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Modulus Waveform Design Based on Manifold ADMM Idea in Dual-Function Radar–Communication System

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Abstract: In this paper, we try to design the joint waveform and passive beamforming within the context of dual-function radar–communication (DFRC) systems. Focusing on the intricate trade-off between stringent radar beampattern constraints and their desired performance, we introduce a novel manifold idea based on the alternating direction method of multipliers (ADMM) framework. Specifically, our proposed method, named DFRC-MA, could address the challenge of constant modulus waveform design in a multiple-input–multiple-output (MIMO) DFRC system. Firstly, our methodology begins by formulating the reference waveform to achieve an optimal radar beamforming pattern. Subsequently, we define the DFRC optimization problem to mitigate the multi-user interference (MUI) under the constant modulus constraint. Through a series of simulations, we evaluate the efficacy of DFRC-MA, where the integrated waveform designed by DFRC-MA exhibits superior performance over some prevalent ones.

Keywords: dual-function radar–communication system; waveform design; manifold optimization; constant modulus; alternating direction multiplier method



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1. Introduction

Waveform design in dual-function radar–communication (DFRC) systems has drawn a lot of attention in the last 5 years [1–3]. Therein, the dual-function integrated waveform of multiple-input–multiple-output (MIMO)-based DFRC outperforming the traditional phased-array has been the focus [4–6]. Typically, in DFRC systems, most communication signals using multicarrier modulation would inevitably lead to high peak average power ratio (PAPR) and great distortion to affect the transmission efficiency. Zhang et al. aimed at optimizing the DFRC waveform with low PAPR and low-range side lobes [7]. The authors in [8] considered DFRC waveform designed with PAPR constraint to minimize the downlink multiuser multi-user interference (MUI), which can be regarded as a quadratically constrained quadratic programs (QCQP) problem and further derived into a convex problem. To solve this, the authors in [9] utilized the constant modulus (CM) waveform and proposed an efficient branch-and-bound (BnB) algorithm, while only minimizing MUI could satisfy the beampattern similarity constraint but with high computational complexity. In addition, the authors in [10] discussed the spatial beamforming problem to achieve the dual-functional base station (BS) detecting and communicating, by defining the achievable performance region and formulating a radar–communication-centric optimization model, respectively. Meanwhile, the authors in [11] developed a space–time coding scheme where their shape approximation and integrated power approximation criteria were adopted, respectively, under constant-envelope constraint. Based on this space–time coding scheme, the direct constellation mapping and phase-rotation constellation mapping methods are proposed. Moreover, Du and Liu et al. characterized the multifold trade-offs between

communication and radar sensing by defining an achievable performance region, and also formulated a fairness profile optimization problem [12].

Different from using radar waveforms to achieve integration, the authors in [13,14] used existing communication waveforms to achieve radar sensing tasks. From the communication perspective, DFRC signals need to have high-quality communication performance, such as a higher communication rate. The communication rate can be improved by minimizing multiuser interference. In [15], a DFRC waveform design model was proposed that reduces MUI under the constraint of similarity between the integrated waveform and the reference radar waveform. That is, under the given reference radar waveform covariance matrix and total power constraints, MUI is minimized to design a DFRC waveform. DFRC waveform design is more challenging when considering scenarios involving multiple communication users and multiple radar-sensing targets. For communication, the SNR of each user needs to be considered. For radar, the SNR of each target also needs to be considered. For the entire DFRC system, the performance of communication and sensing is coupled together, which means that any improvement in communication may deteriorate radar performance, and vice versa. Therefore, in reality, when the DFRC system works in a multiuser and multitarget scenario, it inevitably faces multiple performance trade-offs, including trade-offs between multiple users, multiple targets, and between communication and perception. In order to achieve a good performance trade-off between radar and communication, it is necessary to design a good DFRC waveform [16].

Usually, a DFRC waveform often develops a referenced radar waveform to satisfy radar beamforming patterns by formulating the covariance matrix and other constraints [17]. The authors in [18] investigated the CM waveform design for DFRC systems in the presence of clutter. To minimize the interference power and enhance the target acquisition performance, they tried to use the signal-to-interference-plus-noise-ratio (SINR) as the metric, and enforced a constraint on the synthesis error of every communication signal. To obtain an MIMO-based CM waveform with good beamforming traits, the authors in [19] tried to minimize the minimization square error of the designing beam and the desired one, and proposed an efficient alternating direction method of multipliers (ADMM) algorithm, while it cannot satisfy the strict CM constraint. But this idea gave us some enlightenment. In the context of DFRC systems, various approaches have been proposed to balance radar and communication performance. These approaches differ in the practical constraints they address and in the mathematical techniques they employ, as shown in Table 1.

Table 1. Categorization of these studies.

References	Practical Waveform and System Constraints	Mathematical Optimization Approach	Key Findings
[8]	Low PAPR, low-range side lobes	QCQP	Optimized waveform with low PAPR and low-range side lobes.
[9]	PAPR constraint, minimize MUI	BnB	Satisfied beampattern similarity constraint with high computational complexity.
[11]	Space–time coding, constant-envelope constraint	Constellation mapping methods	Proposed direct constellation mapping and phase-rotation constellation mapping methods.
[12]	Multiple users, multiple targets	Fairness profile optimization	Characterized trade-offs between communication and radar sensing.
[19]	Clutter, SINR Metric	ADMM algorithm	Proposed efficient ADMM algorithm for CM waveform design.
Proposed Work	Constant modulus, multiuser interference	Manifold ADMM	Achieved superior performance with DFRC-MA compared to prevalent methods.

In this paper, we try to design a DFRC waveform based on a manifold-ADMM framework. The main contributions of this paper are summarized as follows:

- **Novel manifold-based ADMM framework:** We introduce the manifold-ADMM framework, a novel approach that leverages manifold optimization techniques within the ADMM framework. This innovation addresses the complex problem of constant modulus waveform design in MIMO DFRC systems, ensuring optimal radar beamforming while maintaining stringent beampattern constraints.
- **Enhanced radar–communication integration:** Our proposed method effectively balances the trade-off between radar and communication performance. By formulating and solving the DFRC optimization problem under the constant modulus constraint, DFRC-MA significantly mitigates MUI, resulting in superior integrated waveform performance compared to existing methods.
- **Superior simulation results:** Through extensive simulations, we demonstrate the efficacy of the DFRC-MA approach. The integrated waveform designed by DFRC-MA not only exhibits excellent matching performance with the desired radar beampattern but also showcases enhanced radar communication integration, outperforming prevalent waveform design methods in both radar and communication functionalities.

The organization of this paper is as follows. The system model and problem formulation are presented in Section 2. The DFRC waveform design by the manifold-ADMM idea is proposed in Section 3. Section 4 presents our numerical results. Finally, our conclusions are drawn in Section 5.

Notation: Lowercase letters $\mathbf{x}$ and uppercase letters $\mathbf{X}$ denote vectors and matrices, respectively. The symbols $(\cdot)^T$, $(\cdot)^H$, and $(\cdot)^*$ stand for the transpose, the conjugate transpose, and the conjugate operators, respectively. The n -th element of a vector $\mathbf{x}$ is written as $\mathbf{x}(n)$. $\text{vec}(\mathbf{X})$ denotes the column vector obtained by stacking columns of matrix $\mathbf{X}$ on top of one another. The set of $N \times N$ complex matrices and the set of n -dimensional complex vectors are denoted by $\mathbb{C}^{N \times N}$ and $\mathbb{C}^n$, respectively. The l_2 norm and the l_1 norm are, respectively, denoted by the symbols $\|\cdot\|_2$ and $\|\cdot\|_1$, and the Frobenius norm is represented by the symbol $\|\cdot\|_F$. $\mathbf{I}_N$ stands for the identity matrix of size $N \times N$. Finally, we use $\text{Re}\{x\}$ and $\text{Im}\{x\}$ to denote the real and imaginary part of x ; $\mathbb{E}(\cdot)$ means the expectation operator; and the symbol $\otimes$ stands for Kronecker product.

2. System Model and Problem Formulation

Considering a narrow-band DFRC system, as shown in Figure 1, the system incorporates an N -antenna uniform linear array (ULA) and serves K single antenna communication users in the downlink while detecting targets simultaneously.

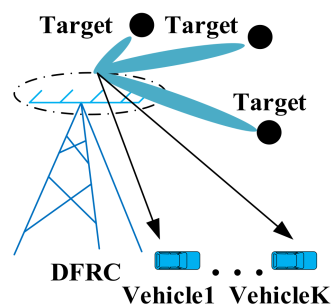


Figure 1. The DFRC system.

2.1. Communication Part

Suppose that $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_K]^T \in \mathbb{C}^{K \times N}$ denotes the downlink channel matrix from the transmitter to K users, and $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_L] \in \mathbb{C}^{K \times L}$ is the noise matrix corresponding to Gaussian distribution with $\mathbf{w}_j \sim \mathcal{CN}(0, N_0 \mathbf{I}_N)$. Defining $\mathbf{x}_l = [x_{1,l}, x_{2,l}, \dots, x_{N,l}]^T$ as the N -dimensional spatial transmission data at time l , $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_L] \in \mathbb{C}^{N \times L}$ represents the transmitted signal matrix, L means the number of emitted waveforms during

one pulse, and the compact formulation of all users' signal matrix within one pulse can be written as

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}. \quad (1)$$

Furthermore, define $\mathbf{S} \in \mathbb{C}^{K \times L}$ as the expected signal for K users; then the received signal matrix of (1) can be represented as

$$\mathbf{Y} = \mathbf{S} + \underbrace{(\mathbf{H}\mathbf{X} - \mathbf{S})}_{\text{MUI}} + \mathbf{W}, \quad (2)$$

The second term on the right-hand side of (2) is called MUI. Additionally, the SINR for the i -th user is given by

$$\eta_i = \frac{E(|s_{i,l}|^2)}{E(\underbrace{|\bar{\mathbf{h}}_i^T \mathbf{x}_l - s_{i,l}|^2}_{P_{\text{MUI}}}) + N_0}, \quad (3)$$

where $|\bar{\mathbf{h}}_i^T \mathbf{x}_l - s_{i,l}|^2$ in the denominator part means the element of MUI. We propose a typical metric to evaluate the communication performance loss, i.e., the achievable sum-rate,

$$R_{\text{com}} = \sum_{i=1}^K \log_2(1 + \eta_i). \quad (4)$$

Namely, by minimizing the MUI energy, we could obtain the expected communication performance. To avoid the nonlinear distortion of power amplifier and maximize the working efficiency of DFRC's transmitter, the CM waveform must be considered [20]. As shown in [18], the classical CM waveform can be formulated as

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{H}\mathbf{X} - \mathbf{S}\|_F^2 \\ \text{s.t.} \quad & \|\mathbf{X} - \mathbf{X}_0\|_F^2 \leq \eta_{\min}, \\ & |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \end{aligned} \quad (5)$$

where $\mathbf{X}_0 \in \mathbb{C}^{N \times L}$ is the referenced waveform with some desirable properties, and P_0 denotes the total transmission power. To enhance the trade-off performance, the permissible square WSE norm $\eta_{\min}$ is proposed.

2.2. Radar Part

Assuming that a far-field target is located at the θ direction, the transmitted signal in the θ direction can be given by

$$\mathbf{y}(\theta) = \mathbf{a}_T^T(\theta)\mathbf{X}, \quad (6)$$

where $\mathbf{a}_T(\theta) = [1, e^{-j\pi \sin \theta}, \dots, e^{-j\pi(N-1) \sin \theta}]^T \in \mathbb{C}^{N \times 1}$ denotes the steering vector. Thus, the transmitted power at θ can be expressed as

$$\begin{aligned} P(\theta) &= \|\mathbf{y}(\theta)\|_2^2 \\ &= \mathbf{x}^H (\mathbf{I}_L \otimes \mathbf{a}_T^T(\theta))^H (\mathbf{I}_L \otimes \mathbf{a}_T(\theta)) \mathbf{x} \\ &= \mathbf{x}^H \mathbf{R}(\theta) \mathbf{x}, \end{aligned} \quad (7)$$

where $\mathbf{x} = \text{vec}(\mathbf{X})$, and $\mathbf{R}(\theta) = (\mathbf{I}_L \otimes \mathbf{a}_T^T(\theta))^H (\mathbf{I}_L \otimes \mathbf{a}_T(\theta))$. The MIMO radar transmission pattern design mostly adopts the pattern matching design principle. In the least squares framework, the waveform is designed to match the transmission pattern with the expected pattern and minimize the cross-correlation pattern [19]. This paper uses weighted least squares as the cost function, which can be expressed as

$$f(\mathbf{x}) = \frac{1}{J} \sum_{j=1}^J \left| \alpha^2 D(\theta_j) - \mathbf{x}^H \mathbf{R}(\theta_j) \mathbf{x} \right|^2, \quad (8)$$

where α is a scaling parameter to be optimized, $D(\theta_k)$ represents the desired beampattern, and J means the number of sampled angles over the region of interest. To simplify notations under CM, Equation (8) can be expressed as

$$\begin{aligned} \min_{\mathbf{z}} f(\mathbf{x}) &= \frac{1}{J} \sum_{j=1}^J |\mathbf{z}^H \mathbf{A}(\theta_j) \mathbf{z}|^2 \\ \text{s.t. } |x_{n,l}| &= \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \end{aligned} \quad (9)$$

where

$$\mathbf{z} \triangleq \begin{pmatrix} \alpha \\ \mathbf{x} \end{pmatrix}, \quad (10)$$

$$\mathbf{A}(\theta_k) \triangleq \begin{pmatrix} D(\theta_k) & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{R}(\theta_k) \end{pmatrix}. \quad (11)$$

Typically, the problem under CM is also a nonconvex fourth-order polynomial. By optimizing the problem of (9), we could obtain the desired $\mathbf{X}_0 \in \mathbb{C}^{N \times L}$ as the referenced waveform.

2.3. DFRC Trade-Off Model between Radar and Communication

Combining the problems of (5) and (8) together, we mainly discuss the trade-off model between radar and communication systems. Namely, the trade-off optimization problem of DFRC is formulated as

$$\begin{aligned} \min_{\mathbf{x}, \alpha} \quad & w \|\mathbf{H}\mathbf{X} - \mathbf{S}\|_F^2 + (1-w) \frac{1}{J} \sum_{j=1}^J |\alpha^2 D(\theta_j) - \mathbf{x}^H \mathbf{R}(\theta_j) \mathbf{x}|^2 \\ \text{s.t.} \quad & \|\mathbf{X} - \mathbf{X}_0\|_F^2 \leq \eta_{\min}, \\ & |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \end{aligned} \quad (12)$$

where $w \in [0, 1]$ is the weighting factor to balance radar and communication performances.

Remark 1: By considering communication performance solely when $w = 1$, and radar performance exclusively when $w = 0$, the judicious selection of weighting factors enables the achievement of a well-balanced performance.

3. DFRC Waveform Design by Alternating Sequential Optimization Manifold-ADMM Idea

To tackle the problem of (12), firstly, we borrow an auxiliary variable $\mathbf{X}_0$ to satisfy the second component $\frac{1}{J} \sum_{j=1}^J |\alpha^2 D(\theta_j) - \mathbf{x}^H \mathbf{R}(\theta_j) \mathbf{x}|^2$ of (12); the original problem could be divided into two parts:

$$\begin{aligned} P_1 \quad & \min_{\mathbf{x}, \mathbf{X}_0, \alpha} \frac{1}{J} \sum_{j=1}^J |\alpha^2 D(\theta_j) - \mathbf{x}^H \mathbf{R}(\theta_j) \mathbf{x}|^2 \\ \text{s.t.} \quad & |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \\ & \mathbf{X}_0 \sim D(\theta_j), \end{aligned} \quad (13)$$

$$\begin{aligned} P_2 \quad & \min_{\mathbf{x}} w \|\mathbf{H}\mathbf{X} - \mathbf{S}\|_F^2 + (1-w) \|\mathbf{X} - \mathbf{X}_0\|_F^2 \\ \text{s.t.} \quad & |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \end{aligned} \quad (14)$$

For the problem of P_1 , its output $\mathbf{X}$ is the referenced waveform $\mathbf{X}_0$. To obtain the reference waveform $\mathbf{X}_0$, a manifold-ADMM algorithm is proposed to achieve optimal radar performance. Then, the CM waveform of (14) would be optimized by the sequential manifold-ADMM algorithm.

3.1. Auxiliary Variable Calculation

To solve the problem of (13), we decouple the fourth-order polynomial by introducing an auxiliary primal variable $\mathbf{h} \in \mathbb{C}^{NL+1}$, which is defined in the manner of the Riemannian manifold [21]; thus, the problem (9) is equivalent to

$$\begin{aligned} \min_{\mathbf{z}, \mathbf{h}} \quad & F(\mathbf{z}, \mathbf{h}) = \frac{1}{K} \sum_{k=1}^K w_k \left| \mathbf{h}^H \mathbf{A}(\theta_k) \mathbf{z} \right|^2 \\ \text{s.t.} \quad & \mathbf{h} - \mathbf{z} = \mathbf{0}, \\ & \mathbf{h} \in S_h, \mathbf{z} \in S_z, \end{aligned} \quad (15)$$

where S_h, S_z denotes the Riemannian manifold sets with CM. Notice that the objective function of (15) is bi-convex, i.e., it is convex in $\mathbf{h}$ for each $\mathbf{z}$ and convex in $\mathbf{z}$ for each $\mathbf{h}$. So that we can use the ADMM idea in Riemannian manifold framework, to solve (15), we introduce one dual variable $\lambda_1 \in \mathbb{C}^{NL+1}$, and then the augmented Lagrange function can be expressed as

$$L(\mathbf{z}, \mathbf{h}, \lambda_1) = F(\mathbf{z}, \mathbf{h}) + \frac{\rho}{2} \|\mathbf{h} - \mathbf{z} + \lambda_1\|_2^2. \quad (16)$$

At the $(m+1)$ -th iteration, the manifold-ADMM consists of the following iterations:

$$\mathbf{h}^{(m+1)} := \underset{\mathbf{h} \in S_h}{\operatorname{argmin}} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}), \quad (17)$$

$$\mathbf{z}^{(m+1)} := \underset{\mathbf{z} \in S_z}{\operatorname{argmin}} L(\mathbf{z}, \mathbf{h}^{(m+1)}, \lambda_1^{(m)}), \quad (18)$$

$$\lambda_1^{(m+1)} := \lambda_1^{(m)} + \mathbf{h}^{(m+1)} - \mathbf{z}^{(m+1)}. \quad (19)$$

Taking (16) into (17), we could obtain

$$\mathbf{h}^{(m+1)} := \underset{\mathbf{h} \in S_h}{\operatorname{argmin}} F(\mathbf{z}^{(m)}, \mathbf{h}) + \frac{\rho}{2} \|\mathbf{h} - \mathbf{z}^{(m)} + \lambda_1^{(m)}\|_2^2, \quad (20)$$

Since the problem of (20) is convex about $\mathbf{h}$, its closed-form solution can be obtained by the first derivative condition

$$\mathbf{0} = \operatorname{grad}_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}), \quad (21)$$

where $\operatorname{grad}_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)})$ is the Riemannian gradient of augmented Lagrange function with respect to $\mathbf{h}^*$ in Riemannian space. S_h is a Riemannian submanifold of Euclidean space $\mathbb{C}^{NL+1 \times 1}$. The relationship between Riemannian gradient and Euclidean gradient can be obtained by orthogonal projection of Euclidean gradient $\nabla_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)})$ to tangent space $T_h S_h$, i.e.,

$$\begin{aligned} \operatorname{grad}_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}) &= \operatorname{Proj}_{\mathbf{h}^*} \left(\nabla_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}) \right) \\ &= \nabla_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}) - \Re \left\{ \nabla_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}) \odot \mathbf{h} \right\} \odot \mathbf{h}^*. \end{aligned} \quad (22)$$

Solving the European gradient, we could have

$$\begin{aligned} \nabla_{\mathbf{h}^*} F(\mathbf{z}^{(m)}, \mathbf{h}) &= \left(\frac{1}{K} \sum_{k=1}^K w_k \mathbf{A}(\theta_k) \mathbf{z}^{(m)} \mathbf{z}^{(m)H} \mathbf{A}^H(\theta_k) \right) \mathbf{h}, \\ \nabla_{\mathbf{h}^*} \left\| \mathbf{h} - \mathbf{z}^{(m)} + \lambda_1^{(m)} \right\|_2^2 &= \mathbf{h} - \left(\mathbf{z}^{(m)} - \lambda_1^{(m)} \right), \end{aligned} \quad (23)$$

$$\nabla_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}) = \mathbf{0} \Rightarrow \operatorname{grad}_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}) = \mathbf{0}, \quad (24)$$

$$\begin{aligned}
& \nabla_{\mathbf{h}^*} L(\mathbf{z}^{(m)}, \mathbf{h}, \lambda_1^{(m)}) \\
&= \nabla_{\mathbf{h}^*} F(\mathbf{z}^{(m)}, \mathbf{h}) + \frac{\rho}{2} \nabla_{\mathbf{h}^*} \left\| \mathbf{h} - \mathbf{z}^{(m)} + \lambda_1^{(m)} \right\|_2^2 \\
&= \left(\left(\frac{1}{K} \sum_{k=1}^K w_k \mathbf{A}(\theta_k) \mathbf{z}^{(m)} \mathbf{z}^{(m)\text{H}} \mathbf{A}^{\text{H}}(\theta_k) \right) + \frac{\rho}{2} \mathbf{I} \right) \mathbf{h} - \left(\frac{\rho}{2} (\mathbf{z}^{(m)} - \lambda_1^{(m)}) \right) \\
&= \mathbf{0}.
\end{aligned} \tag{25}$$

Finally, the closed-form solution can be represented as

$$\tilde{\mathbf{h}}^{(m+1)} = \left(\left(\frac{1}{K} \sum_{k=1}^K w_k \mathbf{A}(\theta_k) \mathbf{z}^{(m)} \mathbf{z}^{(m)\text{H}} \mathbf{A}^{\text{H}}(\theta_k) \right) + \frac{\rho}{2} \mathbf{I} \right)^{-1} \left(\frac{\rho}{2} (\mathbf{z}^{(m)} - \lambda_1^{(m)}) \right). \tag{26}$$

That is, the updating of each point of (26) can be expressed as

$$\mathbf{h}^{(m+1)} = \sqrt{\frac{P_0}{NL}} \left[\sqrt{\frac{NL}{P_0}} \tilde{\mathbf{h}}_1^{(m+1)}, \frac{\tilde{\mathbf{h}}_2^{(m+1)}}{\left\| \tilde{\mathbf{h}}_2^{(m+1)} \right\|_2}, \dots, \frac{\tilde{\mathbf{h}}_{NL+1}^{(m+1)}}{\left\| \tilde{\mathbf{h}}_{NL+1}^{(m+1)} \right\|_2} \right]. \tag{27}$$

We could obtain $\mathbf{z}^{(m+1)} := \underset{\mathbf{z} \in \mathcal{S}_z}{\operatorname{argmin}} F(\mathbf{z}, \mathbf{h}^{(m+1)}) + \frac{\rho}{2} \left\| \mathbf{h}^{(m+1)} - \mathbf{z} + \lambda_1^{(m)} \right\|_2^2$ which is also a convex function problem about $\mathbf{h}$, and its closed-form solution can be obtained by the first derivation. Similarly, the solution process of $\mathbf{z}$ has

$$\begin{aligned}
& \nabla_{\mathbf{z}} L(\mathbf{z}, \mathbf{h}^{(m+1)}, \lambda_1^{(m)}) = \nabla_{\mathbf{z}} F(\mathbf{z}, \mathbf{h}^{(m+1)}) + \frac{\rho}{2} \nabla_{\mathbf{z}} \left\| \mathbf{h}^{(m+1)} - \mathbf{z} + \lambda_1^{(m)} \right\|_2^2 \\
&= \left(\left(\frac{1}{K} \sum_{k=1}^K w_k \mathbf{A}^{\text{H}}(\theta_k) \mathbf{h}^{(m+1)} \mathbf{h}^{(m+1)\text{H}} \mathbf{A}(\theta_k) \right) + \frac{\rho}{2} \mathbf{I} \right) \mathbf{z} - \left(\frac{\rho}{2} (\mathbf{h}^{(m+1)} + \lambda_1^{(m)}) \right) \\
&= \mathbf{0},
\end{aligned} \tag{28}$$

Its closed-form solution has

$$\tilde{\mathbf{z}}^{(m+1)} = \left(\left(\frac{1}{K} \sum_{k=1}^K w_k \mathbf{A}^{\text{H}}(\theta_k) \mathbf{h}^{(m+1)} \mathbf{h}^{(m+1)\text{H}} \mathbf{A}(\theta_k) \right) + \frac{\rho}{2} \mathbf{I} \right)^{-1} \left(\frac{\rho}{2} (\mathbf{h}^{(m+1)} + \lambda_1^{(m)}) \right). \tag{29}$$

Thus, the updating of each point of (29) can be expressed as

$$\mathbf{z}^{(m+1)} = \sqrt{\frac{P_0}{NL}} \left[\sqrt{\frac{NL}{P_0}} \tilde{\mathbf{z}}_1^{(m+1)}, \frac{\tilde{\mathbf{z}}_2^{(m+1)}}{\left\| \tilde{\mathbf{z}}_2^{(m+1)} \right\|_2}, \dots, \frac{\tilde{\mathbf{z}}_{NL+1}^{(m+1)}}{\left\| \tilde{\mathbf{z}}_{NL+1}^{(m+1)} \right\|_2} \right]. \tag{30}$$

Once we have obtained each point $(\mathbf{z}^{(m+1)}, \mathbf{h}^{(m+1)})$ on a given product manifold, we can update λ by

$$\lambda_1^{m+1} = \lambda_1^{(m)} + \mathbf{h}^{(m+1)} - \mathbf{z}^{(m+1)}. \tag{31}$$

We summarize the waveform design Algorithm 1 for pattern-matching (WAPM) as follows.

Algorithm 1: Applying manifold-ADMM to solve the problem (15).

Initialize: $\mathbf{z}^{(0)}, \mathbf{h}^{(0)}, \lambda_1^{(0)}, \rho$, and the tolerance ε .

While the termination criteria are not satisfied **do**.

Update $\mathbf{h}^{(m+1)}$ using (27).

Update $\mathbf{z}^{(m+1)}$ using (30).

Update $\lambda_1^{(m+1)}$ using (31).

$m = m + 1$.

End while

3.2. DFRC Waveform Design

The objective function of (14) can be represented as

$$\begin{aligned} & w \cdot \|\mathbf{H}\mathbf{X} - \mathbf{S}\|_F^2 + (1-w) \cdot \|\mathbf{X} - \mathbf{X}_0\|_F^2 \\ &= \left\| \begin{bmatrix} \sqrt{w}\mathbf{H}^T, \sqrt{1-w}\mathbf{I}_N \end{bmatrix}^T \mathbf{X} - \begin{bmatrix} \sqrt{w}\mathbf{S}^T, \sqrt{1-w}\mathbf{X}_0^T \end{bmatrix}^T \right\|_F^2. \end{aligned} \quad (32)$$

We define $\mathbf{A} = \begin{bmatrix} \sqrt{w}\mathbf{H}^T, \sqrt{1-w}\mathbf{I}_N \end{bmatrix}^T$, $\mathbf{B} = \begin{bmatrix} \sqrt{w}\mathbf{S}^T, \sqrt{1-w}\mathbf{X}_0^T \end{bmatrix}^T$, then (32) can be expressed as

$$\begin{aligned} & \min_{\mathbf{X}} \|\mathbf{A}\mathbf{X} - \mathbf{B}\|_F^2, \\ & \text{s.t. } |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \end{aligned} \quad (33)$$

and we could further have

$$\begin{aligned} \|\mathbf{A}\mathbf{X} - \mathbf{B}\|_F^2 &= \text{tr}((\mathbf{A}\mathbf{X} - \mathbf{B})^H(\mathbf{A}\mathbf{X} - \mathbf{B})) \\ &= \text{tr}(\mathbf{X}^H \mathbf{A}^H \mathbf{A} \mathbf{X}) - \text{tr}(\mathbf{X}^H \mathbf{A}^H \mathbf{B}) - \text{tr}(\mathbf{B}^H \mathbf{A} \mathbf{X}) + \text{tr}(\mathbf{B}^H \mathbf{B}). \end{aligned} \quad (34)$$

Letting $\mathbf{D} = \mathbf{A}^H \mathbf{A}$, $\mathbf{G} = \mathbf{A}^H \mathbf{B}$, (34) can be expressed as

$$\begin{aligned} & \min_{\mathbf{X}} \text{tr}(\mathbf{X}^H \mathbf{D} \mathbf{X}) - 2\Re(\text{tr}(\mathbf{X}^H \mathbf{G})) \\ & \text{s.t. } |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \end{aligned} \quad (35)$$

Similarly, we introduce an auxiliary primal variable $\mathbf{Y} \in \mathbb{C}^{N \times L}$, and (35) can be expressed as

$$\begin{aligned} & \min_{\mathbf{X} \in S_X, \mathbf{Y} \in S_Y} F(\mathbf{X}, \mathbf{Y}) = \text{tr}(\mathbf{Y}^H \mathbf{D} \mathbf{X}) - 2\Re(\text{tr}(\mathbf{Y}^H \mathbf{G})) \\ & \text{s.t. } \mathbf{Y} - \mathbf{X} = \mathbf{0} \\ & |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L, \end{aligned} \quad (36)$$

where

$$\begin{aligned} S_X &= \left\{ \mathbf{X} \in \mathbb{C}^{N \times L} \mid |x_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L \right\}, \\ S_Y &= \left\{ \mathbf{Y} \in \mathbb{C}^{N \times L} \mid |y_{n,l}| = \sqrt{\frac{P_0}{NL}}, n = 1, 2, \dots, N; l = 1, 2, \dots, L \right\}. \end{aligned} \quad (37)$$

Here, we also use the manifold-ADMM algorithm to solve the problem of (36). Obviously, the augmented Lagrange function can be expressed as

$$L(\mathbf{X}, \mathbf{Y}, \lambda) = F(\mathbf{X}, \mathbf{Y}) + \frac{\rho}{2} \|\mathbf{Y} - \mathbf{X} + \lambda\|_2^2. \quad (38)$$

Taking (38) into (17),

$$\mathbf{Y}^{(m+1)} := \arg \min_{\mathbf{Y} \in S_Y} F(\mathbf{X}^{(m)}, \mathbf{Y}) + \frac{\rho}{2} \|\mathbf{Y} - \mathbf{X}^{(m)} + \lambda^{(m)}\|_2^2. \quad (39)$$

Since (39) is a convex function problem about $\mathbf{Y}$, and the closed-form solution can be obtained by the first derivation on $\mathbf{y}$, i.e.,

$$\begin{aligned} \nabla_{\mathbf{y}^*} L(\mathbf{x}^{(m)}, \mathbf{y}, \lambda^{(m)}) &= \nabla_{\mathbf{Y}^*} F(\mathbf{X}^{(m)}, \mathbf{Y}) + \frac{\rho}{2} \nabla_{\mathbf{Y}^*} \|\mathbf{Y} - \mathbf{X}^{(m)} + \lambda^{(m)}\|_2^2 \\ &= \frac{\rho}{2} \mathbf{I} \mathbf{Y} - \left(\frac{\rho}{2} (\mathbf{X}^{(m)} - \lambda^{(m)}) - \mathbf{D} \mathbf{X} + 2\mathbf{G} \right) = \mathbf{0}, \end{aligned} \quad (40)$$

its closed-form solution has

$$\tilde{\mathbf{Y}}^{(m+1)} = \left(\frac{\rho}{2} \mathbf{I} \right)^{-1} \left(\frac{\rho}{2} (\mathbf{X}^{(m)} - \lambda^{(m)}) - \mathbf{D} \mathbf{X} + 2\mathbf{G} \right). \quad (41)$$

Then the updating of each point of (41) can be expressed as

$$\mathbf{Y}^{(m+1)} = \sqrt{\frac{P_0}{NL}} \left[\frac{\tilde{\mathbf{y}}_{n,l}}{\|\tilde{\mathbf{y}}_{n,l}\|_2} \right], n = 1, 2, \dots, N; l = 1, 2, \dots, L. \quad (42)$$

Combining (38) with (18), we have

$$\mathbf{X}^{(m+1)} := \arg \min_{\mathbf{X} \in S_X} F(\mathbf{X}, \mathbf{Y}^{(m+1)}) + \frac{\rho}{2} \|\mathbf{Y}^{(m+1)} - \mathbf{X} + \boldsymbol{\lambda}^{(m)}\|_2^2. \quad (43)$$

$$\begin{aligned} & \nabla_{\mathbf{X}} L(\mathbf{X}, \mathbf{Y}^{(m+1)}, \boldsymbol{\lambda}^{(m)}) \\ &= \nabla_{\mathbf{X}} F(\mathbf{X}, \mathbf{Y}^{(m+1)}) + \frac{\rho}{2} \nabla_{\mathbf{X}} \|\mathbf{Y}^{(m+1)} - \mathbf{X} + \boldsymbol{\lambda}^{(m)}\|_2^2 \\ &= \frac{\rho}{2} \mathbf{I} \mathbf{X} - \left(\frac{\rho}{2} (\mathbf{Y}^{(m+1)} + \boldsymbol{\lambda}^{(m)}) - \mathbf{D}^H \mathbf{Y} \right) \\ &= \mathbf{0}, \end{aligned} \quad (44)$$

The closed-form solution has

$$\tilde{\mathbf{X}}^{(m+1)} = \left(\frac{\rho}{2} \mathbf{I} \right)^{-1} \left(\frac{\rho}{2} (\mathbf{Y}^{(m+1)} + \boldsymbol{\lambda}^{(m)}) - \mathbf{D}^H \mathbf{Y} \right), \quad (45)$$

$$\mathbf{X}^{(m+1)} = \sqrt{\frac{P_0}{NL}} \left[\frac{\tilde{\mathbf{x}}_{n,l}}{\|\tilde{\mathbf{x}}_{n,l}\|_2} \right], n = 1, 2, \dots, N; l = 1, 2, \dots, L. \quad (46)$$

Once we have obtained each point $(\mathbf{X}^{(m+1)}, \mathbf{Y}^{(m+1)}) \in S_{(X,Y)}$ on a given product manifold, we could update $\boldsymbol{\lambda}$ by

$$\boldsymbol{\lambda}^{(m+1)} = \boldsymbol{\lambda}^{(m)} + \mathbf{Y}^{(m+1)} - \mathbf{X}^{(m+1)}. \quad (47)$$

We summarize the DFRC Algorithm 2 based on manifold-ADMM (named DFRC-MA) as follows.

Algorithm 2: Applying manifold-ADMM to solve the problem (36).

Initialize: $\mathbf{X}^{(0)}, \mathbf{Y}^{(0)}, \boldsymbol{\lambda}^{(0)}, \rho$, and the tolerance ε .
While the termination criteria are not satisfied **do**.
 Update $\mathbf{Y}^{(m+1)}$ using (42).
 Update $\mathbf{X}^{(m+1)}$ using (46).
 Update $\boldsymbol{\lambda}^{(m+1)}$ using (47).
 $m = m + 1$.
End while

4. Numeric Examples

In this section, we provide a series of examples to evaluate the performance of CM DFRC waveform designed by manifold-ADMM. In simulations, the uniform linear array (ULA) incorporates $N = 10$ elements with a half-wavelength interelement interval, and each transmitting pulse has $L = 20$ samples. The range of angles is $(-90^\circ, 90^\circ)$ with spacing interval 1° , and the weight for the k -th angle has $w_k = 1, k = 1, 2, \dots, K$. For convenience, we choose the unit-power QPSK alphabet as the constellation for communication users, i.e., the power of each entry in the symbol matrix $\mathbf{S}$ is 1. Firstly, we will discuss the radar waveform design and analysis for beampattern matching. Next, we will design the DFRC waveform for communication performance. Finally, we will analyze the ambiguity function characteristic of the DFRC waveform.

4.1. Radar Waveform Design and Analysis for Beampattern Matching

Consider that the desired beampattern has three main lobes at $\theta = -45^\circ, 0^\circ, 45^\circ$ corresponding to three targets $Q = 3$, their interval width is $\Delta\theta = 20^\circ$, and the penalty parameter of (16) is set as $\rho = 50$. Then, the desired beampattern can be formulated as

$$d(\theta) = \begin{cases} 1, & \theta \in \left[\theta_q - \frac{\Delta\theta}{2}, \theta_q + \frac{\Delta\theta}{2}\right], q = 1, 2, \dots, Q \\ 0, & \text{others} \end{cases} \quad (48)$$

As discussed in [19], we also choose the squared-error cost function as the designing metric. Here, the variables $\mathbf{h}^{(0)}$, $\lambda^{(0)}$ are initialized to be $\mathbf{0}$, and $\mathbf{z}^{(0)}$ is initialized to be a random vector. An illustration of the three main-lobes beampattern is shown in Figure 2, where we also provide results of WAPM and other prevalent ones. For comparison, the beampattern optimized by the semidefinite program (SDP) method in [22] and ADMM in [19] are also considered. Obviously, the beampattern obtained by WAPM is very close to the SDP-based beampattern, where it achieves the transmitting waveform in a direct manner rather than two-step way. Furthermore, compared with the main-lobe beamforming at 0° , WAPM algorithm can generate a higher main lobe than the classical ADMM algorithm, also with a lower side lobe at 20° .

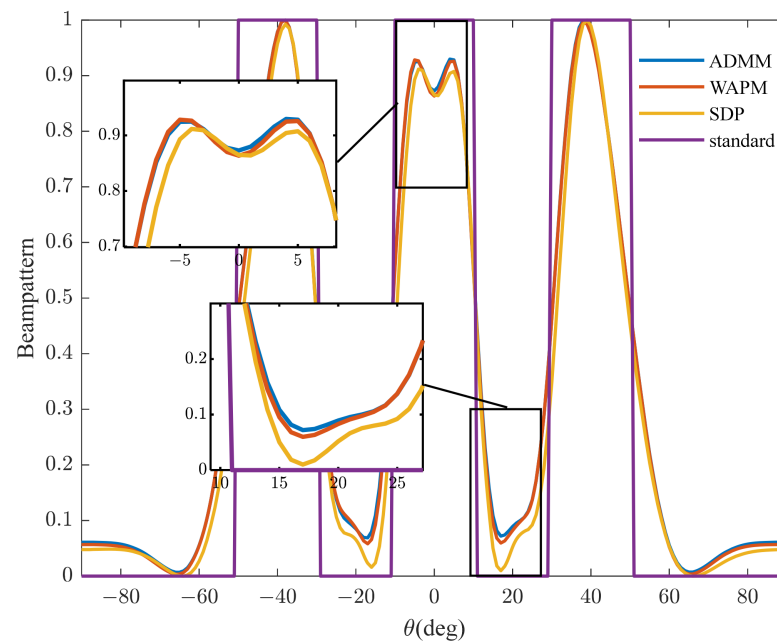


Figure 2. Comparison of three main-lobes' beampatterns in the case of $\Delta\theta = 20^\circ$.

One of the indicators for measuring the constant modulus of a signal is the signal modulus. As can be seen from Figure 3, the waveform designed by the WAPM algorithm under a single beam has a strict constant modulus characteristic, while the waveform designed by the ADMM algorithm converges to the constant modulus. The minimum modulus value of the waveform designed by the ADMM algorithm is 0.9996, and the maximum modulus value is 1.0013. The superior performance of the WAPM algorithm in maintaining a constant modulus can be attributed to its inherent design, which emphasizes consistency and stability in waveform characteristics.

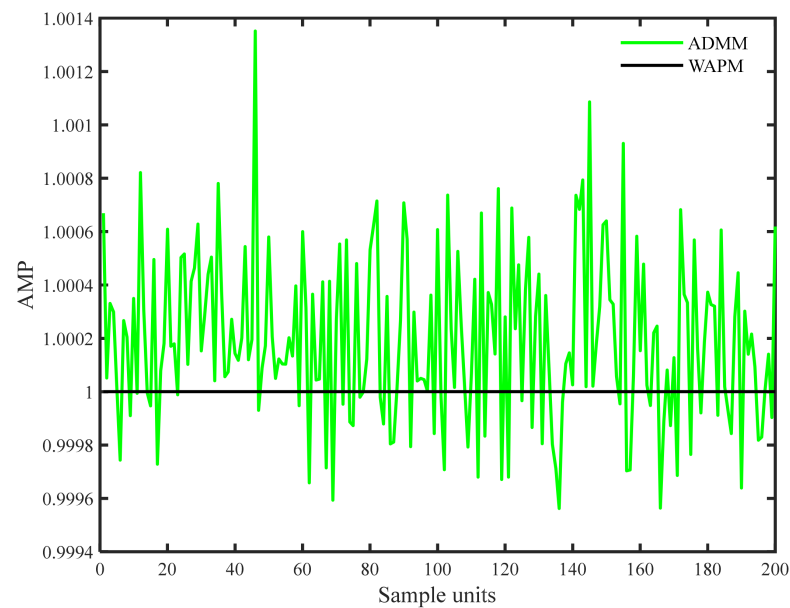


Figure 3. Performance comparison of ADMM and WAPM with the waveform modulus description.

In the five-beam scenario, the detection is located at $\theta = -60^\circ, -30^\circ, 0^\circ, 30^\circ$, and 60° , respectively. The five-beam waveform means that there are five main-lobe beams in the direction of the detection target. Figure 4 describes the five-beam pattern performance of the ADMM algorithm, WAPM algorithm and SDP algorithm, and the results are all normalized. As can be seen from Figure 4, the five beams formed by the above algorithms can form main-lobe beams similar to the standard waveform in the five directions of $-60^\circ, -30^\circ, 0^\circ, 30^\circ$ and 60° . Unlike single-beam and three-beam, there are no side lobes between the main lobes of the five beams. The ADMM algorithm and the WAPM algorithm have the same main-lobe depression peak position at 0° ; that is, both are located at 0° , and the SDP algorithm has a main-lobe depression peak position of 1° at 0° . At the concave position, the normalized peak value of the waveform designed by the ADMM algorithm is 0.866, the normalized peak value of the waveform designed by the WAPM algorithm is 0.884, and the normalized peak value of the waveform designed by the SDP algorithm is 0.767.

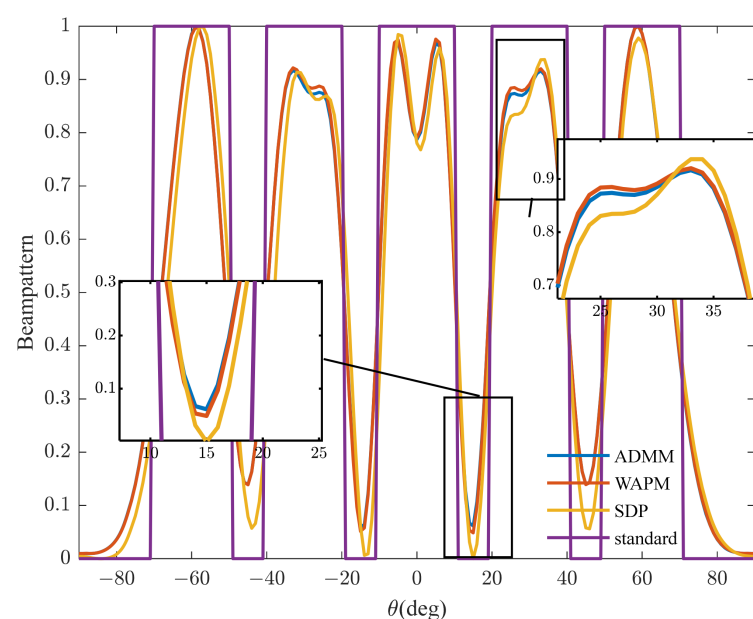


Figure 4. Comparison of five main-lobes beampattern in the case of $\Delta\theta = 20^\circ$.

Furthermore, we also consider a main-lobe beampattern with a width $\Delta\theta = 60^\circ$, as shown in Figure 5. WAPM has a higher main lobe than ADMM at the -53° and lower side lobe at the 0° , which means some excellent beam-matching performance.

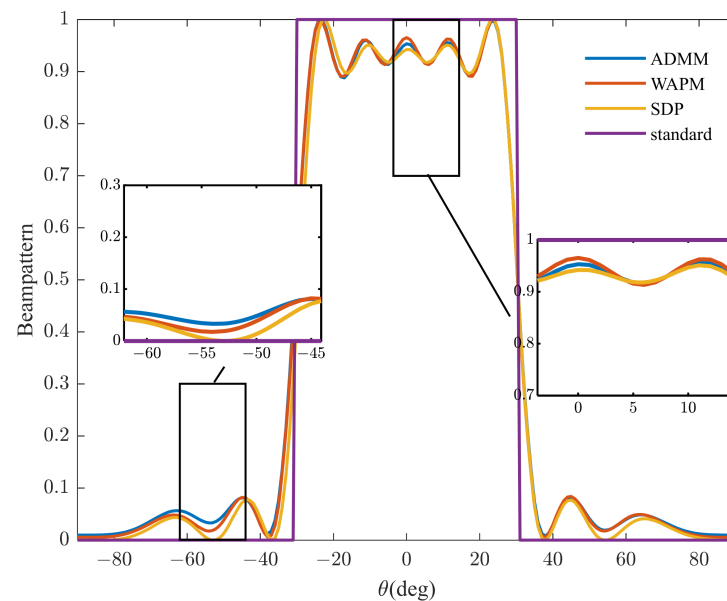


Figure 5. Comparison of only one main-lobe beampattern in the case of $\Delta\theta = 60^\circ$.

The goal of the optimization problem is to maximize or minimize the value of the function to be optimized, and the goal of the beam-matching problem is to minimize the difference between the designed waveform and the standard waveform, that is, the final optimization function converges to a certain value. Figure 6 describes the relationship between the number of iterations of the manifold-ADMM algorithm and the ADMM algorithm and the value of the objective function under the same convergence conditions. From the number of iterations, the number of iterations of the WAPM algorithm is 35 times, and the number of iterations of the ADMM algorithm is 25 times. The number of iterations of the WAPM algorithm is about 1.5 times that of the ADMM algorithm. From the function trend, the algorithm designed by the WAPM algorithm makes the objective function curve approximate to an S-shaped curve, and the WAPM algorithm can make the function objective value have a better change trend.

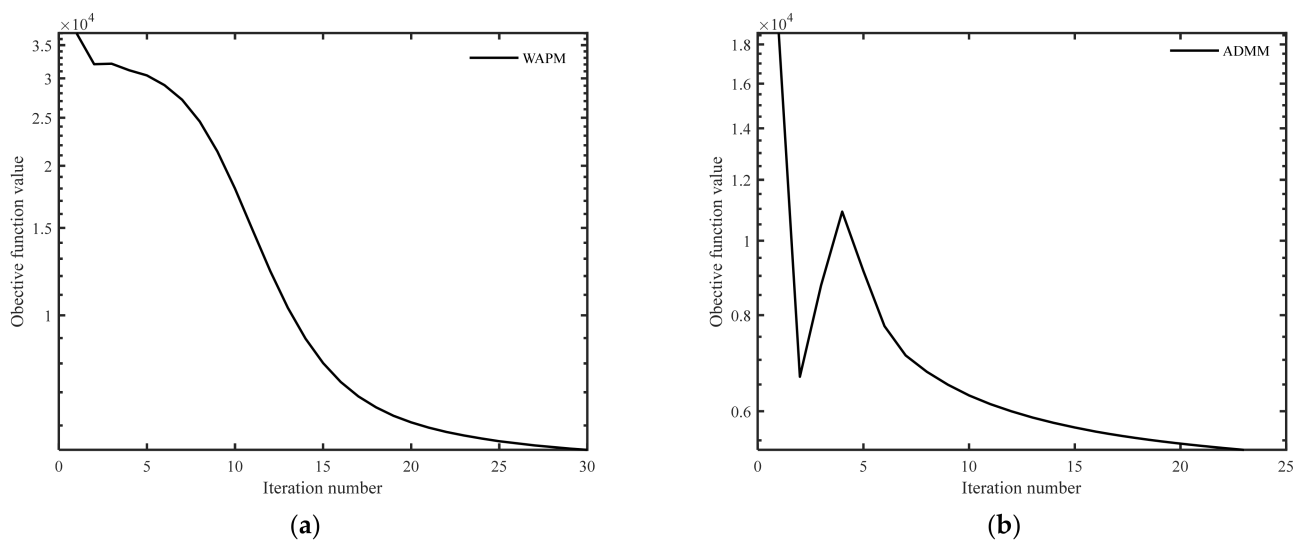


Figure 6. Objective function values under different algorithms. (a) WAPM algorithm; (b) ADMM algorithm.

4.2. DFRC Waveform Design for Communication Performance

The CM waveform obtained by WAPM is used as the radar referenced waveform which would be further fed into the DFRC system. In this section, we will design the integrated DFRC waveform by DFRC-MA where the QPSK modulation signal is selected as the communication-reference signal. The number of communication users is $K = 3$. Other parameters are the same as in the previous section. Variables $\mathbf{Y}^{(0)}$, $\lambda^{(0)}$ are initialized to be $\mathbf{0}$, and variable $\mathbf{X}^{(0)}$ is initialized to be the random matrix. For comparison, the BnB method in [18] is considered to make the comparison. We studied the communication performance vs. SNR with different algorithms and trade-off factors. In Figure 7, when the trade-off factor is set as $w = 0$, the communication performance has been lost, which is consistent with our objective problem. When $w = 0.25$, it can be found that the larger the trade-off factor, the higher the communication rate and the bit error rate will be, while the increase in the trade-off factor will increase the communication rate at the expense of the bit error rate.

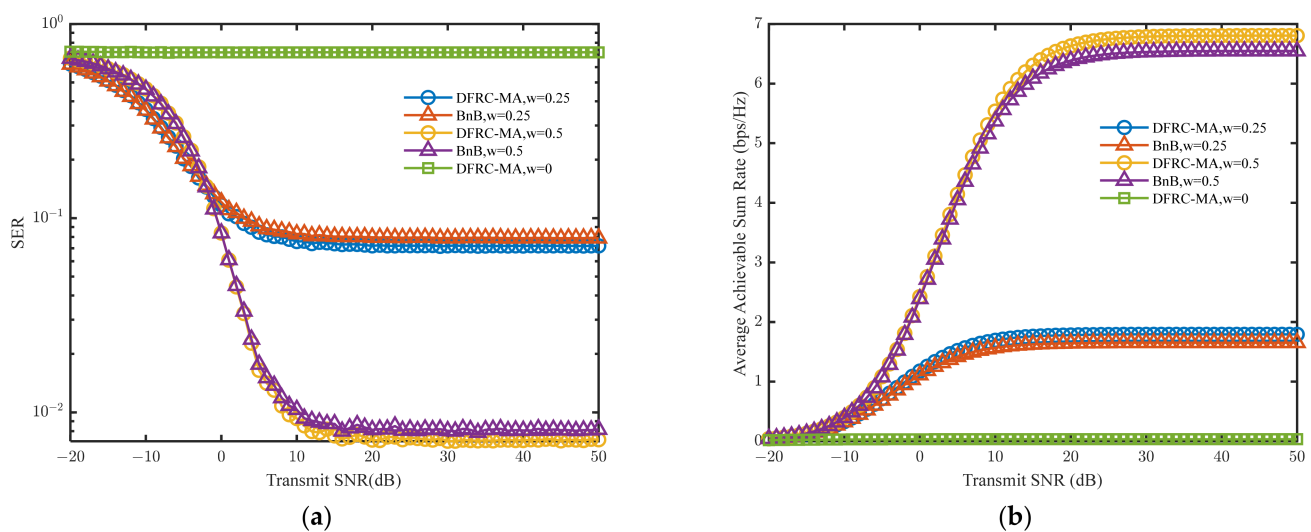


Figure 7. (a) SER vs. transmit SNR. (b) Sum rate vs. transmit SNR.

In Table 2, we also compare the CM performance and consumption time of the DFRC-MA and BnB algorithms. The DFRC-MA algorithm can achieve strict CM constraint, owing to its Riemannian solution space. Our proposed DFRC-MA as an iterative closed optimizing manner consumes less time than the BnB algorithm.

Table 2. Comparison of consumption time under different cases.

Methods	Average Amplitude	Consumption Time/s
DFRC-MA, $w = 0$	1.00	9.87
DFRC-MA, $w = 0.25$	1.00	9.65
DFRC-MA, $w = 0.5$	1.00	9.84
BnB, $w = 0.25$	1.02	13.12
BnB, $w = 0.5$	0.99	13.01

5. Conclusions

In this paper, we studied the joint waveform design and beamforming for DFRC systems and proposed a manifold-ADMM framework to address both strict radar beam pattern constraints and the trade-off between radar and communication performances. Simulations demonstrated that the radar waveform designed by the manifold-ADMM framework exhibits excellent matching performance, while the integrated waveform provides outstanding radar–communication integration. These results underscore the framework’s effectiveness in balancing the dual requirements of radar and communication functionalities in DFRC systems. Future research could explore the use of fully polarimetric systems

to enhance the system's capability in distinguishing different targets and improving overall performance. Additionally, further studies could focus on implementing the manifold-ADMM framework in real-world scenarios, investigating adaptive algorithms for dynamic waveform and beamforming adjustments, and integrating advancements from related fields such as machine learning and artificial intelligence to optimize DFRC system design.

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