

Article

A Low-Complexity DOA Estimation Method for Acoustic Vector Sensors Based on Noise Power Invariance

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Abstract

To reduce the computational burden of conventional spectral search direction-of-arrival (DOA) estimation algorithms for acoustic vector sensor arrays (AVSAs), this paper proposes a low-complexity DOA estimation method based on semi-real-valued noise power invariance (SR-NPI). The proposed method is developed for centrosymmetric AVSAs under the required steering-vector parity and pressure-channel conjugate-symmetry conditions after pressure–velocity (PV) co-processing. The AVSA measurements are first pre-processed through PV co-processing, and a pseudo-data covariance matrix is then reconstructed by exploiting the complex conjugate relationship between the true DOAs and their symmetric virtual DOAs. By introducing a scanning source into the reconstructed covariance matrix, a DOA-dependent spatial spectrum is constructed according to the eigenvalue-ordering behavior. Since the reconstructed matrix contains both the true DOAs and the symmetric virtual DOAs, the angular search range can be reduced to one half of the original domain. Under the tested configuration, the proposed method reduces the computational cost to approximately 1.92% of that of the original NPI algorithm. Simulation and sea trial results indicate that SR-NPI can maintain competitive estimation accuracy while significantly reducing computational complexity under the centrosymmetric-array conditions.

Keywords: direction of arrival (DOA); high-resolution; low-complexity DOA; semi-real-valued noise power invariant (SR-NPI); eigenvalue ordering law

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1. Introduction

With the rapid advancement of underwater stealth technology [1], existing detection techniques now face heightened demands in terms of detection time and parameter estimation accuracy. As one of the core technologies of array signal processing, DOA estimation has important application value in the fields of sonar detection and radar localization [2–4]. Especially in the underwater target detection field, the DOA estimation directly affects the detection performance of the system. After decades of development, the field has formed a more complete theoretical system and produced many innovative research results. The early development of DOA estimation techniques [5,6] can be traced back to the Conventional Beamformer (CBF) method [7]. However, CBF considers only the time delay

differences between array elements without performing deeper processing of the acquired spatial data. As a result, its performance is fundamentally constrained by the Rayleigh resolution limit, rendering it incapable of distinguishing multiple closely spaced sources in the spatial domain [8]. In an effort to overcome the limitations of the CBF and to resolve the DOAs of multiple adjacent sources, Capon introduced the Minimum Variance Distortionless Response (MVDR) algorithm in 1969 [9]. The MVDR algorithm significantly improved the DOA estimation performance for spatially adjacent targets and enhanced angular resolution, thereby overcoming the Rayleigh limit. To further enhance DOA resolution, considerable research efforts have been devoted to the development of more advanced DOA estimation algorithms. Notable among these are the Maximum Entropy Spectral Estimation Method (MEM) [10] and the Linear Prediction (LP) algorithm [11]. These methods have greatly accelerated the evolution of high-resolution techniques toward super-resolution capabilities, and some researchers regard these developments as the initial emergence of super-resolution DOA estimation methodologies.

The concept of true super-resolution was not realized until 1986, when R. O. Schmidt proposed the Multiple Signal Classification (MUSIC) algorithm in his doctoral dissertation [12]. This breakthrough marked a significant leap in DOA estimation from high-resolution to super-resolution techniques, inaugurating a new chapter in spatial spectrum estimation theory. Subsequently, subspace-based algorithms garnered extensive research attention and were thoroughly investigated across various application scenarios. However, the subspace-based algorithms exhibit two rather obvious drawbacks. One is the acquisition of the number of signal sources, while the other is their high computational complexity, which becomes particularly pronounced in two-dimensional DOA (2D-DOA) estimation scenarios [13]. To determine the number of sources, the information-theoretic Maximum Degree of Liberty (MDL) method [3] is commonly employed. The MDL algorithm transforms the source estimation problem into a model selection problem: by identifying the correct number of sources, it maximizes the data fitting term while minimizing the penalty term. Compared to the AIC criterion, the MDL has a stronger penalty term, enabling it to estimate the true number of sources with a probability of 100% under sufficient sample size. Its computational complexity represents a common limitation shared by many spectrum-based estimation algorithms, making the reduction in computational complexity in DOA algorithms a key research focus in this field. Chen [14] et al. proposed a high-resolution subspace algorithm based on AVSAs, which does not require prior knowledge of the number of sources and effectively addresses the dependence of subspace-based algorithms on source number estimation. However, this algorithm requires matrix decomposition at each scanning point, which leads to a sharp increase in computational complexity and limits its practical application. In existing DOA estimation algorithms, computational complexity primarily arises from two aspects: (1) the computation of the sample covariance matrix [15]; (2) the spectral search process [16,17]. To reduce the computational complexity associated with covariance matrix operations, researchers have proposed a series of practical DOA estimation algorithms. Notable among them are the power iteration method introduced by Tufts and Melissinos [18], the classical Propagator Method (PM) [19], and the Fast Subspace Decomposition (FSD) approach [20]. Reference [21] proposes a MUSIC algorithm for arbitrary array configurations that requires only real-valued computations. By employing the concept of virtual arrays, this approach effectively mitigates the strict array structure constraints inherent in traditional real-valued algorithms, achieving a significant leap from complex-valued to real-valued computations. However, due to the virtual array transformation performed by this algorithm, its performance is constrained by multiple factors in practical applications, making it difficult to achieve significant improvements. To this end, WANG [22] et al. proposed a low-complexity DOA estimation method based on sparse Bayesian learning. The core idea of this

approach is to convert the imaginary-valued operations of the sampled covariance matrix into real-valued operations. Since the computational load of real-valued operations is only 25% of that for imaginary-valued operations, this results in a significant reduction in computational complexity. However, this algorithm remains applicable only to specific formations. When formations do not match, the algorithm's performance drops sharply. Nan et al. [23] addressed this by decomposing the sampling covariance matrix into two real matrices (i.e., the real part and the imaginary part), thereby resolving the issue of high array configuration requirements for real-valued algorithms. In order to reduce the amount of computation of spectral searching, some researchers have proposed the concepts of dimensionality reduction search and sharding search. Among them, the method of dimensionality reduction search is to transform the multidimensional DOA search into a one-dimensional DOA search, which reduces the dimensionality of the search and thus significantly reduces the computation brought by the spectral search. Tian et al. [24] derived a theoretical framework for low-complexity DOA algorithms and proposed a novel low-complexity DOA algorithm based on the MVDR algorithm, which offers advantages in estimation resolution compared to other parameter estimation methods. Wang et al. [25] proposed a low-complexity 2D DOA estimation algorithm based on L-arrays to reduce the computational effort of the 2D algorithm. This method reduces the computational effort of the algorithm by calculating the pitch and azimuth angles of the target in a separate one-dimensional subspace-like algorithm. However, the one-dimensional search still requires a lot of computation [26], so many experts focus on another way to reduce the computation, namely, sharding search algorithms. Yan et al. [27] proposed a symmetric compression strategy that incorporates this idea by projecting targets onto a predefined region of interest, thereby significantly reducing the computational complexity. Experimental results demonstrated that this method could reduce computational complexity to approximately 50% of the original algorithm while maintaining comparable estimation accuracy. However, it still requires specific array structures, particularly those with centrosymmetric configurations. To address this limitation, Yan further improved the method by introducing a transform-domain approach, which eliminates the dependency on array geometry and enables the algorithm to be applied to arbitrary array configurations. However, the above algorithms are not developed for acoustic vector sensors (AVSs) [14,28,29], which are the core components of underwater DOA estimation algorithms. Meng [21] et al. proposed a real-valued MUSIC algorithm applicable to arbitrary arrays to address the high computational complexity of the conventional MUSIC algorithm and validated the method through experiments. However, the influence of environmental factors on the algorithm's performance was not considered. In this context, Busacca [30] et al. focused on underwater communication channels, which suffer from severely limited bandwidth and long propagation delays, and proposed a predictive algorithm that achieves a trade-off between accuracy and complexity. The method was validated using real shallow-water measurement data collected in the Tyrrhenian Sea, Italy.

For vector sonar technology, its ability to receive richer acoustic field information significantly enhances underwater detection performance. However, the multi-channel data also causes the time required for sonar to detect targets to increase exponentially. The algorithm proposed in this paper effectively alleviates the challenge of prolonged DOA estimation time in vector sonar technology. The proposed algorithm is primarily applicable to vector sonar systems with symmetrical array configurations, such as towed line arrays and side-scan arrays. Additionally, through certain array virtual transformation techniques, other array structures can be transformed into arrays with symmetric configurations, effectively expanding the application scenarios of the proposed algorithm.

2. Signal Model

As an important component of acoustic target detection, underwater passive detection uses the receiving characteristics of AVSs to acquire richer underwater acoustic field information, thereby improving detection performance. This section presents the AVSA signal model used in the proposed method.

In a stationary, homogeneous, isotropic, and non-viscous fluid medium, Euler's equation can be expressed as

$$\frac{\partial v(r,t)}{\partial t} + v(r,t)^T \nabla v(r,t) = -\frac{\nabla \cdot p(r,t)}{\rho} \quad (1)$$

where $p(r,t)$ represents the sound pressure at r in the sound field at time t . Similarly, $v(r,t)$ the speed of vibration at time ρ is the density of the medium. For plane waves under far-field conditions, the sound pressure function can be rewritten as

$$p(r,t) = f\left(\frac{u^T r}{v} + t\right) \quad (2)$$

where $u^T = [\cos\theta \sin\varphi \ \sin\theta \cos\varphi]$ is the steering vector of AVS. θ and φ represent direction and elevation angles, respectively, and v represents the speed of sound wave propagation. Therefore, the sound pressure gradient function can be expressed as

$$\nabla p(r,t) = \frac{\partial p(r,t)}{\partial t} \cdot \frac{u(\theta)}{v} \quad (3)$$

By neglecting the vibrational acceleration term in Equation (3) and applying linearization, we obtain:

$$v(r,t) = -\frac{p(r,t)}{Z} \cdot u(\theta) \quad (4)$$

We extend the sound pressure sensor signal model to the AVS signal model. Additionally, we consider the two-dimensional AVS model. The received data model for each sound pressure and particle velocity channel can thus be represented as:

$$\begin{cases} V_x = \frac{P}{Z} \cos \theta \\ V_y = \frac{P}{Z} \sin \theta \\ p = p(r,t) \end{cases} \quad (5)$$

where Z represents the wave impedance, which is typically set to 1. Suppose that K -independent far-field narrowband incident signals are observed on an array comprising M array elements. The spacing of the array elements is a half wavelength. Consequently, the signal received by the AVSA can be represented as

$$X(t) = A(\Theta)S(t) + N(t) \quad (6)$$

where $A(\Theta) = [a(\theta_1), a(\theta_2), \dots, a(\theta_K)]$ is the $3M \times K$ steering vector formed by the K source position vectors along their respective source directions, $S(t)$ represents the $K \times L$ signal vector, and $N(t)$ denotes the $3M \times L$ noise vector, where L represents the number of snapshots. Specifically, it can be expressed as

$$A(\theta_k) = a_p(\theta_k) \otimes V(\theta_k) \quad (7)$$

where $a_p(\theta_k) = [e^{j\omega\tau_1(\theta_k)}, e^{j\omega\tau_2(\theta_k)}, \dots, e^{j\omega\tau_M(\theta_k)}]^T$ denotes the $M \times K$ sound pressure steering vector for the K th signal, $\omega = 2\pi f$ represents the angular frequency, τ_m is the delay between the reference array element and the m th array element, f denotes the signal frequency, $k \in \mathbb{S} = \{1, 2, \dots, K\}$, and $\otimes$ denotes the Kronecker product. When $V(\theta_k)$ represents the steering vector of AVS, it can be expressed as

$$\mathbf{V}(\theta_k) = \begin{bmatrix} 1 \\ \cos(\theta_k) \\ \sin(\theta_k) \end{bmatrix} \quad (8)$$

For a uniform AVSs linear array, the steering vector of the acoustic pressure component can be expressed as

$$\mathbf{a}_p(\theta_k) = [1, e^{jkd \sin \theta_k}, e^{j2kd \sin \theta_k}, \dots, e^{j(M-1)kd \sin \theta_k}]^T \quad (9)$$

A uniform linear array is a typical symmetric array, so the steering vector of the acoustic pressure component satisfies $\mathbf{a}_p(-\theta_k) = \mathbf{a}_p^*(\theta_k)$.

3. Problem Formulation

3.1. High-Resolution DOA Estimation Based on Noise Power Invariance

This section proposes a high-resolution DOA estimation algorithm based on the eigenvalue ranking. It is assumed that a scanning source with scanning angle θ as a variable is introduced into the covariance matrix containing the K real targets, and its power is assumed to be β . The matrix $\mathbf{B}(\theta)$ constructed at this point can be expressed as

$$\begin{aligned} \mathbf{B}(\theta) &= \mathbf{R}_{(\mathbf{P}+\mathbf{V}_c)\mathbf{V}_c} + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) \\ &= \mathbf{A}_p \mathbf{R}_s \mathbf{A}_p^H \ell(\theta_g, \theta_k) + \frac{1}{2} \sigma_v^2 \mathbf{I}_M + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) \end{aligned} \quad (10)$$

where $\mathbf{R}_{(\mathbf{P}+\mathbf{V}_c)\mathbf{V}_c}$ can be expressed as

$$\begin{aligned} \mathbf{R}_{(\mathbf{P}+\mathbf{V}_c)\mathbf{V}_c} &= \mathbb{E} \{ [\mathbf{P} + \mathbf{V}_c] \mathbf{V}_c^H \} \\ &= \mathbf{A}_p \mathbf{R}_s \mathbf{A}_p^H \cos(\theta_g - \theta_k) \\ &\quad + \mathbf{A}_p \mathbf{R}_s \mathbf{A}_p^H [\cos(\theta_g - \theta_k) + \cos^2(\theta_g - \theta_k)] \\ &\quad + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \cos^2 \theta_g + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \sin^2 \theta_g \\ &= \mathbf{A}_p \mathbf{R}_s \mathbf{A}_p^H \ell(\theta_g, \theta_k) + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \end{aligned} \quad (11)$$

where $\ell(\theta_g, \theta_k) = \cos(\theta_g - \theta_k) + \frac{1}{2} [1 + \cos(2\theta_g - 2\theta_k)]$, θ_g denotes the reference direction, and θ_k denotes the DOA.

The data $\mathbf{V}_c$ obtained from the combined processing of sound pressure and vibration velocity can be expressed as

$$\begin{aligned} \mathbf{V}_c &= \mathbf{V}_x \cos \theta_g + \mathbf{V}_y \sin \theta_g \\ &= \mathbf{P} \cos \theta_k \cos \theta_g + \mathbf{P} \sin \theta_k \sin \theta_g + \mathbf{N}_x \cos \theta_g + \mathbf{N}_y \sin \theta_g \\ &= \mathbf{A}_p \mathbf{S} \cos(\theta_g - \theta_k) + \mathbf{N}_x \cos \theta_g + \mathbf{N}_y \sin \theta_g \end{aligned} \quad (12)$$

$\mathbf{R}_{(\mathbf{P}+\mathbf{V}_c)\mathbf{V}_c} = \mathbb{E} \{ [\mathbf{P} + \mathbf{V}_c] \mathbf{V}_c^H \}$, and Equation (10) can be further rewritten as

$$\mathbf{B} = \sum_{i=1}^K \sigma_i^2 \mathbf{a}_p(\theta_i) \mathbf{a}_p^H(\theta_i) \ell(\theta_g, \theta_i) + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \quad (13)$$

Assuming that the observation direction θ_g is set to be consistent with the scanning angle θ , the scanning angle θ is divided into two cases: (1) the scanning angle is consistent with the target angle $\theta \in \Theta = \{\theta_1, \theta_2, \dots, \theta_K\}$; (2) the scanning angle is inconsistent with the target angle $\theta \notin \Theta = \{\theta_1, \theta_2, \dots, \theta_K\}$.

When the scanning angle is consistent with the target DOAs $\theta \in \Theta$, the scanning angle θ is consistent with the DOA of the K th target $\theta = \theta_k$. Equation (13) can be rewritten as

$$\begin{aligned}
\mathbf{B} &= \sum_{i=1; i \neq k}^K \sigma_i^2 \mathbf{a}_p(\theta_i) \mathbf{a}_p^H(\theta_i) \ell(\theta_k, \theta_i) \\
&\quad + (\sigma_k^2 \ell(\theta_k, \theta_k) + \beta) \mathbf{a}_p(\theta_k) \mathbf{a}_p^H(\theta_k) + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \\
&= \sum_{i=1; i \neq k}^K \sigma_i^2 \mathbf{a}_p(\theta_i) \mathbf{a}_p^H(\theta_i) \ell(\theta_k, \theta_i) \\
&\quad + (2\sigma_k^2 + \beta) \mathbf{a}_p(\theta_k) \mathbf{a}_p^H(\theta_k) + \frac{1}{2} \sigma_v^2 \mathbf{I}_M
\end{aligned} \tag{14}$$

Right-multiplying both sides of Equation (14) by the noise subspace $\mathbf{U}_N$, derived from the decomposition of the matrix $\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}$, yields:

$$\begin{aligned}
\mathbf{B}\mathbf{U}_N &= \sum_{i=1; i \neq k}^K \sigma_i^2 \mathbf{a}_p(\theta_i) \mathbf{a}_p^H(\theta_i) \ell(\theta_k, \theta_i) \mathbf{U}_N \\
&\quad + (2\sigma_k^2 + \beta) \mathbf{a}_p(\theta_k) \mathbf{a}_p^H(\theta_k) \mathbf{U}_N + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \mathbf{U}_N \\
&= \frac{1}{2} \sigma_v^2 \mathbf{I}_M \mathbf{U}_N
\end{aligned} \tag{15}$$

When both sides of Equation (15) are right-multiplied by $\mathbf{U}_N$, we can see that no additional signal components are introduced, since the signals represented by the matrix $\mathbf{B}$ are intrinsic to the matrix $\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}$.

Consequently, the orthogonality between the signal and noise subspaces is preserved, leading to the following result:

$$\begin{aligned}
&\sum_{i=1; i \neq k}^K \sigma_i^2 \mathbf{a}_p(\theta_i) \mathbf{a}_p^H(\theta_i) \ell(\theta_k, \theta_i) \mathbf{U}_N \\
&\quad + (2\sigma_k^2 + \beta) \mathbf{a}_p(\theta_k) \mathbf{a}_p^H(\theta_k) \mathbf{U}_N \\
&= 0
\end{aligned} \tag{16}$$

According to the definition of eigenvalues and based on Equation (15), it can be seen that the noise eigenvalues corresponding to matrix $\mathbf{B}$ and matrix $\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}$ satisfy the following relationship:

$$\left\{ \lambda_i(\mathbf{B}) = \lambda_i(\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}) \mid i \in \xi, \theta \in \Theta \right\} \tag{17}$$

where $\xi = [K+1, \dots, M]$, and when $\theta \notin \Theta$, the steering vector of the scanning angle can be written as:

$$\mathbf{a}_p(\theta) = \mathbf{C}_S \mathbf{U}_S + \mathbf{C}_N \mathbf{U}_N \tag{18}$$

By removing $\mathbf{C}_N \mathbf{U}_N$ from the noise subspace $\mathbf{U}_N$, the resulting $M-K-1$ column vectors are subsequently normalized and orthogonalized to $\mathbf{C}_N \mathbf{U}_N$. Therefore, $\mathbf{v}_i \perp \mathbf{C}_N \mathbf{U}_N$ ($i = 2, 3, \dots, M-K$) can be obtained, where $\mathbf{v}_i$ denotes an arbitrary column vector from the noise subspace $\mathbf{U}_N$, and $\mathbf{U}_S$ denotes the signal subspace. A new subspace can thus be constructed from $\mathbf{v}_i$ and $\mathbf{C}_N \mathbf{U}_N$:

$$\mathbf{V}_N = [\mathbf{C}_N \mathbf{U}_N \ \mathbf{v}_2 \ \dots \ \mathbf{v}_{M-K}] \tag{19}$$

When $\theta \notin \Theta$, while right-multiplying the matrix $\mathbf{B}$ by the matrix $\mathbf{V}$, where $\mathbf{V} = [\mathbf{v}_2 \ \dots \ \mathbf{v}_{M-K}]$, the following equation can be obtained:

$$\begin{aligned}
\mathbf{B}\mathbf{V} &= \sum_{i=1}^K \sigma_i^2 \mathbf{a}_p(\theta_i) \mathbf{a}_p^H(\theta_i) \ell(\theta, \theta_i) \mathbf{V} \\
&\quad + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) \mathbf{V} + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \mathbf{V}
\end{aligned} \tag{20}$$

Since $V \perp C_N U_N$ holds, it is straightforward to derive $\text{span}(V_N) \perp \text{span}(U_S)$ and $\text{span}(V) \perp \text{span}(U_S)$. The expression for $\text{span}(V) \perp \text{span}(U_S, C_N U_N)$ can be obtained. Accordingly, Equation (20) can be rewritten as follows:

$$\begin{aligned} BV &= \sum_{i=1}^K \sigma_i^2 a_p(\theta_i) a_p^H(\theta_i) \ell(\theta, \theta_i) V + \beta a_p(\theta) a_p^H(\theta) V + \frac{1}{2} \sigma_v^2 I_M V \\ &= \mathbf{0}_{M \times (M-K-1)} + \beta a_p(\theta) a_p^H(\theta) V + \frac{1}{2} \sigma_v^2 I_M V \\ &= \beta a_p(\theta) a_p^H(\theta) V + \frac{1}{2} \sigma_v^2 I_M V \end{aligned} \quad (21)$$

From Equation (21), it can be obtained that when $\theta \notin \Theta$, the eigenvalues corresponding to the noise in the matrix $B(\theta)$ are related to those in the matrix $R_{(P+V_c)V_c}$ as follows:

$$\{\lambda_i(B) = \lambda_i(R_{(P+V_c)V_c}) \mid i \in \xi - [K+1], \theta \notin \Theta\} \quad (22)$$

Equation (22) indicates that even when $\theta \notin \Theta$, the last $M - K - 1$ eigenvalues of matrix $B(\theta)$ and matrix $R_{(P+V_c)V_c}$ must be identical. The analysis below focuses on the $K + 1$ th eigenvalue:

$$\begin{aligned} BC_N U_N &= \sum_{i=1}^K \sigma_i^2 a_p(\theta_i) a_p^H(\theta_i) \ell(\theta, \theta_i) C_N U_N \\ &\quad + \beta a_p(\theta) a_p^H(\theta) C_N U_N \\ &\quad + \frac{1}{2} \sigma_v^2 I_M C_N U_N \\ &= \beta (C_N U_N + C_S U_S) \end{aligned} \quad (23)$$

When $C_S = \mathbf{0}$, $BC_N U_N = \beta C_N U_N$, i.e., $\{\lambda_i(B) \neq \lambda_i(R_{PV}) \mid i = K+1\}$; when $C_S \neq \mathbf{0}$, $BC_N U_N = \sigma_v^2 (C_S U_S + C_N U_N) \neq \alpha (C_N U_N)$, where α represents a constant.

Therefore, it follows that when $\theta \notin \Theta$, the eigenvalues of B satisfy the following:

$$\begin{cases} \lambda_i(B) \neq \lambda_i(R_{PV}); & i = K+1 \\ \lambda_i(B) = \lambda_i(R_{PV}); & \forall i \in \xi - [K+1] \end{cases} \quad (24)$$

From Equation (24), it is observed that the eigenvalues corresponding to the noise are always divergent. Therefore, a sampling covariance matrix is constructed to model the ideal behavior of the noise eigenvalues, with the construction rule given as follows:

$$R_N = U \hat{\Sigma} U^H \quad (25)$$

where $\hat{\Sigma}$ is a diagonal matrix of eigenvalues, with each diagonal element constructed as follows:

$$\begin{cases} \hat{\lambda}_i = \lambda_i(R_{(P+V_c)V_c}) & ; \forall i \in S \\ \hat{\lambda}_i = \frac{\lambda_K(R_{(P+V_c)V_c}) + \lambda_{K+1}(R_{(P+V_c)V_c})}{2} & ; \forall i \in \xi \end{cases} \quad (26)$$

Based on the above derivation, it is evident that when $\theta \in \Theta$, the number of targets in the matrix B remains at K , the noise eigenvalues of matrices B and $R_{(P+V_c)V_c}$ are equal. However, when $\theta \notin \Theta$, due to the change in the number of targets in the matrix B from K to $K + 1$, the $K + 1$ th eigenvalue becomes unequal.

Therefore, when R_N constructed using Equations (25) and (26) is used to replace $R_{(P+V_c)V_c}$, the eigenvalue patterns of matrix R_N and matrix B should also satisfy the following equation, i.e.,

$$\begin{cases} \lambda_{K+1}(B) \neq \lambda_{K+1}(R_N); & \forall \theta \notin \Theta \\ \lambda_{K+1}(B) = \lambda_{K+1}(R_N); & \forall \theta \in \Theta \end{cases} \quad (27)$$

To make the two types of eigenvalues more distinguishable, we use Equation (26) to construct the eigenvalues of the new matrix $\mathbf{R}_N$. The eigenvector matrix $\mathbf{U}$ is taken as the eigenvector matrix of the covariance matrix $\mathbf{R}_{(P+V_c)V_c}$ obtained after pressure–velocity joint processing. The advantage of this construction is that all effective angular information contained in the sampled covariance matrix of the vector array can be retained.

Therefore, $\mathbf{R}_N$ constructed by Equation (26) can be used to replace $\mathbf{R}_{(P+V_c)V_c}$ for DOA estimation. In this case, $\mathbf{B}(\theta)$ becomes

$$\mathbf{B}(\theta) = \mathbf{R}_N + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) \quad (28)$$

Since only the eigenvalue magnitudes are reconstructed when constructing $\mathbf{R}_N$, while the form of the eigenvectors remains unchanged, it follows from the relevant subspace theory that the signal subspace and the steering-vector span are equivalent. Therefore, $\mathbf{B}(\theta)$ can be rewritten as

$$\begin{aligned} \mathbf{B}(\theta) &= \mathbf{R}_N + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) \\ &= \boldsymbol{\alpha}_1 \mathbf{A}_p \mathbf{R}_s \mathbf{A}_p^H \ell(\theta_g, \theta_k) + \frac{1}{2} \sigma_v^2 \boldsymbol{\alpha}_2 \mathbf{I}_M + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) \end{aligned} \quad (29)$$

where $\boldsymbol{\alpha}_1 = \text{diag}(\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{1,K})$ and $\boldsymbol{\alpha}_2 = \text{diag}(\alpha_{2,K+1}, \alpha_{2,K+2}, \dots, \alpha_{2,M})$. According to Equation (26), the noise eigenvalues have the same magnitude; therefore, their scaling factors are identical, namely $\alpha_{2,K+1} = \alpha_{2,K+2} = \dots = \alpha_{2,M}$. It can thus be seen that the matrix $\mathbf{B}(\theta)$ constructed using $\mathbf{R}_N$ is not essentially different from the matrix $\mathbf{B}(\theta)$ constructed using $\mathbf{R}_{(P+V_c)V_c}$. The difference lies only in a certain scaling of the signal and noise eigenvalues. Therefore, according to Equation (24), the eigenvalue relationship of the matrix $\{\mathbf{B}(\theta) | \theta \notin \Theta\}$ constructed using $\mathbf{R}_N$ can also be obtained as

$$\begin{cases} \lambda_i(\mathbf{B}) \neq \lambda_i(\mathbf{R}_N); & i = K+1 \\ \lambda_i(\mathbf{B}) = \lambda_i(\mathbf{R}_N); & \forall i \in \xi - [K+1] \end{cases} \quad (30)$$

Rewriting the above relationship gives

$$\begin{cases} \lambda_i(\mathbf{B}) \neq \hat{\lambda}_i; & i = K+1 \\ \lambda_i(\mathbf{B}) = \hat{\lambda}_i; & \forall i \in \xi - [K+1] \end{cases} \quad (31)$$

Similarly, according to Equation (17), the eigenvalue relationship of the matrix $\{\mathbf{B}(\theta) | \theta \notin \Theta\}$ constructed using $\mathbf{R}_N$ can be obtained as

$$\lambda_{K+1}(\mathbf{B}) = \hat{\lambda}_{K+1} \quad (32)$$

Therefore, the new spatial-spectral function is constructed by imitating the spatial-spectral form of the MUSIC algorithm as follows:

$$P(\theta) = \frac{1}{\lambda_{K+1}(\mathbf{B}) - \hat{\lambda}_{K+1}} \quad (33)$$

However, due to the influence of noise, the difference between the two eigenvalues will approach zero. Therefore, a peak is formed.

As a result, the $K+1$ th eigenvalue of matrices $\mathbf{B}$ and $\mathbf{R}_N$ will not be equal. The difference between the two eigenvalues will not approach zero, leading to a flat sidelobe.

Assume two independent and equal-power far-field plane wave signals incident on a ULA (Uniform Linear Array) of four AVS with DOAs of 15° and 20° , which at SNR = 0 dB, $\beta = \text{tr}(\mathbf{R}_{(P+V_c)V_c})/M$ and a snapshot number of $N = 100$. A performance comparison between the classical MUSIC algorithm and the algorithms proposed in this section is illustrated in Figure 1. From Figure 1, the classical MUSIC algorithm generates only one peak in the spatial spectrum when the DOAs are close, failing to resolve the two adjacent

targets. In contrast, the proposed algorithm accurately distinguishes the DOA of the two sources.

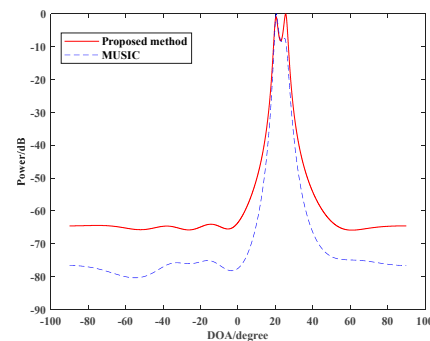


Figure 1. Performance comparison of the MUSIC and the proposed algorithm.

However, in order to obtain the $K + 1$ th eigenvalue of the matrix $\mathbf{B}$ at each scanning angle, the proposed algorithm requires eigenvalue decomposition at all search angles, which inevitably increases computational complexity.

3.2. Half-Spectrum-Search-Based DOA Estimation with Noise Power Invariance

To address the high computational complexity caused by eigenvalue decomposition at each search angle, this paper proposes a high-resolution algorithm based on sharding search, which reduces the computational complexity by narrowing the spatial spectrum search range and optimizes the proposed algorithm accordingly. The key idea of the proposed half-spectrum search is to construct a covariance matrix in which the true DOAs and their symmetric virtual DOAs coexist. For a scalar pressure array with centrosymmetric geometry, the pressure-channel steering vector can satisfy a conjugate-symmetry relationship. However, for an AVSA, the complete steering vector contains pressure and particle velocity components. Because the pressure component, the cosine-velocity component, and the sine-velocity component have different parity properties, the complete AVSA steering vector cannot be directly treated in the same way as the scalar pressure steering vector. This is the main reason why conventional symmetric-search methods cannot be directly applied to AVSA data. To make half-spectrum search applicable to AVSAs, the proposed method first performs PV co-processing and then uses the conjugate relationship of the pressure-channel response to construct a pseudo-data covariance matrix. For a ULA whose sensor positions are symmetrically distributed with respect to the array center, two incident directions with opposite angular signs produce mutually conjugate pressure-phase responses. Therefore, applying complex conjugation to the PV-co-processed covariance matrix can introduce the covariance information associated with the corresponding symmetric virtual direction.

Assuming the presence of K identical symmetric targets at the symmetric position of the real target, the DOA of the symmetric target DOA can be denoted as $-\Theta$. The signal covariance matrix of the received signal at the symmetric position can be denoted as

$$\mathbf{R}_v = \mathbf{A}_p(-\Theta)\mathbf{R}_s\mathbf{A}_p^H(-\Theta) + \frac{1}{2}\sigma_v^2\mathbf{I}_M \quad (34)$$

For clarity, the notation of the true and symmetric virtual DOAs is first defined. Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_k\}$ denote the set of true target DOAs, and let $-\Theta = \{-\theta_1, -\theta_2, \dots, -\theta_k\}$ denote the set of symmetric virtual DOAs. The union set is defined as $J = \Theta \cup (-\Theta)$.

For an AVS uniform linear array, the acoustic pressure component satisfies the following:

$$\begin{aligned}
 \mathbf{a}_p^*(\theta) &= [\exp(j\pi d_1 \sin(\theta)), \dots, \exp(j\pi d_M \sin(\theta))]^T \\
 &= [\exp(-j\pi d_1 \sin(-\theta)), \dots, \exp(-j\pi d_M \sin(-\theta))]^T \\
 &= \mathbf{a}_p(-\theta)
 \end{aligned} \quad (35)$$

where $d_m = (m-1)d$, and d represents array element spacing. In addition, according to Equation (35), we have

$$\begin{aligned}
 \mathbf{R}_V(\Theta) &= \mathbf{A}_p(-\Theta) \mathbf{R}_S \mathbf{A}_p^H(-\Theta) \ell(-\theta, \theta_i) + \frac{1}{2} \sigma_v^2 \mathbf{I}_M \\
 &= \mathbf{A}_p^*(\Theta) \mathbf{R}_S (\mathbf{A}_p^*(\Theta))^H \ell^*(\theta, \theta_i) + \frac{1}{2} (\sigma_v^2 \mathbf{I}_M)^* \\
 &= \mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}^*
 \end{aligned} \quad (36)$$

where $\mathbf{A}_p(\theta) = [a_p(\theta_1), a_p(\theta_2), \dots, a_p(\theta_K)]$. For a centrosymmetric ULA, the pressure-channel steering vector satisfies the conjugate-symmetry relationship $a_p(-\theta) = a_p^*(\theta)$.

Consequently, when the steering vector $\mathbf{a}_p(\theta)$ is an odd function, the complex conjugate operation on $\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}$ results in a matrix that can be regarded as a covariance matrix with symmetric target DOA information. It is evident that when the complex conjugate operation is performed on $\mathbf{R}_N$, the $\mathbf{R}_N^*$ contains the symmetric angle information. Given that the DOA information of the symmetric target is not a reality, the matrix is to be designated a pseudo-data covariance matrix.

This property is utilized to construct a matrix $\mathbf{C}$, which incorporates both the target angle information and the corresponding symmetric angle information. The expression for $\mathbf{C}$ is given as follows: $\mathbf{C} = \mathbf{R}_N + \mathbf{R}_N^*$. It can be shown that matrices $\mathbf{R}_N$ and $\mathbf{R}_N^*$ share the same eigenvalues, and their eigenvectors exhibit a complex conjugate relationship. As a result, matrix $\mathbf{C}$ can be further simplified as:

$$\begin{aligned}
 \mathbf{C} &= \mathbf{R}_N + \mathbf{R}_N^* \\
 &= \mathbf{U}_S \widehat{\boldsymbol{\Sigma}}_S \mathbf{U}_S^H + \mathbf{U}_N \widehat{\boldsymbol{\Sigma}}_N \mathbf{U}_N^H \\
 &\quad + \mathbf{U}_S^* \widehat{\boldsymbol{\Sigma}}_S^* (\mathbf{U}_S^*)^H + \mathbf{U}_N^* \widehat{\boldsymbol{\Sigma}}_N^* (\mathbf{U}_N^*)^H \\
 &= \mathbf{U}_S \widehat{\boldsymbol{\Sigma}}_S \mathbf{U}_S^H + \mathbf{U}_S^* \widehat{\boldsymbol{\Sigma}}_S^* (\mathbf{U}_S^*)^H + \sigma_v^2 \mathbf{I} \\
 &= \text{Re}[\mathbf{A}_p(\Theta) \mathbf{R}_S \mathbf{A}_p^H(\Theta) \ell(\theta, \theta_i) + \mathbf{A}_p(-\Theta) \mathbf{R}_S \mathbf{A}_p^H(-\Theta) \ell(-\theta, \theta_i)] \\
 &\quad + \sigma_v^2 \mathbf{I}_M
 \end{aligned} \quad (37)$$

$$\text{where } \widehat{\boldsymbol{\Sigma}}_S = \begin{bmatrix} \lambda_1 - \frac{\lambda_K(\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}) + \lambda_{K+1}(\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c})}{2} & & \\ & \ddots & \\ & & \lambda_K - \frac{\lambda_K(\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c}) + \lambda_{K+1}(\mathbf{R}_{(\mathbf{p}+\mathbf{v}_c)\mathbf{v}_c})}{2} \end{bmatrix} \text{ is the diagonal matrix,}$$

and $\widehat{\boldsymbol{\Sigma}}_S$ and $\widehat{\boldsymbol{\Sigma}}_N$ correspond to the signal eigenvalue and noise eigenvalue of matrix $\mathbf{R}_N$, respectively. As shown in Equation (37), the signal subspace of the reconstructed matrix $\mathbf{C}$ is expanded from the original K columns to $2K$ columns, which means that the number of signals is also expanded from the original K to $2K$, and its noise subspace is also reduced by K columns accordingly.

Imitating Equation (10) to construct the matrix $\mathbf{D}$, the expression is as follows:

$$\mathbf{D} = \mathbf{C} + \beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta) \quad (38)$$

Due to the coexistence of real-valued data $\mathbf{C}$ and complex operations $\beta \mathbf{a}_p(\theta) \mathbf{a}_p^H(\theta)$ within the matrix $\mathbf{D}$, the proposed method is termed the semi-real-valued noise power invariant (SR-NPI) algorithm, highlighting its partially real-valued structure.

The $2K+1$ th eigenvalue of matrix $\mathbf{D}$ and $\mathbf{C}$ at this point should satisfy the following relationship:

$$\begin{cases} \lambda_{2K+1}(\mathbf{D}(\theta)) = \lambda_{2K+1}(\mathbf{C}); & \theta \in \Theta \cup (-\Theta) \\ \lambda_{2K+1}(\mathbf{D}(\theta)) \neq \lambda_{2K+1}(\mathbf{C}); & \theta \notin \Theta \cup (-\Theta) \end{cases} \quad (39)$$

From Equation (39), it can be deduced that when the scanning angle θ belongs to the set of target angles Θ or the set of symmetric angles $-\Theta$, the difference between the $2K+1$ th eigenvalue of $\mathbf{D}$ and $\mathbf{C}$ tends to be close to 0. Conversely, when the scanning angle θ does not belong to the set of target angles Θ or the set of symmetric angles $-\Theta$, the difference between the $2K+1$ th eigenvalue of $\mathbf{D}$ and $\mathbf{C}$ does not tend to be close to 0. It is possible to construct a new spatial spectrum function by utilizing this property:

$$f(\theta) = \frac{1}{\lambda_{2K+1}(\mathbf{D}) - \lambda_{2K+1}(\mathbf{C})} \quad (40)$$

According to the spatial spectrum function described by Equation (40), when $\theta \in \Theta$, the difference between the $\lambda_{2K+1}(\mathbf{D}) - \lambda_{2K+1}(\mathbf{C})$ approaches 0, taking its inverse spectrum, and the function will form a sharp spectrum peak. Similarly, when the scanning angle θ belongs to $-\Theta$, the difference between $\lambda_{2K+1}(\mathbf{D}) - \lambda_{2K+1}(\mathbf{C})$ also approaches 0, so a spectrum peak will also be formed at its symmetrical position.

Based on the above, it can be concluded that Equation (40) retains a peak at the target angle during the spatial spectrum search. Additionally, a peak is also formed at the symmetric position of the target. Therefore, by utilizing the symmetry of the y -axis, it is sufficient to search only the left or right half of the y -axis to obtain the complete directional information.

It should be noted that $\mathbf{C}$ is independent of the scanning angle. Therefore, $\lambda_{2K+1}(\mathbf{C})$ only needs to be computed once before the spectral search. In this work, the SR-NPI spectrum is constructed directly by using $\lambda_{2K+1}(\mathbf{C})$, avoiding any unproven eigenvalue replacement between $\mathbf{C}$ and $\mathbf{R}_N$. Thus, Equation (40) can be rewritten, and the spatial spectrum of the SR-NPI algorithm is given by:

$$f(\theta) = \frac{1}{\lambda_{2K+1}(\mathbf{D}(\theta)) - 2\lambda_{2K+1}(\mathbf{R}_N)} \quad (41)$$

As demonstrated in Equation (39), the DOA information obtained encompasses both the target real DOA information and the symmetric DOA information.

To account for special cases, such as the presence of 0° and symmetrical angles relative to the true angle, we can employ algorithms like MDL to estimate the number of sources in matrix $\mathbf{C}$, thereby obtaining the number of sources $\bar{K}$ in $\mathbf{C}$. This approach can replace Equation (41). The specific implementation is as follows:

$$f(\theta) = \frac{1}{\lambda_{\bar{K}+1}(\mathbf{D}) - \lambda_{\bar{K}+1}(\mathbf{C})} \quad (42)$$

Consequently, it is imperative to discriminate the obtained DOA information. Due to the MUSIC algorithm, which can form spectral peaks at the target, the MUSIC algorithm's spatial-spectral function can be utilized to discriminate the obtained DOA information within the half-spectrum. Therefore, the following discrimination rules can be utilized for the selection of the obtained DOA information:

$$\begin{cases} f_{MUSIC}(\theta_k) \gg f_{MUSIC}(-\theta_k), & \Theta = \Theta \cup \{\theta_k\} \\ f_{MUSIC}(\theta_k) \ll f_{MUSIC}(-\theta_k), & \Theta = \Theta \cup \{-\theta_k\} \\ f_{MUSIC}(\theta_k) \approx f_{MUSIC}(-\theta_k), & \Theta = \Theta \cup \{\theta_k, -\theta_k\} \end{cases} \quad (43)$$

The DOA information θ_k obtained from either the left or right half of the y -axis spectrum is substituted into Equation (42). Since the MUSIC algorithm's spatial spectrum function is derived from the eigenvalue decomposition of $\mathbf{R}$ into its noise subspace, it contains only the true target DOA information, excluding the symmetric DOA information. Substituting the scanning angle θ_k into Equation (43), when $f_{MUSIC}(\theta_k) \gg f_{MUSIC}(-\theta_k)$, it indicates that θ_k is the true DOA, and $-\theta_k$ represents the

virtual symmetric target. Conversely, when $f_{MUSIC}(\theta_k) \ll f_{MUSIC}(-\theta_k)$, $-\theta_k$ is the true DOA, and θ_k is the virtual symmetric target. When $f_{MUSIC}(\theta_k) \approx f_{MUSIC}(-\theta_k)$, it indicates that both targets are real DOA targets. Thus, using the discrimination rule in Equation (40), the true target DOA and symmetric DOA can be accurately identified.

3.3. Applicability and Limitations

It should be noted that the proposed half-spectrum search is valid only when the required symmetry conditions are satisfied. Specifically, the AVSA geometry should be centrosymmetric, and the PV-co-processed pressure-channel steering vector should satisfy the conjugate-symmetry relationship between θ and $-\theta$. Under these conditions, the reconstructed pseudo-data covariance matrix can represent both the true DOAs and their symmetric virtual DOAs, which provides the basis for the half-spectrum search.

If the array geometry is non-symmetric, or if sensor-position errors, channel mismatch, or sound-speed mismatch destroy the steering-vector symmetry, the reconstructed pseudo-data covariance matrix may no longer correctly represent the symmetric virtual DOAs. In addition, when coherent sources are present, the covariance matrix may suffer from rank deficiency, and the expected eigenvalue-ordering relationship may be weakened or invalid. Therefore, SR-NPI is mainly applicable to centrosymmetric AVSA configurations or arrays that can be reliably transformed into equivalent symmetric virtual arrays. For coherent sources or severe model mismatch, additional preprocessing, such as calibration or decorrelation, may be required.

3.4. Algorithm Implementation and Computational Analysis

Based on the above analysis, the SR-NPI algorithm generates sharp spectral peaks at both the true DOA and its symmetric angle. As a result, during the spectral search, only half of the angular space (with the y -axis as the axis of symmetry) needs to be searched, significantly reducing the search range. This effectively controls the computational complexity and enables fast DOA estimation. The main processing flow of the SR-NPI algorithm is shown in Table 1.

Table 1. The proposed SR-NPI algorithm procedure.

Step 1. Acquisition of vector array of N snap output signals $X(t)$, where $t = 1, 2, \dots, N$.
Step 2. Perform joint estimation of PVs to obtain the sampling covariance matrix $R_{(P+V_c)V_c}$.
Step 3. Construct matrices C and D according to Equations (37) and (38).
Step 4. Construct the SR-NPI algorithm spatial-spectral function $f(\theta)$ based on Equation (42).
Step 5. Search for $f(\theta)$ in the range $[-\pi, 0]$ or $[0, \pi]$ and give the angle θ_k corresponding to the position of its spectral peak.
Step 6. Decide based on the criteria in Equation (43) $\pm\theta_k$ and find the true DOA.

For SR-NPI, the computational cost consists of five parts: PV co-processing, covariance matrix estimation, pseudo-data covariance reconstruction, construction of the scanning matrix, and the half-spectrum search. In particular, the pseudo-data reconstruction includes the complex conjugation of the covariance matrix, the construction of $C = R_N + R_N^*$, and the construction of D at each scanning angle. These operations are included in the following complexity evaluation.

The computational complexity of the SR-NPI and NPI (noise power invariant) [23] algorithms are analyzed. For the NPI algorithm, the complexity of matrix inversion is $O(3M)^{3m}$, the eigenvalue decomposition is $O(3M)^3$ [24], and the spectral search is

$\mathcal{O}(J)$, where J represents the number of search points. Therefore, the overall complexity of the NPI algorithm is $\mathcal{O}[J(27M^3)+J+27(2m-1)M^3]$. Similarly, the complexity of the SR-NPI algorithm is $\mathcal{O}[J(M^3+1)/2+M^2K]$. After including the pseudo-data reconstruction cost, the computational cost of SR-NPI is approximately 1.92% of that of the NPI algorithm under the tested configuration.

4. Simulations

4.1. Correctness Verification of the Proposed Algorithm

Assume that K mutually independent far-field plane-wave signals of equal power are incident on a ULA consisting of six AVSs of half-wavelength spacing with DOA angles of Θ . The SNR = 0 dB, the number of snapshots $N = 500$ and the search step size is 0.01° . During simulation testing, we set the frequency for the single target to 1000 Hz, while for the dual targets, we set frequencies of 1000 Hz and 1020 Hz. The method for selecting the

$$\text{array element spacing was } d = \frac{1}{2} \frac{c}{\frac{1}{K} \sum_{i=1}^K f_i}.$$

A comparative experiment is conducted using the NPI, SR-NPI, and classical MUSIC algorithms, with the results shown in Figure 2. As seen in Figure 2a, when $K = 1$ and the DOA is 40° , both the NPI and MUSIC algorithms form sharp spectral peaks at 40° . In contrast, the SR-NPI algorithm not only forms a sharp peak at 40° , but also forms a peak at -40° , indicating that the proposed algorithm can obtain both the true target source and its symmetric sources, respectively. This validates the correctness of the SR-NPI algorithm. To further validate its effectiveness in multi-target scenarios, the number of targets $K = 2$ with DOAs of 20° and 40° , and the simulation results are shown in Figure 2b. Both the NPI and classical MUSIC algorithms form sharp peaks at 20° and 40° , indicating that these algorithms can accurately estimate the target DOAs. Meanwhile, the SR-NPI spatial spectrum not only retains the true target peaks at 20° and 40° , but also forms sharp peaks at the symmetric positions, i.e., -20° and -40° . This demonstrates that the SR-NPI algorithm's spectrum allows for the selection of either the left or right half of the y -axis search space to obtain the real DOA.

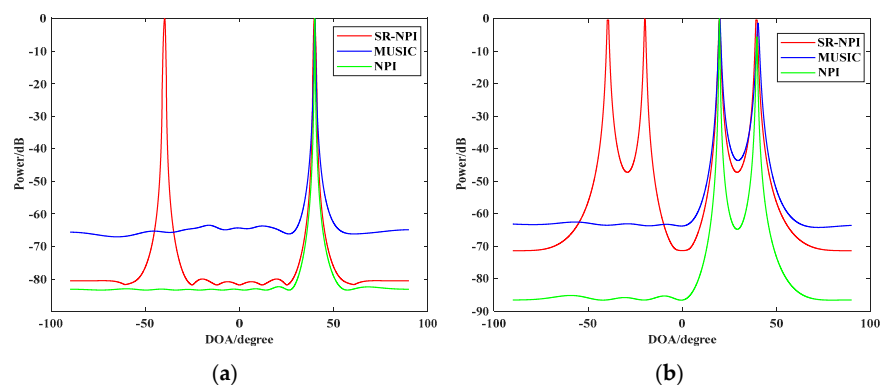


Figure 2. Spatial spectrum comparison: (a) $K = 1$ $\Theta = [40^\circ]$; (b) $K = 2$ $\Theta = [20^\circ, 40^\circ]$.

To further examine the robustness of the proposed method under more challenging source conditions, three unequal-power far-field signals were considered. Unless otherwise stated, the power ratio was set to 1:0.85:0.6, the number of snapshots was 500, and the number of sources was $K = 3$. As shown in Figure 3, unequal source powers lead to different spectral-peak amplitudes in the SR-NPI spectrum. The dominant source is more stable, whereas weaker sources are more sensitive to low SNR and angular separation. In

particular, when weak sources are closely spaced, adjacent peaks may merge into a single peak, indicating that close spacing and power imbalance jointly degrade the resolvability of the proposed method.

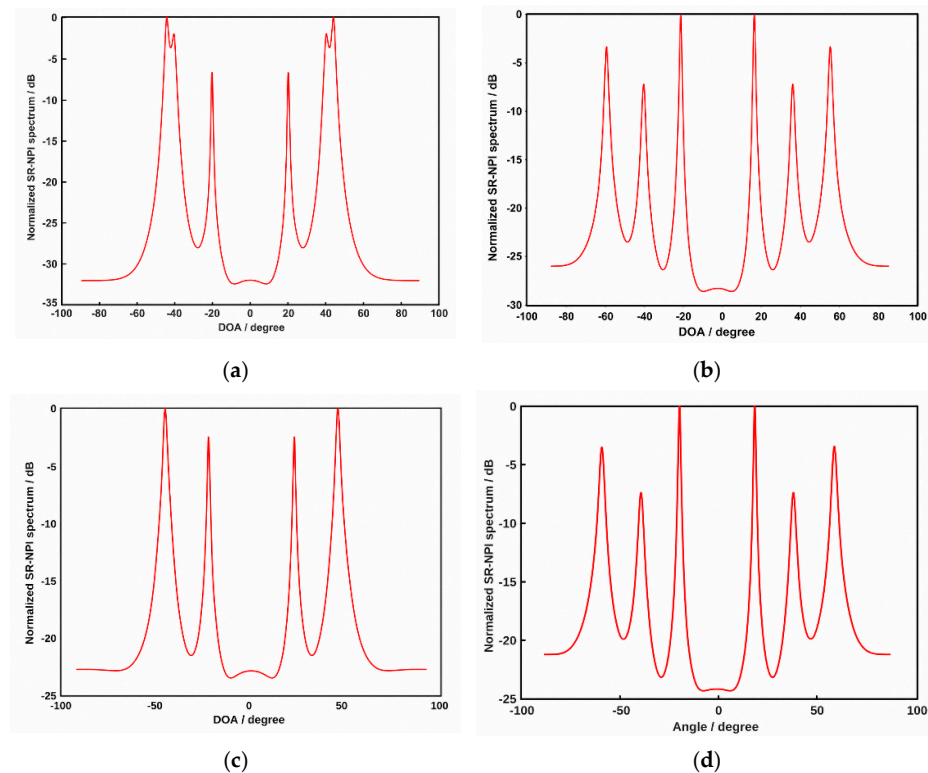


Figure 3. Spatial spectrum of SR-NPI under unequal-power three-source conditions with power ratio of 1:0.85:0.6: (a) SNR = 0 dB, DOAs = [20°, 40°, 45°]; (b) SNR = 0 dB, DOAs = [20°, 40°, 60°]; (c) SNR = −5 dB, DOAs = [20°, 40°, 45°]; (d) SNR = −5 dB, DOAs = [20°, 40°, 60°].

To further validate the correctness of the proposed algorithm, we keep the other experimental conditions unchanged, and the relationship between the eigenvalues of $\mathbf{D}$ and $\mathbf{R}_N$ is analyzed. Figure 4a presents the case when the scanning angle $\theta \notin \mathcal{G}$, where $\mathcal{G}$ denotes the set of true and symmetric DOA. It can be seen that a relatively large eigenvalue in $\mathbf{R}_N$ indicates the presence of a single signal source. In contrast, $\mathbf{D}$ exhibits three larger eigenvalues, which indicates that three signal sources are present in $\mathbf{D}$ under this condition, corresponding to the true target, the symmetric target, and the scanning source. This is due to the fact that when the scanning angle θ does not belong to the set $\mathcal{G}$, an additional scanning source is introduced. In this case, the $2K + 1$ th eigenvalue of $\mathbf{D}$ corresponds to a signal eigenvalue and is significantly different from the $2K + 1$ th eigenvalue of $\mathbf{R}_N$. As a result, a sharp spectral peak cannot be formed. Similarly, when the scanning angle $\theta \in \mathcal{G}$, as shown in Figure 4b, $\mathbf{D}$ contains two larger eigenvalues among multiple eigenvalues, indicating that the number of signals is 2, i.e., the target signal and the symmetric target. This occurs because the scanning angle coincides with $\mathcal{G}$, leading to signal power reinforcement rather than the introduction of an additional source. Therefore, the $2K + 1$ th eigenvalue of matrix $\mathbf{D}$ closely matches the $2K + 1$ th eigenvalue of matrix $\mathbf{R}_N$. This consistency allows a sharp spectral peak to be formed, thereby enabling accurate DOA estimation.

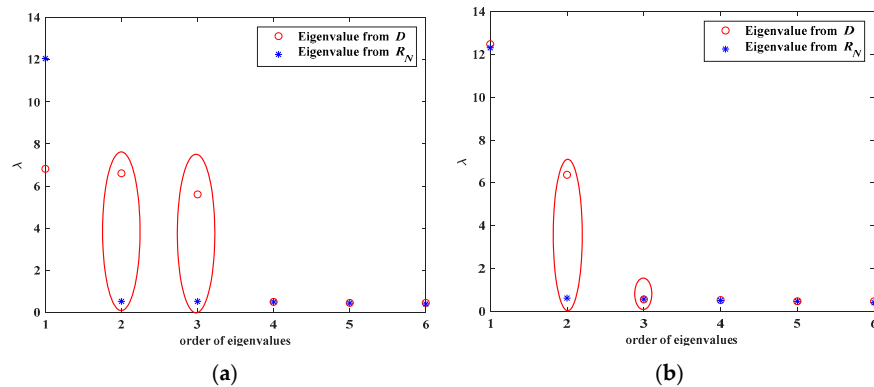


Figure 4. Eigenvalue ranking diagram for a single target: (a) $K = 1$ $\theta \notin \mathcal{G}$; (b) $K = 1$ $\theta \in \mathcal{G}$.

To validate the theoretical effectiveness under multiple target conditions, as seen in Figure 5, the relationship between the eigenvalues of matrix $\mathbf{D}$ and matrix $\mathbf{R}_N$ is analyzed when $K = 2$. Figure 5a presents the case when the scanning angle $\theta \notin \mathcal{G}$. It can be seen that five relatively large eigenvalues are in the matrix $\mathbf{R}_N$, which indicates that five signal sources are present in $\mathbf{D}$ under this condition, corresponding to the two true targets, the two symmetric targets, and one scanning source. In this case, the $2K + 1$ th eigenvalue of $\mathbf{D}$ corresponds to a signal eigenvalue and is significantly different from the $2K + 1$ th eigenvalue of $\mathbf{R}_N$. As a result, a sharp spectral peak cannot be formed. Similarly, when the scanning angle $\theta \in \mathcal{G}$, as shown in Figure 5b, $\mathbf{D}$ contains four larger eigenvalues among multiple eigenvalues, i.e., two target signals and two symmetric targets. It is further shown that the $2K + 1$ th eigenvalue corresponds to the noise eigenvalue. Therefore, the $2K + 1$ th eigenvalue of matrix $\mathbf{D}$ closely matches the $2K + 1$ th eigenvalue of matrix $\mathbf{R}_N$. This consistency allows a sharp spectral peak to be formed, thereby enabling accurate DOA estimation.

By jointly analyzing Figures 4 and 5, it can be concluded that the $2K + 1$ th eigenvalue of $\mathbf{D}$ corresponds to a signal eigenvalue when $\theta \notin \mathcal{G}$; the $2K + 1$ th eigenvalue of $\mathbf{D}$ corresponds to the noise eigenvalue, and the eigenvalue corresponding to the noise of $\mathbf{D}$ is almost the same size as the noise eigenvalue of $\mathbf{R}_N$ when $\theta \in \mathcal{G}$. These results are consistent with the above eigenvalue-ordering analysis and provide finite-sample numerical evidence for the construction of the SR-NPI spectrum.

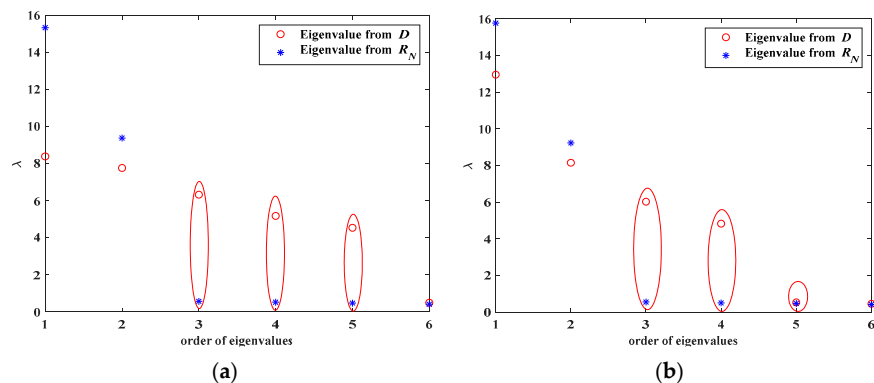


Figure 5. Eigenvalue ranking diagram for dual targets: (a) $K = 2$; $\theta \notin \mathcal{G}$; (b) $K = 2$; $\theta \in \mathcal{G}$.

It should be noted that the ideal eigenvalue equality holds for the population covariance matrix. In finite-snapshot cases, the sample covariance matrix contains estimation errors, and the equality becomes an approximate eigenvalue-consistency condition.

Therefore, the spectral peak is formed when the eigenvalue difference reaches a local minimum rather than when the two eigenvalues are exactly identical.

4.2. Performance Verification of the Proposed Algorithm

To evaluate the root mean square error (RMSE) performance of the SR-NPI algorithm, 300 independent Monte Carlo simulations were conducted while keeping the other experimental conditions unchanged. The number of snapshots is $N = 500$, the number of sources is $K = 1$, and the DOA is 40° . The SNR varied from -10 dB to 10 dB. Before presenting the comparative simulation figures, we explained the reference methods involved in the comparison. CRLB denotes the Cramer–Rao lower bound, which is used as the theoretical accuracy benchmark rather than as an estimation algorithm. MUSIC denotes the classical Multiple Signal Classification algorithm. NPI denotes the noise-power-invariance-based algorithm and is used as the direct baseline method. SR-NPI denotes the proposed semi-real-valued noise-power-invariance algorithm. LCDSR denotes the low-complexity sparse-reconstruction-based DOA estimation method and is used as an additional low-complexity benchmark.

The statistical performance of the MUSIC, NPI, and SR-NPI algorithms are illustrated in Figure 6. As shown in Figure 6, the RMSE of all three algorithms decreases as the SNR increases. Overall, the proposed NPI algorithm outperforms both the MUSIC and SR-NPI algorithms in terms of RMSE. The slight degradation in SR-NPI performance compared to NPI is attributed to the inclusion of symmetric virtual sources, which increases the number of targets. Despite this, the SR-NPI algorithm offers a significant advantage by substantially reducing the computational complexity of spatial spectrum search, while still maintaining performance superior to that of the conventional MUSIC algorithm.

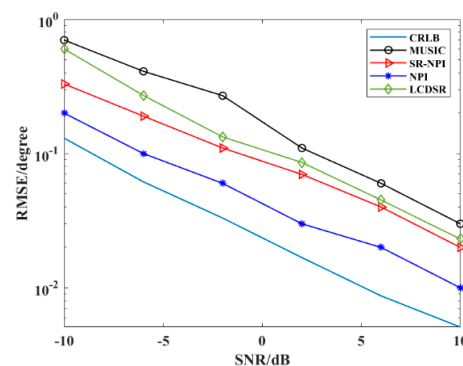


Figure 6. RMSE versus SNR, $K = 1$.

To further validate the RMSE performance under multiple targets, the number of sources was set to $K = 2$ with DOAs at 20° and 40° . As shown in Figure 7, the proposed SR-NPI algorithm continues to outperform the conventional MUSIC algorithm, LCDSR [23], and maintains performance close to that of the NPI algorithm. This result further justifies the effectiveness of the SR-NPI algorithm. Comparing Figures 6 and 7 reveals that as the number of sources increases, the RMSE of all four algorithms including MUSIC, LCDSR, NPI and SR-NPI is also increased. This indicates that the number of targets is a critical factor influencing the estimation accuracy of DOA algorithms.

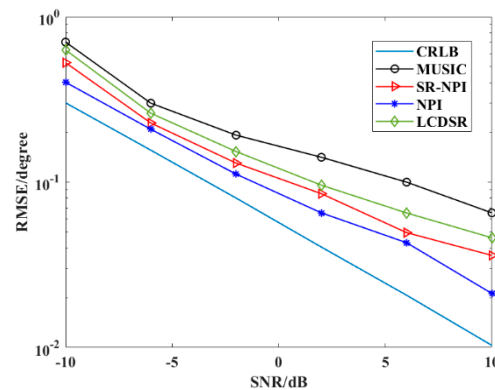


Figure 7. RMSE versus SNR, $K = 2$.

Furthermore, five array elements and 500 snapshots were used to evaluate the RMSE performance under unequal-power sources with DOAs of 20° , 40° , and 45° . The power ratio remained at 1:0.85:0.6. As shown in Figure 8, the RMSE of the proposed algorithm decreases as the SNR increases. Compared with the single-source and two-source cases, a certain performance degradation is observed because unequal powers and close angular spacing increase the difficulty of eigenvalue discrimination. Nevertheless, SR-NPI remains close to NPI and outperforms MUSIC and LCDSR under the tested conditions.

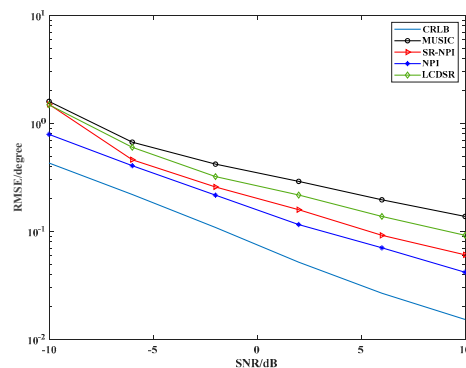
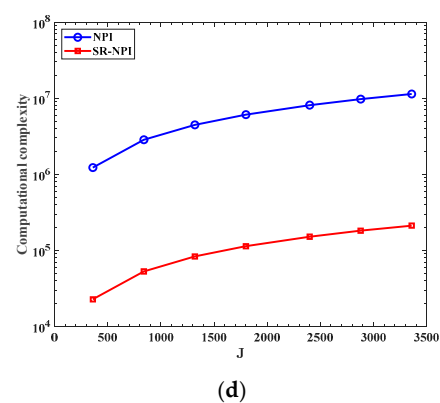
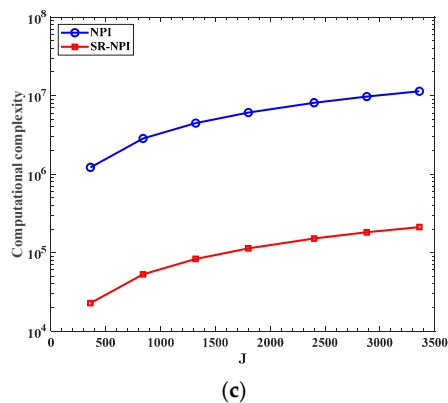
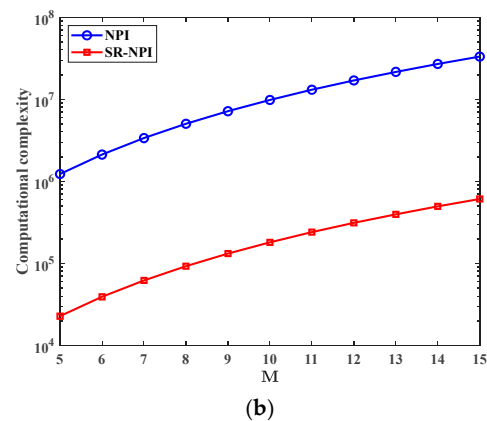
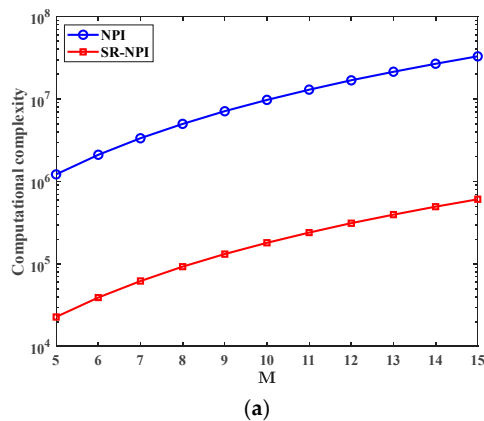


Figure 8. RMSE versus SNR for unequal-power signals.

To provide a fair comparison, the computational complexity of the SR-NPI algorithm is evaluated by including the main processing steps, namely PV co-processing, covariance matrix estimation, pseudo-data covariance reconstruction, matrix construction, and the half-spectrum search. The corresponding computational cost of the NPI algorithm is also evaluated under the same parameter settings. The simulation focuses on three key factors affecting computational complexity: the number of array elements M , the power index m , and the number of search points J . The simulation focuses on three key factors affecting computational complexity: the number of array elements M , the power index m , and the number of search points J .

When the number of search points is fixed at $J = 361$, and the power index is set to $m = 1$, the number of AVSs is varied from 5 to 15. As shown in Figure 9a, the computational complexity of both algorithms increases with the number of array elements. The SR-NPI algorithm consistently maintains approximately two orders of magnitude lower complexity compared to the NPI algorithm. When the power index is increased to $m = 2$ while keeping other parameters constant, Figure 9b shows that the computational complexity gap between the SR-NPI and NPI algorithms remains nearly unchanged. This is because the SR-NPI algorithm is independent of the power index m , and thus its computational

complexity does not increase with m . In contrast, the complexity of the NPI algorithm grows with m , although the impact is relatively small. As a result, the difference in computational complexity between the two algorithms does not significantly expand. When $m = 1$ and $M = 5$, the number of search points J is varied from 361 to 3610. Figure 9c demonstrates that the computational complexity of two algorithms increases with J . Similarly, when $m = 2$ and $M = 5$, Figure 9d shows that the relative difference in complexity remains stable. As shown in Figure 9e, when the number of search points is set to $J = 361$ and the number of array elements is $M = 5$, the computational complexity of the NPI algorithm increases with the power index m (ranging from 1 to 10), though the growth is not significant. In contrast, the complexity of the SR-NPI algorithm remains constant due to its independence from m . When the number of array elements is further increased to $M = 15$, as shown in Figure 9f, the computational advantage of SR-NPI remains evident, with the complexity gap between the two algorithms consistently around two orders of magnitude. Overall, Figure 9a–f demonstrate that although the complexity of both algorithms increases with M , J , and m , the SR-NPI algorithm consistently exhibits superior computational efficiency.



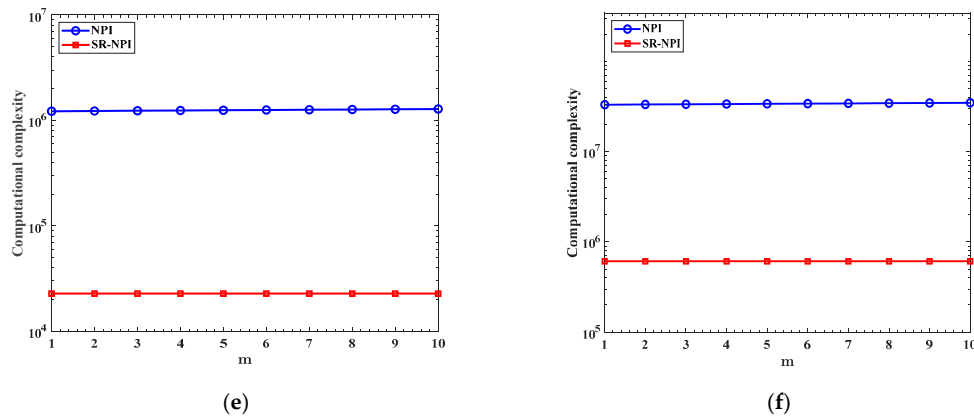


Figure 9. Comparison chart of complexity of algorithms versus M , m , and J : (a) $m = 1$, $J = 361$; (b) $m = 2$, $J = 361$; (c) $m = 1$, $M = 5$; (d) $m = 2$, $M = 5$; (e) $M = 5$, $J = 361$; (f) $M = 15$, $J = 361$.

We set DOAs = $[20^\circ, 40^\circ]$ and SNR = 2 dB and kept the number of elements constant to conduct RMSE simulations under different snapshots. As shown in Figure 10, the RMSE decreases as the number of snapshots increases, and the proposed SR-NPI algorithm demonstrates performance comparable to the NPI algorithm while outperforming algorithms such as MUSIC and LCDSR.

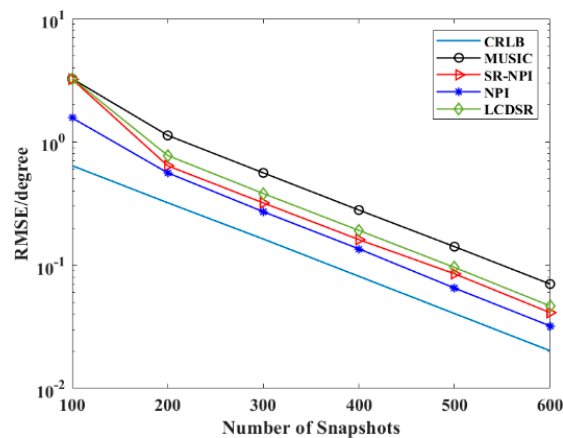


Figure 10. RMSE performance versus the number of snapshots for MUSIC, NPI, LCDSR, and SR-NPI under the two-source scenario.

To evaluate the influence of the snapshot number, the SNR was fixed at 2 dB and the number of snapshots varied from 100 to 600. As shown in Figure 11, the estimation performance of SR-NPI improves as the number of snapshots increases. The proposed method remains comparable to NPI and performs better than MUSIC and LCDSR, which indicates that SR-NPI maintains robustness under unequal-power multiple-source conditions.

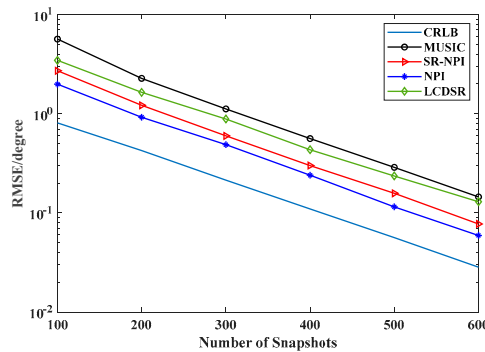


Figure 11. RMSE versus snapshots for unequal-power signals.

In order to further verify the effectiveness of the algorithm, the SR-NPI and NPI algorithms are tested on the same hardware, and the time they take to test is counted. The number of array elements is set M to 10, the SNR = 10 dB, and the number of snapshots N is set to 2000. The efficiency of the two algorithms under different power indexes and search points is counted. The results are shown in Table 2. It can be seen that as the number of search points increases, the time spent by the SR-NPI and NPI algorithms also increases. However, the computational cost of SR-NPI is much smaller than that of NPI.

That is, when the power index m in NPI is 1, the difference between the two is two orders of magnitude, and when the power index m in NPI is 2, the difference between the two will not expand. This shows that when the number of array elements and the number of snapshots are fixed, SR-NPI has higher computational efficiency than NPI.

Table 2. The time required for the algorithm under different search points.

J	180	360	720	1440	2880
SR-NPI (s)	0.01263	0.02373	0.04713	0.08972	0.17062
NPI ($m = 1$) (s)	1.151	2.235	4.683	9.103	16.985
NPI ($m = 2$) (s)	1.211	2.242	4.6911	9.126	17.015

Set the number of search points J to 240, SNR = 10 dB, and snapshots N to 2000. The efficiency of the two algorithms under different power indexes and array element counts is counted. The results are shown in Table 3. It can be observed that as the number of array elements increases, the time spent by the SR-NPI and NPI algorithms also increases. However, the computational cost of SR-NPI is much smaller than that of NPI. That is, when the power index m in NPI is 1, the difference between the two is two orders of magnitude, and when the power index m in NPI is 2, the difference between the two will not expand. It can be seen that the SR-NPI algorithm is more suitable for multiple array elements compared to the NPI algorithm.

Table 3. The time required for the algorithm with different numbers of array elements.

M	5	7	9	11	13
SR-NPI (s)	0.00923	0.01373	0.01713	0.02113	0.02613
NPI ($m = 1$) (s)	0.971	1.356	1.689	2.116	2.5185
NPI ($m = 2$) (s)	1.003	1.365	1.708	2.196	2.5213

Setting the number of search points J as 240, the number of array elements M is set to 3, the SNR = 10 dB, and the number of snapshots N is set to 200. The efficiency of the two algorithms under different power indexes is counted. The results are shown in Table 4. It can be observed that as the number of power indices increases, the time spent by the SR-

NPI remains essentially unchanged, and the NPI algorithms increase. Therefore, the amount of calculation of SR-NPI is much smaller than that of NPI.

Table 4. The time required for the algorithm to run under different power exponents.

m	1	2	3	4	5
SR-NPI (s)	0.00164	0.00163	0.00165	0.00162	0.00163
NPI (s)	0.1619	0.1719	0.1801	0.1821	0.1906

To further validate the computational complexity advantages of the proposed algorithm over the LCDSR algorithm, we conducted computational comparison tests by setting Ω to 0.1° , 1° , 10° , and 30° respectively. The results are shown in Figure 12.

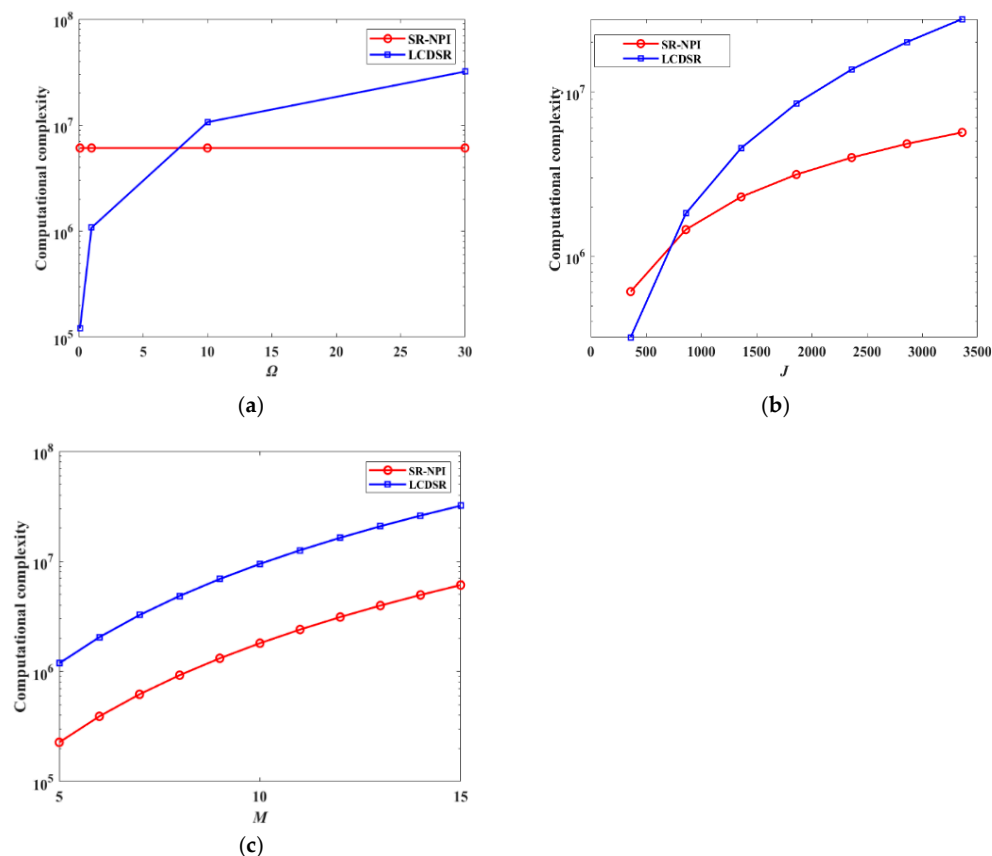


Figure 12. Comparison of computational complexity under different M , J , and Ω : (a) $M = 15, J = 3610$; (b) $\Omega = 30, M = 15$; (c) $\Omega = 30, J = 3610$.

As shown in Figure 12, when Ω is large, the computational complexity of the proposed algorithm demonstrates a clear advantage. Therefore, this demonstrates that the proposed algorithm is more suitable than LCDSR for scenarios involving multiple elements and large angle diffusion intervals.

5. Experiment

To validate the performance of the proposed algorithm in practical scenarios, we conducted a field experiment. The specific details of the experimental setup are as follows:

In the sea trial experiment, the eight AVSs used (one out of nine AVS was damaged) (Self-developed, Harbin, China) are all capsule-shaped, and the vibration velocity data

along the z-axis is not considered, meaning they are treated as two-dimensional AVSs. The deployment configuration of the uniform linear vector array used in the experiment is shown in Figure 13a, with the distance between two adjacent array elements being 0.075 m. Figure 13b shows the UW350 transducer used in the experiment, which was fixed to the steering rod at the bow of the ship to ensure that the sound source does not shift with ocean currents. The receiving end of the experimental system includes the following equipment: a preamplifier circuit (Self-developed, Harbin, China) and a signal conditioning circuit (the vector array consists of a 9-channel sound pressure circuit amplification module and a 27-channel vibration velocity circuit amplification module) (Self-developed, Harbin, China), a programmable filter amplifier (PF28000), and a data acquisition device (NI) (Self-developed, Harbin, China). Among these, the preamplifier circuit conditioning module provides an amplification factor of 26 dB for both sound pressure and vibration velocity channels. The programmable filter amplifier PF28000 has a bandpass filter selection range of 100 Hz to 10 kHz and provides an amplification factor of 20 dB for both sound pressure and vibration velocity channels. The operating range of the NI data acquisition device is 12 V, and its sampling frequency is set to 50 kHz.

In the experiment, a single-frequency signal with a frequency of 5 kHz was used, with an integration time of 1 s. Owing to fixed-point transmission, the angle between the transmitting transducer and the AVSs is -90° . It can be observed from Figure 14b that the SR-NPI algorithm can stably track the target angle, and the DOA angle does not exhibit significant fluctuations. Its bearing-time record (BTR) shows not only a bright yellow line near -90° but also another bright yellow line near 90° , indicating that the algorithm can form spectral peaks not only at the target location but also at its symmetric position. Therefore, in practical searching, the spectrum scan can be performed only within the range of -180° to 0° or 0° to 180° , thereby reducing the search range. Figure 14a and 14c show the bearing-time records of the MUSIC algorithm and the NPI algorithm, respectively. From Figure 14c, we can see that compared with MUSIC and SR-NPI, the NPI algorithm exhibits a cleaner background, indicating its stronger noise-suppression capability. However, neither MUSIC nor NPI forms corresponding spectral peaks at the symmetric positions, so they cannot perform half-spectrum searching to obtain the target location. The above experimental data verify the feasibility of the proposed low-complexity algorithm in practical field trials.

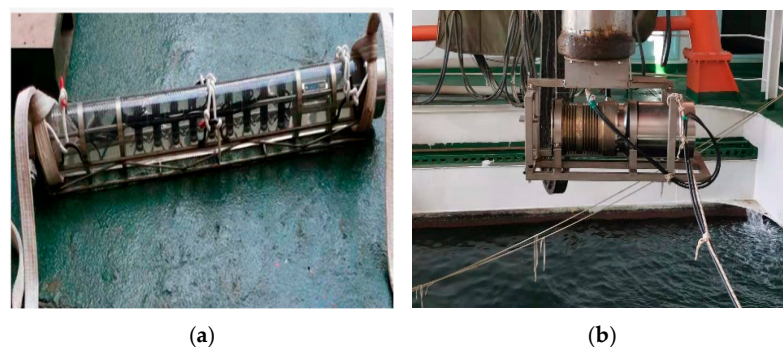


Figure 13. Sea trial equipment: (a) uniform linear acoustic vector sensor array; (b) transmitting transducer.

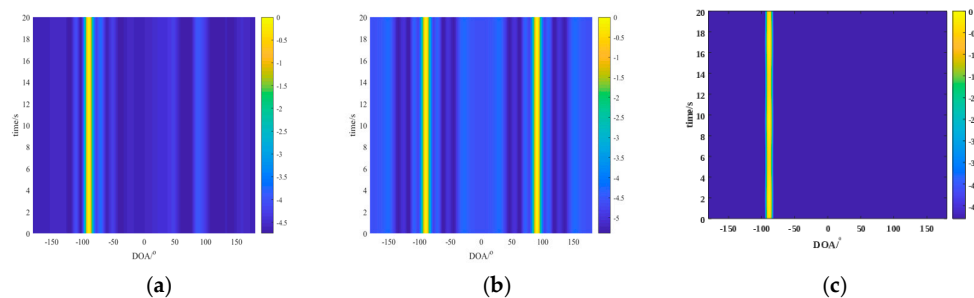


Figure 14. Algorithm time position history diagram: (a) MUSIC; (b) SR-NPI; (c) NPI.

To further verify the statistical performance of the algorithms, we conducted RMSE statistical analysis using the above data. As shown in the Figure 15, all three curves remain relatively stable during the observation interval, indicating that the DOA estimation results do not fluctuate significantly with time. MUSIC exhibits the largest RMSE, whereas SR-NPI achieves a much lower error level. NPI provides the smallest RMSE among the three methods, showing the best estimation accuracy in this experiment. However, compared with NPI, SR-NPI can greatly reduce the computational complexity through the half-spectrum search mechanism while maintaining acceptable estimation accuracy. Therefore, SR-NPI provides a favorable trade-off between estimation performance and computational efficiency.

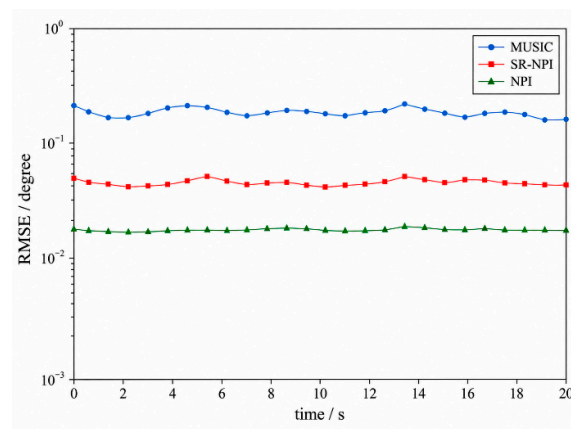


Figure 15. RMSE performance comparison of MUSIC, SR-NPI, and NPI over time. The y -axis is shown on a logarithmic scale to highlight the difference in estimation accuracy among the three algorithms.

6. Conclusions

This paper has proposed a low-complexity DOA estimation method for acoustic vector sensor arrays, referred to as SR-NPI. The proposed method first exploits the symmetry properties of the array steering vector through pressure–velocity (PV) co-processing. Then, a pseudo-data covariance matrix is constructed by using the original covariance matrix and its complex conjugate. Based on the eigenvalue-ordering property of the reconstructed matrix, SR-NPI constructs the spatial spectrum by observing the eigenvalue variation caused by the introduced scanning source. In this way, DOA estimation can be achieved through half-spectrum search. Compared with MUSIC and the original NPI algorithm, the proposed SR-NPI algorithm exploits the eigenvalue-ordering property of the reconstructed covariance matrix, enabling half-spectrum search for acoustic vector sensor arrays (AVSAs). In contrast, MUSIC and the original NPI algorithm generally require full-angular-domain spectral search to obtain accurate DOA estimates. Compared with the LCDSR algorithm, the proposed method does not rely on sparse dictionary construction

or sparse reconstruction, thereby avoiding the computational burden and potential grid mismatch associated with dictionary-based methods. Instead, it provides a direct eigenvalue-analysis-based low-complexity framework for acoustic vector sensor arrays.

However, the proposed algorithm also has certain limitations. Although SR-NPI can significantly reduce the computational complexity of AVSA-based DOA estimation, its implementation relies on the required parity and conjugate-symmetry properties of the array steering vector after PV co-processing. When the array steering vector does not satisfy the required odd-function or conjugate-symmetry relationship, the reconstructed pseudo-data covariance matrix cannot correctly represent the symmetric virtual DOAs, and the proposed half-spectrum search mechanism cannot be effectively implemented. In addition, when coherent sources are present, the covariance matrix may suffer from rank deficiency. This rank deficiency reduces the number of effective signal eigenvalues and destroys the expected eigenvalue-ordering structure.

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