforded experimenting and hands-on learning mathematics—see, e.g., Sriraman (2009), and Leikin et al. (2025) for the state of the art on creativity, curiosity, and adjacent topics.

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Institutional Review Board Statement: The study did not involve humans and animals.

Data Availability Statement: Links to all the applets used in the paper are provided.

Conflicts of Interest: The author declares no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

PM	Precision Mathematics
AM	Approximation Mathematics
EM	Experimental Mathematics
DT	Digital Technologies
WM	Wolfram Mathematica

Appendix A

Appendix A.1. Derivation of Equation (1)

In isosceles triangle ABC ($AC = BC$), the base is trisected; s is the difference between the middle angle and each one of the side angles; in notations of Figure A1, $s = \angle DCE - \alpha$:

$$AD = DE = CB = t; \quad \angle ACD = \angle ECB = \alpha; \quad \angle DCE = \alpha + s;$$

Then,

$$\beta = \frac{180^\circ - (3\alpha + s)}{2};$$

$$\alpha + \beta = 90^\circ - \frac{\alpha + s}{2};$$

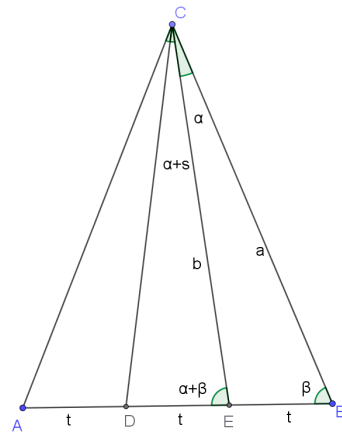


Figure A1. $\angle ECB = \alpha$.

By the Law of Sines applied to $\triangle ECB$ and $\triangle DCE$, $\frac{\sin \alpha}{t} = \frac{\sin \beta}{b}$, $\frac{\sin(\alpha+s)}{t} = \frac{\sin(\alpha+\beta)}{b}$.
Therefore, from the first equality, $\frac{t}{b} = \frac{\sin \alpha}{\sin \beta}$, and, from the second one, $\frac{t}{b} = \frac{\sin(\alpha+s)}{\sin(\alpha+\beta)}$.

Thus, $\frac{\sin(\alpha+s)}{\sin(\alpha+\beta)} = \frac{\sin \alpha}{\sin \beta}$.

The substitution of expressions above implies

$$\sin \beta = \sin \frac{180^\circ - (3\alpha + s)}{2} = \sin \left(90^\circ - \frac{3\alpha + s}{2} \right) = \cos \frac{3\alpha + s}{2}$$

$$\sin(\alpha + \beta) = \sin \left(90^\circ - \frac{\alpha + s}{2} \right) = \cos \frac{\alpha + s}{2};$$

$$\text{Finally, } \frac{\sin(\alpha+s)}{\sin \alpha} = \frac{\cos \frac{\alpha+s}{2}}{\cos \frac{3\alpha+s}{2}}.$$

Appendix A.2. Proof of the Claim: If, in an Isosceles Triangle, the Base Is Trisected, the Middle Angle (The Angle Formed by Trisecting Segments) Is Always Larger than the Side Angles (Those Formed by a Trisecting Segment and the Adjacent Side of the Triangle)

In $\triangle ACB$, $AC = CB = a$; the base is trisected by points D and E:

$$AD = DE = DB = t;$$

Prove that $\angle ECD \geq \angle ECB$, or, in notations of Figure A2, $\beta \geq \alpha$.

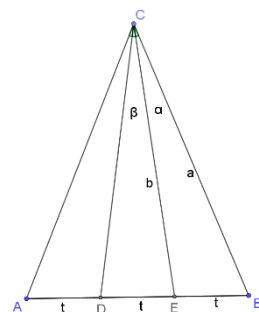


Figure A2. Construction for the proof of inequality $\beta > \alpha$.

For proof, one must make sure that $a > b$. In Figure A3, extend CE to G so that $EG = CE$. Then, DCBG is a parallelogram; hence $DC = a$. The triangle inequality applied to $\triangle CDG$ implies $a + b > 2b$; therefore, $a > b$.

To prove that $\beta > \alpha$, apply the Law of Sines to triangles DCE and ECB:

$$\frac{\sin \angle CED}{b} = \frac{\sin \beta}{t}, \quad \frac{\sin \angle CEB}{a} = \frac{\sin \alpha}{t};$$

$$\sin \angle CED = \sin \angle CEB \Rightarrow b \sin \beta = a \sin \alpha;$$

$$a > b \Rightarrow \sin \beta > \sin \alpha.$$

Since both α, β are acute angles, the last inequality implies $\beta > \alpha$. Q.E.D.

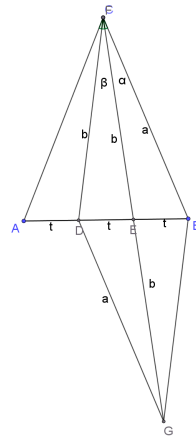


Figure A3. $EG = CE = b$. Auxiliary construction for Proof that $\beta > \alpha$.

Appendix A.3. Parabola Is Tangent to Any Perpendicular Bisector Between Its Focus F and a Point P on Its Directrix

Proof. Let A and P be points on the directrix of a parabola, and F its focus. The intersection point C of the parabola with a perpendicular CP to the directrix is equidistant from F and P ($i = h$, Figure A4a), by the definition of a parabola. Therefore, it is located on the perpendicular bisector to PF, by its property as the locus.

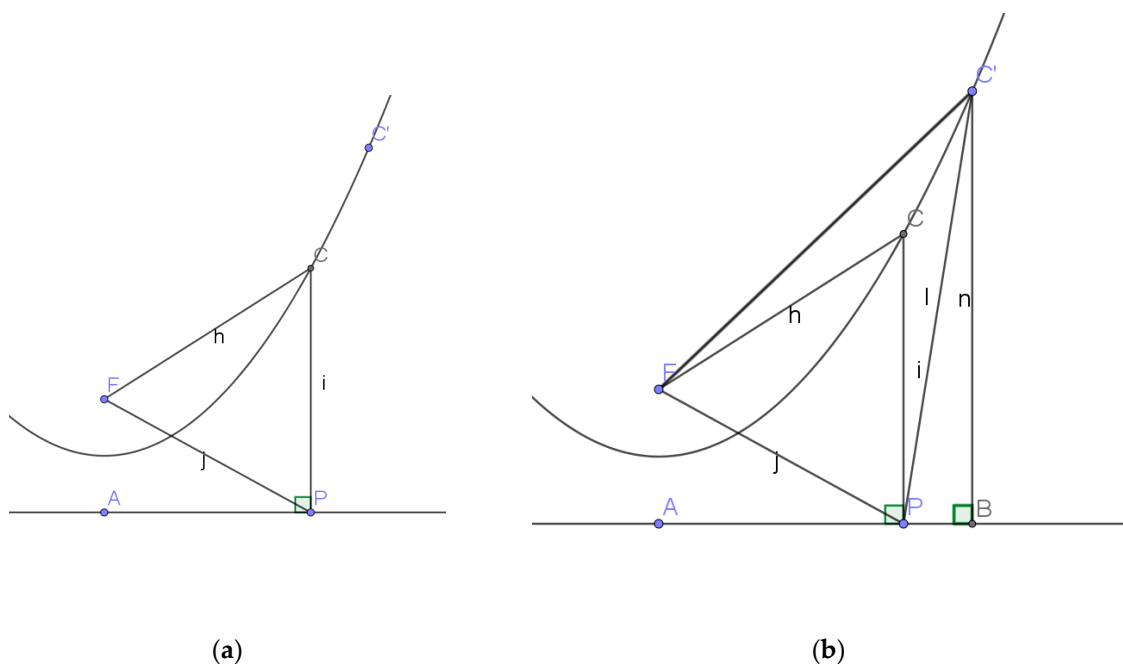


Figure A4. Constructions for Proof of A3. (a) $CP \perp AP$; C—a point on the parabola, AP—its directrix. (b) C' —another point on the parabola, $CB \perp AP$

To show that the perpendicular bisector to PF is tangent to the parabola, one has to show that there are no other common points to it and to the parabola (since the parabola is a second-degree curve, it may have no more than two points in common with a straight line). Suppose that another point C' lies on both the parabola and the same perpendicular bisector to PF (Figure A4b). Then, $C'F = C'P$ (C' being a point on the bisector) and $C'B = C'F$ (C' being a point of the parabola; B is a point on the directrix AP, so that $C'B \perp AP$; see Figure A4b); thus, $C'P = C'B$, and the triangle $C'PB$ should be isosceles with a right angle at the basis, which is impossible. Therefore, there exists a single point common to the parabola and perpendicular bisector to PF; thus, the perpendicular bisector is tangent to the parabola.

Note, strictly speaking, it is necessary to mention that the perpendicular bisector to FP is not perpendicular to the directrix, to claim that it is a tangent. $\square$

Appendix A.4. Solutions by WM for Two One-Parametric Curves Forming Envelopes of Circles ($A(x, x^2)$, a)

Here, a stands for ϵ , Denoted as eps in GeoGebra Applets

```

In[ ]:= f[x_, y_] := (x - c)^2 + (y - c^2)^2
Out[ ]:= (-c + x)^2 + (-c^2 + y)^2

In[ ]:= (-c + x)^2 + (-c^2 + y)^2

In[ ]:= Solve[(x - c)^2 + (y - c^2)^2 == a^2 && -2(-c + x) - 4c(-c^2 + y) == 0, {x, y}]

Out[ ]:= {{x -> c + 2 c^3 - (2 c^3)/(1 + 4 c^2) - (8 c^5)/(1 + 4 c^2) + (2 c Sqrt[a^2 (1 + 4 c^2)])/(1 + 4 c^2), y -> (c^2 + 4 c^4 - Sqrt[a^2 + 4 a^2 c^2])/ (1 + 4 c^2)}, {x -> c + 2 c^3 - (2 c^3)/(1 + 4 c^2) - (8 c^5)/(1 + 4 c^2) - (2 c Sqrt[a^2 (1 + 4 c^2)])/(1 + 4 c^2), y -> (c^2 + 4 c^4 + Sqrt[a^2 + 4 a^2 c^2])/ (1 + 4 c^2)}}

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Appendix A.5. Focal Property of the Parabola

A ray parallel to the parabola axis, when reflected by the parabola, passes through its focus F.

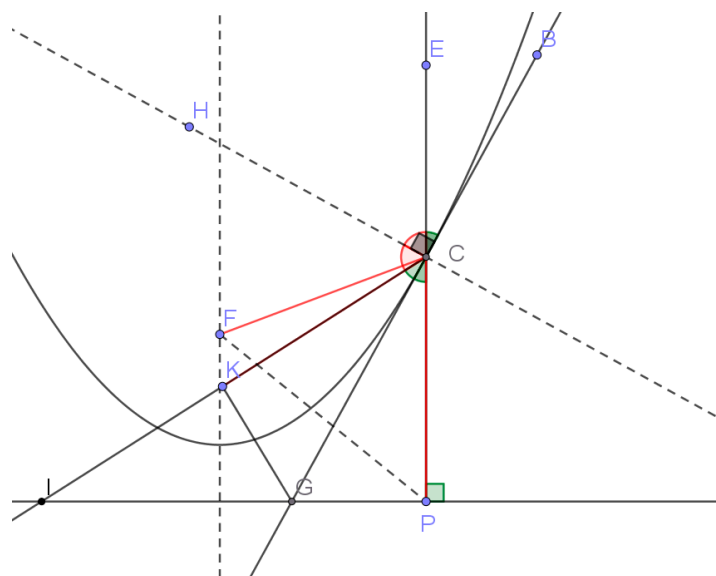


Figure A5. Proof of the focal property of parabola.

Proof. In Figure A5, the ray EC is parallel to the axis of the parabola (and, therefore, perpendicular to the directrix), and C is its reflection point from the parabola. F is the focus.

BG is tangent to the parabola at Point C. $HC \perp BG$. By the reflection law of optics, $\angle ECH = \angle HCI$ (red angles in the Figure A5); P is the intersection point of line EC with the directrix. CI is the reflected ray that intersects with the axis at Point K. Our aim is to show that K coincides with F.

$\angle ECB = \angle GCP$, as vertical angles. $\angle ECB$ and $\angle ICG$ complete the equal angles $\angle ECH$ and $\angle HCI$ to a right angle; therefore, $\angle ECB = \angle ICG$; thus, CG is the angle bisector of $\angle ICP$.

By the locus property of the tangent as the bisector between F and P, $CF = CP$, so that the triangle FCP is isosceles, and CG contains a median to FP; thus, it is also the angle bisector of $\angle FCP$, which implies that $\angle FCP = \angle ICP$; thus, the points F and K coincide. $\square$

Appendix A.6. An EM–PM Analytic Method of Finding Out Whether a Curve Is a Parabola Whose Directrix Is Parallel to the x-Axis

The equation of such a parabola is $y = ax^2 + bx + c$. Any such parabola is fixed by three points on it. Choosing any three points A, B, and C whose coordinates satisfy the equation of the curve to be tested, one obtains a 3X3 system of linear equations for the coefficients a, b, and c. It is always solvable, for the points are not collinear.

Thus, if there is a line graphically represented in a computer and looking like a parabola, to check whether it really is a parabola, choose any three points A, B, and C on it, and solve the system:

$$(x(A))^2 a + x(A)b + c = y(A)$$

$$(x(A))^2 a + x(B)b + c = y(B)$$

$$(x(C))^2 a + x(C)b + c = y(C)$$

for a, b, and c and plot the graph of $y = ax^2 + bx + c$ with the values you found. If the two graphs coincide, the line in question is a parabola.

This method implies several tedious processes, which one probably will not wish to do by hand, but which are rather straightforward using DT (<https://www.geogebra.org/worksheet/edit/id/ne5keagc> (accessed on 14 May 2026)). Furthermore, the points A, B, and C are arbitrary and may be moved. The solution of the system should not be affected by this, due to the symbolic representation. Here, again, the essential features of a symbolic representation vs. a numerical one are crucial both in the formulation of the problem and in preserving invariants indispensable for the proposed DT-based reasoning.

Nevertheless, the final decision depends on the reliability of the symbolic input of coordinates and on the visual output and is within the affordances of DT. To make sure that the two lines coincide, one has to follow the values of x along the number axis to make sure that, when x ascends in absolute value, the graphs do not part, and to change resolution of the display to make sure that the lines really coincide with each other. This is subject to the limitations of a platform, though these limitations are broadly extended by a platform like GeoGebra which can afford the zoom-in and zoom-out of the graphic representation. To sum up, the method is PM-based, but any realization is EM–DT-dependent, and subject to precision considerations which belong to the AM realm.

A purely geometric method of finding the directrix and focus of the parabola by its three points exists and presents a fine problem, but it is not within the focus of this paper.

Notes

- While platforms like WM are professional-grade packages for vast scientific applications, their basic tools remain accessible for educational use, as we shall see here. However, unlike the free and easily installed Desmos or GeoGebra, WM requires a

- subscription. For broad educational implementation, “availability”, defined as being cost-free and user-friendly, is a critical factor.
2. The original applet was downloaded from <https://www.geogebra.org/m/qqvrtwzd> (accessed on 19 July 2026). Changing its parameters, the author unintentionally came upon this situation and used it for the purpose of the discussion relevant for the paper.
 3. Even if they were finite but longer than the default rounding, the output would provide an approximate value, without any indication that the decimal is finite.
 4. Should such a construction be possible, the triangle ABC would have two different heights to the base BC, since triangles ACD and ECB would both be isosceles.
 5. These values vary with dragging the vertices. Yet, as long as the example holds, the ratios are of similar orders of magnitude.
 6. To be precise, Klein wrote several descriptions, e.g., that the empirical curve is of the form $f(x) \pm \varepsilon$. But such a definition does not form a stripe, being just a vertical translation of the curve. For example, the width of such a stripe measured in the x-direction tends to 0 along the parabola, which is not consistent with what he wrote further. Therefore, we understand the $f(x) \pm \varepsilon$ description as a metaphoric idea clarified further on by the description of the stripe, and the “ $\pm \varepsilon$ ” as possible “ $\pm \varepsilon$ -fluctuations” from the curve that form a stripe of width ε around it.
 7. An EM–PM approach to ascertain that a curve parabola is outlined in Appendix A.6.

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