

Article

Resolution-Induced Collapse in Quantized Nonlinear Dynamics: A Finite-Horizon Structural Framework

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Abstract

Finite-precision implementation fundamentally changes nonlinear dynamical systems by replacing continuous-state evolution with deterministic dynamics on a finite set of representable states. This study examines when that change becomes structurally important over a finite observation horizon. Quantization is treated as a resolution constraint, and an operational separation scale $\delta_{\text{sep}}(T_0, T; \varepsilon)$ is introduced to compare the implementation resolution with the attractor detail exposed by a reference trajectory. The ratio $\eta = \Delta / \delta_{\text{sep}}$ and its associated critical bit width $b_c(T)$ are used as protocol-dependent measures for precision screening. Experiments on the Hénon map, a Lorenz system integrated by fixed-step fourth-order Runge–Kutta, and the Logistic map show strong system dependence. Hénon exhibits broad, non-monotonic finite-state reshaping across bit width, whereas Lorenz remains in a low-complexity regime over a wider low-bit range before recovering more complex recurrent behavior. Results from 100 selected occupied quantized attractor positions show that entropy alone is insufficient to characterize collapse; recurrence and transient lengths provide complementary information about orbit organization. A 21-horizon Lorenz study produces stepwise changes and a long plateau in $b_c(T)$, rather than a smooth linear scaling law. For Hénon, the largest-horizon crossing is resolved at $b_c = 25.457$, corresponding to a minimum integer bit width of 26 under the declared estimator. An alternative estimator gives materially different crossing values, while the Logistic map also shows strongly non-monotonic behavior. Overall, η and $b_c(T)$ provide useful implementation-oriented screening measures, but they are not universal thresholds or hardware guarantees.

Keywords: finite precision; quantized nonlinear dynamics; finite-state systems; digital chaos; implementation resolution; Hénon map; Lorenz system; entropy analysis; recurrence analysis; finite-horizon dynamics



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1. Introduction

Nonlinear dynamical systems exhibit rich behaviors such as bifurcation, attractor formation, and sensitive dependence on initial conditions. Canonical models such as the Lorenz system and the Hénon map remain central because they show how simple deterministic rules can generate geometrically intricate long-term behavior and practical limits on predictability [1–4]. In the classical setting, these systems are typically analyzed in continuous or effectively infinite-resolution state spaces using tools such as Jacobian linearization, Lyapunov exponents, invariant measures, and bifurcation analysis [3–6]. These tools explain how instability and stretching generate complex dynamics, but they do not directly address the question that becomes unavoidable in digital realization: what happens when finite implementation resolution is no longer negligible relative to attractor structure?

That question is no longer peripheral. Nonlinear dynamical systems are now routinely simulated or implemented under explicit finite-precision constraints in digital hardware, embedded controllers, FPGA-based designs, and low-precision numerical pipelines [7–9]. In these settings, quantization is not merely a small perturbation of a continuous model. It converts the original system into a finite-state deterministic dynamical system whose trajectories must eventually repeat, often through short cycles or fixed points. Recent work on low-precision chaotic simulation has made this point particularly clear: deterministic finite-precision representations inevitably produce eventual periodic orbits even when the underlying continuous dynamics are chaotic [10,11]. The central problem is therefore not simply whether quantization changes the dynamics, but how to predict when that change becomes structurally dominant.

A substantial literature has documented finite-precision degradation in digital chaos and related implementations. Earlier implementation-oriented studies showed that finite-word-length effects can alter orbit structure, shorten periods, distort statistical behavior, and reduce practical usability in hardware realizations [10,12–15]. More recent work has reached the same broad conclusion from complementary perspectives, including finite-precision simulation fidelity [11], functional-graph analysis of degraded digital maps [10], and finite-precision rounding effects in nonlinear circuits and systems [16]. Taken together, these studies show that finite-precision degradation is real, measurable, and practically important.

However, most of the digital-chaos literature has taken one of two directions. One is descriptive: it documents degradation in specific systems, compares output statistics, or analyzes state-mapping structure for particular maps and implementations [10,12,16]. The other is remedial: it develops anti-degradation mechanisms intended to prolong complexity, extend period length, or resist collapse under low precision [17–23]. These contributions are valuable, but they do not fully answer the predictive question. Observing that degradation occurs is not the same as identifying when structural collapse begins. Likewise, delaying degradation is not the same as establishing a general criterion for when finite resolution first destroys trajectory distinguishability.

This distinction motivates the present work. Rather than treating quantization as an implementation nuisance or focusing on how to repair degraded digital chaos after collapse begins, we develop a resolution-limited perspective on quantized nonlinear dynamics. In this view, replacing the nominal update map $f(x)$ with the quantized update $Q(f(x))$ is a structural operation that limits the effective distinguishability of nearby trajectories. Under this interpretation, the transition from chaotic-like behavior to low-complexity periodic or fixed-point behavior is not merely a numerical artifact. It is the dynamical consequence of imposing finite-state resolution on an attractor whose distinguishable structure may continue to refine over a finite observation horizon.

This perspective is organized around an operational finite-horizon separation scale $\delta_{\text{sep}}(T_0, T; \varepsilon)$, the dimensionless ratio

$$\eta(b; T_0, T; \varepsilon) = \frac{\Delta(b)}{\delta_{\text{sep}}(T_0, T; \varepsilon)}, \quad (1)$$

and its reference crossing $b_c(T)$. Here, $\Delta(b)$ is the quantization spacing; fixed protocol arguments are suppressed when no ambiguity arises. Estimator labels are retained whenever alternative definitions produce different numerical values. The ratio is a normalized comparison coordinate, not a universal sharp threshold.

The observation horizon T defines the finite window over which attractor structure is operationally resolved and trajectory distinguishability is assessed. The continuous crossing $b_c(T)$ and its integer ceiling are therefore protocol-dependent outputs, not one system-wide threshold.

The framework also distinguishes rigorous statements from empirical ones. Eventual periodicity and the quantization-error bounds are deterministic consequences of the finite realized state space and quantizer. By contrast, the horizon relation is only an empirical ansatz, and the experiments do not validate it as a general linear law or identify its fitted slope with a Lyapunov exponent. Operational separation estimates can be geometry- and estimator-sensitive; that dependence is measured explicitly rather than hidden behind one diagnostic.

The contribution is deliberately separated from established facts. Eventual periodicity of a deterministic finite-state system, uniform quantization bounds, and the qualitative degradation of digital chaos are not claimed as new. The contribution is the combined operational workflow: horizon-indexed separation estimates, estimator-specific normalized resolution coordinates, entropy and first-recurrence diagnostics under one declared quantizer, and an empirical bit-width screening procedure whose estimator dependence is measured rather than suppressed. The resulting crossings are protocol outputs, not system invariants.

The remainder of the paper is organized as follows. Section 2 reviews the relevant literature and clarifies the gap addressed here. Subsequent sections formalize quantized nonlinear dynamics, introduce the operational separation scale and the finite-horizon collapse interpretation, define the normalized resolution ratio, and present experiments on representative systems. The overarching objective is not to propose yet another anti-degradation mechanism, but to develop a framework for understanding resolution-induced collapse itself.

2. Related Work

2.1. Classical Chaos, Attractors, and Instability Measures

Classical nonlinear dynamics provides the conceptual backdrop for the present study. Foundational models such as the Lorenz equations and the Hénon map established paradigmatic examples of deterministic chaos, geometric attractors, sensitive dependence on initial conditions, and discrete-time complexity generation [1,2,24]. Standard tools—including Jacobian linearization, Lyapunov exponents, bifurcation theory, symbolic descriptions, and invariant-measure viewpoints—have long been used to characterize stability and complexity in continuous or effectively infinite-resolution state spaces [3–6]. These foundations are essential for understanding chaos in idealized mathematical systems, but they generally assume effectively infinite-state resolution and therefore do not directly address how finite-resolution implementations reshape attractor structure and stability properties.

2.2. Finite-Precision Degradation and Digital Implementations

A second line of work examines what happens when chaotic systems are realized in digital environments. In this setting, finite precision alters the state space itself [7,10,11]. The system no longer evolves on a continuum but on a finite set of representable values, so eventual periodicity is unavoidable. This issue has been studied from several perspectives, including hardware implementations, state-mapping analysis, and numerical-simulation fidelity. Wang et al. developed a theoretical and FPGA-oriented framework for higher-dimensional digital chaotic systems and explicitly identified finite-precision degradation as a practical obstacle to digital realization [12]. Similar implementation-oriented concerns appear in earlier FPGA-based Hénon studies, where finite-word-length effects were shown to influence orbit structure, stability characteristics, and control-oriented behavior in practical realizations [13–15]. Li et al. analyzed digital chaotic maps through state-mapping networks, showing that the graph structure of the finite-precision state-transition map provides a natural way to study collapse, period structure, and degeneration in discrete-time

maps [25]. Fan and Ding further developed a functional-graph approach for analyzing degradation and period distribution, again emphasizing that limited computational precision can strongly reshape the dynamics [10]. From the numerical-analysis side, Antonelli et al. highlighted the broader effects of finite precision and rounding in nonlinear systems and circuits, including cases in which computed dynamics can differ qualitatively from the intended model [16]. More recently, Klöwer et al. showed that deterministic finite-precision simulations necessarily yield periodic orbits and examined how this phenomenon depends on low-precision number formats and model dimension [11]. Taken together, these studies establish that finite precision can substantially reshape chaotic dynamics in ways that are both measurable and practically important. Yet, most of this literature remains primarily descriptive: it documents degradation, analyzes output properties, or characterizes particular classes of digital maps. The present work builds on that foundation but asks a different question: not only how degradation appears, but when finite resolution becomes structurally dominant.

2.3. Anti-Degradation Design and Complexity-Preserving Remedies

A third line of work aims to counteract or delay dynamical degradation once it is recognized as a practical problem. This literature includes perturbation, coupling, hybridization, feedback, and higher-dimensional design strategies intended to prolong complexity, extend period lengths, or recover stronger pseudorandom properties. Liu and collaborators introduced shift-coupling style remedies for enhancing digital chaotic behavior under finite precision [17]. Zheng et al. used symbolic-dynamics ideas to construct effective anti-degradation schemes and argued that such remedies should be designed with structural dynamical considerations in mind rather than output tests alone [18]. Merah et al. proposed a self-perturbation mechanism that improves degradation behavior in practice without requiring an external entropy source [19]. Zhang and Liu developed delayed-exponent coupling and related constructions intended to enhance the chaotic behavior of digital maps under limited precision [20,21]. Li et al. studied delay-based enhancement mechanisms in the same general spirit [22]. Huang et al. extended this direction further through a higher-dimensional delayed cyclic-coupled model with time-varying parameter control, explicitly targeting finite-precision degradation in digital chaotic systems [23].

This literature is highly relevant because it shows that degradation is not a marginal numerical curiosity, but a serious enough problem to motivate an entire family of remedy-oriented designs. However, remedy-oriented work and collapse-prediction work are not the same. A method that delays collapse does not by itself provide a system-comparable criterion for when collapse begins in the unmodified system. One of the main goals of the present paper is therefore to separate these two problems conceptually: anti-degradation design concerns how to resist collapse, whereas the present framework concerns how to analyze collapse onset itself.

2.4. Quantization as Structural Constraint in Control and Hybrid Systems

A fourth line of work comes from quantized and hybrid control. In this setting, quantization is treated as a structural nonlinearity associated with finite data rates, saturating measurements, or switched/hybrid closed-loop evolution [8,9,26]. Brockett and Liberzon's work on quantized feedback stabilization is a canonical example, showing that quantization can fundamentally reshape stability analysis and controller design [26]. The conceptual relevance of this literature is not that it studies chaos directly, but that it reinforces a broader principle: finite resolution is often an intrinsic dynamical constraint rather than an incidental implementation detail.

The present work is aligned with that principle, but its objective is different. Quantized-control studies typically prioritize stabilization, boundedness, or performance under limited information. Here, the central concerns are attractor structure, trajectory distinguishability, and collapse from chaotic-like behavior to low-complexity recurrence. This shift in objective makes it necessary to define a separation scale and to analyze bit-width requirements in relation to finite-horizon distinguishability.

2.5. Gap

Taken together, the literature supports three robust observations. First, classical chaos theory provides the language of instability, attractors, and long-term structure [3–6]. Second, digital and finite-precision studies show that implementation constraints can induce degradation, short cycles, and eventual periodicity [7,10–12,16]. Third, anti-degradation research demonstrates that the problem is important enough to warrant active countermeasures [17–23]. What is still missing is a unified finite-horizon criterion that treats quantization as a structural resolution constraint and explains when dynamically distinct structures cease to be distinguishable as bit width decreases.

That is the gap addressed in this paper. The proposed framework is centered on the operational separation scale $\delta_{\text{sep}}(T_0, T; \varepsilon)$, the ratio $\eta = \Delta/\delta_{\text{sep}}$, and a finite-horizon interpretation of collapse. The extracted crossing depends on the horizon and declared estimator rather than defining a universal threshold.

3. Problem Setup and Theory

3.1. Quantized Nonlinear Dynamics

Consider a nonlinear discrete-time update law implemented under finite-precision quantization,

$$x_{t+1} = F_b(x_t) := Q_b(f(x_t)), \quad x_t \in \mathcal{A} \subset \mathbb{R}^d, \quad (2)$$

where $f : \mathcal{A} \rightarrow \mathbb{R}^d$ is the nominal high-precision map, Q_b is the declared b -bit quantizer with clipping, and $\mathcal{A}$ is the effective bounded state domain. The notation F_b makes the dependence of the realized dynamics on resolution explicit.

For the theoretical development, let $R > 0$ denote the effective clipped scalar range associated with a state component. The corresponding quantization step is

$$\Delta(b) = \frac{R}{2^b - 1}. \quad (3)$$

Here, $R = x_{\max} - x_{\min}$ is the span of the declared closed range. The endpoint-inclusive grid has exactly 2^b reconstruction levels indexed by $0, \dots, 2^b - 1$, so its index uses exactly b unsigned bits. Quantization and clipping therefore replace continuous-state evolution by deterministic dynamics on a finite representable set [8–11,26].

3.2. Eventual Periodicity and Finite-State Bounds

The following elementary observation fixes the asymptotic background.

Proposition 1 (Eventual periodicity of finite-state trajectories). *Suppose the realized update law F_b in (2) evolves on a finite-state set $\mathcal{X}_b \subset \mathcal{A}$. Then, for every initial condition $x_0 \in \mathcal{X}_b$, the trajectory $\{x_t\}_{t \geq 0}$ is eventually periodic. That is, there exist integers $0 \leq i < j \leq |\mathcal{X}_b|$ such that $x_i = x_j$, and consequently*

$$x_{t+P} = x_t, \quad t \geq i, \quad P = j - i. \quad (4)$$

Proof. The sequence $(x_0, x_1, \dots, x_{|\mathcal{X}_b|})$ contains $|\mathcal{X}_b| + 1$ state occurrences, but the finite-state set $\mathcal{X}_b$ contains only $|\mathcal{X}_b|$ distinct admissible states. By the pigeonhole principle, some

states must repeat. Since F_b is deterministic, identical states have identical futures, which implies eventual periodicity. $\square$

For a deterministic realized map on $\mathcal{X}_b$, the detected transient and recurrence lengths satisfy

$$L_{\text{tr}} + L_{\text{rec}} \leq |\mathcal{X}_b|. \quad (5)$$

For coordinatewise nearest-lattice rounding with no active clipping,

$$\|Q_b(u) - u\|_{\infty} \leq \frac{\Delta}{2}, \quad \|Q_b(u) - u\|_2 \leq \frac{\sqrt{d} \Delta}{2}. \quad (6)$$

Two mapped states merge in one realized update exactly when $Q_b(f(x)) = Q_b(f(y))$, for example, when the mapped states lie in the same quantization cell. Clipping can add saturation error; moreover, $\|u - v\| < \Delta$ alone does not guarantee a common cell, and pre-map closeness does not guarantee post-map merging. These exact results establish eventual recurrence and quantization bounds, but they do not locate finite-horizon structural loss.

3.3. Operational Separation and Crossing

Let T_0 denote the discarded transient and T the observation horizon. Given a partition $\mathcal{P}_{\varepsilon} = \{C_1, \dots, C_M\}$ with $\text{diam}(C_i) \leq \varepsilon$ and a high-precision reference trajectory $\{x_t^*\}_{t \geq 0}$, define the occupied cells by

$$\mathcal{O}_{\varepsilon}(T_0, T) := \{C_i \in \mathcal{P}_{\varepsilon} \mid \exists t, T_0 \leq t < T_0 + T, x_t^* \in C_i\}. \quad (7)$$

For a regular box partition, let $\text{dist}(C_i, C_j)$ be the Euclidean distance between cell centers. The operational separation scale is

$$\delta_{\text{sep}}(T_0, T; \varepsilon) := \min_{\substack{C_i, C_j \in \mathcal{O}_{\varepsilon}(T_0, T) \\ i \neq j}} \text{dist}(C_i, C_j). \quad (8)$$

This cell-center form is denoted as δ_{cell} in numerical comparisons. The trajectory-minimum and blocked alternatives are denoted as δ_{traj} and δ_{block} and are defined below.

3.4. Resolution Ratio and Bit-Width Extraction

With fixed protocol arguments suppressed, define

$$\eta(b; T) = \frac{\Delta(b)}{\delta_{\text{sep}}(T)}. \quad (9)$$

The value $\eta = 1$ is only an operational reference crossing. With the endpoint-inclusive quantizer,

$$b_c(T) = \log_2 \left(1 + \frac{R}{\delta_{\text{sep}}(T)} \right), \quad b_{\min}(T) = \lceil b_c(T) \rceil. \quad (10)$$

Here, $b_c(T)$ is a continuous resolution index and $b_{\min}(T)$ is the smallest integer bit width satisfying $\eta \leq 1$. Estimator subscripts are added only when alternative definitions yield different values. Neither quantity is a system invariant or a sufficient collapse condition.

3.5. Empirical Horizon Ansatz

To test horizon dependence, we use the falsifiable empirical ansatz

$$\delta_{\text{traj}}(T_0, T) \approx Ce^{-\kappa T}, \quad (11)$$

where $C > 0$ and κ are protocol-dependent fit quantities. If this ansatz were adequate, the exact endpoint-inclusive crossing would give

$$b_{c,\text{traj}}(T) = \log_2\left(1 + \frac{R}{C}e^{\kappa T}\right) \approx \log_2 \frac{R}{C} + \frac{\kappa}{\ln 2}T \quad (12)$$

The large-argument approximation applies only when $(R/C)e^{\kappa T} \gg 1$. Here, κ is not identified with a Lyapunov exponent. The observed step changes, long plateau, structured residuals, and estimator dependence show that Equation (12) is not validated as a general scaling law.

3.6. Scope

Eventual periodicity, the finite-state and unclipped quantization-error bounds, the 2^b -level spacing formula, and the algebraic conversion in Equation (10) are exact under the declared realized system and quantizer. By contrast, δ_{sep} , η , the entropy and recurrence diagnostics, the horizon ansatz, and the extracted crossings are protocol-dependent empirical quantities. They do not prove preservation of the continuous attractor, establish a universal collapse threshold, or provide a hardware guarantee; their role is to support a reproducible screening procedure with estimator dependence stated explicitly.

4. Method

Table 1 summarizes the principal symbols and their status.

Table 1. Principal symbols and their status. Estimator subscripts are retained where numerical values differ.

Symbol	Meaning	Status
b, Δ	bit width and endpoint-inclusive spacing $R/(2^b - 1)$	implementation
T_0, T	discarded transient and retained observation length	protocol
ε	regular-cell scale	protocol
δ_{cell}	minimum occupied-cell-center separation	estimate
δ_{traj}	global trajectory-point minimum with temporal exclusion	estimate
δ_{block}	mean of three smallest block minima	estimate
η	$\Delta/\delta_{\text{sep}}$ for the declared estimator	operational ratio
H_{norm}	normalized empirical state-occupation entropy	estimate
$L_{\text{rec}}, L_{\text{tr}}$	first-recurrence and pre-recurrence transient lengths	exact for realized orbit
$b_c(T), b_{\text{min}}(T)$	continuous $\eta = 1$ crossing and its integer ceiling	empirical extraction

4.1. Systems, Integration, and Quantization

The Hénon map is $x_{k+1} = 1 - 1.4x_k^2 + y_k$, $y_{k+1} = 0.3x_k$, with initial state $(0.1, 0)$ and clipping range $[-2, 2]$ in each coordinate. The Lorenz equations use $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, initial state $(1, 1, 1)$, and range $[-30, 30]$. The Lorenz flow is integrated by the classical fourth-order Runge–Kutta method with fixed step $h = 0.01$. All four internal stages are evaluated in double precision; quantization is applied once after the complete RK4 update.

One additional benchmark is included: the Logistic map $x_{k+1} = 4x_k(1 - x_k)$, $x_0 = 0.123456789$, on $[0, 1]$. It tests transferability to a geometrically different one-dimensional map; it is not used to claim universality.

For all three systems, the implemented quantizer first clips each coordinate to the closed range $[x_{\min}, x_{\max}]$, where $R = x_{\max} - x_{\min}$, and then rounds to the nearest of exactly 2^b endpoint-inclusive levels

$$q_i = x_{\min} + i \frac{R}{2^b - 1}, \quad i = 0, \dots, 2^b - 1. \quad (13)$$

The spacing is therefore $\Delta = R/(2^b - 1)$. Exact half-way cases are assigned to the higher integer index, and values outside the range are clipped to an endpoint. This is an endpoint-inclusive uniform nearest-neighbor quantizer, neither a canonical mid-rise nor a canonical mid-tread quantizer. Its index requires exactly b unsigned bits. Quantization occurs after every Hénon or Logistic update and after each complete RK4 step.

4.2. Diagnostics and Estimators

After discarding T_0 updates, T states are retained. If n_i retained states equal lattice state i , then

$$p_i = \frac{n_i}{T}, \quad H = - \sum_i p_i \log p_i, \quad H_{\text{norm}} = \frac{H}{\log |\mathcal{X}_b|}, \quad |\mathcal{X}_b| = (2^b)^d. \quad (14)$$

This is a finite-horizon occupation statistic, not a topological entropy or proof of chaos. A hash table records the first repeated complete quantized state. If a state first appears at s and repeats at t , then $L_{\text{tr}} = s - 1$ and $L_{\text{rec}} = t - s$. Searches stop at 50,000 updates and unresolved runs are reported.

The occupied-cell-center estimator δ_{cell} is retained for fixed-horizon baseline and sensitivity comparisons, with $\varepsilon = \|\text{span}(X)\|/64$; the corresponding ratio is $\eta_{\text{cell}} = \Delta/\delta_{\text{cell}}$. Horizon scans use the global trajectory-point minimum

$$\delta_{\text{traj}}(T) = \min_{\substack{1 \leq i < j \leq T \\ j-i \geq 20}} \|x_i^* - x_j^*\|_2. \quad (15)$$

One alternative is the mean of the three smallest minimum separations from overlapping 500-state blocks with stride 250, denoted as δ_{block} . For each declared estimator, $\eta = \Delta/\delta$ and Equation (10) are evaluated directly; no linear interpolation of η against b is used. Numerical statements distinguish $b_{c,\text{traj}}$ from $b_{c,\text{block}}$ and report the corresponding integer ceilings where relevant.

4.3. Experiment Design

The single-position baseline uses $T_0 = 2000$, $T = 12,000$, and $b = 3, \dots, 16$. The deterministic ensemble uses $b = 6, \dots, 16$. For every system/bit-width group, a 20,000-state unquantized reference attractor is quantized, unique occupied lattice states are retained in trajectory order, and 100 evenly spaced distinct occupied positions are selected as initial conditions. Main-text uncertainty summaries are descriptive: median, IQR, mean, standard deviation, and range across the selected ensemble. A deterministic 2000-resample bootstrap interval for the median, conditional on these selected 100 positions and bootstrap RNG seed 20260715, is retained in the underlying tabulated data only; it is not interpreted as a population confidence interval.

The Lorenz horizon scan uses 21 horizons $T = 6000, 7500, \dots, 36,000$ and $b = 3, \dots, 24$. The Hénon scan uses $T \in \{6000, 18,000, 36,000\}$ and extends to $b = 32$. A one-factor sensitivity study uses $b \in \{8, 10, 12\}$ and varies $T \in \{6000, 12,000, 18,000\}$, $T_0 \in \{1000, 2000, 4000\}$, ε ratio in $\{1/128, 1/64, 1/32\}$, and range scale in $\{0.8, 1.0, 1.2\}$.

4.4. Computational Complexity

For state dimension d , trajectory generation costs $O(Td)$ time and $O(Td)$ stored memory. Sorting rows for occupation entropy costs approximately $O(T \log T)$ comparisons and $O(Td)$ memory. The implemented global trajectory-minimum search is brute force, $O(T^2d)$ time with $O(T)$ working-distance memory in addition to the trajectory. For B blocks of length m , the alternative estimator costs $O(Bm^2d)$. Hash recurrence detection has expected $O(Ld)$ time and $O(L)$ stored keys through the detected repeat or cap L ; the ensemble multiplies the per-position cost by 100. No tree-based or approximate nearest-neighbor method is implemented, so the quadratic separation search is the main barrier to high-dimensional or very long-horizon use.

5. Results and Discussion

Under the declared 2^b -level quantizer, Lorenz at $b = 12$ has $H_{\text{norm}} = 0.1948$ with forward Euler and 0.2213 with RK4. The results below use RK4 and are organized around the single-position baseline, deterministic 100-position variability, dense horizon scans, estimator and parameter sensitivity, and one additional benchmark.

5.1. Single-Position Baseline

For Hénon, the realized orbit is fixed-point-like at $b = 3$ and 4, has $H_{\text{norm}} = 0.3580$ and $L_{\text{rec}} = 53$ at $b = 8$, and remains non-monotone through $H_{\text{norm}} = 0.2678$ and $L_{\text{rec}} = 380$ at $b = 16$. For RK4 Lorenz, the realized orbit is a fixed point from $b = 3$ through $b = 8$; nonzero occupation entropy first appears at $b = 9$, where $H_{\text{norm}} = 0.2222$ and $L_{\text{rec}} = 64$. The periods are 812 and 1749 at $b = 15$ and 16. These patterns distinguish broad finite-state reshaping from the sharper low-bit Lorenz collapse, but neither defines a universal estimator-specific ratio threshold. Figure 1 summarizes the baseline metrics, while Figures 2 and 3 show the corresponding Hénon and Lorenz phase projections.

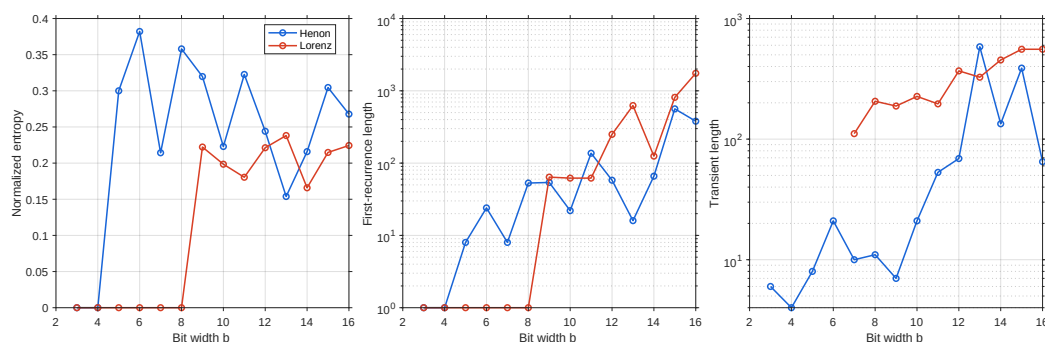


Figure 1. Single-position baseline at $T_0 = 2000$, $T = 12,000$. The three panels report normalized occupation entropy, first-recurrence length, and transient length for $b = 3, \dots, 16$. Lorenz uses classical RK4 with $h = 0.01$; recurrence and transient axes are logarithmic.

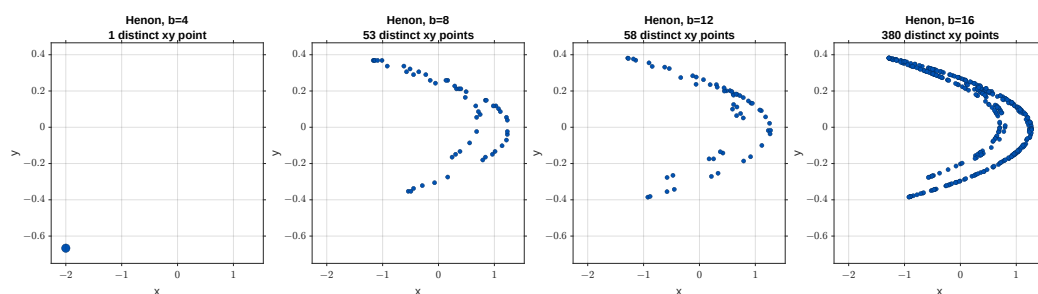


Figure 2. Hénon x - y phase projections at $b = 4, 8, 12, 16$. Each panel shows the distinct occupied points in the retained trajectory; the single point at $b = 4$ is the realized fixed-state collapse.

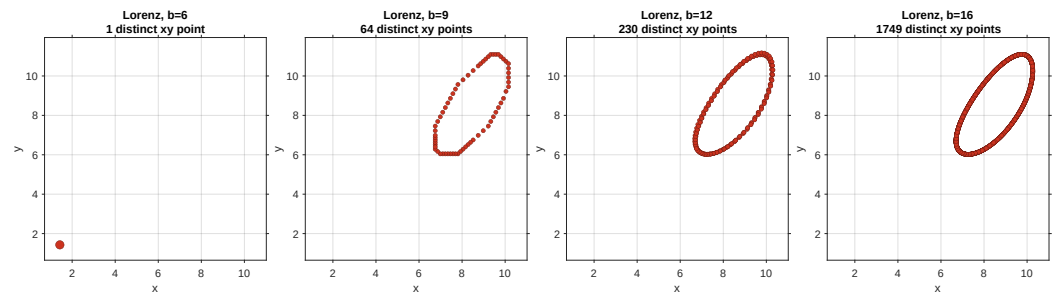


Figure 3. RK4 Lorenz x - y phase projections at $b = 6, 9, 12, 16$. Each panel shows the distinct occupied points in the retained projection; the single point at $b = 6$ is the realized fixed-state collapse.

5.2. Variability Across 100 Distinct Initial Quantized Positions

All 2200 deterministic-ensemble runs found a recurrence before the 50,000-state cap. Figure 4 shows medians and interquartile bands across the selected occupied positions. At $b = 8$, the Lorenz ensemble remains fixed-point-like in entropy and recurrence ($H_{\text{norm}} = 0$, $L_{\text{rec}} = 1$), while the transient median is 181 with IQR 144.75–197. Hénon at the same precision has $H_{\text{norm}} = 0.3580$, $L_{\text{rec}} = 53$, and transient median 8.5 with IQR 2–16. At $b = 16$, Hénon transient length has median 537, IQR 288–795.5, mean 538.62, and standard deviation 302.36; Lorenz has median 518.5, IQR 384–892.5, mean 653.42, and standard deviation 327.10. Conditional bootstrap intervals remain available in the underlying tabulated data but are not used as population confidence intervals. Selected numerical summaries are listed in Table 2.

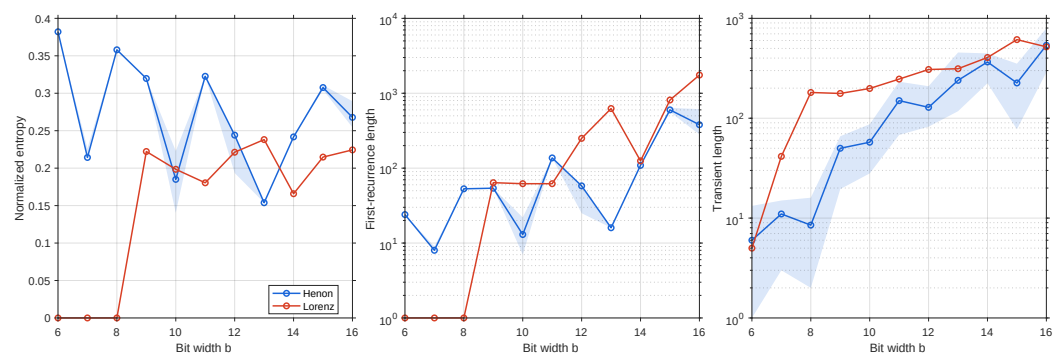


Figure 4. Variability across 100 distinct occupied quantized attractor positions per system and bit width ($b = 6, \dots, 16$). Lines are medians and shaded regions are IQRs for normalized entropy, first-recurrence length, and transient length. All 2200 recurrence searches resolved before 50,000 states.

Table 2. Selected deterministic 100-position ensemble medians [IQR]. Means, standard deviations, ranges, and conditional bootstrap intervals are provided in the underlying tabulated data; the intervals are not population confidence intervals.

System	b	H_{norm}	L_{rec}	L_{tr}
Hénon	8	0.3580 [0.3580, 0.3580]	53 [53, 53]	8.5 [2, 16]
Hénon	12	0.2441 [0.1935, 0.2441]	58 [25, 58]	128.5 [82.25, 209.25]
Hénon	16	0.2678 [0.2544, 0.2894]	380 [282, 613]	537 [288, 795.5]
Lorenz	8	0 [0, 0]	1 [1, 1]	181 [144.75, 197]
Lorenz	12	0.2213 [0.2213, 0.2213]	250 [250, 250]	307.5 [236.5, 370.25]
Lorenz	16	0.2244 [0.2244, 0.2244]	1749 [1749, 1749]	518.5 [384, 892.5]

The degenerate entropy and period intervals at several bit widths are themselves informative: different selected initial lattice states converge to the same finite recurrent component after variable transients. The deterministic ensemble therefore characterizes variability across the selected occupied positions; it is not an independent or statistically representative population sample and does not prove universality.

5.3. Dense Horizon Scan and Estimator Dependence

All 21 RK4 Lorenz horizons yield an in-scan trajectory-estimator crossing, but the exact sequence is stepwise: $b_{c, \text{traj}} = 10.083$ at $T = 6000$, 10.629 at $T = 7500$ and 9000 , 10.891 at $T = 10,500$ – $13,500$, 10.978 at $T = 15,000$, and 14.736 for every tested horizon from $T = 16,500$ to $36,000$. The corresponding integer minimum is $b_{\min, \text{traj}} = 11$ before the jump and 15 on the plateau. A descriptive ordinary-least-squares (OLS) fit gives slope 1.737×10^{-4} bits/step, with a nominal 95% OLS slope interval of 1.174×10^{-4} to 2.300×10^{-4} under standard regression assumptions, intercept 9.748 , $R^2 = 0.687$, adjusted $R^2 = 0.670$, RMSE 1.066 bits, maximum absolute residual 2.123 bits, and Durbin–Watson statistic 0.563 . The positive slope summarizes the endpoints, but the long plateau and structured residuals limit OLS inference and do not support a smooth linear scaling law.

For the Hénon scan through $b = 32$, all three exact trajectory-estimator crossings are resolved: $b_{c, \text{traj}} = 22.580$, 23.093 , and 25.457 for $T = 6000$, $18,000$, and $36,000$, with $b_{\min, \text{traj}} = 23$, 24 , and 26 . The largest-horizon point is bracketed by $b = 25$ and 26 . Figure 5 shows the Lorenz horizon sequence, residuals, and the extended Hénon scan.

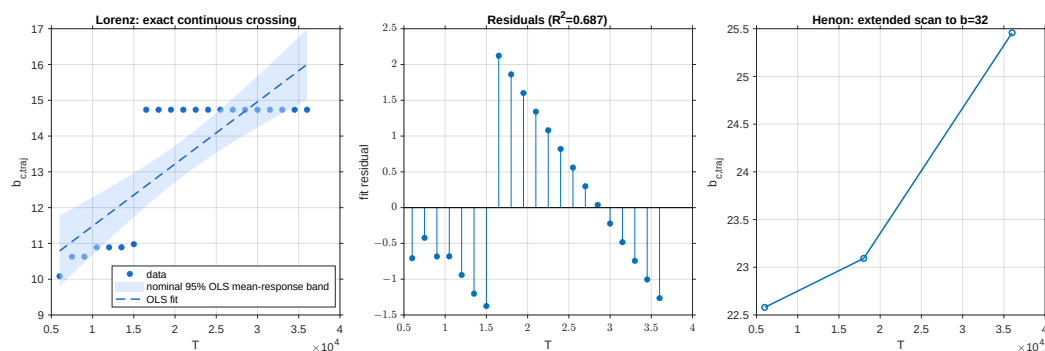


Figure 5. Exact continuous trajectory-estimator crossings. (Left): 21 Lorenz values, descriptive OLS fit, and nominal 95% OLS mean-response band under standard regression assumptions. (Center): structured fit residuals ($R^2 = 0.687$). (Right): Hénon values for the scan through $b = 32$, including $b_{c, \text{traj}} = 25.457$ and $b_{\min, \text{traj}} = 26$ at $T = 36,000$.

Estimator comparison further limits interpretation. At $T = 36,000$, Hénon has $b_{c, \text{traj}} = 25.457$ ($b_{\min, \text{traj}} = 26$) but $b_{c, \text{block}} = 19.150$ ($b_{\min, \text{block}} = 20$); Lorenz has 14.736 (15) and 9.397 (10), respectively. The trajectory estimator contains sharp record-minimum jumps, whereas the blocked estimate changes more gradually. Thus, collapse localization and trend magnitude are not robust to estimator choice. Figure 6 compares the trajectory and blocked estimators.

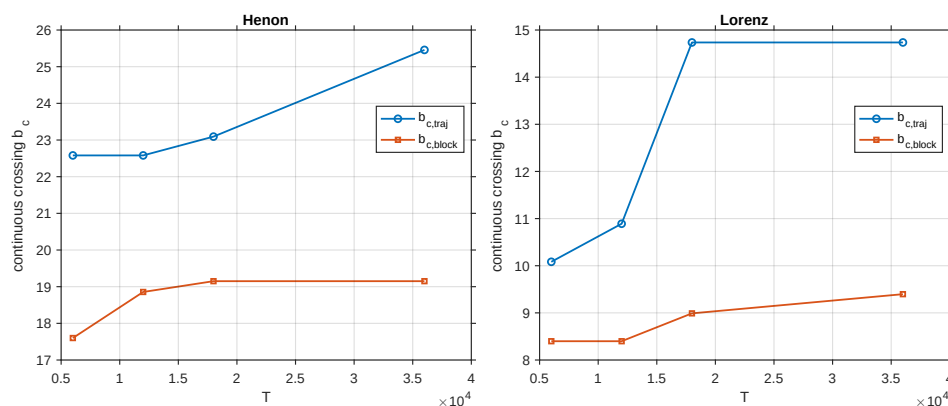


Figure 6. Estimator dependence at four horizons. The curves show exact continuous crossings $b_{c, \text{traj}}$ and $b_{c, \text{block}}$; the global trajectory minimum and the mean of the three smallest overlapping-block minima produce materially different levels and horizon trends.

5.4. Parameter Sensitivity and Additional Benchmark

The one-factor study shows that the tested H_{norm} values are nearly unchanged by T , T_0 , and ε once the finite recurrent component is fully sampled. Changing ε nevertheless changes η_{cell} inversely, demonstrating that this ratio inherits partition choice even when the realized orbit is unchanged. Quantization range is more consequential because it changes both spacing and clipping. For example, Lorenz at $b = 10$ is fixed-point-like when the range is scaled to 0.8, has $H_{\text{norm}} = 0.1985$ at the baseline range, and has 0.2063 at scale 1.2. Hénon at $b = 10$ changes from 0.3614 to 0.2230 and 0.2196 across the same range factors. These are local sensitivity results, not global robustness guarantees. Figure 7 summarizes the one-factor sensitivity results.

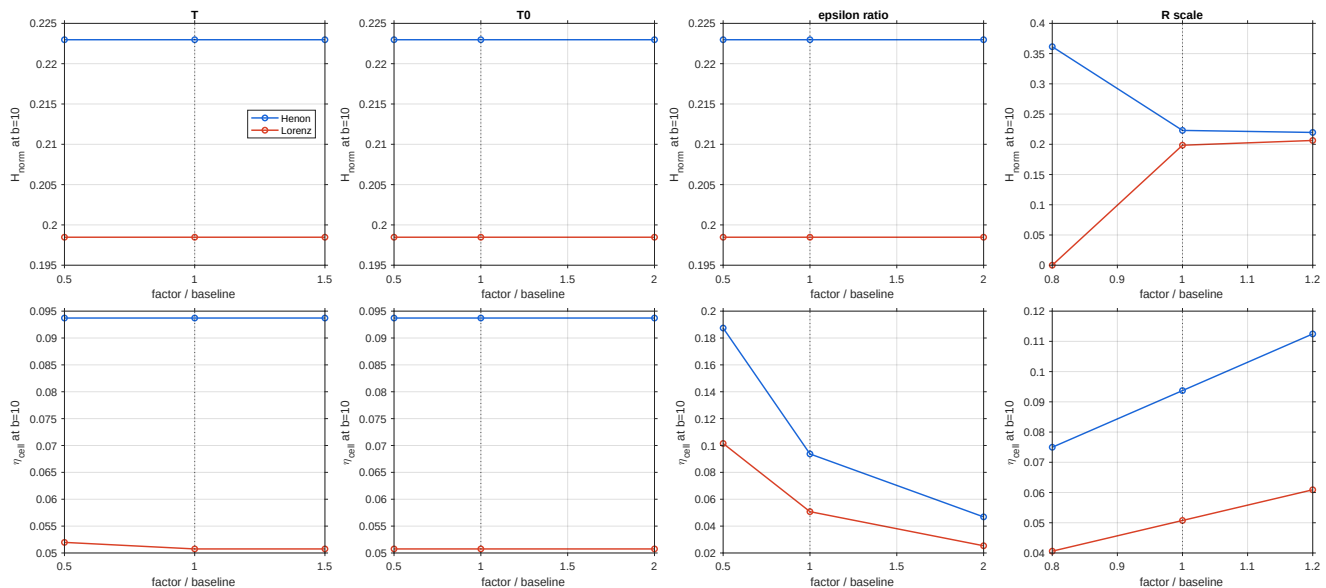


Figure 7. One-factor sensitivity at $b = 10$. (**Top row**): normalized entropy. (**Bottom row**): η_{cell} . Columns vary T , T_0 , ε , and range by the declared lower/baseline/upper factors. Range changes the realized orbit; ε changes η_{cell} without changing the quantized trajectory.

The Logistic map supplies a deliberately bounded third-system test. It is fixed-point-like for $b = 3$ –5, has $H_{\text{norm}} = 0.1667$ and $L_{\text{rec}} = 2$ at $b = 6$, rises to $H_{\text{norm}} = 0.4942$ and $L_{\text{rec}} = 11$ at $b = 7$, and returns to a fixed point at $b = 8$. Further collapses occur at $b = 12, 14$, and 15, whereas $b = 16$ has $H_{\text{norm}} = 0.4548$ and $L_{\text{rec}} = 155$. This strongly non-monotone sequence shows that the workflow transfers to another map while the extracted finite-state behavior does not follow a universal transition pattern. Figure 8 reports the Logistic-map benchmark.

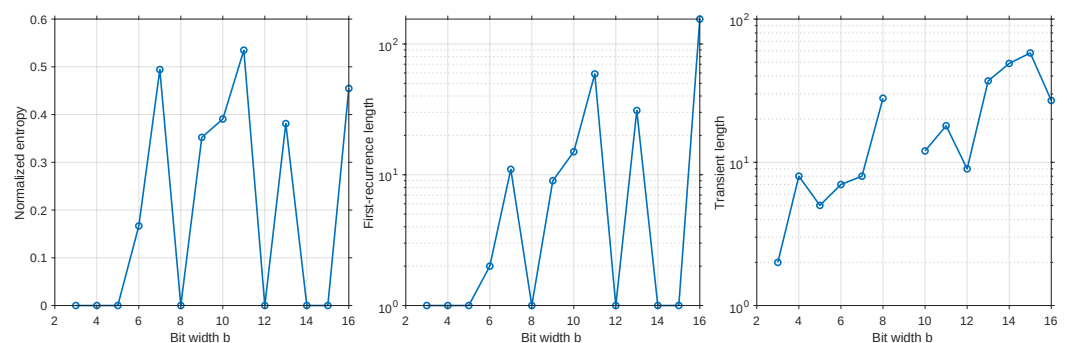


Figure 8. Additional Logistic-map benchmark ($r = 4$, $T_0 = 2000$, $T = 12,000$). Entropy, recurrence, and transient length vary non-monotonically over $b = 3, \dots, 16$, including repeated fixed-point collapse at both low and selected higher bit widths.

Across all three systems, the ratio $\eta = \Delta/\delta$ remains useful as a bookkeeping coordinate relating to range, bit width, and the declared separation estimate. The estimator must be named in numerical comparisons because the values differ materially. No such ratio removes cell-boundary effects, recurrence organization, clipping, or estimator dependence.

5.5. Engineering Use and Limitations

A practical pre-implementation procedure is to:

1. Define the application horizon and acceptable transient;
2. Set the representable range and clipping rule from expected states;
3. Choose and document a separation estimator;
4. Sweep candidate bit widths and compute entropy plus recurrence diagnostics;
5. Locate an empirical transition band rather than one universal threshold;
6. Add an application-specific precision margin above that band;
7. Validate the selected precision over multiple initial quantized states.

For a protocol-specific numerical example, the Hénon trajectory-estimator protocol at $T = 36,000$ gives $b_{c,\text{traj}} = 25.457$; hence, $b_{\min,\text{traj}} = 26$. An engineering screening exercise would begin at 26 bits, add a declared application-specific margin, and rerun the deterministic 100-position entropy and recurrence checks at the selected range. This is a numerical bit-width-selection example, not an FPGA implementation or guarantee.

The evidence remains limited in four ways. First, δ_{traj} and δ_{block} give materially different crossings. Second, the Lorenz horizon sequence contains one record-minimum jump and a long plateau, so its fitted slope is not a stable invariant. Third, the 100 initial positions are a deterministic selection of occupied reference-attractor lattice states rather than random or statistically representative population draws. Fourth, no HDL, synthesis, device, timing, throughput, power, or high-dimensional benchmark is reported. These limits bound the allowed engineering interpretation.

6. Conclusions

The exact claims in this study are limited to eventual finite-state recurrence, the declared quantization bounds and spacing formula, and the algebraic conversion from a separation estimate to b_c and its integer ceiling. All separation estimates, ratios, and crossings are protocol-dependent empirical outputs.

Across 100 selected occupied initial positions, entropy alone did not capture transient and recurrence variability. Hénon and Logistic behavior was strongly non-monotonic, whereas RK4 Lorenz remained fixed-point-like over a broad low-bit range. In the scan through $b = 32$, the Hénon trajectory crossing at $T = 36,000$ was $b_{c,\text{traj}} = 25.457$, giving $b_{\min,\text{traj}} = 26$. The 21-point Lorenz sequence was stepwise with a long plateau; its descriptive fit ($R^2 = 0.687$), structured residuals, and disagreement with the blocked estimator do not support a general linear scaling law.

The practical result is a reproducible screening workflow combining declared range, horizon, estimator, exact continuous-to-integer crossing conversion, and multi-position entropy and recurrence checks. The outputs are screening coordinates, not universal thresholds or hardware guarantees. Future work should accelerate the separation search, test estimator families prospectively, and validate selected bit widths in HDL or embedded hardware; no synthesis, timing, throughput, power, or device result is claimed here.

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