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An Efficient Analytical Approach for the Solution of Certain Fractional-Order Dynamical Systems

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Abstract: Mostly, it is very difficult to obtained the exact solution of fractional-order partial differential equations. However, semi-analytical or numerical methods are considered to be an alternative to handle the solutions of such complicated problems. To extend this idea, we used semi-analytical procedures which are mixtures of Laplace transform, Shehu transform and Homotopy perturbation techniques to solve certain systems with Caputo derivative differential equations. The effectiveness of the present technique is justified by taking some examples. The graphical representation of the obtained results have confirmed the significant association between the actual and derived solutions. It is also shown that the suggested method provides a higher rate of convergence with a very small number of calculations. The problems with derivatives of fractional-order are also solved by using the present method. The convergence behavior of the fractional-order solutions to an integer-order solution is observed. The convergence phenomena described a very broad concept of the physical problems. Due to simple and useful implementation, the current methods can be used to solve problems containing the derivative of a fractional-order.

Keywords: Homotory perturbation method; Shehu transform; Burger equation; Caputo operator

1. Introduction

Coupled schemes of fractional-order partial differential equations (PDEs) are commonly applied in phenomena that occur in biomechanics and engineering. Various implementations of coupled PDE schemes arise in the modeling of electrical movement of the heart in biomechanics (see, for instance, [1–3]). They similarly occur when modeling other problems in biochemical and physical engineering, such as a device that includes a continuous stirred boiler container and a series plug or container [4,5]. The coupled FPDEs can be used for the combination of different-deformable objects with a fractional-order continuum of standard lightly surfaces [6,7]. Coupled PDE schemes also occur in modeling several significant gravitational and electromagnetic problems (see, for instance, [8–13]).

In 1965, Harry Bateman introduced a differential equation [14], which was later renamed as the Burger equation [15]. In science and engineering, the Burger equation has several implementations, particularly in problems that have the structure of non-linear problems. The Burger equation has interesting and important applications and defines various types of physical processes such as dynamic modeling, turbulence, acoustic waves heat transfer, and several others [16–18]. In many other cases, this type of non-linear PDE should be addressed utilizing special techniques because it does not support analytical approaches. In modern years, several scholars and mathematicians have developed an analytical technique for the solution of fractional-order problems such as the high order spectral volume formulation of Kannan et al. [19–23], homotopy perturbation (HPM), differential transformation, homotopy analysis, variational iteration and Adomian decomposition methods [24–28].

Recently, researchers have shown a greater interest in the study of fractional-calculus and Fractional differential equations (FDEs). Several important implementations have been explored in a number of different fields [29–33]. Researchers have also shown that several engineering and practical phenomena can be described well by FDEs systems as compared to classical differential equation systems and that equivalent FDEs and fractional integral equations give better precise and practical insights into the systems under discussion [34–38]. Many of these engineering challenging problems are addressed by using deterministic mathematical models that are represented by either partial differential equations of integer order or fractional-order. These mathematical models can further be classified into a scheme of ordinary differential equations, integro differential equations, and partial differential equations [39,40]. The existence of fractional differential equations is also discussed in [41]. In 1998, He [42,43] introduced HPM. In this technique, the solution is assumed to be in series form with a large number of terms that converge quickly towards the actual derived solution. The technique has the capability to solve nonlinear PDEs adequately. The HPTM results were compared with the actual solution to the problems and confirmed a higher degree of accuracy. This technique has also been used to solve address non-linear wave equations [44], bifurcation of nonlinear problems [45], and boundary value problems [46].

In the present research work, an efficient analytical technique is utilized to solve fractional-order Burger equations. The current is found to be very effective for the systems of FDEs. The present methodology is very attractive and has less computational cost. The present technique has shown a sufficient degree of accuracy.

2. Preliminaries

In this section, we present fractional calculus definitions along with properties of Laplace and Shehu transform theory.

Definition 1. *The Rieman–Liouville fractional integral is defined by* [47–49]

$$I_0^{\gamma} h(\tau) = \frac{1}{\Gamma(\gamma)} \int_0^{\tau} (\tau - s)^{\gamma-1} h(s) ds, \tag{1}$$

showing that the integral on the right side converges.

Definition 2. *Caputo's fractional-order derivative of $h(\tau)$ is given as* [47–49]

$$D_{\tau}^{\gamma} h(\tau) = \begin{cases} I^{n-\gamma} f^n, \ n-1 < \gamma < n, & n \in \mathbb{N} \\ \frac{d^n}{d\tau_n} h(\tau), \quad \gamma = n, & n \in \mathbb{N}. \end{cases} \tag{2}$$

Definition 3. *Shehu transformation is new and similar to other integral transformation which is defined for functions of exponential order. We take a function in the set A define by* [50–53]

$$A = \{\nu(\tau) : \exists, \rho_1, \rho_2 > 0, |\nu(\tau)| < Me^{\frac{|\tau|}{\rho_i}}, \ if \ \ \tau \in [0, \infty), \tag{3}$$

The Shehu transformation which is defined by $S(.)$ for a function $\nu(\tau)$ is expressed as

$$S\{\nu(\tau)\} = V(s, \mu) = \int_0^{\infty} \nu(\tau) e^{\frac{-s\tau}{\mu}} \nu(\tau) d\tau, \ \ \tau > 0, \ \ s > 0. \tag{4}$$

The Shehu transformation of a function $\nu(\tau)$ is $V(s,\mu)$: then $\nu(\tau)$ is called the inverse of $V(s,\mu)$ which is defined as

$$S^{-1}\{V(s,\mu)\} = \nu(\tau), for\ \tau \geq 0,\ \ S^{-1} \text{ is inverse Shehu transformation.} \tag{5}$$

Definition 4. *Shehu transform for nth derivatives. The Shehu transformation for nth derivatives is defined as* [50–53]

$$S\Big\{\nu^{(n)}(\tau)\Big\} = \frac{s^n}{u^n}V(s,u) - \sum_{k=0}^{n-1}\Big(\frac{s}{u}\Big)^{n-k-1}\nu^{(k)}(0). \tag{6}$$

Definition 5 (Shehu transform for fractional order derivatives [50–53])**.** *The Shehu transformation for the fractional order derivatives is expressed as*

$$S\Big\{\nu^{(\gamma)}(\tau)\Big\} = \frac{s^\gamma}{u^\gamma}V(s,u) - \sum_{k=0}^{n-1}\Big(\frac{s}{u}\Big)^{\gamma-k-1}\nu^{(k)}(0), \quad 0 < \beta \leq n, \tag{7}$$

3. Homotopy Perturbation Shehu Transform Method

In this section, we explain the main idea of Homotopy Perturbation Shehu Transform Method [50–53].

$$\begin{aligned} &D_\tau^\gamma \xi(\nu,\tau) + M\xi(\nu,\tau) + N\xi(\nu,\tau) = h(\nu,\tau), \quad \tau > 0, \quad 0 < \gamma \leq 1, \\ &\xi(\nu,0) = g(\nu), \quad \nu \in \Re. \end{aligned} \tag{8}$$

where $D_\tau^\gamma = \frac{\partial^\gamma}{\partial \tau^\gamma}$ is Caputo's derivative, M, N are the linear and nonlinear operators in ν and $h(\nu,\tau)$ represents source terms.

Using Shehu transform, we can write Equation (8) as [50–53]

$$\begin{aligned} &S[D_\tau^\gamma \xi(\nu,\tau) + M\xi(\nu,\tau) + N\xi(\nu,\tau)] = S[h(\nu,\tau)], \quad \tau > 0, 0 < \gamma \leq 1, \\ &R(\nu,s,u) = \frac{g(\nu)}{s} + \frac{u^\gamma}{s^\gamma}S[h(\nu,\tau)] - \frac{u^\gamma}{s^\gamma}S[M\xi(\nu,\tau) + N\xi(\nu,\tau)]. \end{aligned} \tag{9}$$

Now, by taking inverse Shehu transform, we get [50–53]

$$\xi(\nu,\tau) = F(\nu,\tau) - S^{-1}\Big(\frac{u^\gamma}{s^\gamma}S[M\xi(\nu,\tau) + N\xi(\nu,\tau)]\Big), \tag{10}$$

where

$$F(\nu,\tau) = S^{-1}[\frac{g(\nu)}{s} + \frac{u^\gamma}{s^\gamma}S[h(\nu,\tau)]] = g(\nu) + S^{-1}\Big[\frac{u^\gamma}{s^\gamma}S[h(\nu,\tau)]\Big]. \tag{11}$$

Now, perturbation technique having parameter ϵ in the form of power series is given as

$$\xi(\nu,\tau) = \sum_{k=0}^{\infty}\epsilon^k \xi_k(\nu,\tau), \tag{12}$$

where ϵ is perturbation parameter and $\epsilon \in [0,1]$.

The nonlinear term can be expressed as

$$N\xi(\nu,\tau) = \sum_{k=0}^{\infty}\epsilon^k H_k(\xi_k), \tag{13}$$

where H_n are He's polynomials in term of $\xi_0, \xi_1, \xi_2, ..., \xi_n$, and can be determined as

$$H_n(\xi_0, \xi_1, \cdots, \xi_n) = \frac{1}{\gamma(n+1)} D_\epsilon^k \left[N\left(\sum_{k=0}^{\infty} \epsilon^k \xi_k \right) \right]_{\epsilon=0}, \tag{14}$$

where $D_\epsilon^k = \frac{\partial^k}{\partial \epsilon^k}$.

Putting Equations (13) and (14) in Equation (10) and introducing the Homotopy, we get the couple of HPSTM as

$$\sum_{k=0}^{\infty} \epsilon^k \xi_k(\nu, \tau) = F(\nu, \tau) - \epsilon \times \left(S^{-1}\left[\frac{u^\gamma}{s^\gamma} S\{M \sum_{k=0}^{\infty} \epsilon^k \xi_k(\nu, \tau) + \sum_{k=0}^{\infty} \epsilon^k H_k(\xi_k)\} \right] \right). \tag{15}$$

On comparing coefficient of ϵ on both sides, we obtain

$$\begin{aligned}
&\epsilon^0 : \xi_0(\nu, \tau) = F(\nu, \tau), \\
&\epsilon^1 : \xi_1(\nu, \tau) = S^{-1}\left[\frac{u^\gamma}{s^\gamma} S(M\xi_0(\nu, \tau) + H_0(\xi)) \right], \\
&\epsilon^2 : \xi_2(\nu, \tau) = S^{-1}\left[\frac{u^\gamma}{s^\gamma} S(M\xi_1(\nu, \tau) + H_1(\xi)) \right], \\
&\vdots \\
&\epsilon^k : \xi_k(\nu, \tau) = S^{-1}\left[\frac{u^\gamma}{s^\gamma} S(M\xi_{k-1}(\nu, \tau) + H_{k-1}(\xi)) \right], \quad k > 0, k \in N.
\end{aligned} \tag{16}$$

The component $\xi_k(\nu, \tau)$ can be calculated easily, which leads us to the convergent series rapidly. By taking $\epsilon \to 1$, we obtain

$$\xi(\nu, \tau) = \lim_{M \to \infty} \sum_{k=1}^{M} \xi_k(\nu, \tau). \tag{17}$$

Similarly, the procedure of the Laplace transform as special case for $u = 1$ of Shehu transform is used to derived similar results as Shehu transformation.

4. Applications

In this section, the solutions of numerical examples are presented to confirm the validity of the suggested methods.

Example 1. *Consider the following system of fractional-order Burger's equations* [54–56]

$$\begin{aligned}
&\xi_\tau^\gamma - \xi_{\nu\nu} - 2\xi\xi_\nu + (\xi\zeta)_\nu = 0 \\
&\zeta_\tau^\gamma - \zeta_{\nu\nu} - 2\zeta\zeta_\nu + (\xi\zeta)_\nu = 0
\end{aligned} \tag{18}$$

with initial conditions

$$\xi(\nu, 0) = \sin(\nu), \quad \zeta(\nu, 0) = \sin(\nu), \tag{19}$$

Taking the Shehu Transform of Equation (18), we have

$$\begin{aligned}
&\frac{s^\gamma}{u^\gamma} S[\xi(\nu, \tau)] = \xi^{(0)}(\nu, 0) \frac{s^{\gamma-1}}{u^\gamma} + S(\xi_{\nu\nu} + 2\xi\xi_\nu - (\xi\zeta)_\nu), \\
&\frac{s^\gamma}{u^\gamma} S[\zeta(\nu, \tau)] = \zeta^{(0)}(\nu, 0) \frac{s^{\gamma-1}}{u^\gamma} + S(\zeta_{\nu\nu} + 2\zeta\zeta_\nu - (\xi\zeta)_\nu),
\end{aligned} \tag{20}$$

$$\begin{aligned}
&S[\xi(\nu, \tau)] = \frac{1}{s} \sin(\nu) + \frac{u^\gamma}{s^\gamma} [S(\xi_{\nu\nu} + 2\xi\xi_\nu - (\xi\zeta)_\nu)], \\
&S[\zeta(\nu, \tau)] = \frac{1}{s} \sin(\nu) + \frac{u^\gamma}{s^\gamma} [S(\zeta_{\nu\nu} + 2\zeta\zeta_\nu - (\xi\zeta)_\nu)].
\end{aligned} \tag{21}$$

Taking Inverse Shehu Transform, we obtain

$$\begin{aligned}\xi(\nu,\tau) &= \sin(\nu) + S^{-1}\Big[\frac{u^{\gamma}}{s^{\gamma}}\{S(\xi_{\nu\nu} + 2\xi\xi_{\nu} - (\xi\zeta)_{\nu})\}\Big],\\ \zeta(\nu,\tau) &= \sin(\nu) + S^{-1}\Big[\frac{u^{\gamma}}{s^{\gamma}}\{S(\zeta_{\nu\nu} + 2\zeta\zeta_{\nu} - (\xi\zeta)_{\nu})\}\Big].\end{aligned} \tag{22}$$

By applying homotopy perturbation method as in Equation (16), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\epsilon^{k}\xi_{k}(\nu,\tau) &= \sin(\nu) + \epsilon\Bigg[S^{-1}\Bigg[\frac{u^{\gamma}}{s^{\gamma}}S\Bigg[\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\xi_{k}(\nu,\tau)\Bigg)_{\nu\nu} + 2\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\xi_{k}(\nu,\tau)\Bigg)\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\zeta_{k}(\nu,\tau)\Bigg)_{\nu}\\ &\quad -\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\xi_{k}(\nu,\tau)\sum_{k=0}^{\infty}\epsilon^{k}\zeta_{k}(\nu,\tau)\Bigg)_{\nu}\Bigg]\Bigg]\Bigg]\\ \sum_{k=0}^{\infty}\epsilon^{k}\zeta_{k}(\nu,\tau) &= \sin(\nu) + \epsilon\Bigg[S^{-1}\Bigg[\frac{u^{\gamma}}{s^{\gamma}}S\Bigg[\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\zeta_{k}(\nu,\tau)\Bigg)_{\nu\nu} + 2\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\xi_{k}(\nu,\tau)\Bigg)\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\zeta_{k}(\nu,\tau)\Bigg)_{\nu}\\ &\quad -\Bigg(\sum_{k=0}^{\infty}\epsilon^{k}\xi_{k}(\nu,\tau)\sum_{k=0}^{\infty}\epsilon^{k}\zeta_{k}(\nu,\tau)\Bigg)_{\nu}\Bigg]\Bigg]\Bigg]\end{aligned} \tag{23}$$

On comparing coefficient of ϵ on both sides, we obtain

$$\begin{aligned}&\epsilon^{0}: \xi_{0}(\nu,\tau) = \sin(\nu)\\ &\epsilon^{0}: \zeta_{0}(\nu,\tau) = \sin(\nu)\\ &\epsilon^{1}: \xi_{1}(\nu,\tau) = S^{-1}\Big(\frac{u^{\gamma}}{s^{\gamma}}S[\xi_{0\nu\nu} + 2\xi_{0}\xi_{0\nu} - (\xi_{0}\zeta_{0})_{\nu}]\Big) = -\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\sin(\nu)\\ &\epsilon^{1}: \zeta_{1}(\nu,\tau) = S^{-1}\Big(\frac{u^{\gamma}}{s^{\gamma}}S[\zeta_{0\nu\nu} + 2\zeta_{0}\zeta_{0\nu} - (\xi_{0}\zeta_{0})_{\nu}]\Big) = -\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\sin(\nu)\\ &\epsilon^{2}: \xi_{2}(\nu,\tau) = S^{-1}\Big(\frac{u^{\gamma}}{s^{\gamma}}S[\xi_{1\nu\nu} + 2(\xi_{1}\xi_{0\nu} + \xi_{0}\xi_{1\nu}) - (\xi_{1}\zeta_{0} + \xi_{0}\zeta_{1})_{\nu}]\Big) = \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\sin(\nu)\\ &\epsilon^{2}: \zeta_{2}(\nu,\tau) = S^{-1}\Big(\frac{u^{\gamma}}{s^{\gamma}}S[\zeta_{1\nu\nu} + 2(\zeta_{1}\zeta_{0\nu} + \zeta_{0}\zeta_{1\nu}) - (\xi_{1}\zeta_{0} + \xi_{0}\zeta_{1})_{\nu}]\Big) = \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\sin(\nu)\\ &\vdots\end{aligned} \tag{24}$$

Thus, by taking $\epsilon \to 1$ we get convergent series form solution as

$$\begin{aligned}\xi(\nu,\tau) &= \xi_{0} + \xi_{1} + \xi_{2} + \cdots\\ &= \sin(\nu) - \frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\sin(\nu) + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\sin(\nu) + \cdots = \sin(\nu)\Big(1 - \frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} + \cdots\Big)\\ \zeta(\nu,\tau) &= \zeta_{0} + \zeta_{1} + \zeta_{2} + \cdots\\ &= \sin(\nu) - \frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\sin(\nu) + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\sin(\nu) + \cdots = \sin(\nu\Big(1 - \frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} + \cdots\Big)\end{aligned} \tag{25}$$

Particularly, putting $\gamma = 1$, we get the exact solution

$$\begin{aligned}\xi(\nu,\tau) &= \exp^{-\tau}\sin(\nu)\\ \zeta(\nu,\tau) &= \exp^{-\tau}\sin(\nu)\end{aligned} \tag{26}$$

The homotopy perturbation Laplace transform method which is the special case for $u = 1$ of the homotopy perturbation Shehu transform method is used to obtain the same results of Example 1.

In Figure 1, the graphs a and b represent the exact and HPSTM solutions of Example 1. It is observed that the exact and HPSTM solutions are in closed contact and justify the validity of the proposed method. In Figure 2, the sub-graphs a and b have shown the plot of HPSTM solutions at various fractional-order of the derivatives in two and one dimensions of Example 1 respectively. The convergence phenomena of the fractional-order solutions towards integer-order solution is observed by using sub-graphs a and b.

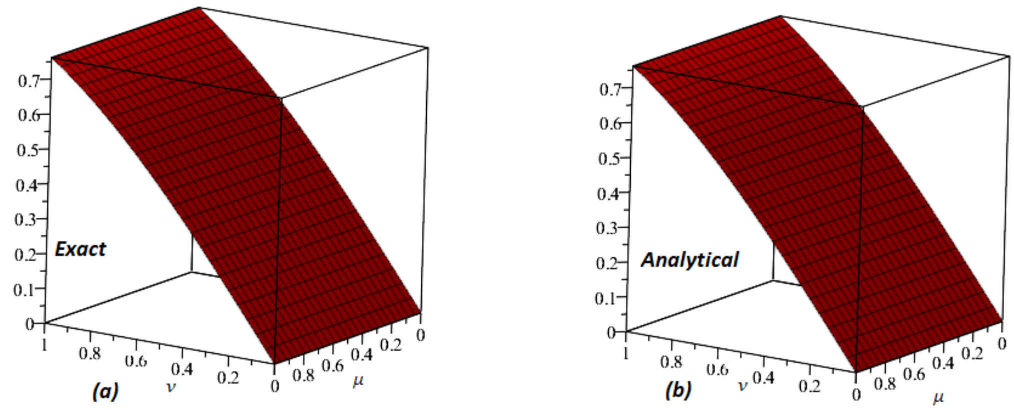

Figure 1. Plot of (**a**) Exact (**b**) HPSTM solutions of ξ $\gamma = 1$ for Example 1.

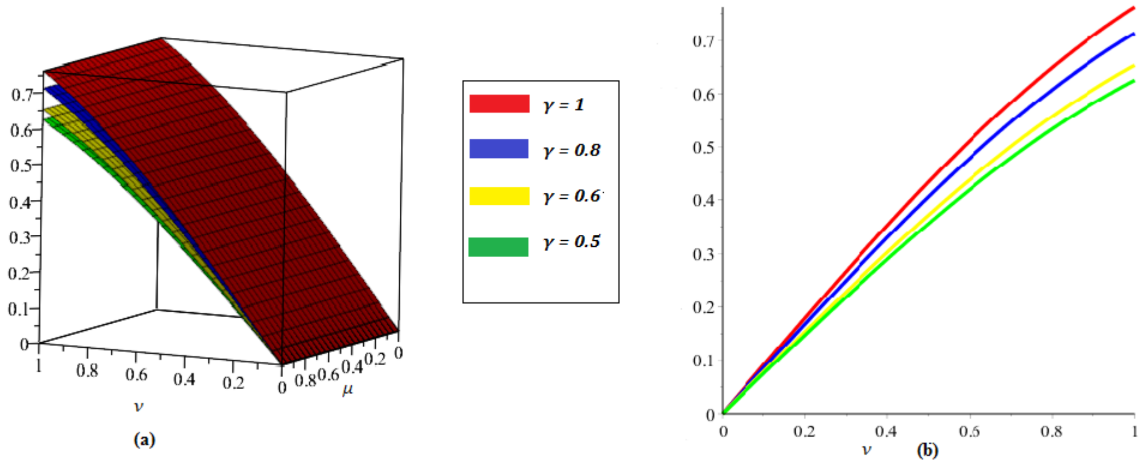

Figure 2. The plot of HPSTM solutions of ζ example 1 at (**a**) various values of γ (**b**) $\tau = 0.5$.

In Table 1, the solutions of Example 1 at fractional-orders $\gamma = 0.8, 1$ have been investigated. For this purpose, the homotopy perturbation method (HPM) with two different transformations is implemented to obtain the solutions. The results of HPM, homotopy perturbation Laplace transform method (HPLTM) and homotopy perturbation Shehu transform method (HPSTM) are compared in Table 1 for the variable ξ and ζ. The comparison has confirmed the best contact among the solutions of the suggested methods. The comparisons have been done in terms of absolute error. It is analyzed from the table that the proposed techniques have the desire degree of accuracy towards the exact solution of the problems.

Table 1. HPLTM, HPSTM and HPM solutions comparison of Example 1 at $\xi(\nu,\tau)$ and $\zeta(\nu,\tau)$ for different fractional-order of γ absolute error.

		HPLTM	**HPLTM**	**HPSTM**	**HPSTM**	**HPM [54]**	**HPM [54]**
τ	ν	$\gamma = 0.8$	$\gamma = 1$	$\gamma = 0.8$	$\gamma = 1$	$\gamma = 0.8$	$\gamma = 1$
	1	8.19373×10^{-5}	7.06835×10^{-8}	8.19373×10^{-5}	7.06835×10^{-8}	8.19373×10^{-5}	7.06835×10^{-8}
	2	8.85419×10^{-5}	7.63809×10^{-8}	8.85419×10^{-5}	7.63809×10^{-8}	8.85419×10^{-5}	7.63809×10^{-8}
0.1	3	1.37414×10^{-5}	1.18540×10^{-8}	1.37414×10^{-5}	1.18540×10^{-8}	1.37414×10^{-5}	1.18540×10^{-8}
	4	7.36928×10^{-5}	6.35714×10^{-8}	7.36928×10^{-5}	6.35714×10^{-8}	7.36928×10^{-5}	6.35714×10^{-8}
	5	9.33742×10^{-5}	8.05496×10^{-8}	9.33742×10^{-5}	8.05496×10^{-8}	9.33742×10^{-5}	8.05496×10^{-8}
	1	1.40490×10^{-4}	2.32077×10^{-6}	1.40490×10^{-4}	2.32077×10^{-6}	1.40490×10^{-4}	2.32077×10^{-6}
	2	1.51814×10^{-4}	2.50784×10^{-6}	1.51814×10^{-4}	2.50784×10^{-6}	1.51814×10^{-4}	2.50784×10^{-6}
0.2	3	2.35611×10^{-4}	3.89208×10^{-6}	2.35611×10^{-4}	3.89208×10^{-6}	2.35611×10^{-4}	3.89208×10^{-6}
	4	1.26354×10^{-4}	2.08726×10^{-6}	1.26354×10^{-4}	2.08726×10^{-6}	1.26354×10^{-4}	2.08726×10^{-6}
	5	1.60100×10^{-4}	2.64471×10^{-6}	1.60100×10^{-4}	2.64471×10^{-6}	1.60100×10^{-4}	2.64471×10^{-6}
	1	1.92364×10^{-4}	1.79300×10^{-5}	1.92364×10^{-4}	1.79300×10^{-5}	1.92364×10^{-4}	1.79300×10^{-5}
	2	2.07869×10^{-4}	1.93753×10^{-5}	2.07869×10^{-4}	1.93753×10^{-5}	2.07869×10^{-4}	1.93753×10^{-5}
0.3	3	3.22607×10^{-4}	3.00698×10^{-5}	3.22607×10^{-4}	3.00698×10^{-5}	3.22607×10^{-4}	3.00698×10^{-5}
	4	1.73008×10^{-4}	1.61259×10^{-5}	1.73008×10^{-4}	1.61259×10^{-5}	1.73008×10^{-4}	1.61259×10^{-5}
	5	2.19214×10^{-4}	2.04327×10^{-5}	2.19214×10^{-4}	2.04327×10^{-5}	2.19214×10^{-4}	2.04327×10^{-5}

Example 2. *Consider the following system of fractional PDEs* [47]

$$\begin{aligned} \xi_\tau^\gamma + \zeta_\nu \eta_\mu - \zeta_\mu \eta_\nu &= -\xi \\ \zeta_\tau^\gamma + \eta_\nu \xi_\mu - \xi_\nu \eta_\mu &= \zeta \\ \eta_\tau^\gamma + \xi_\nu \zeta_\mu - \xi_\mu \zeta_\nu &= \eta, \end{aligned} \tag{27}$$

with initial conditions

$$\begin{aligned} \xi(\nu,\mu,0) &= \exp^{\nu+\mu} \\ \zeta(\nu,\mu,0) &= \exp^{\nu-\mu} \\ \eta(\nu,\mu,0) &= \exp^{\mu-\nu} \end{aligned} \tag{28}$$

Taking Shehu Transform of Equation (27), we have

$$\begin{aligned} \frac{s^\gamma}{u^\gamma} S[\xi(\nu,\mu,\tau)] &= \xi^{(0)}(\nu,\mu,0)\frac{s^{\gamma-1}}{u^\gamma} + S\Big(-\zeta_\nu\eta_\mu + \zeta_\mu\eta_\nu - \xi\Big). \\ \frac{s^\gamma}{u^\gamma} S[\zeta(\nu,\mu,\tau)] &= \zeta^{(0)}(\nu,\mu,0)\frac{s^{\gamma-1}}{u^\gamma} + S\Big(-\eta_\nu\xi_\mu + \xi_\nu\eta_\mu + \zeta\Big). \\ \frac{s^\gamma}{u^\gamma} S[\eta(\nu,\mu,\tau)] &= \eta^{(0)}(\nu,\mu,0)\frac{s^{\gamma-1}}{u^\gamma} + S\Big(-\xi_\nu\zeta_\mu + \xi_\mu\zeta_\nu + \eta\Big). \end{aligned} \tag{29}$$

$$\begin{aligned} S[\xi(\nu,\mu,\tau)] &= \frac{1}{s}\exp^{\nu+\mu} + \frac{u^\gamma}{s^\gamma}\Big[S\Big(-\zeta_\nu\eta_\mu + \zeta_\mu\eta_\nu - \xi\Big)\Big] \\ S[\zeta(\nu,\mu,\tau)] &= \frac{1}{s}\exp^{\nu-\mu} + \frac{u^\gamma}{s^\gamma}\Big[S\Big(-\eta_\nu\xi_\mu + \xi_\nu\eta_\mu + \zeta\Big)\Big] \\ S[\eta(\nu,\mu,\tau)] &= \frac{1}{s}\exp^{\mu-\nu} + \frac{u^\gamma}{s^\gamma}\Big[S\Big(-\xi_\nu\zeta_\mu + \xi_\mu\zeta_\nu + \eta\Big)\Big] \end{aligned} \tag{30}$$

Taking Inverse Shehu Transform, we get

$$\begin{aligned} \xi(\nu,\mu,\tau) &= \exp^{\nu+\mu} + S^{-1}\left[\frac{u^\gamma}{s^\gamma}\Big\{S\Big(-\zeta_\nu\eta_\mu + \zeta_\mu\eta_\nu - \xi\Big)\Big\}\right]. \\ \zeta(\nu,\mu,\tau) &= \exp^{\nu-\mu} + S^{-1}\left[\frac{u^\gamma}{s^\gamma}\Big\{S\Big(-\eta_\nu\xi_\mu + \xi_\nu\eta_\mu + \zeta\Big)\Big\}\right]. \\ \eta(\nu,\mu,\tau) &= \exp^{\mu-\nu} + S^{-1}\left[\frac{u^\gamma}{s^\gamma}\Big\{S\Big(-\xi_\nu\zeta_\mu + \xi_\mu\zeta_\nu + \eta\Big)\Big\}\right]. \end{aligned} \tag{31}$$

By applying homotopy perturbation method as in Equation (16), we get

$$\begin{aligned} \sum_{k=0}^{\infty}\epsilon^k\xi_k(\nu,\mu,\tau) =& e^{\nu+\mu} + \epsilon\Bigg[S^{-1}\Bigg[\frac{u^\gamma}{s^\gamma}S\Bigg[-\Bigg(\bigg(\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\nu,\mu,\tau)\bigg)_\nu\bigg(\sum_{k=0}^{\infty}\epsilon^k\eta_k(\nu,\mu,\tau)\bigg)_y\Bigg) \\ &+\Bigg(\bigg(\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\nu,\mu,\tau)\bigg)_\mu\bigg(\sum_{k=0}^{\infty}\epsilon^k\eta_k(\nu,\mu,\tau)\bigg)_\nu\Bigg) - \sum_{k=0}^{\infty}\epsilon^k\xi_k(\nu,\mu,\tau)\Bigg]\Bigg]\Bigg], \\ \sum_{k=0}^{\infty}\epsilon^k\zeta_k(\nu,\mu,\tau) =& e^{\nu-\mu} + \epsilon\Bigg[S^{-1}\Bigg[\frac{u^\gamma}{s^\gamma}S\Bigg[-\Bigg(\bigg(\sum_{k=0}^{\infty}\epsilon^k\eta_k(\nu,\mu,\tau)\bigg)_\nu\bigg(\sum_{k=0}^{\infty}\epsilon^k\xi_k(\nu,\mu,\tau)\bigg)_\mu\Bigg) \\ &+\Bigg(\bigg(\sum_{k=0}^{\infty}\epsilon^k\xi_k(\nu,\mu,\tau)\bigg)_\nu\bigg(\sum_{k=0}^{\infty}\epsilon^k\eta_k(\nu,\mu,\tau)\bigg)_\mu\Bigg) + \sum_{k=0}^{\infty}\epsilon^k\zeta_k(\nu,\mu,\tau)\Bigg]\Bigg]\Bigg], \\ \sum_{k=0}^{\infty}\epsilon^k\eta_k(\nu,\mu,\tau) =& e^{\mu-\nu} + \epsilon\Bigg[S^{-1}\Bigg[\frac{u^\gamma}{s^\gamma}S\Bigg[-\Bigg(\bigg(\sum_{k=0}^{\infty}\epsilon^k\xi_k(\nu,\mu,\tau)\bigg)_\nu\bigg(\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\nu,\mu,\tau)\bigg)_\mu\Bigg) \\ &+\Bigg(\bigg(\sum_{k=0}^{\infty}\epsilon^k\xi_k(\nu,\mu,\tau)\bigg)_\mu\bigg(\sum_{k=0}^{\infty}\epsilon^k\zeta_k(\nu,\mu,\tau)\bigg)_\mu\Bigg) + \sum_{k=0}^{\infty}\epsilon^k\eta_k(\nu,\mu,\tau)\Bigg]\Bigg]\Bigg]. \end{aligned} \tag{32}$$

On comparing coefficient of ϵ on both sides, we obtain

$$
\begin{aligned}
&\epsilon^0 : \xi_0(\nu,\mu,\tau) = \exp^{\nu+\mu} \\
&\epsilon^0 : \zeta_0(\nu,\mu,\tau) = \exp^{\nu-\mu} \\
&\epsilon^0 : \eta_0(\nu,\mu,\tau) = \exp^{\mu-\nu} \\
&\epsilon^1 : \xi_1(\nu,\mu,\tau) = S^{-1}\Big(\frac{u^\gamma}{s^\gamma} S[-\zeta_{0\nu}\eta_{0\mu} + \zeta_{0\mu}\eta_{0\nu} - \xi_0]\Big) = -\frac{\tau^\gamma}{\Gamma(\gamma+1)} \exp^{\nu+\mu} \\
&\epsilon^1 : \zeta_1(\nu,\mu,\tau) = S^{-1}\Big(\frac{u^\gamma}{s^\gamma} S[-\eta_{0\nu}\xi_{0\mu} + \xi_{0\nu}\eta_{0\mu} + \zeta_0]\Big) = \frac{\tau^\gamma}{\Gamma(\gamma+1)} \exp^{\nu-\mu} \\
&\epsilon^1 : \eta_1(\nu,\mu,\tau) = S^{-1}\Big(\frac{u^\gamma}{s^\gamma} S[-\xi_{0\nu}\zeta_{0\mu} + \xi_{0\mu}\zeta_{0\nu} + \eta_0]\Big) = \frac{\tau^\gamma}{\Gamma(\gamma+1)} \exp^{-\nu+\mu} \\
&\epsilon^2 : \xi_2(\nu,\mu,\tau) = S^{-1}\Big(\frac{u^\gamma}{s^\gamma} S[(\zeta_{1\mu}\eta_{0\nu} + \zeta_{0\mu}\eta_{1\nu}) - (\zeta_{1\nu}\eta_{0\nu} + \zeta_{0\nu}\eta_{1\mu}) - \xi_1]\Big) = \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} \exp^{\nu+\mu} \\
&\epsilon^2 : \zeta_2(\nu,\mu,\tau) = S^{-1}\Big(\frac{u^\gamma}{s^\gamma} S[\zeta_1 - (\eta_{1\nu}\xi_{0\mu} + \eta_{0\nu}\xi_{1\mu}) - (\xi_{1\nu}\eta_{0\mu} + \xi_{0\nu}\eta_{1\mu})]\Big) = \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} \exp^{\nu-\mu} \\
&\epsilon^2 : \eta_2(\nu,\mu,\tau) = S^{-1}\Big(\frac{u^\gamma}{s^\gamma} S[\eta_1 - (\xi_{1\nu}\zeta_{0\mu} + \xi_{0\nu}\zeta_{1\mu}) - (\xi_{1\mu}\zeta_{0\nu} + \xi_{0\mu}\zeta_{1\nu})]\Big) = \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} \exp^{\mu-\nu} \\
&\vdots
\end{aligned} \tag{33}
$$

Thus, by taking $\epsilon \to 1$ we get convergent series form solution as

$$
\begin{aligned}
&\xi(\nu,\mu,\tau) = \xi_0 + \xi_1 + \xi_2 + \cdots \\
&= \exp^{\nu+\mu} - \frac{\tau^\gamma}{\Gamma(\gamma+1)} \exp^{\nu+\mu} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} \exp^{\nu+\mu} + \cdots = \exp^{\nu+\mu}\Big(1 - \frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} + \cdots\Big) \\
&\zeta(\nu,\mu,\tau) = \zeta_0 + \zeta_1 + \zeta_2 + \cdots \\
&= \exp^{\nu-\mu} + \frac{\tau^\gamma}{\Gamma(\gamma+1)} \exp^{\nu-\mu} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} \exp^{\nu-\mu} + \cdots = \exp^{\nu-\mu}\Big(1 + \frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} + \cdots\Big) \\
&\eta(\nu,\mu,\tau) = \eta_0 + \eta_1 + \eta_2 + \cdots \\
&= \exp^{\mu-\nu} + \frac{\tau^\gamma}{\Gamma(\gamma+1)} \exp^{\mu-\nu} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} \exp^{\mu-\nu} + \cdots = \exp^{\mu-\nu}\Big(1 + \frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} + \cdots\Big)
\end{aligned} \tag{34}
$$

particularly, putting $\gamma = 1$, we get the exact solution of Equation (27)

$$
\begin{aligned}
\xi(\nu,\mu,\tau) &= \exp^{\nu+\mu-\tau} \\
\zeta(\nu,\mu,\tau) &= \exp^{\nu-\mu+\tau} \\
\eta(\nu,\mu,\tau) &= \exp^{\mu-\nu+\tau}
\end{aligned} \tag{35}
$$

Using the Laplace homotopy perturbation method, the same results are derived for Example 2, because Laplace transformation is the special case for $u = 1$ of Shehu transformation.

In Figures 3–5 the sub-graphs a and b are respectively the graphs of the exact and HPSTM solutions at $\gamma = 1$ of example 2 for variables ξ, ζ and η. The graphical representation has confirmed the closed contact of the exact solution with HPSTM solution. In Figure 6, the sub-graphs a and b have shown the plot of HPSTM solutions at various fractional-order of the derivatives in two dimensions of Example 2 for variables ξ and ζ respectively. In Figure 7, the sub-graphs a and b have shown the plot of HPSTM solutions at various fractional-order of the derivatives in two and one dimensions of Example 2 for variable η respectively. The convergence phenomena of the fractional-order solutions towards integer-order solution is observed by using sub-graphs a and b.

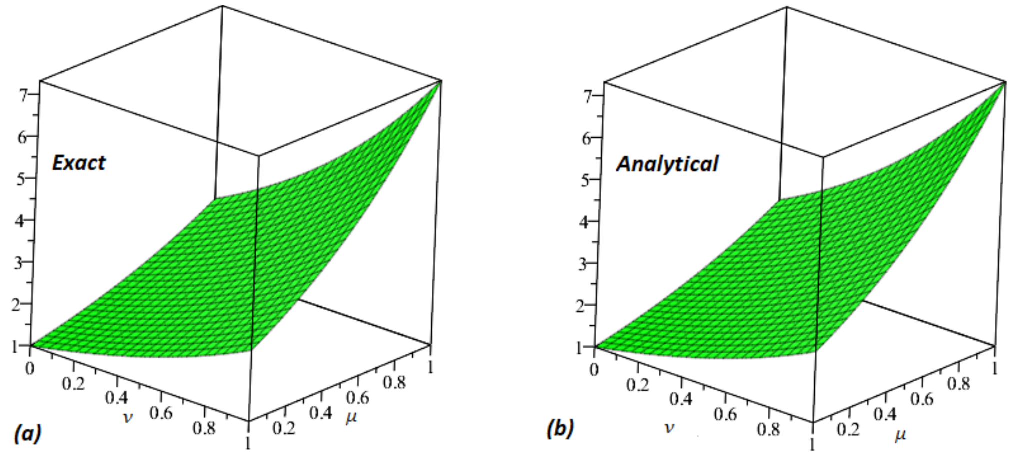

Figure 3. (**a**) Exact (**b**) HPSTM solution graph of ξ Example 2, at $\gamma = 1$.

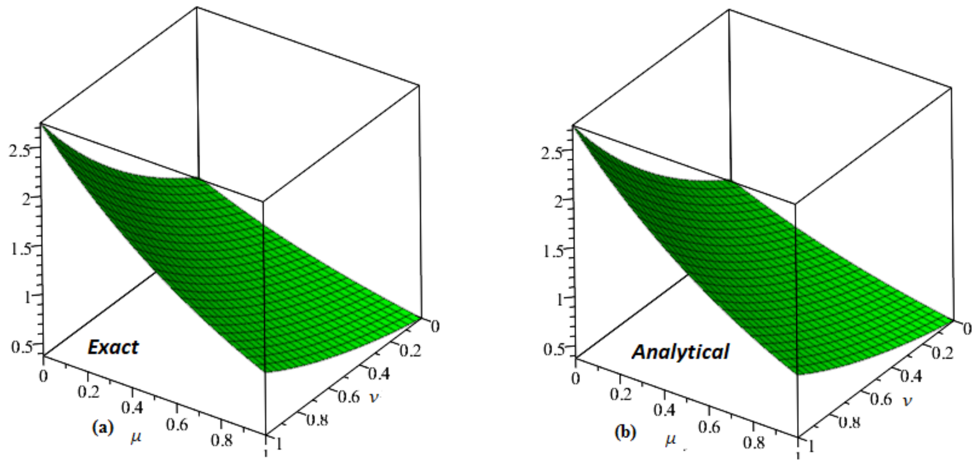

Figure 4. (**a**) Exact (**b**) HPSTM solution graph of ζ Example 2, at $\gamma = 1$.

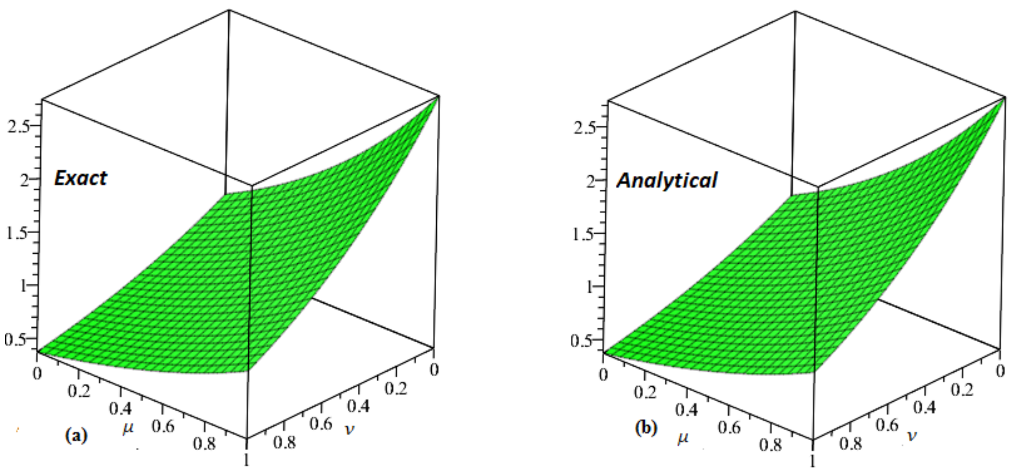

Figure 5. (**a**) Exact (**b**) HPSTM solution graph of η Example 2, $\gamma = 1$.

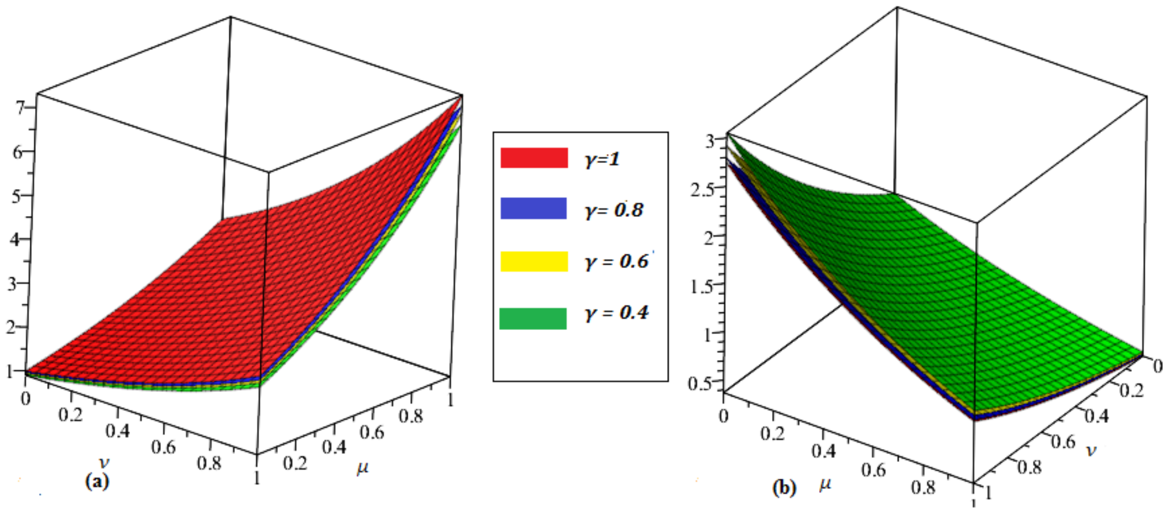

Figure 6. The HPSTM solutions plot are represented by (**a**,**b**) for variables ξ and ζ respectively at $\gamma = 1$, 0.8, 0.6 and 0.5 of $\xi\ (\nu, \mu, \tau)$ and $\zeta\ (\nu, \mu, \tau)$.

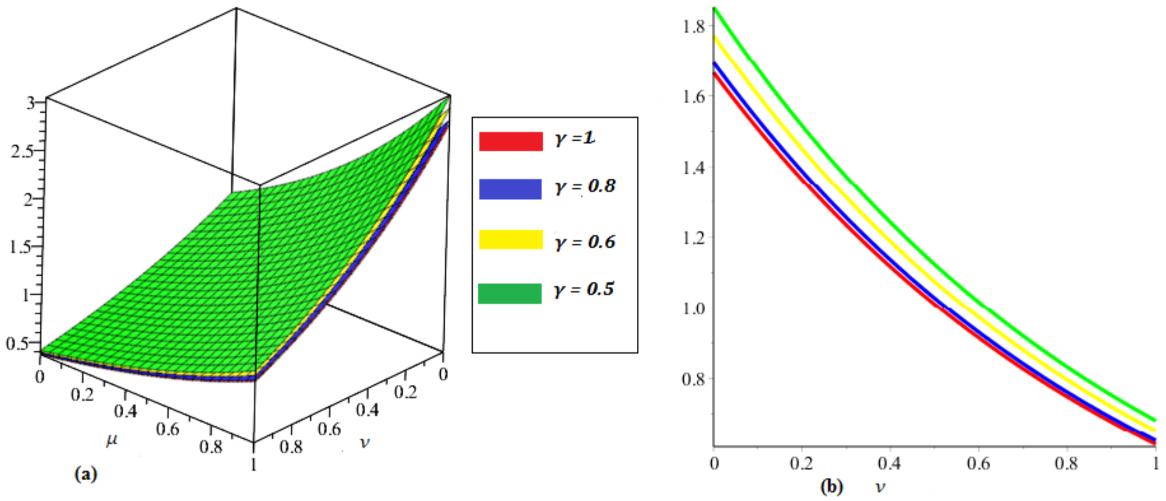

Figure 7. The (**a**) represents HPSTM solution of example 2 at different fractional orders of γ and (**b**) $\tau = 0.5$ of $\eta\ (\nu, \mu, \tau)$.

5. Conclusions

In this paper, some systems of FPDEs are solved by the homotopy perturbation method along with Laplace and Shehu transformations. The derivatives with fractional-order are expressed in term of the Caputo operator. The suggested technique is implemented to find the solution of certain numerical examples. The solutions of these illustrative examples are determined for derivatives at different fractional-orders. The significant extent between the actual and approximate solutions is observed. Furthermore, fractional solutions are found to be convergent to integer-order solution for every targeted problem. It is observed that the proposed methods are simple, straightforward, have low computational cost, and can be modified for the solutions of FPDEs in science and engineering. In future, the proposed method can be extended to find the analytical solutions of nonlinear higher dimension fractional partial differential equations and systems of fractional partial differential equations.

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