Implementation and Experimental Verification of Resistorless Fractional-Order Basic Filters

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Abstract: Novel structures of fractional-order differentiation and integration stages are presented in this work, where passive resistors are not required for their implementation. This has been achieved by considering the inherent resistive behavior of fractional-order capacitors. The implementation of the presented stages is performed using a current feedback operational amplifier as active element and fractional-order capacitors based on multi-walled carbon nano-tubes. Basic filter and controller stages are realized using the introduced fundamental blocks, and their behavior is evaluated through experimental results.

Keywords: fractional-order differentiators; fractional-order integrators; fractional-order filters; analog filters; inverse filters; resistorless filters; current feedback operational amplifiers

1. Introduction

Integer-order calculus deals with differentiation described by the expression \( D^nf(t) \equiv \frac{d^n f(t)}{dt^n} \), where \( n \) is the (integer) order of the operation \( (n \in \mathbb{Z}) \). In fractional calculus, the differentiation is described by a formula of the same form, \( D^\alpha f(t) \equiv \frac{d^\alpha f(t)}{dt^\alpha} \), but the order is now a real \( (\alpha \in \mathbb{R}) \) number. The Laplace transform of the fractional-order derivative of a function \( f(t) \), for zero initial conditions, in the fractional domain is: \( \mathcal{L}_{[0}D^\alpha f(t)] = s^\alpha F(s) \), with \( s^\alpha \) being the fractional-order Laplacian operator [1]. Using this, the impedance of a fractional-order capacitor is given by (1)

\[
Z = \frac{1}{C_\alpha s^\alpha},
\]

where the variable \( 0 < \alpha < 1 \) represents the order of the element and the variable \( C_\alpha \) is the pseudo-capacitance in Farad/sec\(^{1-\alpha}\) [2,3].

Fractional-order capacitors play a crucial role in the implementation of stages, which perform signal processing in the fractional domain. They have been used for implementing fractional-order controllers, filters, oscillators, biological tissues models, etc. [4–11]. Compared to their conventional integer-order counterparts, they offer the benefit of the extra degree of freedom in the design; this comes from the fact that the order is different from the other. As a result, the derived fractional-order controller structures offer better open-loop response shaping than that offered by their integer-order counterparts; fine adjustment of the slope of the attenuation gradient can be performed in the case of fractional-order filters; oscillators with extremely low/high frequency of oscillation can be implemented.

The implementation of fractional-order transfer functions can be performed in the following ways:
Substituting the capacitors of the conventional (i.e., integer order) stages by fractional-order capacitors. At this point, there are two alternatives: (a) utilizing real fractional-order capacitors [12–20] or (b) utilizing RC networks for emulating the behavior of fractional-order capacitors [21–27], and their derivation is based on the utilization of approximation tools such as Charef, Oustaloup, Matsuda, continued fraction expansion, and partial fraction expansion [28–32]. The first alternative is straightforward, but the main obstacle is the absence of commercially available fractional-order capacitors. The second one is more practical, but requires an increased number of passive components.

Approximating fractional-order transfer functions by rational-integer-order transfer functions, which are derived using the aforementioned approximation tools [33,34]. This solution suffers from the increased circuit complexity in both active and passive components count and increased power demand.

Fractional-order differentiation and integration stages play a fundamental role in the synthesis of systems, which perform signal processing based on the concept of fractional calculus [5]. Implementations are based on the substitution of integer-order capacitors in the corresponding active RC stages by their fractional-order counterparts. This approach has been successfully followed in [12–17,19,20] for implementing fractional-order differentiators/integrators as well as simple low pass filter stages. As the resistors remain unaffected, a practical problem related to the values of the resistors occurs in cases where large time constants are required, for example in biomedical or control systems, due to their low frequency nature. This can be overcome through the utilization of active elements whose behavior is described by the transconductance parameter (\(g_m\)), but this solution encounters the obstacle of being valid only under small-signal operation conditions. As a consequence, the dynamic range of the derived systems is limited.

The main contributions made in this work are the following:

(a) Novel scheme for implementing fractional-order differentiator/ integrator and filter stages are proposed in this work, where the presented structures are constructed only by capacitors and, therefore, avoiding the utilization of passive resistors.

(b) The realized time constants are formed as a ratio of a conventional and a pseudo-capacitance, overcoming the restriction of small-signal operation.

The work is organized as follows: the differentiation and integration stages are presented in Section 2, where their benefits with regards to the conventional structures are also discussed. The filter design examples are provided in Section 3. The performance of the introduced structures is evaluated through experimental results in Section 4, where a fractional-order capacitor fabricated using multi-walled carbon nanotubes (MWCNT) [18] and a current feedback operational amplifier (CFOA) as an active element, have been utilized.

2. Novel Differentiation and Integration Stages

2.1. Proposed Topologies

Let us consider the typical CFOA-based fractional-order differentiator structure in Figure 1a, where the realized transfer function is

\[ H(s) = R_1 C_\alpha s^\alpha, \]

with \(C_\alpha\) being the pseudo-capacitance (in Farad/s\(^{-1}\)) and \(0 < \alpha < 1\) being the order of the differentiator [5]. The associated time constant has the form:

\[ \tau = (R_1 C_\alpha)^{1/\alpha}. \]

In the case of the fractional-order integrator, the corresponding topology is depicted in Figure 1b and the transfer function is given by (4)
\[ H(s) = \frac{1}{R_1 C_\alpha s^\alpha}, \quad (4) \]

while the realized time constant is still given by \( (3) \).

Novel resistorless structures, alternatives to those in Figure 1, are proposed in this work. The transfer function realized by the topology in Figure 2a is

\[ H(s) = \frac{C_1}{C_\alpha} s^{1-\alpha}, \quad (5) \]

with the order of the differentiator being equal to \( 1-\alpha \). The associated time constant is

\[ \tau = \left( \frac{C_1}{C_\alpha} \right)^{\frac{1}{1-\alpha}}. \quad (6) \]

The topology in Figure 2b implements the transfer function of an integrator

\[ H(s) = \frac{1}{C_\alpha s^{1-\alpha}}, \quad (7) \]

with the associated time constant still given by the expression in \( (6) \).

![Figure 1. CFOA-based implementations of fractional-order (a) differentiator, and (b) integrator.](image)

![Figure 2. Proposed resistorless implementations of fractional-order (a) differentiator, and (b) integrator.](image)

Setting \( s^\alpha = \omega^\alpha \cdot [\cos(\alpha \pi/2) + j \sin(\alpha \pi/2)] \), the expression in \( (1) \) expands as follows

\[ Z = \frac{\cos(\alpha \pi/2)}{C_\alpha \omega^\alpha} + \frac{\sin(\alpha \pi/2)}{j C_\alpha \omega^\alpha}. \quad (8) \]

According to \( (8) \), the implementation of the time constant without using an actual resistor originates from the fact that the frequency behavior of a fractional-order capacitor is determined by both a “resistive” and a “capacitive” part, connected in series. Another important feature is that the implementation of the time constants in the resistorless structures in Figure 2 is performed without assuming small-signal operation, as it has been performed in the case of the topologies where the small-signal transconductance parameter \( (g_m) \) is employed. This is achieved at the expense of losing the electronic tuning capability, offered by the aforementioned structures.
2.2. Comparison with the Conventional Implementations

For comparison purposes, let us consider that a time constant equal to 1 s will be realized. Assuming that the fractional-order capacitor has a pseudo-capacitance equal to \(1 \text{nF/s}^{1-\alpha}\), then according to (3) the value of the resistor must be equal to \(1 \text{G\Omega}\), which is actually non-practical. Using (6) it is derived that an integer-order capacitor equal to \(1\text{nF}\) is required and this is a reasonable value of capacitance.

Due to the absence of commercially available fractional-order capacitors, these elements can be approximated through the utilization of suitable passive RC networks such as the Foster and Cauer structures. In the cases, where the order has a relatively high value (i.e., \(\alpha > 0.5\)) the practical problem, which arises is related to the increased spread of element values. A solution for overcoming this obstacle has been proposed in the literature, where the transfer function of the differentiator/integrator is formed by the product of the transfer functions of an integrator/differentiator of order equal to \(1-\alpha\) and of an integer-order differentiator/integrator, significantly increasing (almost doubling) the circuit complexity. The topologies in Figure 2 do not suffer from this problem because the order of capacitors for implementing differentiators/integrators of orders greater that 0.5 should be less than 0.5, as it is readily concluded from the transfer functions in (5) and (7). In the opposite cases, the topologies in Figure 1 are more preferable than the proposed ones. For demonstration purposes, let us consider the approximation of a capacitor of order \(\alpha = 0.8\) and pseudo-capacitance \(C_\alpha = 1 \mu\text{F/s}^{0.2}\). Using the Oustaloup approximation in the range \([1, 10^4]\) rad/s, the passive elements values of the corresponding Foster-I network, rounded to the E48 series defined in IEC 60063, are summarized in Table 1. The corresponding ones in the case of a capacitor with the same value of the pseudo-capacitance and order \(\alpha = 0.2\) are provided in Table 2, where the reduction in the spread of the component values is evident.

| Table 1. Passive elements of a Foster-I network for approximating a fractional-order capacitor with pseudo-capacitance \(C_\alpha=1 \mu\text{F}\) and order \(\alpha = 0.8\). |
|---|---|---|
| Element | Value | Element | Value |
| \(R_p\) | 100 \text{\Omega} | \(R_1\) | 348 \text{\Omega} |
| \(R_2\) | 3.48 \text{k\Omega} | \(C_1\) | 332 \text{nF} |
| \(R_3\) | 31.6 \text{k\Omega} | \(C_1\) | 562 \text{nF} |
| \(R_4\) | 301 \text{k\Omega} | \(C_1\) | 953 \text{nF} |
| \(R_5\) | 5.9 \text{M\Omega} | \(C_1\) | 1.62 \text{\mu\text{F}} |

| Table 2. Passive elements of a Foster-I network for approximating a fractional-order capacitor with pseudo-capacitance \(C_\alpha=1 \mu\text{F}\) and order \(\alpha = 0.2\). |
|---|---|---|
| Element | Value | Element | Value |
| \(R_p\) | 100 \text{\text{k\Omega}} | \(R_1\) | 71.5 \text{\Omega} |
| \(R_2\) | 127 \text{\text{k\Omega}} | \(C_1\) | 0.75 \text{nF} |
| \(R_3\) | 215 \text{\text{k\Omega}} | \(C_1\) | 6.81 \text{nF} |
| \(R_4\) | 383 \text{\text{k\Omega}} | \(C_1\) | 61.9 \text{nF} |
| \(R_5\) | 681 \text{\text{k\Omega}} | \(C_1\) | 562 \text{nF} |

3. Filter Design Examples

3.1. Standard Filter Functions

The structures in Figure 2 will be utilized for realizing low pass and high pass filters. The resulting topologies are demonstrated in Figure 3a,b, respectively, with the realized transfer functions being

\[
H_{LP}(s) = \frac{1}{(\tau s)^{1-\alpha} + 1},
\]
for the low pass filter, and

$$H_{LP}(s) = \frac{(\tau s)^{1-\alpha}}{(\tau s)^{1-\alpha} + 1}, \quad (10)$$

for the high pass filter \[22\]. The time constant in (9) and (10) is given by (6) and is associated with the pole frequency ($\omega_0$) according to the formula: $\tau = 1/\omega_0$.

The half-power frequency ($\omega_{h}$) is different from the pole frequency and is calculated according to (11)

$$\omega_{h,LP} = \frac{1}{\tau} \left[ \sqrt{1 + \cos^2 \left( \frac{(1-\alpha)\pi}{2} \right)} - \cos \left( \frac{(1-\alpha)\pi}{2} \right) \right]^{\frac{1}{1-\alpha}} \quad (11)$$

for the low pass filter, with the phase at this frequency given by (12)

$$\angle H_{LP}(j\omega)_{\omega = \omega_{h,LP}} = -\tan^{-1} \left[ \frac{\sin \left( \frac{(1-\alpha)\pi}{2} \right)}{2 \cos \left( \frac{(1-\alpha)\pi}{2} \right) + \sqrt{1 + \cos^2 \left( \frac{(1-\alpha)\pi}{2} \right)}} \right]. \quad (12)$$

The corresponding expressions of the high pass filter are provided by (13) and (14)

$$\omega_{h,HP} = \frac{1}{\tau} \left[ \sqrt{1 + \cos^2 \left( \frac{(1-\alpha)\pi}{2} \right)} + \cos \left( \frac{(1-\alpha)\pi}{2} \right) \right]^{\frac{1}{1-\alpha}} \quad (13)$$

$$\angle H_{HP}(j\omega)_{\omega = \omega_{h,HP}} = \frac{(1-\alpha)\pi}{2} - \tan^{-1} \left[ \frac{\sin \left( \frac{(1-\alpha)\pi}{2} \right)}{\sqrt{1 + \cos^2 \left( \frac{(1-\alpha)\pi}{2} \right)}} \right]. \quad (14)$$

Figure 3. Implementations of fractional-order (a) low pass and (b) high pass filters using the structures in Figure 2.

The filter structures in Figure 3 have both a maximum gain equal to one; in cases where a value different from one is required, the topologies in Figure 4 are suitable for this purpose.
3.2. Inverse Filter Functions

Inverse filters can be used for recovering distorted signals in signal processing systems, because they have reciprocal frequency characteristics with regards to their conventional counterparts. Originated from their non-integer nature, fractional-order inverse filters offer fine adjustment of the loop shaping, performed through the adjustment of the order of the filter; also, the realized time constants can be a scaled version of those realized by their integer-order counterparts. Fractional-order inverse filters have been introduced using various types of active elements and approximating the behavior of fractional-order capacitors by suitable RC networks [23–27]. The approximation of inverse fractional-order filters at transfer function level has been performed in [35], resulting in topologies of increased complexity paying the price of the increased flexibility. Electronically tunable fractional-order inverse filters structures have also been introduced in [36].

Using a fractional-order capacitor, the topology in Figure 5a implements the transfer function of a fractional-order inverse low pass filter, given by (17)

$$H_{I,LP}(s) = K \cdot \left[ 1 + (\tau s)^{1-\alpha} \right],$$  \hspace{1cm} (17)

with $K = C_2/C_1$, and the time constant given by (6) [24].

The filter at low-frequencies has gain equal to $K$, while at the high-frequency range behaves as a fractional-order differentiator of order equal to $1-\alpha$.

The characteristic frequency ($\omega_{c,LP}$), where a 3 dB increase from the minimum gain of the filter is observed, is given by (11), while the phase at this frequency is the opposite of that given in (12) [23,25,36].

The transfer function of the scheme in Figure 5b is [24]

$$H_{I,HP}(s) = K \cdot \frac{1 + (\tau s)^{1-\alpha}}{(\tau s)^{1-\alpha}}. \hspace{1cm} (18)$$

The filter at low frequencies behaves as integrator of order equal to $1-\alpha$, while at high frequencies has gain equal to $K = C_1/C_2$. The characteristic frequency ($\omega_{c,HP}$), where a 3 dB drop from the maximum gain of the filter is observed, is given by (13), with the associated phase being the opposite of that in (14).
Figure 5. Implementations of fractional-order inverse (a) low pass, and (b) high pass filters.

4. Experimental Results

The performance of the presented topologies has been evaluated using solid state fractional-order capacitors. They have been fabricated using multi-walled carbon nanotubes (MWCNTs), and their order is determined by the percentage of the carbon nano-tubes (CNTs) into the epoxy resin [18]. The AD844 discrete component IC, biased at \( \pm 15 \text{ V} \), has been used as CFOA [37]. The experimental setup is demonstrated in Figure 6. Using a sample of fractional-order capacitor of order \( \alpha = 0.74 \) and pseudocapacitance \( C_\alpha = 5.25 \text{ nF/s}^{0.26} \), which has a frequency range of operation \([10 \text{ kHz}, 100 \text{ kHz}]\), the value of the integer-order capacitor, calculated through (6), for realizing a unity gain frequency equal to 20 kHz, was \( C_1 = 248 \text{ pF} \). The actual value of the capacitance, which has been used in the experiment, selected from the E48 series, was 249 pF, as it was selected from the E48 series defined in IEC 60063. The input and output waveforms in the case of the differentiator in Figure 2a stimulated by a sinusoidal of amplitude and frequency \( (3 \text{ V}_{p-p}, 20 \text{ kHz}) \), derived using the Matlab software (R2022b), are demonstrated in Figure 7a, and correspond to the theoretically predicted ones. The corresponding experimental waveforms, observed using the Agilent DSO 6034A oscilloscope, are depicted in Figure 7b. The measured gain was 1.03 and the phase difference was 22.1°, close to the theoretically predicted values 1 and 23.4°, respectively. In the case of a square wave stimulation, the corresponding plots are provided in Figure 8a,b, respectively.

Figure 6. Experimental setup for evaluating the performance of the proposed filters.
Figure 7. Input and output waveforms of a fractional-order differentiator of order 0.26, realized by the topology in Figure 2a, for a sinusoidal ($3\,V_{p-p}$, 20 kHz) stimulus (a) theoretical, and (b) experimental.

Figure 8. Input and output waveforms of a fractional-order differentiator of order 0.26, realized by the topology in Figure 2a, for a square wave ($3\,V_{p-p}$, 20 kHz) stimulus (a) theoretical and (b) experimental.

The corresponding waveforms of a fractional-order integrator with the same order and unity gain frequency, stimulated by the same signals, are depicted in Figures 9a,b and 10a,b. The measured values of gain was 1.02 and the phase difference was $-21.4^\circ$, close to the theoretically predicted ones that are equal to 1 and $-23.4^\circ$, respectively.

Figure 9. Input and output waveforms of a fractional-order integrator of order 0.26, realized by the topology in Figure 2b, for a sinusoidal ($3\,V_{p-p}$, 20 kHz) stimulus (a) theoretical, and (b) experimental.
Figure 10. Input and output waveforms of a fractional-order integrator of order 0.26, realized by the topology in Figure 2b, for a square wave (3 V_{pp}, 20 kHz) stimulus (a) theoretical, and (b) experimental.

Considering the low pass filter structure in Figure 4a with half-power frequency ($f_{h,LP}$) equal to 20 kHz and maximum gain equal to 0.5, the values of the integer-order capacitors will be $C_1 = 109$ pF and $C_2 = 218$ pF, as they are derived from (6), (11), and (15). The values of capacitors employed in experiments were 110 pF and 215 pF, respectively. In the case of the high pass filter in Figure 4b, with half-power frequency ($f_{h,HP}$) equal to 20 kHz and maximum gain equal to 0.5, the calculated values of both integer-order capacitors through the utilization of (6), (13) and (16), were $C_1 = 564$ pF and $C_2 = 282$ pF. The actual values of capacitors employed in experiments were 562 pF and 287 pF, respectively. The obtained theoretical and experimental gain and phase frequency responses are provided in Figure 11a,b; it must be mentioned at this point that measurements in the range [10 kHz, 100 kHz] are available due to the aforementioned limited frequency range of the fractional-order capacitor. The time-domain behavior of the filters has been evaluated through the stimulation of them by a 3 V_{pp} sinusoidal signal and frequency equal to the half-power frequency (i.e., 20 kHz). The derived waveforms are provided in Figure 12a,b, and the measured gain and phase of the low pass filter were 0.39 and $-7^\circ$, with the corresponding theoretical values being 0.35 and $-7.1^\circ$, respectively. The corresponding results of the high pass filter were 0.37 and 8.2°, close to the theoretically predicted ones 0.35 and 7.1°. The DC power dissipation of the filters was 0.6 mW.

Figure 11. Frequency responses of filters of order 0.26, realized by (a) the topology in Figure 4a, and (b) the topology in Figure 4b.
Figure 12. Input and output waveforms of filters of order 0.26, realized by (a) the topology in Figure 4a, and (b) the topology in Figure 4b, stimulated by a (3 V<sub>p-p</sub>, 20 kHz) sinusoidal input.

In the case of the inverse filters topologies in Figure 5, the same procedure as above has been followed. Considering a minimum/maximum gain equal to one and a characteristic frequency of 20 kHz for both filters, then the required integer-order capacitors will be both $C_1 = C_2 = 109$ pF (110 pF was the employed value) for the inverse low pass filter and $C_1 = C_2 = 564$ pF (562 pF the employed value) for the inverse high pass filter. The corresponding frequency and time-domain responses are depicted in Figures 13a,b and 14a,b, respectively. The set of values of gain and phase, obtained from the waveforms in Figure 14a,b, are $(1.39, 7.8^\circ)$ and $(1.48, -7.9^\circ)$, while the theoretical ones are $(1.41, 7.1^\circ)$ and $(1.41, -7.1^\circ)$, respectively.
Figure 14. Input and output waveforms of inverse filters of order 0.26, realized by (a) the topology in Figure 5a, and (b) the topology in Figure 5b stimulated by a (3 Vp-p, 20 kHz) sinusoidal input.

In order to evaluate the effect of the choice of the series of capacitor values, let us consider the case that the capacitances are chosen from the E12 series. In the obtained sets are {100 pF, 220 pF}, {560 pF, 270 pF}, {100 pF, 10 pF}, and {560 pF, 560 pF} for the low pass, high pass, inverse low pass, and inverse high pass filters, respectively. The corresponding frequency responses are provided in Figure 15a–d, with the half-power frequencies being 23.5 KHz, 19.3 kHz, 24.1 kHz, and 20.1 kHz, respectively. Taking into account that the theoretically expected value is 20 KHz, it seems that the filter topologies have reasonable sensitivity characteristics.

Figure 15. Cont.
Figure 15. Evaluation of the effect of the error in the capacitors values on the frequency response of the (a) low pass, (b) high pass, (c) inverse low pass, and (d) inverse high pass filters.

5. Conclusions

The resistorless structures of basic signal processing blocks in the fractional calculus domain have been experimentally tested using real fractional-order capacitors and their behavior is in excellent agreement with the theory. The proposed concept is independent of the type of the fractional capacitor used, as well as the type of active elements, making it suitable for large-signal operation. Limitations are imposed by the frequency range of operation and by the linearity of the samples of fractional-order capacitors, both determined by the fabrication process used for their implementation, and by the employed active elements. This technique is also particularly promising for implementing systems suitable for low-frequency applications, including control, biology/bio-medicine applications due to the nature of the formed time-constants expressions. Further research steps include the comparison of the performance of the proposed structures with the corresponding switched-capacitor fractional-order stages, which are another type of large-signal-based resistorless implementation of fractional-order filters [38].

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Abbreviations
The following abbreviations are used in this manuscript:

CFOA Current Feedback Operational Amplifier
CNT Carbon Nano-Tubes
MWCNT Multi-Walled Carbon Nano-Tubes
RC Resistor Capacitor
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