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Robust Flexibility Provision from DERs: A Network-Constrained Multi-Step Optimization Approach

Shinya Sekizaki *  and Tomohiro Hayashida 

Graduate School of Advanced Science and Engineering, Hiroshima University, Hiroshima 739-8527, Japan;
hayashida@hiroshima-u.ac.jp

* Correspondence: sekizaki@hiroshima-u.ac.jp

Abstract

This paper proposes a network-constrained multi-step optimization approach for robust flexibility provision from distributed energy resources (DERs) under photovoltaic (PV) generation uncertainty. The proposed method optimally schedules day-ahead battery operations based on PV output predictions and confidence intervals, while strictly satisfying network constraints. To ensure operational feasibility, the AC power flow is modeled using DistFlow equations. To bridge the gap between computational tractability and exact AC feasibility, a multi-step solution strategy is introduced. First, a linearized DistFlow (LinDistFlow) model is employed within a column-and-constraint generation algorithm to efficiently identify the worst-case scenario. Subsequently, the robust day-ahead battery schedule is determined by solving a second-order cone (SOC)-relaxed DistFlow model under the identified scenario. Finally, a post-processing exact recovery step is executed by solving the original non-convex DistFlow equations under the fixed battery schedule and the worst-case scenario. This crucial step compensates for approximation errors introduced by the LinDistFlow and SOC models, significantly enhancing practical operational AC feasibility under the identified critical condition. Extensive case studies on the IEEE 33-bus test system verify the effectiveness of the proposed multi-step approach in balancing computational efficiency and robust flexibility provision.

Keywords: robust optimization; flexibility provision; distributed energy resources

1. Introduction

Driven by accelerating efforts to achieve carbon neutrality, the grid integration of distributed energy resources (DERs) is rapidly advancing. The capability of DERs to provide grid services by generating or consuming power in response to commands is referred to as flexibility [1] (i.e., the ability to adjust active and reactive power). By utilizing DERs owned by prosumers within the distribution system to provide this flexibility, it is expected to contribute to mitigating network congestion and maintain the supply–demand balance [2]. Recent studies have increasingly focused on flexibility procurement, proposing various frameworks such as distributionally robust chance-constrained optimization [3], bi-level market clearing models for TSO-DSO coordination [4], and decentralized flexibility allocation schemes [5,6].

In the provision of flexibility, network congestion and constraint violations will occur unless the DER group is appropriately controlled while taking into account capacities and appropriate voltage/current ranges. Therefore, it is necessary to aggregate the required amount of flexibility from the DER group while strictly satisfying network



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constraints [7,8]. Many previous studies employ AC optimal power flow (AC-OPF) to account for these constraints, widely utilizing the LinDistFlow equations [2,9] and the non-convex DistFlow equations [10–12]. Because the non-convex DistFlow equations are difficult to handle directly, optimization problems are often solved by first linearizing them [13] or transforming them into second-order cone (SOC) constraints via convex relaxation. However, solutions obtained by relaxing the DistFlow equations into SOC constraints are not always exact, and strict sufficient conditions are required to guarantee exactness [14]. These conditions are too specific to be easily satisfied in typical real-world distribution networks [15]. In active distribution networks (ADNs) with high shares of renewable energy, the expected occurrence of reverse power flows further impacts the exactness of the convex relaxation [16]. When a large number of DERs are interconnected and reverse power flows are expected, it becomes necessary to compensate for the inexactness caused by the SOC relaxation to recover feasibility [4,17].

Furthermore, because renewable energy-based DERs involve uncertainties in the flexibility they can provide, it is essential to optimize DERs while accounting for these uncertainties. Among the optimization methods under uncertainty, robust optimization does not suffer from out-of-sample issues [18]. It can be solved efficiently by decomposing the problem using methods such as the column-and-constraint generation (C&CG) algorithm [19]. Although robust optimization is often criticized for being overly conservative, the trade-off between robustness and conservativeness can be controlled by utilizing an uncertainty budget [20,21]. However, robust optimization inherently requires solving a min-max-min problem, and the introduction of binary variables—required to linearize bilinear terms—significantly increases the computational burden [22]. Consequently, reducing the computational burden of solving robust optimization problems remains a critical challenge from a practical perspective.

To address these computational challenges and accommodate dynamic operational environments, advanced robust and stochastic optimization frameworks have been actively explored. For instance, multi-period adaptive robust optimization models, such as the planner–attacker–operator framework proposed by Du et al. [23], have been developed to coordinate dynamic decision-making under severe uncertainties using decomposition algorithms. Similarly, advanced AC-OPF frameworks, such as the distributionally robust chance-constrained approach in [24], elegantly combine nominal exact AC formulations with incremental linear approximations to handle fluctuations. However, these chance-constrained methods inherently accept a pre-defined probability of physical limit violations during real-time operations, and incorporating multi-period time-coupling constraints—such as the State-of-Charge (SoC) dynamics of energy storage—alongside non-convex AC limits remains computationally prohibitive for such single-period stochastic formulations. Consequently, a critical review of the existing literature reveals a fundamental gap in network-constrained robust scheduling: the trade-off between multi-interval computational tractability and exact AC power flow feasibility under severe uncertainty.

To bridge this gap, this paper proposes a network-constrained multi-temporal four-step optimization approach for robust flexibility provision from DERs. Instead of attempting to solve the intractable non-convex min-max-min problem directly, the proposed strategy systematically decouples the robust scheduling process into distinct, tractable stages. First, a linearized DistFlow (LinDistFlow) model is utilized within a C&CG algorithm to efficiently identify the worst-case scenarios of PV generation. Subsequently, the robust day-ahead battery schedules are determined by solving a second-order cone (SOC)-relaxed DistFlow model under the identified scenarios. Finally, a post-processing exact recovery step—using the original non-convex DistFlow equations—is executed under the predetermined battery schedule and the identified critical scenario. This crucial step

eliminates the approximation and relaxation errors, acting as a rigorous feasibility recovery mechanism. By decoupling the time-coupling dynamics from the non-convex spatial AC equations, this novel multi-step architecture significantly enhances practical operational AC feasibility for the identified critical conditions without heavily compromising computational efficiency. While many existing robust AC-OPF studies are confined to single-period optimizations, our model systematically handles multi-step temporal constraints (e.g., dynamic SoC for BESS and demand-side flexibility limits) simultaneously with spatial distribution network constraints under uncertainty, enabling a robust and dynamic scheduling of DER flexibilities over the entire operational horizon.

The main contributions of this paper are summarized as follows:

1. An optimization framework is developed to procure the flexibility requested by the external network from a fleet of DERs, while satisfying network constraints in a distribution system subject to renewable energy output uncertainties.
2. Robust DER control scheduling against uncertainties is achieved within a practical computation time by employing a linear approximation and a multi-stage optimization approach to the non-convex AC power flow equations.
3. Practical operational AC feasibility for the identified critical conditions is guaranteed by formulating a post-processing exact recovery step that compensates for and eliminates the approximation errors introduced by the linearized and SOC-relaxed models.

2. Problem Statement

The problem addressed in this paper is formulated as a two-stage robust optimization problem (min-max-min problem) under specific assumptions and limitations. The determination of the battery charge/discharge schedule on the day-ahead corresponds to the first-stage problem, while the adjustment of each control amount on the operation day corresponds to the second-stage problem. The conceptual figure of the target problem is shown in Figure 1.

The main assumptions of the proposed model are summarized as follows:

- **Distribution Network Topology:** The target distribution network is assumed to have a strictly radial topology, which is a fundamental prerequisite for applying the DistFlow branch flow equations.
- **Types of Distributed Energy Resources (DERs):** The DERs providing flexibility in the distribution system are assumed to be photovoltaic (PV) systems, PV power conditioning systems (PCSs), batteries, and flexible loads (FLs) [1].
- **Controllability of DERs:** It is assumed that the grid-tied PV inverters (PCSs) can control their reactive power output. Furthermore, FLs are assumed to be capable of controlling both their active and reactive power consumption.
- **Deterministic Parameters Given Day-Ahead:** The framework proposed in this paper assumes TSO-DSO coordination at the day-ahead scheduling stage. Consequently, the forecasted PV generation, its confidence interval (uncertainty bounds), and the flexibility requirement from the external grid are treated as predetermined, deterministic parameters. Specifically, the flexibility requirement is not an arbitrary constant but a deterministic requirement pre-allocated to the DSO by the TSO. It is derived from historical load and renewable generation prediction values in the bulk power system to secure balancing capacity for critical periods on the following day (e.g., peak PV generation hours or evening load ramps). Developing dynamic or endogenous uncertainty sets is beyond the scope of this study.
- **Definition of the Worst-Case Scenario:** In the proposed robust optimization framework, the “worst-case scenario” is defined as the realization of uncertain available PV gen-

eration that maximizes the operational cost. The day-ahead battery charge/discharge schedule is determined to minimize the operational cost even under this severe condition.

- **Intra-Day Operational Constraints:** During actual operation on the current day (Day D), the actual battery operation must adhere strictly to the schedule fixed in the day-ahead plan. Subsequently, the PV curtailment amount, the reactive power output of the PCS, and the active/reactive control amount of the FLs are adjusted in response to the actual realized PV output.

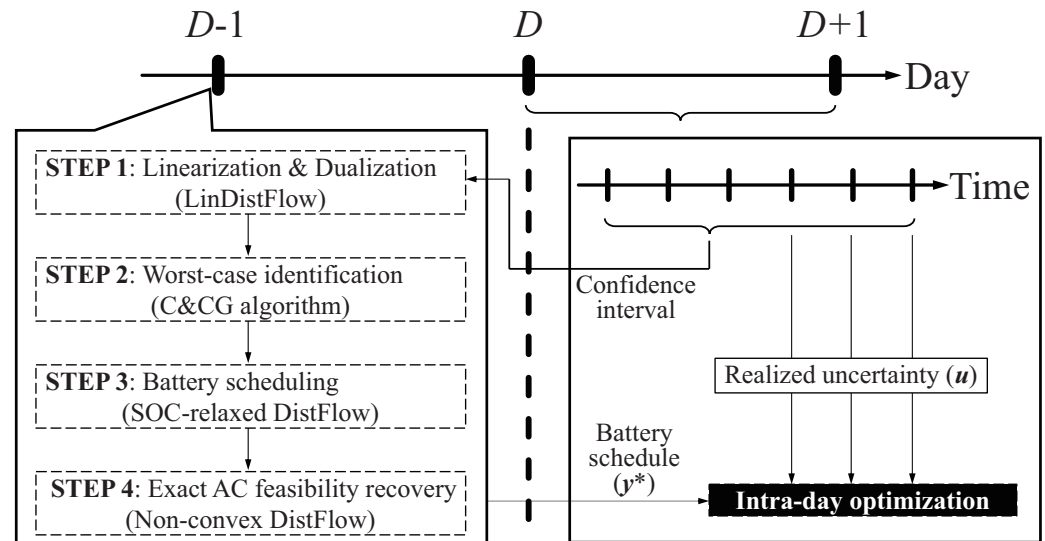


Figure 1. An overview of the proposed multi-step robust optimization framework. In the day-ahead stage ($D - 1$), a four-step solution procedure is executed to determine the robust battery schedule (y^*): identifying the worst-case scenario using the LinDistFlow model and C&CG algorithm (Steps 1 and 2), optimizing the battery schedule via SOC relaxation (Step 3), and recovering exact AC feasibility using the original non-convex DistFlow equations (Step 4). In the intra-day operational stage (D), real-time DER adjustments are performed in response to the realized uncertainty (u) while adhering to the fixed battery schedule.

3. Formulation

3.1. DER Models

3.1.1. Battery

The battery model is formulated as in (1):

$$0 \leq P_{i,t}^{\text{ch}} \leq \bar{P}_i^{\text{bt}}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (1a)$$

$$0 \leq P_{i,t}^{\text{dch}} \leq \bar{P}_i^{\text{bt}}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (1b)$$

$$E_{i,t+1}^{\text{bt}} = E_{i,t}^{\text{bt}} + \eta^{\text{ch}} P_{i,t}^{\text{ch}} \tau - \frac{1}{\eta^{\text{dch}}} P_{i,t}^{\text{dch}} \tau, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T} \setminus \{T\}, \quad (1c)$$

$$\underline{E}_i^{\text{bt}} \leq E_{i,t}^{\text{bt}} \leq \bar{E}_i^{\text{bt}}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}. \quad (1d)$$

Equations (1a) and (1b) denote the upper and lower bound constraints for the charging and discharging power. Equation (1c) expresses the relationship between the charging/discharging power and the state of energy (SOE). Equation (1d) represents the upper and lower bound constraints of the SOE.

Note that this paper does not consider the reactive power control by batteries because it focuses on the robust charge/discharge scheduling.

3.1.2. PV

The uncertain available PV power generation is modeled using an uncertainty budget [20]. The uncertainty budget can control the robustness (conservativeness) against risks caused by uncertainty by adjusting the parameter Γ . In practice, the forecasted available PV power $\tilde{P}_{i,t}^{\text{PV}}$ and its corresponding uncertainty bounds, $\bar{P}_{i,t}^{\text{PV}}$ and $\underline{P}_{i,t}^{\text{PV}}$, are determined utilizing data-driven approaches based on historical forecast error distributions (e.g., as applied in [25]). By leveraging these historical datasets, realistic upper and lower limits of the confidence intervals can be established. Using the auxiliary variables $z_t^+ \in [0.0, 1.0]$ and $z_t^- \in [0.0, 1.0]$, and the forecasted available PV power $\tilde{P}_{i,t}^{\text{PV}} \in [\underline{P}_{i,t}^{\text{PV}}, \bar{P}_{i,t}^{\text{PV}}]$, the available PV power $\hat{P}_{i,t}^{\text{PV}}$ is calculated by Equation (2a), and the actual PV power generation $P_{i,t}^{\text{PV}}$ is constrained by Equation (2b). Assuming that the grid-tied PV inverter can control its reactive power output $Q_{i,t}^{\text{PV}}$, its power factor is constrained by Equation (2c).

$$\hat{P}_{i,t}^{\text{PV}} = \tilde{P}_{i,t}^{\text{PV}} + z_t^+ (\bar{P}_{i,t}^{\text{PV}} - \tilde{P}_{i,t}^{\text{PV}}) + z_t^- (\underline{P}_{i,t}^{\text{PV}} - \tilde{P}_{i,t}^{\text{PV}}), \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (2a)$$

$$0 \leq P_{i,t}^{\text{PV}} \leq \hat{P}_{i,t}^{\text{PV}}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (2b)$$

$$-P_{i,t}^{\text{PV}} \tan \theta \leq Q_{i,t}^{\text{PV}} \leq P_{i,t}^{\text{PV}} \tan \theta, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}. \quad (2c)$$

Using the uncertainty budget model, the uncertainty set $\mathcal{U}$ of the available PV power $\hat{P}_{i,t}^{\text{PV}}$ is defined by Equation (3):

$$\mathcal{U} = \left\{ \hat{P}_{i,t}^{\text{PV}}, \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \mid (2a), \sum_{t \in \mathcal{T}} (z_t^+ + z_t^-) \leq \Gamma, z_t^+, z_t^- \in [0, 1] \right\}. \quad (3)$$

To quantify the uncertainty, the upper and lower bounds of the PV generation are constructed using a data-driven approach based on historical distributions. This study implements a Distributionally Robust Bound (DRB) [25] based on the Wasserstein metric. The resulting intervals are designed to ensure a specified probability, employing a confidence level of $\alpha = 0.05$ (a 95% confidence interval).

3.1.3. Flexible Load

We assume a FL whose power consumption (active and reactive power) is controllable. The FL control variables and their upper and lower bounds are expressed by Equation (4):

$$P_{i,t}^{\text{fl}} = P_{i,t}^{\text{fl}+} - P_{i,t}^{\text{fl}-}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (4a)$$

$$Q_{i,t}^{\text{fl}} = Q_{i,t}^{\text{fl}+} - Q_{i,t}^{\text{fl}-}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (4b)$$

$$0 \leq P_{i,t}^{\text{fl}+} \leq \bar{P}_{i,t}^{\text{fl}+}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (4c)$$

$$0 \leq P_{i,t}^{\text{fl}-} \leq \bar{P}_{i,t}^{\text{fl}-}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (4d)$$

$$0 \leq Q_{i,t}^{\text{fl}+} \leq \bar{Q}_{i,t}^{\text{fl}+}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (4e)$$

$$0 \leq Q_{i,t}^{\text{fl}-} \leq \bar{Q}_{i,t}^{\text{fl}-}, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (4f)$$

where $P_{i,t}^{\text{fl}+}$, $P_{i,t}^{\text{fl}-}$, $Q_{i,t}^{\text{fl}+}$, and $Q_{i,t}^{\text{fl}-}$ are non-negative continuous variables representing the FL control amounts. $P_{i,t}^{\text{fl}+}$ and $P_{i,t}^{\text{fl}-}$ represent the grid demand reduction and demand increase resulting from the active power control of the FL, respectively. Furthermore, $Q_{i,t}^{\text{fl}+}$ and $Q_{i,t}^{\text{fl}-}$ represent the reactive power supply (leading) and reactive power consumption (lagging) to the grid resulting from the reactive power control of the FL, respectively.

3.2. AC Power Flow Model

3.2.1. DistFlow Equations

Let node 0 be the TSO-DSO connection node, and the distribution system is assumed to be radial. The AC power flow equations used in the formulation in the proposed method are formulated as Equation (5) using the DistFlow equations [1,3,4,9–11,26]:

$$-\sum_{k \in \sigma(0)} P_{0k,t} + P_{0,t}^{\text{fl}} + P_{0,t}^{\text{pv}} - P_{0,t}^{\text{ch}} + P_{0,t}^{\text{dch}} = P_{0,t}^{\text{db}}, \quad \forall t \in \mathcal{T}, \quad (5a)$$

$$(P_{ij,t} - R_{ij}l_{ij,t}) - \sum_{k \in \sigma(j)} P_{jk,t} + P_{j,t}^{\text{fl}} + P_{j,t}^{\text{pv}} - P_{j,t}^{\text{ch}} + P_{j,t}^{\text{dch}} = P_{j,t}^{\text{db}}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (5b)$$

$$-\sum_{k \in \sigma(0)} Q_{0k,t} + Q_{0,t}^{\text{fl}} + Q_{0,t}^{\text{pv}} = Q_{0,t}^{\text{db}}, \quad \forall t \in \mathcal{T}, \quad (5c)$$

$$(Q_{ij,t} - X_{ij}l_{ij,t}) - \sum_{k \in \sigma(j)} Q_{jk,t} + Q_{j,t}^{\text{fl}} + Q_{j,t}^{\text{pv}} = Q_{j,t}^{\text{db}}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (5d)$$

$$v_{j,t} = v_{i,t} - 2(R_{ij}P_{ij,t} + X_{ij}Q_{ij,t}) + (R_{ij}^2 + X_{ij}^2)l_{ij,t}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (5e)$$

$$P_{ij,t}^2 + Q_{ij,t}^2 = v_{i,t}l_{ij,t}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (5f)$$

$$v_{0,t} = v_t^*, \quad \forall t \in \mathcal{T}, \quad (5g)$$

$$\underline{v}_i \leq v_{i,t} \leq \bar{v}_i, \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T}, \quad (5h)$$

$$l_{ij,t} \leq \bar{l}_{ij}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}. \quad (5i)$$

Equations (5a)–(5f) are the DistFlow equations, and Equation (5g) specifies the voltage $v_{0,t}$ at the TSO-DSO connection node. Equations (5h) and (5i) represent the allowable range for the voltage and the distribution line current, respectively.

3.2.2. Second-Order Cone (SOC)-Relaxed DistFlow Equations

As will be discussed later, Equation (5f) is non-convex, and solving a problem that includes it as a constraint incurs an extremely high computational burden. Therefore, in some formulations used in the proposed method, we apply a convex relaxation to the equality constraint in Equation (5f), transforming it into Equation (6):

$$P_{ij,t}^2 + Q_{ij,t}^2 \leq v_{i,t}l_{ij,t}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}. \quad (6)$$

Since the right-hand side of Equation (6) is a bilinear term, making it difficult to handle directly, it is converted into the SOC constraint (7a),

$$\|2P_{ij,t} \quad 2Q_{ij,t} \quad (v_{i,t} - l_{ij,t})\|_2 \leq v_{i,t} + l_{ij,t}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (7a)$$

$$(5a)–(5e), (5g)–(5i), \quad (7b)$$

where $\|\cdot\|_2$ in Equation (7a) represents the Euclidean norm. The power flow equations in which only Equation (5f) is second-order cone (SOC)-relaxed are defined as Equation (7).

3.2.3. Linearized DistFlow Equations

Although Equation (7) is convex, the second-order cone constraint (7a) must be linearized to solve the robust optimization problem introduced later. As shown in Appendix C, when employing the polyhedral approximation technique [27], a larger approximation parameter R improves accuracy but increases the computational burden. Since applying polyhedral approximation to the second-order cone constraints incurs a high computational

cost that is undesirable for scalability, the power flow equations, including the linearized DistFlow equations, are defined as Equation (8):

$$P_{ij,t} - \sum_{k \in \sigma(j)} P_{jk,t} + P_{j,t}^{\text{fl}} + P_{j,t}^{\text{pv}} - P_{j,t}^{\text{ch}} + P_{j,t}^{\text{dch}} = P_{j,t}^{\text{db}}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (8a)$$

$$Q_{ij,t} - \sum_{k \in \sigma(j)} Q_{jk,t} + Q_{j,t}^{\text{fl}} + Q_{j,t}^{\text{pv}} = Q_{j,t}^{\text{db}}, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (8b)$$

$$v_{j,t} = v_{i,t} - 2(R_{ij}P_{ij,t} + X_{ij}Q_{ij,t}), \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (8c)$$

$$(5a), (5c), (5g), (5h), \quad (8d)$$

where Equation (8) is obtained by removing the terms related to the squared current $l_{ij,t}$ from Equation (5), and omitting Equations (5f) and (5i). Since Equation (8) does not consider the current, it is inferior to Equation (5) in terms of accuracy; however, an improvement in computational efficiency can be expected.

3.3. Imbalance Against the Flexibility Requirement

In this framework, the required flexibility P_t^{req} from the external network is modeled as a predetermined fixed parameter. In real-world applications, this requirement is typically derived from the forecast errors of the broader system load or predetermined through day-ahead coordination between the transmission system operator (TSO) and the DSO. The flexibility procured across the entire distribution system is defined as the difference between the active power flowing through the TSO-DSO connection node and the baseline. Let $P_t^{\text{bl}}, \forall t \in \mathcal{T}$ denote the baseline, which serves as the reference for the flexibility. Let P_t^{pimb} be a non-negative auxiliary variable representing the absolute value of the difference (i.e., the imbalance amount) between the procured flexibility and the flexibility requirement P_t^{req} . The imbalance against the flexibility requirement is modeled by (9),

$$-P_t^{\text{pimb}} \leq (P_t^{\text{bl}} + P_t^{\text{req}}) - \sum_{j \in \sigma(0)} P_{0j,t} \leq P_t^{\text{pimb}}, \quad \forall t \in \mathcal{T}. \quad (9a)$$

$$P_t^{\text{pimb}} \geq 0, \quad \forall t \in \mathcal{T}, \quad (9b)$$

where $\sum_{j \in \sigma(0)} P_{0j,t}$ is the active power flowing through the TSO-DSO connection node. Since the objective function (introduced later) is proportional to P_t^{pimb} , this term is minimized in the optimal solution under constraint (9a). Furthermore, as the specific method for determining the baseline P_t^{bl} is beyond the primary scope of this paper, it is omitted from the main text; however, the details are provided in Appendix B.

3.4. Objective Function

Equation (10) defines the battery operation cost C^{bt} , the PV curtailment cost C^{pv} , the FL control cost C^{fl} , the penalty cost for the imbalance between the flexibility requirement and the procured amount C^{penal} , the distribution line loss cost C^{loss} , the net total cost C^{total} , and $C^{\text{total}'}$, which represents the net total cost C^{total} minus the distribution line loss cost C^{loss} . Similar to the flexibility requirement, the imbalance penalty parameter c_t^{penal} is treated as a given constant. This parameter reflects the economic cost or agreed-upon penalty rate determined in the day-ahead flexibility market to compensate for the operational risk associated with forecast errors.

$$C^{\text{bt}} = \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} c_t^{\text{bt}} (p_{i,t}^{\text{ch}} + p_{i,t}^{\text{dch}}) \tau, \quad (10a)$$

$$C^{\text{pv}} = \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} c_t^{\text{pv,cut}} (\hat{p}_{i,t}^{\text{pv}} - p_{i,t}^{\text{pv}}) \tau, \quad (10b)$$

$$C^{\text{fl}} = \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} (c_{i,t}^{\text{fl}+} p_{i,t}^{\text{fl}+} + c_{i,t}^{\text{fl}-} p_{i,t}^{\text{fl}-} + c_{i,t}^{\text{qfl}+} Q_{i,t}^{\text{fl}+} + c_{i,t}^{\text{qfl}-} Q_{i,t}^{\text{fl}-}) \tau, \quad (10c)$$

$$C^{\text{penal}} = \sum_{t \in \mathcal{T}} c_t^{\text{penal}} p_t^{\text{imb}} \tau, \quad (10d)$$

$$C^{\text{loss}} = \sum_{t \in \mathcal{T}} \left(\lambda_t^{\text{price}} \sum_{\substack{j \in \mathcal{N} \setminus \{0\}, \\ i = \pi(j)}} R_{ij} l_{ij,t} \tau \right), \quad (10e)$$

$$C^{\text{total}} = C^{\text{bt}} + C^{\text{pv}} + C^{\text{fl}} + C^{\text{penal}} + C^{\text{Loss}}, \quad (10f)$$

$$C^{\text{total}'} = C^{\text{bt}} + C^{\text{pv}} + C^{\text{fl}} + C^{\text{penal}}. \quad (10g)$$

where $C^{\text{total}'}$ is used when solving the problem in Step 2 of the proposed method (described later in Section 4).

3.5. Robust Optimization Problem

As depicted in Figure 1, the first stage optimizes the battery charge and discharge schedules, whereas the second stage determines the optimal DER control once the uncertainty is realized. The first-stage feasible set $\mathcal{Y}^{\text{1st}}$, which governs these scheduling decisions, is defined by Equation (11):

$$\mathcal{Y}^{\text{1st}} = \left\{ p_{i,t}^{\text{ch}}, p_{i,t}^{\text{dch}}, \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \mid (1), (5) \right\}. \quad (11)$$

For a given battery schedule $\mathbf{y} \in \mathcal{Y}^{\text{1st}}$ and an uncertainty realization $\hat{\mathbf{u}}$, the feasible region $\mathcal{X}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}})$ of the second-stage minimization problem is given by Equation (12):

$$\mathcal{X}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}}) = \left\{ p_{i,t}^{\text{pv}}, Q_{i,t}^{\text{pv}}, p_{i,t}^{\text{fl}+}, p_{i,t}^{\text{fl}-}, Q_{i,t}^{\text{fl}+}, Q_{i,t}^{\text{fl}-}, p_{ij,t}, Q_{ij,t}, v_{i,t}, l_{ij,t}, p_t^{\text{imb}}, \forall i, j \in \mathcal{N}, \forall t \in \mathcal{T} \mid (2), (3), (4), (5), (9) \right\}. \quad (12)$$

Hereafter, to avoid notational complexity, the robust optimization problem is simply expressed as in (13).

$$\underbrace{\underset{\text{First-stage}}{\underset{\mathbf{y} \in \mathcal{Y}^{\text{1st}}}{\text{minimize}}}} \left\{ \underbrace{\underset{\text{Second-stage}}{\underset{\hat{\mathbf{u}} \in \mathcal{U}}{\text{maximize}} \left(\underset{\mathbf{x} \in \mathcal{X}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}})}{\text{minimize}} C^{\text{total}} \right)}} \right\}. \quad (13)$$

In (13), $\mathbf{y} \in \mathbb{R}^{n_y \times 1}$, $\hat{\mathbf{u}} \in \mathbb{R}^{n_u \times 1}$, and $\mathbf{x} \in \mathbb{R}^{n_x \times 1}$ represent the decision-variable vectors in the first and second stages. The variables n_y , n_u , and n_x denote the dimensions of $\mathbf{y}$, $\hat{\mathbf{u}}$, and $\mathbf{x}$, respectively. Equation (13) features a min-max-min structure designed to determine a robust battery schedule $\mathbf{y}$ and other DER control variables $\mathbf{x}$ under the worst-case available PV generation scenario $\hat{\mathbf{u}}$. While Equation (13) provides an exact formulation, direct solution is computationally intractable owing to the non-convex AC power flow Equation (5f). Consequently, we solve this problem using the proposed methodology introduced in Section 4.

4. Proposed Robust Optimization Method

In order to robustly procure flexibility while satisfying network constraints against the uncertainty set $\mathcal{U}$ of available PV generation power, it is necessary to solve the robust optimization problem (13) and obtain its optimal solution $(\mathbf{y}^*, \hat{\mathbf{u}}^*, \mathbf{x}^*)$. If the feasible region $\mathcal{X}^{2\text{nd}}(\mathbf{y}, \hat{\mathbf{u}})$ is linear, the second-stage problem can be solved using the C&CG method [21]; however, $\mathcal{X}^{2\text{nd}}(\mathbf{y}, \hat{\mathbf{u}})$ contains the non-convex DistFlow Equation (5f). To solve the min-max-min problem using the C&CG method, we reformulate the problem using the convex-relaxed and linearly approximated power flow Equations (7) or (8) instead of the original DistFlow equations, and we propose an optimization method that compensates for the errors in the optimal solution caused by these relaxations and approximations. In the proposed method, rather than solving the robust optimization problem (13) directly, a step-by-step solution and error compensation approach is performed.

4.1. Procedure

Because the formulated min-max-min multi-stage robust optimization problem (13) includes the non-convex AC power flow Equation (5) in the constraints of the second-stage problem, strong duality cannot be applied directly, making it difficult to solve. Therefore, to achieve both computational tractability and exactness of the final solution, we propose a step-by-step solution approach consisting of the following four steps. The procedure of the proposed method is described below.

- Step 1 To avoid the computational intractability caused by non-convexity, the linearized AC power flow equations (LinDistFlow) (8) are introduced. This reformulates the min-max-min problem (13) into the min-max problem (19).
- Step 2 The C&CG algorithm is applied to solve the min-max problem (19) formulated in Step 1. This efficiently identifies the worst-case scenario set $\hat{\mathcal{U}}^*$ of the available PV generation power that maximizes the operational cost.
- Step 3 Under the worst-case scenario set $\hat{\mathcal{U}}^*$ identified in Step 2, the battery charging and discharging schedules are determined using a more accurate network model. Specifically, the minimization problem (20) containing the second-order cone (SOC)-relaxed DistFlow Equation (7) is solved to determine the robust battery schedule $\mathbf{y}^*$.
- Step 4 To eliminate errors associated with the approximations and relaxations and to obtain an optimal solution that satisfies the network constraints, the DER control problem (21) including the non-convex DistFlow Equation (5) is solved under the battery schedule $\mathbf{y}^*$ determined in Step 3. This derives the practically feasible robust solution $(\mathbf{y}^*, \mathbf{x}^*)$.

The overall procedure of the proposed method is summarized in Algorithm 1. The following section details the processes in Step 1, 2, 3, and 4.

4.2. Step 1

Let $\mathcal{X}_{\text{Lin}}^{2\text{nd}}(\mathbf{y}, \hat{\mathbf{u}})$ be the feasible region of the second-stage problem using the linearized AC power flow Equation (5). The linearized problem of the robust optimization problem (13) is defined by Equation (14):

$$\underset{\mathbf{y} \in \mathcal{Y}^{1\text{st}}}{\text{minimize}} \left\{ \underset{\hat{\mathbf{u}} \in \mathcal{U}}{\text{maximize}} \left(\underset{\mathbf{x} \in \mathcal{X}_{\text{Lin}}^{2\text{nd}}(\mathbf{y}, \hat{\mathbf{u}})}{\text{minimize}} C^{\text{total}'} \right) \right\}, \quad (14a)$$

where

$$\mathcal{X}_{\text{Lin}}^{2\text{nd}}(\mathbf{y}, \hat{\mathbf{u}}) = \left\{ P_{i,t}^{\text{PV}}, Q_{i,t}^{\text{PV}}, P_{i,t}^{\text{fl}+}, P_{i,t}^{\text{fl}-}, Q_{i,t}^{\text{fl}+}, Q_{i,t}^{\text{fl}-}, P_{ij,t}, Q_{ij,t}, v_{i,t}, P_t^{\text{imb}}, \forall i, j \in \mathcal{N}, \forall t \in \mathcal{T}, \mid \right. \\ \left. (2), (3), (4), (8), (9) \right\}. \quad (14b)$$

Algorithm 1 The procedure of the proposed robust optimization method.

Require: Predicted PV outputs, uncertainty confidence intervals, and network parameters.

Ensure: A practically feasible robust solution: $(\mathbf{x}^*, \mathbf{y}^*)$

// Step 1: Linearization and dualization

- 1: Linearize the innermost second-stage min problem using the LinDistFlow Equation (8).
- 2: Take the dual of the inner min problem to convert it into a max problem, merging it with the uncertainty max problem into a single max subproblem.
- 3: Reformulate the bilinear terms to formulate the max subproblem as a mixed-integer linear programming problem (18), resulting in an overall min-max structure.

// Step 2: Worst-case scenario identification

- 4: Solve the reformulated min-max problem (14) via the C&CG algorithm (Algorithm A1).

- 5: **return** The identified set of worst-case scenarios $\mathcal{U}^*$.

// Step 3: Battery scheduling via SOC relaxation

- 6: Under the identified worst-case scenarios, solve the relaxed problem (20) incorporating the SOC-relaxed DistFlow Equation (7).
- 7: **return** The robust day-ahead battery schedules $\mathbf{y}^*$.

// Step 4: Exact DER control recovery

- 8: Fix the battery schedules to $\mathbf{y}^*$ and assume the worst-case scenario.
 - 9: Solve the deterministic operational problem (21) with the original non-convex AC DistFlow equations (5) to eliminate any approximation/relaxation errors.
 - 10: **return** The exact DER control variables $\mathbf{x}^*$.
-

The objective function in Equation (14) is C^{total} because the current-related terms are absent due to the use of the linearized power flow Equation (8). To solve the min-max-min problem (14), it is first necessary to derive the dual problem of the min problem in the second-stage problem. Here, let the decision-variable vector $\mathbf{x}$ of the min problem in the second-stage problem be $\mathbf{x} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top$. $\mathbf{x}_1 \in \mathbb{R}_{\geq 0}^{n_{x1} \times 1}$ is a non-negative decision-variable vector, $\mathbf{x}_2 \in \mathbb{R}^{n_{x2} \times 1}$ is a free decision-variable vector, and $n_x = n_{x1} + n_{x2}$. Describing the linearized robust optimization problem (14) using linear notation yields Equation (15):

$$\underset{\mathbf{y} \in \mathcal{Y}^{\text{1st}}}{\text{minimize}} \left\{ \underset{\hat{\mathbf{u}} \in \mathcal{U}}{\text{maximize}} \left(\underset{\mathbf{x} \in \mathcal{X}_{\text{Lin}}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}})}{\text{minimize}} \quad \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 \right) \right\}, \quad (15a)$$

where

$$\mathcal{X}_{\text{Lin}}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}}) = \left\{ \mathbf{x} \mid \mathbf{F}_1 \mathbf{x}_1 + \mathbf{F}_2 \mathbf{x}_2 \geq \mathbf{d} - \mathbf{D}\mathbf{y} - \mathbf{E}\hat{\mathbf{u}}, \quad (: \boldsymbol{\xi}), \quad \mathbf{J}_1 \mathbf{x}_1 + \mathbf{J}_2 \mathbf{x}_2 = \mathbf{e} - \mathbf{G}\mathbf{y} - \mathbf{H}\hat{\mathbf{u}}, \quad (: \boldsymbol{\pi}) \right. \\ \left. \mathbf{x}_1 \geq \mathbf{0}, \mathbf{x}_2 : \text{free} \right\}. \quad (15b)$$

In Equation (15), $\mathbf{c}_1 \in \mathbb{R}^{n_{x1} \times 1}$ and $\mathbf{c}_2 \in \mathbb{R}^{n_{x2} \times 1}$ are constant vectors, and let $\mathbf{c} = (\mathbf{c}_1^\top, \mathbf{c}_2^\top)^\top$. In Equation (15b), $\mathbf{F}_1 \in \mathbb{R}^{m_i \times n_{x1}}$, $\mathbf{F}_2 \in \mathbb{R}^{m_i \times n_{x2}}$, $\mathbf{D} \in \mathbb{R}^{m_i \times n_y}$, $\mathbf{E} \in \mathbb{R}^{m_i \times n_u}$, $\mathbf{J}_1 \in \mathbb{R}^{m_e \times n_{x1}}$, $\mathbf{J}_2 \in \mathbb{R}^{m_e \times n_{x2}}$, $\mathbf{G} \in \mathbb{R}^{m_e \times n_y}$, and $\mathbf{H} \in \mathbb{R}^{m_e \times n_u}$ are coefficient matrices, while $\mathbf{d} \in \mathbb{R}^{m_i \times 1}$ and $\mathbf{e} \in \mathbb{R}^{m_e \times 1}$ are constant vectors. Additionally, $\mathbf{y} \in \mathbb{R}^{n_y \times 1}$ and $\hat{\mathbf{u}} \in \mathbb{R}_{\geq 0}^{n_u \times 1}$. $\boldsymbol{\xi} \in \mathbb{R}^{m_i \times 1}$ and $\boldsymbol{\pi} \in \mathbb{R}^{m_e \times 1}$ are the dual variable vectors corresponding to the inequality constraint $\mathbf{F}_1 \mathbf{x}_1 + \mathbf{F}_2 \mathbf{x}_2 \geq \mathbf{d} - \mathbf{D}\mathbf{y} - \mathbf{E}\hat{\mathbf{u}}$ and the equality constraint $\mathbf{J}_1 \mathbf{x}_1 + \mathbf{J}_2 \mathbf{x}_2 = \mathbf{e} - \mathbf{G}\mathbf{y} - \mathbf{H}\hat{\mathbf{u}}$, respectively. The min problem of the second-stage problem, $\min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x}$, is a linear programming problem, and its optimal value is $S_P(\mathbf{y}, \hat{\mathbf{u}})$ in Equation (16a). The dual problem of the right-hand side of Equation (16a) corresponds to the right-hand side of Equation (16b), with its optimal value being $S_D(\mathbf{y}, \hat{\mathbf{u}})$. If both the primal and dual problems have feasible solutions, Equation (16c) holds by the strong duality theorem. The original max-min problem is transformed into a max problem using the dual problem and is then solved.

$$S_P(\mathbf{y}, \hat{\mathbf{u}}) = \min_{\mathbf{x}} \{ \mathbf{c}^\top \mathbf{x} \mid \mathbf{x} \in \mathcal{X}_{\text{Lin}}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}}), \text{ (15b)} \}, \quad (16a)$$

$$S_D(\mathbf{y}, \hat{\mathbf{u}}) = \max_{\xi, \pi} \{ (d - D\mathbf{y} - E\hat{\mathbf{u}})^\top \xi + (e - G\mathbf{y} - H\hat{\mathbf{u}})^\top \pi \mid \\ F_1^\top \xi + J_1^\top \pi \leq c_1, F_2^\top \xi + J_2^\top \pi = c_2, \xi \geq 0, \pi : \text{free}, \}, \quad (16b)$$

$$S_P(\mathbf{y}, \hat{\mathbf{u}}) = S_D(\mathbf{y}, \hat{\mathbf{u}}). \quad (16c)$$

Since the objective function $(d - D\mathbf{y} - E\hat{\mathbf{u}})^\top \xi + (e - G\mathbf{y} - H\hat{\mathbf{u}})^\top \pi$ in Equation (16b) contains bilinear terms, it is transformed into a MILP using the big-M method [21]. Since $H = \mathbf{0}$ (see Equation (2) in this paper and (8)), we have $(e - G\mathbf{y} - H\hat{\mathbf{u}})^\top = (e - G\mathbf{y})^\top$. Furthermore, by utilizing the coefficient matrices $\mathbf{U}^+, \mathbf{U}^- \in \mathbb{R}^{n_u \times n_z}$ and the n_z -dimensional vectors $\mathbf{z}^+$ and $\mathbf{z}^-$, we obtain $\hat{\mathbf{u}} = \tilde{\mathbf{u}} + \mathbf{U}^+ \mathbf{z}^+ + \mathbf{U}^- \mathbf{z}^- \in \mathcal{U}$ from Equation (2a). However, in the optimal solution of the linearized problem, $z_i^+, z_i^- \in \{0, 1\}$ holds. Therefore, the dual problem (16b) can be reformulated as Equation (17).

$$R(\mathbf{y}) = \max_{\xi, \pi, \Phi^+, \Phi^-, \mathbf{z}'^+, \mathbf{z}'^-} (d - D\mathbf{y} - E\hat{\mathbf{u}})^\top \xi - (EU^+ \mathbf{1})^\top \Phi^+ - (EU^- \mathbf{1})^\top \Phi^- + (e - G\mathbf{y})^\top \pi, \quad (17a)$$

$$\text{subject to } F_1^\top \xi + J_1^\top \pi \leq c_1, \quad (17b)$$

$$F_2^\top \xi + J_2^\top \pi = c_2, \quad (17c)$$

$$\xi \geq 0, \pi : \text{free}, \quad (17d)$$

$$0 \leq \Phi^+ \leq M \circ \mathbf{z}'^+, \quad (17e)$$

$$\xi - M \circ (\mathbf{1} - \mathbf{z}'^+) \leq \Phi^+ \leq \xi, \quad (17f)$$

$$0 \leq \Phi^- \leq M \circ \mathbf{z}'^-, \quad (17g)$$

$$\xi - M \circ (\mathbf{1} - \mathbf{z}'^-) \leq \Phi^- \leq \xi, \quad (17h)$$

$$\mathbf{z}'^+ \in \{0, 1\}^{m_i \times 1}, \mathbf{z}'^- \in \{0, 1\}^{m_i \times 1}, \quad (17i)$$

$$\mathbf{z}'^+ + \mathbf{z}'^- \leq \mathbf{1}, \quad (17j)$$

$$(\mathbf{z}'^+)^\top \mathbf{1} + (\mathbf{z}'^-)^\top \mathbf{1} \leq \Gamma, \quad (17k)$$

where $\Phi^+, \Phi^- \in \mathbb{R}^{m_i \times 1}$ are auxiliary variable vectors, $\mathbf{M} = (M_1, M_2, \dots, M_{m_i})^\top$ is a vector composed of sufficiently large positive constants, and $\mathbf{z}'^+, \mathbf{z}'^- \in \{0, 1\}^{m_i}$ are m_i -dimensional binary variable vectors, respectively. The symbol “ $\circ$ ” denotes the Hadamard product (element-wise product) of vectors. In addition, $\mathbf{1} \in \mathbb{R}^{n_z \times 1}$ is a vector with all elements equal to 1. The reason the values of the elements M_i ($i = 1, 2, \dots, m_i$) of the constant vector $\mathbf{M}$ are set individually rather than uniformly is that when the optimization solver applies the branch-and-bound method, an unnecessarily large M_i (big-M parameter) deteriorates the tightness of the linear relaxation problem, leading to numerical instability and a significant decrease in search efficiency. Therefore, M_i is set according to the upper bound of each auxiliary variable in Equation (17). Furthermore, to reduce the number of constraint equations involving the constant vector $\mathbf{M}$ and to improve computational efficiency and stability, new variables $\xi^+, \xi^- \in \mathbb{R}^{m_i \times 1}$ are introduced, and the problem is reformulated as Equation (18):

$$R(\mathbf{y}) = \max_{\substack{\xi, \xi^+, \xi^-, \pi, \\ \mathbf{z}^{'+}, \mathbf{z}'^-}} (d - D\mathbf{y} - E\hat{\mathbf{u}})^\top \xi - (EU^+ \mathbf{1})^\top \xi^+ - (EU^- \mathbf{1})^\top \xi^- + (e - G\mathbf{y})^\top \pi \quad (18a)$$

$$\text{subject to } (17b), (17c), (17d), (17j), (17i), (17k), \quad (18b)$$

$$0 \leq \xi^+ \leq M \circ \mathbf{z}^{'+}, \quad (18c)$$

$$0 \leq \xi^- \leq M \circ \mathbf{z}'^-, \quad (18d)$$

$$\xi \geq \xi^+ + \xi^-, \quad (18e)$$

$$\xi - \xi^+ - \xi^- \leq M \circ (\mathbf{1} - \mathbf{z}^{'+} - \mathbf{z}'^-). \quad (18f)$$

It can be observed from (17) and (18) that (18) has a smaller number of constraints involving the vector M . The proof of this reformulation is provided in Appendix D. Based on the above, Equation (15) can be reformulated as Equation (19):

$$\underset{\mathbf{y} \in \mathcal{Y}^{1st}}{\text{minimize}} \quad \underset{\substack{\xi, \xi^+, \xi^-, \pi, \\ \mathbf{z}^{'+}, \mathbf{z}'^-}}{\text{maximize}} \quad R(\mathbf{y}) \quad (19a)$$

$$\text{subject to } (18a), (18b), (18c), (18d), (18e), (18f), \quad (19b)$$

where Equation (19) is a min-max problem in which the second-stage problem is a MILP.

4.3. Step 2

In Step 2, problem (19) is solved by the C&CG method [19] to obtain the worst-case scenario set $\mathcal{U}^*$. Details of the C&CG method used for the solution are provided in Appendix E. In the C&CG method, solving the subproblem at the m -th iteration yields a new scenario (corresponding to $\mathbf{u}^{*(m+1)}$ in Algorithm A1). Therefore, the set of these scenarios is defined as $\hat{\mathcal{U}}^* = \{\mathbf{u}^{*(1)}, \mathbf{u}^{*(2)}, \dots\}$. Furthermore, the worst-case scenario obtained upon convergence of the C&CG method is denoted as $\mathbf{u}^*$.

4.4. Step 3

In Step 3, under the scenario set $\hat{\mathcal{U}}^*$ obtained in Step 2, the battery charge/discharge schedule $\mathbf{y}^*$ is determined using the SOC-relaxed DistFlow Equation (7). This optimization problem is defined by Equation (20):

$$\underset{\mathbf{y} \in \mathcal{Y}^{1st}, \mathbf{x} \in \mathcal{X}_{\text{Rel}}^{2nd}(\mathbf{y}, \hat{\mathbf{u}})}{\text{minimize}} \quad C^{\text{total}} \quad (20a)$$

where

$$\mathcal{X}_{\text{Rel}}^{2nd}(\mathbf{y}, \hat{\mathbf{u}}) = \left\{ P_{i,t}^{\text{pv}}, Q_{i,t}^{\text{pv}}, P_{i,t}^{\text{fl}+}, P_{i,t}^{\text{fl}-}, Q_{i,t}^{\text{fl}+}, Q_{i,t}^{\text{fl}-}, P_{ij,t}, Q_{ij,t}, v_{i,t}, l_{ij,t}, P_t^{\text{imb}}, \mid \right. \\ \left. \forall i, j \in \mathcal{N}, \forall t \in \mathcal{T}, \hat{\mathbf{u}} \in \hat{\mathcal{U}}^*, (2), (4), (7), (9) \right\}. \quad (20b)$$

Since Equation (20) includes terms related to the current $l_{ij,t}$ as well as the current constraints (5i), the accuracy is improved compared to the optimal solution obtained in Step 2. Moreover, because the current is taken into account, the objective function becomes C^{total} . Equation (20) can be solved efficiently by optimization solvers capable of handling second-order cone constraints.

4.5. Step 4

Let $\mathbf{y}^*$ be the robust optimal solution obtained in Step 3. Additionally, the feasible set $\mathcal{X}_{\text{Exa}}^{2nd}(\mathbf{y}, \hat{\mathbf{u}})$ is defined using the equality constraints (5f). Using the worst-case scenario $\mathbf{u}^*$

obtained in Step 2, the DER control problem (21) is solved without applying the convex relaxation of the equality constraints (5f).

$$\underset{x \in \mathcal{X}_{\text{Exa}}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}})}{\text{minimize}} \quad C^{\text{total}}, \quad (21a)$$

where

$$\mathcal{X}_{\text{Exa}}^{\text{2nd}}(\mathbf{y}, \hat{\mathbf{u}}) = \left\{ P_{i,t}^{\text{pv}}, Q_{i,t}^{\text{pv}}, P_{i,t}^{\text{fl}+}, P_{i,t}^{\text{fl}-}, Q_{i,t}^{\text{fl}+}, Q_{i,t}^{\text{fl}-}, P_{ij,t}, Q_{ij,t}, v_{i,t}, l_{ij,t}, P_t^{\text{imb}}, \mid \right. \\ \left. \forall i, j \in \mathcal{N}, \forall t \in \mathcal{T}, \hat{\mathbf{u}} = \hat{\mathbf{u}}^*, \mathbf{y} = \mathbf{y}^*, (2), (4), (5), (9) \right\}. \quad (21b)$$

Although (21) includes the non-convex DistFlow Equation (5f), where the constraint $P_{ij,t}^2 + Q_{ij,t}^2 = v_{i,t} l_{ij,t}$ is modeled directly as a quadratic equality, the resulting problem can be formulated as a non-convex quadratically constrained program. Such problems can be handled by the Gurobi Optimizer, which applies a spatial branch-and-bound algorithm to compute a globally optimal solution up to a specified optimality tolerance.

It should be noted that the “exactness” achieved in this step refers to the operational solution and the AC power flow feasibility under the specific worst-case scenario identified in Step 2. Because the worst-case scenario is identified using the LinDistFlow approximation, it may not perfectly align with the absolute worst-case scenario of the original non-convex robust problem. Therefore, Step 4 functions as a rigorous post-processing recovery stage to eliminate approximation errors and obtain practically feasible robust solution for the identified critical scenarios, rather than theoretically proving the global optimal solution of the entire min-max-min robust optimization framework.

5. Case Study

5.1. Conditions

The effectiveness of the proposed method is verified using a modified IEEE 33-bus distribution system model [28] incorporating PVs and batteries. The optimization horizon is 24 h with a 30-min resolution, and the set of time steps is defined as $\mathcal{T} = \{1, 2, \dots, 48\}$ (i.e., $t = 1$ corresponds to [0:00, 0:29], and $t = 48$ corresponds to [23:30, 23:59]). The acceptable range for node voltages is set to [0.90, 1.10] p.u., and the maximum allowable line current is uniformly set to 250 A to simulate strict network constraints against reverse power flows caused by the massive integration of PVs. Furthermore, the base capacity for the per-unit (p.u.) system is set to $S_{\text{base}} = 1$ MVA. Other detailed parameters of the network model are provided in Appendix A.

Figure 2a shows the time variations of the total base active power demand $P_{i,t}^{\text{db}}$ and base reactive power demand $Q_{i,t}^{\text{db}}$ across the entire distribution network. Additionally, Figure 2b illustrates the predicted available PV generation $\tilde{P}_{i,t}^{\text{PV}}$ at each node (assuming clear weather conditions) and its total sum, while Figure 2c presents the upper and lower bounds of the confidence interval for the predicted values, $\bar{P}_{i,t}^{\text{PV}}$ and $\underline{P}_{i,t}^{\text{PV}}$ (p.u. values normalized by the rated capacity). The thin gray lines in Figure 2b represent the outputs of individual PVs (refer to Appendix A for connection locations and capacities), and the thick black line represents the total value. Note that the confidence intervals shown in Figure 2c were calculated using a data-driven approach [25].

Regarding the flexibility requirement P_t^{req} from the external network, three cases shown in Table 1 are assumed. We assume that flexibility requests primarily occur during the following two time periods. The first is during the daytime ($t = 20, 21, \dots, 26$), when the PV output peaks, requiring downward flexibility (an apparent load increase) to absorb surplus power. The second is during the evening ($t = 36, 37, \dots, 43$, see Figure 2a) when the PV output drops sharply, and demand increases rapidly (the duck curve phenomenon),

requiring upward flexibility (corresponding to an apparent generation increase) to supplement the power supply. Based on these considerations, Case 1 simulates a severe situation in which flexibility is required both during the day and in the evening. Case 2 sets a scenario in which downward flexibility is required during the daytime, and Case 3 sets a scenario in which upward flexibility is required in the evening.

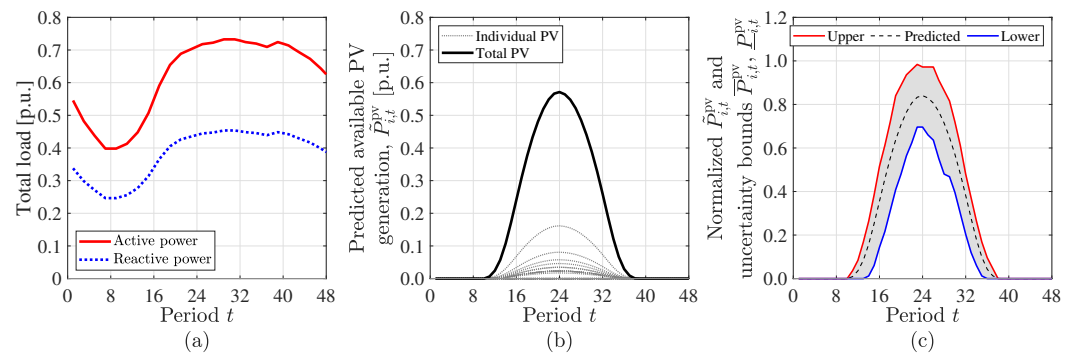


Figure 2. (a) Total active ($P_{i,t}^{db}$) and reactive ($Q_{i,t}^{db}$) power demands aggregated over all nodes. (b) Individual and aggregated predicted PV generation profiles, $\hat{P}_{i,t}^{PV}$. (c) Normalized predicted PV generation $\hat{P}_{i,t}^{PV}$ along with its uncertainty bounds $\bar{P}_{i,t}^{PV}$ and $\underline{P}_{i,t}^{PV}$.

Table 1. The flexibility requirement P_t^{req} across different time periods for three operational cases. Positive values indicate a requirement for downward flexibility (corresponding to apparent load increase), whereas negative values indicate a requirement for upward flexibility (corresponding to apparent generation increase).

Period t	P_t^{req} [p.u.]			Time of Day	Time
	Case 1	Case 2	Case 3		
1, 2, ..., 19	0.0	0.0	0.0	Night/Morning	00:00–09:30
20, 21, ..., 26	0.1	0.1	0.0	Daytime	09:30–13:00
27, 28, ..., 35	0.0	0.0	0.0	Afternoon	13:00–17:30
36, 37, ..., 43	−0.1	0.0	−0.1	Evening	17:30–21:30
44, 45, ..., 48	0.0	0.0	0.0	Night	21:30–24:00

All computations in the case study were implemented in MATLAB R2024b, with AMPL as the modeling language and the Gurobi Optimizer (version 13.0.1) as the optimization solver. The hardware specifications of the execution environment are as follows: OS: Windows 10 Pro; CPU: AMD Ryzen 9 5950X (3.40 GHz); and RAM: 64.0 GB. The convergence tolerance ϵ_{cog} for the C&CG algorithm in Step 2 is set to 0.05%, and the relative MIP gap for the optimization solver solving the subproblem, which is the mixed-integer linear programming problem (17a)–(17k), is set to 10^{-4} .

5.2. DER Operation in the Worst Scenario

Figure 3 shows the flexibility procured from the DERs using the proposed method, and Figure 4 shows the total charge and discharge power of the batteries, $P_{i,t}^{ch}$ and $P_{i,t}^{dch}$. Additionally, Figure 5 illustrates the total PV generation $P_{i,t}^{PV}$ under the worst-case scenario u^* . The target requirement P_t^{req} is superimposed in Figure 3. Due to space limitations, Figure 3 only presents the results for $\Gamma = 1$ and 6. Similarly, Figures 4 and 5 only display the results for $\Gamma = 1$ and 6. In Figures 3 and 4, subfigures (a) and (b) represent the results for Case 1, (c) and (d) for Case 2, and (e) and (f) for Case 3. In both Figures 3 and 4, the positive direction of the vertical axis indicates downward flexibility (corresponding to an apparent load increase), whereas the negative direction indicates upward flexibility (corresponding to an apparent generation increase).

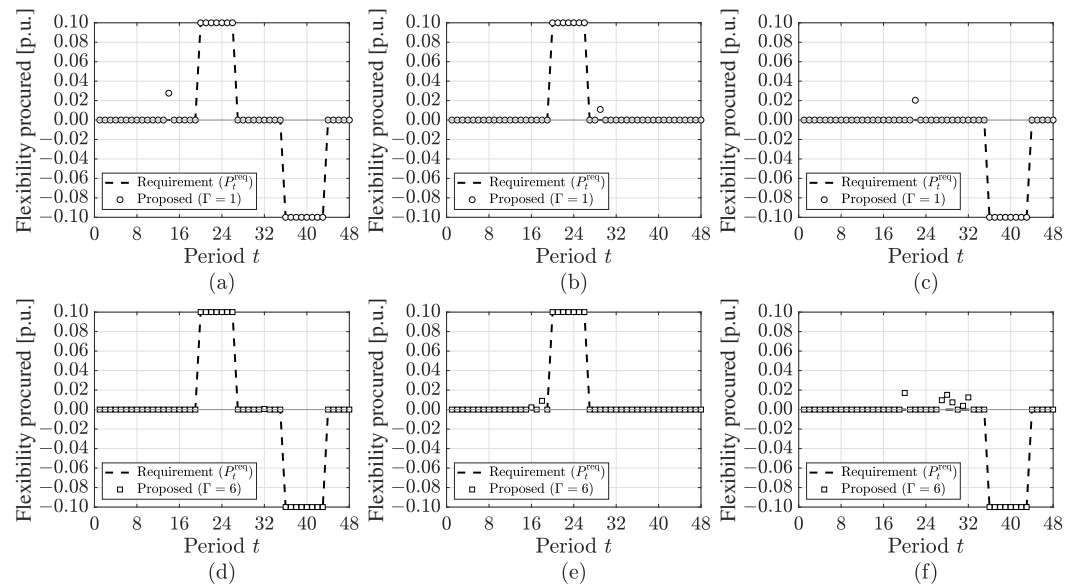


Figure 3. Procured flexibility compared to the requirement (P_t^{req}) under different robustness levels. The rows show the results of the proposed method with different uncertainty budgets: (a–c) $\Gamma = 1$ and (d–f) $\Gamma = 6$. The columns correspond to: (a,d) Case 1; (b,e) Case 2; and (c,f) Case 3.

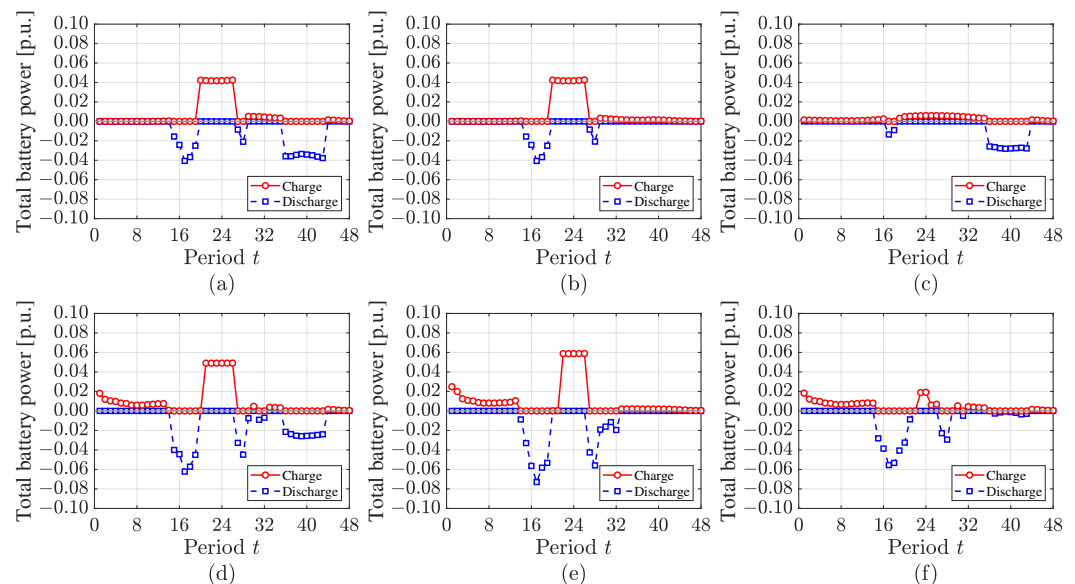


Figure 4. Total charge and discharge power of all batteries, $p_{i,t}^{\text{ch}}$ and $p_{i,t}^{\text{dch}}$. The rows show the results of the proposed method with different uncertainty budgets: (a–c) $\Gamma = 1$ and (d–f) $\Gamma = 6$. The columns correspond to: (a,d) Case 1; (b,e) Case 2; and (c,f) Case 3. Positive values indicate charging, while negative values indicate discharging.

As seen in Figure 3, the procured flexibility matches the requirement P_t^{req} during most time periods in all cases. Although imbalances P_t^{imb} between the requirement and the procured amount exist in some periods, this is to accommodate the uncertainty in available PV generation. Furthermore, the reason the imbalance occurrence is exclusively upward in all cases—meaning it is slightly larger in the direction of load increase compared to the required amount P_t^{req} —is that the worst-case scenario obtained by solving the robust optimization problem involves a “sudden drop in available PV generation from the predicted value”, as shown in Figure 5. In other words, the sudden decrease in available PV generation in the worst-case scenario results in the actual generated power falling below the flexibility requirement. Indeed, the periods where imbalances occur in Figure 3 coincide with the periods where PV generation sharply declines in Figure 5.

The batteries are charged and discharged in preparation for this worst-case scenario. In all cases, the battery scheduling strategy is to “discharge before the daytime when PV generation increases to lower the SOE, and then charge during the daytime to absorb the surplus power.” In Figure 4e, discharging is not performed at $t = 22$ when the available PV generation drops abruptly; this is because discharging must be reserved for the evening ($t = 36, 37, \dots, 43$), when the flexibility requirement becomes significantly larger, to satisfy the requirement. In Figure 4f, although discharging is not always performed during the period of a sharp decline in available PV generation, the activation of FLs instead satisfies the flexibility requirement.

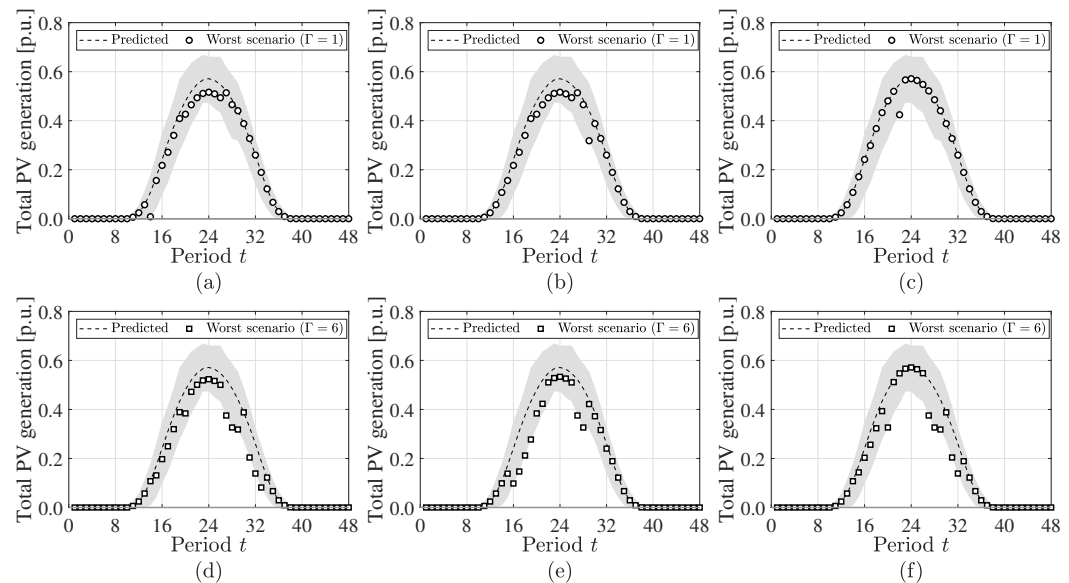


Figure 5. Total PV generation profiles with uncertainty bounds. Dashed lines show the predicted total PV generation $\sum_{i \in \mathcal{N}} \bar{p}_{i,t}^{PV}$, while shaded regions indicate the uncertainty bounds. Markers represent worst-case scenarios identified by the robust optimization for $\Gamma = 1$ and $\Gamma = 6$. (a–c) $\Gamma = 1$ for Cases 1–3; (d–f) $\Gamma = 6$ for Cases 1–3.

5.3. Monte Carlo Simulation

Multiple scenarios of available PV generation are generated to evaluate the robustness of the optimal solution $\mathbf{y}^*$ obtained by the proposed method via Monte Carlo simulation. Let the generated scenarios be denoted by $\hat{\mathbf{u}}^{\text{Monte}}$. In the Monte Carlo simulation, Equation (22) is solved N^{sce} times under the robust optimal solution $\mathbf{y}^*$.

$$\underset{\mathbf{x} \in \mathcal{X}_{\text{Monte}}^{2\text{nd}}(\mathbf{y}, \hat{\mathbf{u}})}{\text{minimize}} \quad C^{\text{total}} \quad (22a)$$

where

$$\mathcal{X}_{\text{Monte}}^{2\text{nd}}(\mathbf{y}, \hat{\mathbf{u}}) = \left\{ p_{i,t}^{PV}, Q_{i,t}^{PV}, p_{i,t}^{\text{fl}+}, p_{i,t}^{\text{fl}-}, Q_{i,t}^{\text{fl}+}, Q_{i,t}^{\text{fl}-}, P_{ij,t}, Q_{ij,t}, v_{i,t}, l_{ij,t}, p_t^{\text{imb}}, \forall t \in \mathcal{N}, \forall i \in \mathcal{T}, \right. \\ \left. \hat{\mathbf{u}} = \hat{\mathbf{u}}^{\text{Monte}}, \mathbf{y} = \mathbf{y}^*, (2), (4), (5), (9) \right\}. \quad (22b)$$

Although solving Equation (22) is not as computationally demanding as Equation (14), it still entails a substantial computational burden even with state-of-the-art optimization solvers because it includes the non-convex AC power flow Equation (5). From the perspective of computational efficiency, we set $N^{\text{sce}} = 200$ in this paper.

The distribution of the generated PV generation scenarios is shown in Figure 6. Each gray line in Figure 6a represents the PV output of an individual scenario (normalized with respect to the rated capacity). Additionally, Figure 6b illustrates the deviation rate of the

scenarios exceeding the uncertainty bounds (upper and lower limits) at each time step, where “Upper violation” and “Lower violation” correspond to the violation rates from the upper and lower bounds, respectively. As can be seen from Figure 6b, the available PV generation scenarios used in the Monte Carlo simulation are not strictly confined within $[\underline{P}_{i,t}^{pv}, \bar{P}_{i,t}^{pv}]$; instead, data with a maximum violation of a few percent at each time step is utilized. This is because if the uncertainty bounds are widened excessively to enclose extreme outliers with very low probabilities of occurrence completely, the solution obtained via robust optimization becomes overly conservative. Conversely, if the bounds are set too narrow, robustness control using the uncertainty budget does not function effectively. Therefore, the Monte Carlo simulation employs data that encompasses the vast majority of scenarios while allowing violations of up to a few percent. As a benchmark for the proposed method, we consider an expected value model in which the available PV generation $\hat{P}_{i,t}^{pv}$ is assumed to be equal to its forecasted value $\bar{P}_{i,t}^{pv}$ (i.e., $\hat{P}_{i,t}^{pv} = \bar{P}_{i,t}^{pv}$), and we compare the two methods.

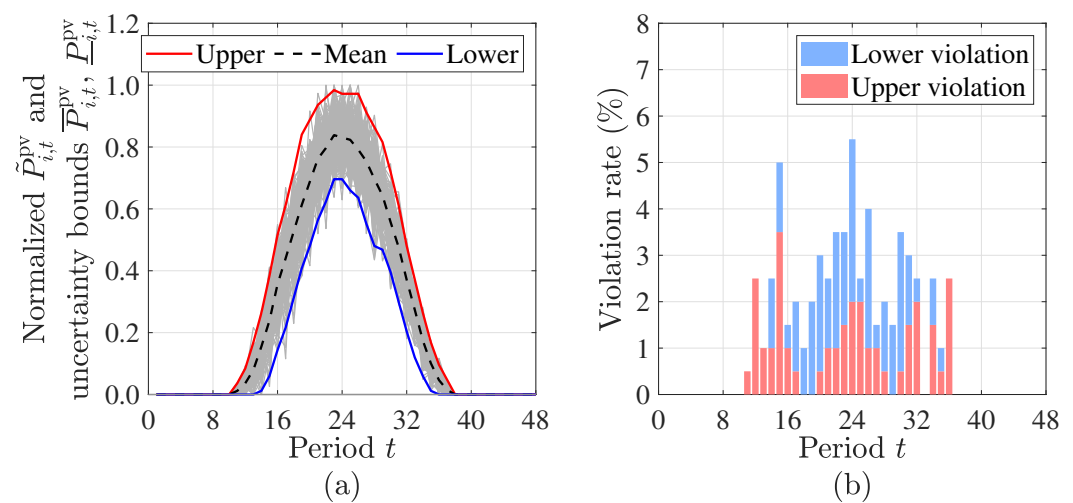


Figure 6. Monte Carlo simulations of PV output profiles and uncertainty-bound violations. (a) Generated scenarios (gray), their sample mean (black dashed), and predefined upper and lower bounds (red and blue). (b) Rates of scenarios violating the upper and lower bounds.

Tables 2 and 3 present the Monte Carlo simulation results when the uncertainty budget Γ is varied as $\Gamma = 1, 2, 3, 4, 5, 6$. Table 2 shows the average values of C^{total} , C^{penal} , C^{bt} , C^{pv} , and C^{fl} across the 200 scenarios, whereas Table 3 shows the conditional value at risk (CVaR) and the maximum and minimum values of C^{total} and C^{penal} over the 200 scenarios. The confidence level α for the CVaR is set to $\alpha = 0.05$. Values in parentheses indicate the relative ratio normalized to the expected model. Note that feasible solutions were obtained in all trials. In all cases, (a), (b), and (c) of Table 2, the average imbalance cost C^{penal} was minimized by the proposed method when $\Gamma = 6$. Although the trends in the average values of the total cost C^{total} and the battery cost C^{bt} differ across Cases 1, 2, and 3, the imbalance cost C^{penal} is consistently smaller for the proposed method regardless of the value of Γ . Compared to the expected value model, the imbalance cost C^{penal} of the proposed method is reduced by 44.6–76.4% in Case 1, 52.1–84.3% in Case 2, and 11.6–60.5% in Case 3, confirming that a more robust battery schedule is achieved by the proposed method than with the expected value model. In every case, $\Gamma = 6$ results in the most significant reduction in the imbalance cost C^{penal} . Since increasing Γ generally increases C^{bt} while simultaneously reducing C^{penal} , the robustness can be effectively controlled by adjusting the value of Γ .

Table 2. Average cost evaluated over 200 Monte Carlo simulation scenarios. The table breaks down the average total cost into imbalance penalties, battery usage/degradation, PV curtailment, and flexible load (FL) operation costs. The proposed method is compared against the expected value model (“Expected”). Values in parentheses indicate the relative ratio normalized to the “Expected” model. The gray-shaded and bold values highlight the minimum (best) average cost in each column.

Method	Γ	Average Cost [JPY]				
		Total C^{total} [$\times 10^5$]	Imbalance C^{penal} [$\times 10^3$]	Battery C^{bt} [$\times 10^3$]	PV C^{pv} [$\times 10^4$]	FL C^{fl} [$\times 10^4$]
(a) Case 1						
Expected	–	1.28 (1.000)	4.66 (1.000)	4.75 (1.000)	3.31 (1.000)	5.09 (1.000)
	1	1.25 (0.977)	2.20 (0.472)	8.57 (1.804)	3.10 (0.937)	4.89 (0.961)
	2	1.26 (0.984)	2.58 (0.554)	8.64 (1.819)	3.13 (0.946)	4.92 (0.967)
Proposed	3	1.26 (0.984)	1.96 (0.421)	8.82 (1.857)	3.16 (0.955)	4.99 (0.980)
	4	1.31 (1.023)	1.85 (0.397)	9.40 (1.979)	3.35 (1.012)	5.28 (1.037)
	5	1.35 (1.055)	1.61 (0.345)	9.85 (2.074)	3.49 (1.054)	5.50 (1.081)
	6	1.39 (1.086)	1.10 (0.236)	10.62 (2.236)	3.61 (1.091)	5.71 (1.122)
(b) Case 2						
Expected	–	0.99 (1.000)	4.66 (1.000)	1.61 (1.000)	3.35 (1.000)	2.19 (1.000)
	1	0.96 (0.970)	2.15 (0.461)	5.40 (3.354)	3.17 (0.946)	1.98 (0.904)
	2	0.97 (0.980)	2.23 (0.479)	5.33 (3.311)	3.17 (0.946)	2.02 (0.922)
Proposed	3	0.98 (0.990)	1.89 (0.406)	5.92 (3.677)	3.23 (0.964)	2.05 (0.936)
	4	1.02 (1.030)	1.45 (0.311)	7.38 (4.584)	3.48 (1.039)	2.20 (1.005)
	5	1.09 (1.101)	0.85 (0.182)	9.28 (5.764)	3.79 (1.131)	2.46 (1.123)
	6	1.11 (1.121)	0.73 (0.157)	9.98 (6.199)	3.98 (1.188)	2.34 (1.068)
(c) Case 3						
Expected	–	1.22 (1.000)	6.64 (1.000)	3.99 (1.000)	1.38 (1.000)	6.50 (1.000)
	1	1.23 (1.008)	5.87 (0.884)	3.81 (0.955)	1.45 (1.051)	6.62 (1.018)
	2	1.23 (1.008)	5.65 (0.851)	3.65 (0.915)	1.48 (1.072)	6.67 (1.026)
Proposed	3	1.26 (1.033)	4.58 (0.690)	3.42 (0.857)	1.63 (1.181)	6.95 (1.069)
	4	1.33 (1.090)	3.38 (0.509)	3.25 (0.815)	1.91 (1.384)	7.44 (1.145)
	5	1.40 (1.148)	2.79 (0.420)	5.53 (1.386)	2.13 (1.543)	7.81 (1.202)
	6	1.43 (1.172)	2.62 (0.395)	5.78 (1.449)	2.23 (1.616)	7.96 (1.225)

As shown in Table 3, across Cases 1, 2, and 3, the proposed method minimizes the CVaR and maximum values of C^{total} and C^{penal} when Γ is set appropriately. In Cases 1 and 3, the expected value model may yield a smaller maximum value of C^{total} depending on Γ ; however, this is likely due to the relatively small number of scenarios N^{sce} . Regarding the CVaR in Cases 1 and 2, the total cost C^{total} of the proposed method is consistently lower than that of the expected value model, irrespective of the value of Γ . Even in Case 3, an appropriate choice of Γ ensures that the total cost C^{total} is not worsened compared to the expected value model. Therefore, from the perspective of the tail risk of the total cost, the proposed method is capable of determining a risk-averse battery charge/discharge schedule. Furthermore, for the imbalance cost C^{penal} , the proposed method outperforms the expected value model in both the CVaR and maximum values regardless of Γ . Note that the minimum value of the imbalance cost C^{penal} is 0 in both models.

Figure 7 illustrates the cumulative probability distributions of the total cost C^{total} and the imbalance cost C^{penal} across Cases 1, 2, and 3, under the assumption that the scenarios used in the Monte Carlo simulation are equiprobable (i.e., $1/200$). To enhance readability, only the expected value model and the proposed method with $\Gamma = 1$ and 6 are plotted. The results for C^{total} are shown in Figure 7a–c, and those for C^{penal} are shown in Figure 7d–f, corresponding to Case 1 (a, d), Case 2 (b, e), and Case 3 (c, f), respectively. The cumulative probability distributions of the total cost C^{total} in Figure 7a–c demonstrate that a smaller Γ brings the distribution of the proposed method closer to that of the expected value model, whereas a larger Γ worsens the distribution of C^{total} for the proposed method. In contrast, as seen in the cumulative probability distributions of the imbalance cost C^{penal}

in Figure 7d–f, the proposed method yields a better distribution than the expected value model irrespective of Γ . In Case 3 when $\Gamma = 1$, the distributions of the two models appear similar, but the proposed method performs slightly better, as supported by Table 2c.

Table 3. Statistical analysis of the costs over the 200 Monte Carlo simulation scenarios. The imbalance cost C^{penal} represents the penalty incurred to satisfy the flexibility requirement. To robustly evaluate the risk profile, “CVaR” reports the Conditional Value-at-Risk at the 95% confidence level (i.e., the expected shortfall beyond the 95% Value-at-Risk threshold), while “max.” and “min.” denote the worst-case and best-case costs, respectively. As in Table 2, the proposed method is compared against the expected value model (“Expected”), with values in parentheses indicating the relative ratio. The gray-shaded and bold values highlight the minimum (best) value in each column.

Method	Γ	Total Cost C^{total} [JPY $\times 10^5$]			Imbalance Cost C^{penal} [JPY $\times 10^4$]		
		CVaR	Max.	Min.	CVaR	Max.	Min.
(a) Case 1							
Expected	–	1.66 (1.000)	1.74 (1.000)	1.06 (1.000)	2.83 (1.000)	4.29 (1.000)	0
	1	1.57 (0.946)	1.72 (0.989)	1.03 (0.972)	1.87 (0.661)	3.69 (0.860)	0
	2	1.58 (0.952)	1.74 (1.000)	1.01 (0.953)	1.97 (0.696)	3.67 (0.855)	0
Proposed	3	1.55 (0.934)	1.62 (0.931)	1.07 (1.009)	1.64 (0.580)	2.76 (0.643)	0
	4	1.60 (0.964)	1.68 (0.966)	1.09 (1.028)	1.60 (0.565)	2.60 (0.606)	0
	5	1.64 (0.988)	1.85 (1.063)	1.14 (1.075)	1.57 (0.555)	3.50 (0.816)	0
	6	1.64 (0.988)	1.69 (0.971)	1.19 (1.123)	1.22 (0.431)	1.85 (0.431)	0
(b) Case 2							
Expected	–	1.37 (1.000)	1.44 (1.000)	0.76 (1.000)	2.82 (1.000)	4.29 (1.000)	0
	1	1.28 (0.934)	1.42 (0.986)	0.74 (0.974)	1.85 (0.656)	3.69 (0.860)	0
	2	1.28 (0.934)	1.43 (0.993)	0.75 (0.987)	1.92 (0.681)	3.83 (0.893)	0
Proposed	3	1.27 (0.927)	1.38 (0.958)	0.74 (0.974)	1.67 (0.592)	3.23 (0.753)	0
	4	1.29 (0.942)	1.41 (0.979)	0.80 (1.053)	1.50 (0.532)	2.46 (0.573)	0
	5	1.32 (0.964)	1.37 (0.951)	0.87 (1.145)	1.06 (0.376)	1.71 (0.399)	0
	6	1.32 (0.964)	1.41 (0.979)	0.91 (1.197)	0.98 (0.348)	1.99 (0.464)	0
(c) Case 3							
Expected	–	1.61 (1.000)	1.88 (1.000)	0.92 (1.000)	3.41 (1.000)	6.35 (1.000)	0
	1	1.61 (1.000)	1.88 (1.000)	0.95 (1.033)	3.19 (0.935)	6.07 (0.956)	0
	2	1.61 (1.000)	1.88 (1.000)	0.96 (1.043)	3.12 (0.915)	6.02 (0.948)	0
Proposed	3	1.62 (1.006)	1.87 (0.995)	1.01 (1.098)	2.78 (0.815)	4.42 (0.696)	0
	4	1.66 (1.031)	1.87 (0.995)	1.11 (1.207)	2.32 (0.680)	4.48 (0.706)	0
	5	1.71 (1.062)	1.89 (1.005)	1.21 (1.315)	2.03 (0.595)	3.75 (0.591)	0
	6	1.72 (1.068)	1.89 (1.005)	1.23 (1.337)	1.93 (0.566)	3.54 (0.557)	0

As the above results indicate, the battery charge/discharge schedule determined by the proposed method exhibits superior robustness regarding flexibility procurement compared to the expected value model. By appropriately setting Γ , the proposed method can suppress the maximum total cost compared to the expected value model, achieving risk aversion, and, depending on the case, can also reduce the average total cost.

5.4. Computation Time

Table 4 presents the total computation time of the proposed method and its breakdown. In the table, “Total” indicates the overall computation time of the proposed method, “C&CG” denotes the computation time for Step 2, and “Other” represents the computation time for Steps 3 and 4. The values in parentheses in the “C&CG” row indicate the proportion of the total computation time spent solving the C&CG method. Across Cases 1, 2, and 3, the computation time for the C&CG method accounts for 61% to 94% of the total computation time, representing the primary computational bottleneck. Furthermore, as the

uncertainty parameter Γ increases, a general upward trend in the total computation time can be observed, although it is not strictly monotonic.

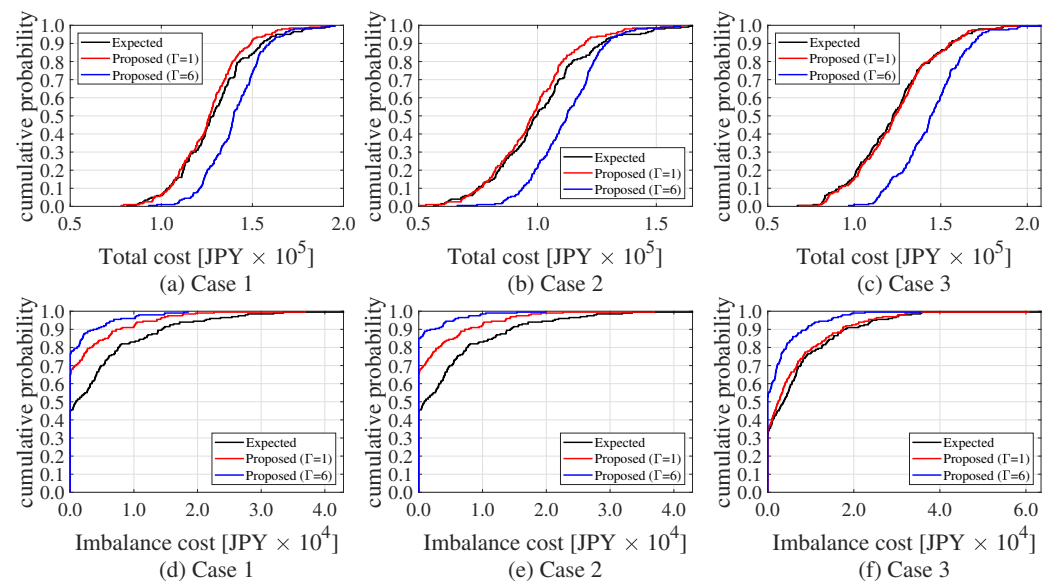


Figure 7. Cumulative probability distributions of the evaluated costs over 200 Monte Carlo simulation scenarios. The top row presents the total operational costs (a–c), while the bottom row shows the imbalance costs (d–f), which represent the penalty incurred to satisfy the flexibility requirement. The columns correspond to the results under Case 1 (a,d), Case 2 (b,e), and Case 3 (c,f) scenarios, respectively. These distributions illustrate the risk profiles and the robustness of the proposed method against uncertainties compared to the expected value model.

Next, we evaluate the computational efficiency of the proposed method using the linearized AC power flow Equation (8). We compare the computation time of the proposed method with that of a method that applies a polyhedral approximation (see Appendix C for details) to the SOC-relaxed power flow equations. Table 5 summarizes the computation times for each method when a time limit of 3600 s is set for solving the master problem and subproblems of the C&CG method. In the table, “TL” indicates that the computation was terminated due to exceeding the time limit, and the value in parentheses next to “TL” represents the optimality gap (%) between the upper and lower bounds of the objective function obtained by the C&CG method at the time of termination. Additionally, the third row for each case shows the ratio of the proposed method’s computation time to that of the polyhedral SOC method. As shown in Table 5, while the proposed method successfully solves all instances within 1650 s, the polyhedral SOC method exceeds the time limit in most cases. It is confirmed that the proposed method is at least 11.7 times faster (requiring at most 1/11.7 of the computation time) compared to the polyhedral SOC method. Although the proposed method introduces an approximation error by using the linearized AC power flow Equation (8) in Step 2, it can significantly reduce computation time. Ensuring scalability in response to the expansion of distribution systems is important, and the high practical utility of the proposed method for application to large-scale distribution systems has been demonstrated.

Table 4. Dominance of the C&CG procedure in the total computation time (in seconds). Percentages in parentheses indicate the proportion of time spent in the C&CG procedure relative to the total computation time.

Case	Uncertainty Budget Γ	Computation Time (s)		
		Total	C&CG	Other
1	1	143	101 (71%)	42
	2	232	169 (73%)	63
	3	234	167 (71%)	67
	4	353	266 (75%)	87
	5	445	380 (85%)	65
	6	302	252 (83%)	50
2	1	108	66 (61%)	42
	2	93	84 (90%)	9
	3	198	130 (66%)	68
	4	231	167 (72%)	64
	5	376	297 (79%)	79
	6	320	217 (68%)	103
3	1	121	82 (68%)	39
	2	146	107 (73%)	39
	3	242	197 (81%)	45
	4	290	220 (76%)	70
	5	919	860 (94%)	59
	6	1753	1650 (94%)	103

Table 5. Comparison of computation time between the proposed method and the baseline method (in seconds). The proposed method is based on the LinDistFlow equations, while the baseline (polyhedral SOC) uses a polyhedral approximation of the SOC-relaxed DistFlow equations. A time limit of 3600 s is imposed. For instances that hit the time limit, we report “TL” along with the final optimality gap in parentheses. The “Ratio” column shows the ratio of the proposed method’s computation time to the baseline method. Ratios are omitted (indicated by “-”) when the baseline hits the time limit. The gray-shaded and bold values highlight the minimum (best) computation time between the two methods.

Case	Uncertainty Budget Γ	Method		
		Proposed	Polyhedral SOC	Ratio
1	1	101	2226	1/22.0
	2	169	TL (0.23%)	-
	3	167	3437	1/20.6
	4	266	TL (2.88%)	-
	5	380	TL (9.00%)	-
	6	252	TL (9.76%)	-
2	1	66	2505	1/38.0
	2	84	TL (9.39%)	-
	3	130	TL (5.35%)	-
	4	167	TL (5.43%)	-
	5	297	TL (4.46%)	-
	6	217	TL (3.30%)	-
3	1	82	958	1/11.7
	2	107	1419	1/13.3
	3	197	TL (2.14%)	-
	4	220	TL (10.42%)	-
	5	860	TL (12.10%)	-
	6	1650	TL (14.28%)	-

5.5. AC Feasibility

To verify the necessity of the final step (Step 4: feasibility recovery via non-convex AC power flow) in the proposed architecture, we evaluated the maximum relaxation gap at the conclusion of Step 3 (SOCP relaxation). Under the severe conditions of Case 1 ($\Gamma = 4$) where reverse power flow occurs, a maximum relaxation gap (error in the equality constraints) of 4.45×10^{-1} p.u. was observed at the end of Step 3, meaning the physical AC power flow equations were not strictly satisfied. However, by directly solving the non-convex DistFlow model in Step 4, this error was completely eliminated down to the solver's numerical tolerance level of 0.27×10^{-7} p.u., successfully recovering a strictly AC-feasible solution. Furthermore, as shown in Table 4, the additional computation time required in Step 4 to recover this exactness (included in the "Other" category in Table 4) averaged between 9 and 103 s. This confirms that high physical exactness can be guaranteed with an exceedingly negligible computational cost in practical operations.

Table 6 shows the second-order cone (SOC) relaxation errors of the solutions immediately after Step 3 (without AC feasibility recovery) and immediately after Step 4 (with AC feasibility recovery). Due to space limitations, the worst values obtained across all time periods and all nodes are reported. The SOC error for each worst-case scenario identified by the C&CG method is calculated by $Gap_{rep} = \max_{t \in T, (i,j) \in \mathcal{N}} \{v_{i,t} l_{ij,t} - (P_{ij,t}^2 + Q_{ij,t}^2)\}$, where the subscript *rep* denotes the index of each worst-case scenario within the identified scenario set $\hat{\mathcal{U}}^*$. Table 6 shows the maximum error across all these scenarios, $\max_{rep \in \hat{\mathcal{U}}^*} Gap_{rep}$. From the table, it can be seen that the worst SOC error of the proposed method (after Step 4) is much smaller than that after Step 3.

Table 6. Second-order cone relaxation errors of the solutions obtained immediately after Step 3 (without AC feasibility recovery) and after Step 4 (with AC feasibility recovery). The values represent the worst-case errors across all time periods and nodes. The smaller error for each case and budget is highlighted in gray and bold.

Case	Uncertainty Budget Γ	Values	
		After Step3 [$\times 10^{-1}$]	After Step 4 [$\times 10^{-7}$]
1	1	2.51	8.93
	2	3.24	8.57
	3	4.16	0.27
	4	4.45	0.27
	5	3.07	0.27
	6	4.36	7.04
2	1	2.62	9.25
	2	2.65	9.82
	3	2.86	8.79
	4	3.96	8.00
	5	4.29	8.22
	6	4.40	6.71
3	1	1.12	4.10
	2	0.85	4.74
	3	1.60	4.74
	4	2.24	0.27
	5	2.85	2.40
	6	3.18	0.27

5.6. Discussions

5.6.1. Influence of the Uncertain Budget

As shown in Table 2, variations in the uncertainty budget Γ result in fluctuations ranging from a few percent to slightly under 20% in the total cost C^{total} , yet they exhibit extremely high sensitivity regarding its internal components: the imbalance cost C^{penal} and the battery operation cost C^{bt} . Specifically, increasing Γ leads to a substantial reduction in C^{penal} by several tens of percent; conversely, cases were also observed where C^{bt} increased by a maximum of 100% or more due to the higher charge and discharge frequency of the battery. Additionally, as indicated in Table 3, an increase in Γ significantly mitigates the CVaR (tail risk) of both C^{total} and C^{penal} in Cases 1 and 2. However, in Case 3, while the CVaR of C^{penal} is successfully reduced, the CVaR of C^{total} exhibits an increasing trend. This highlights that under certain flexibility requirement profiles, an excessive pursuit of robustness to avoid penalties can inadvertently worsen the worst-case total cost due to surging battery operation costs. Regarding computation time, as discussed in Section 5.4, the computational burden tends to grow as Γ increases. Based on the above, it can be concluded that the uncertainty budget Γ is a crucial design parameter for controlling the significant trade-off between “penalty avoidance (robustness)” and “battery operation cost (economic efficiency)” while accommodating the computational load.

5.6.2. Influence of the Reactive Power Control by Batteries

Failing to consider reactive power control by battery storage makes it difficult to satisfy voltage constraints, which in turn increases the voltage control costs associated with the grid-tied PV inverters and flexible loads. Therefore, accounting for reactive power control by battery storage is crucial for procuring more adaptable flexibility and achieving cost reductions. However, it should be noted that incorporating the capacity limits of grid-connected battery inverters requires linearization techniques, such as polyhedral approximation, to maintain a linear formulation. This consequently leads to an increase in both the computational burden and approximation errors.

5.6.3. TSO-DSO Coordination Scheme

In practical operations, each DSO executes this framework to verify whether it can secure the day-ahead requirement (P_t^{req}) while satisfying the network constraints of its own distribution system. Simultaneously, based on the aforementioned concept of the flexibility envelope, the DSO calculates the “unused flexibility potential that can be safely provided” and offers it to the TSO. The TSO then aggregates the available flexibility potential offered by multiple DSOs and utilizes it for continuous balancing.

5.7. Limitations

The practical boundaries of the proposed approach are defined by the following limitations:

- **Limitation on the Exactness of Robust Optimality:** The exact AC power flow feasibility achieved in the final post-processing step (Step 4) is guaranteed specifically for the worst-case scenario identified by the linearized model (LinDistFlow) in Step 2. Because the identified scenario may not perfectly align with the absolute worst-case scenario of the original highly complex non-convex problem, the proposed method ensures practical operational feasibility for the identified critical condition but does not theoretically prove the global optimal solution of the entire min-max-min robust optimization framework.
- **Limitation on Uncertainty Modeling:** The flexibility requirement (P_t^{req}), the imbalance penalty cost, and the uncertainty bounds for PV generation are treated as fixed,

predetermined parameters based on historical forecast errors. Developing dynamic uncertainty sets endogenously or modeling the complex stochastic behaviors of flexibility requirements and market prices is beyond the current scope of this work.

- **Limitation on Computational Scalability:** Although the proposed multi-step approach significantly reduces the computational burden compared to directly solving the non-convex min-max-min problem, the C&CG algorithm used in Step 2 involves solving a mixed-integer linear programming (MILP) subproblem. As demonstrated in the case study, the C&CG procedure accounts for the majority of the computation time. Consequently, as the network size or the uncertainty budget increases, the computation time can grow significantly, presenting a potential scalability limitation for extremely large, multi-feeder real-world distribution networks. Directly applying the two-stage robust optimization model proposed in this study to larger-scale systems (e.g., IEEE 69-bus or 123-bus systems) poses a significant challenge from the perspective of computational complexity. In particular, when multi-period time-coupling constraints (such as the State-of-Charge (SoC) of batteries) are included, the number of binary variables required to linearize the bilinear terms (using the big-M method) in the subproblem of the C&CG algorithm explodes exponentially as the spatial (number of nodes) and temporal scales increase. Consequently, solving this directly in distribution networks with over 100 nodes becomes practically intractable. Applying this approach to large-scale systems requires further spatial clustering techniques or an extension to distributed optimization frameworks using methods such as the Alternating Direction Method of Multipliers (ADMM), which remains a crucial topic for future research.
- **Limitation regarding the omission of reactive power control by batteries:** Since the proposed method does not consider the reactive power control by batteries, the batteries do not contribute to meeting the voltage allowable range constraints.
- **Limitations and Practical Implementations:** We identify two challenges regarding the assumptions of the proposed method in practical operations. First, this study assumes that the battery schedule is fixed at the day-ahead planning stage. Although updating the schedule in real-time is physically possible in actual operations, ad hoc modifications to battery operations, which involve time-coupling constraints, risk causing network constraint violations or a lack of flexibility in subsequent periods. Implementing a dynamic and real-time robust optimization framework to prevent this remains a future challenge from the perspective of computational burden. Second, we assume the simultaneous control of active and reactive power (P/Q) by flexible loads. While reactive power control utilizing the available capacity of inverter-interfaced devices is technically highly feasible, the lack of advanced control and communication infrastructure at the residential end-user level may pose a barrier to implementation for the time being.

6. Conclusions

This paper proposes an optimization method for battery charge/discharge scheduling that is robust to the uncertainty of PV power generation, aiming to procure flexibility from DERs in distribution systems with high PV penetration. The proposed method provides an optimization framework for controlling a cluster of DERs to meet flexibility requirements while satisfying network constraints in the distribution system under uncertainty. As a specific solution approach, we introduced a linearization approximation for the non-convex AC power flow equations and applied a multi-stage robust optimization method, thereby constructing a framework that guarantees practical operational AC feasibility under severe uncertainty while mitigating the computational burden. Furthermore, the conventional issue of robust optimization—namely, yielding overly conservative

operation plans due to extreme worst-case assumptions—is addressed by introducing an uncertainty budget (Γ) into the uncertainty set. This design allows system operators to flexibly manage the trade-off between robustness and economic efficiency according to the system’s risk tolerance. Case studies via Monte Carlo simulations demonstrated that, compared to a deterministic expected value model, the proposed method can significantly mitigate the risk of unfulfilled flexibility (i.e., imbalance costs) due to uncertain PV generation fluctuations. Simultaneously, in terms of computational time, a clear advantage over the SOC-relaxation approach using a polyhedral approximation was demonstrated, confirming its practicality for future applications in larger-scale distribution systems.

Future work includes integrating ΔkW flexibility—which requires faster response times (i.e., value as operating reserves and frequency regulation)—into the model. Considering ΔkW requirements for an immediate response to real-time generation fluctuations will enable the expansion of the proposed framework into a more comprehensive and practical DER control platform. While quantifying the “unused flexibility potential” using the flexibility envelope is conceptually feasible, fully integrating it into the optimization model proposed in this paper as a dynamic mechanism to address continuously fluctuating flexibility requirements is beyond the scope of this paper. Detailed formulation and validation of this dynamic integration remain an important topic for future research.

Additionally, integrating a wider variety of DERs, such as electric vehicles and heat pumps, is an important prospect. Furthermore, while the proposed multi-step method significantly reduces the computational burden for the 33-bus system, evaluating its practical scalability and identifying potential computational bottlenecks on much larger, real-world distribution feeders remains a critical direction for future research.

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Data Availability Statement: The data presented in this study, including the parameters of the modified IEEE 33-bus test system, DER specifications, and other relevant economic and technical data supporting the reported results, are available within the article and its Appendix A.

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

The nomenclature used in this paper is shown below:

Sets

$\mathcal{N}$	Set of nodes in the distribution system, $\mathcal{N} = \{0, 1, \dots, N\}$, $N = \mathcal{N} - 1$
$\pi(j)$	Set of indices of parent nodes of node j (a singleton since it is a radial distribution system: $ \pi(j) = 1$)
$\sigma(j)$	Set of indices of child nodes of node j
$\mathcal{T}$	Set of periods, $\mathcal{T} = \{1, 2, \dots, T\}$, $T = \mathcal{T} $
$\mathcal{Y}^{1st}$	Feasible set in the first-stage problem
$\mathcal{X}^{2nd}$	Feasible set in the second-stage problem
$\mathcal{X}_{Lin}^{2nd}$	Feasible set in the second-stage problem using the linearized LinDistFlow equations
$\mathcal{X}_{Rel}^{2nd}$	Feasible set in the second-stage problem using the second-order cone (SOC)-relaxed DistFlow equations
$\mathcal{X}_{Exa}^{2nd}$	Feasible set in the second-stage problem using the DistFlow equations with parameterized uncertainty sets

$\mathcal{U}$	Uncertainty set of available PV generation power
Variables	
$P_{i,t}^{ch}$	Battery charging power at node i in period t [p.u.]
$P_{i,t}^{dch}$	Battery discharging power at node i in period t [p.u.]
$E_{i,t}^{bt}$	State of energy (SOE) of the battery at node i in period t
$P_{i,t}^{pv}$	PV generation power at node i in period t [p.u.]
$\hat{P}_{i,t}^{pv}$	Available PV generation power at node i in period t [p.u.]
$P_{i,t}^{fl}/Q_{i,t}^{fl}$	Flexible load (FL) control amount (active/reactive power) at node i in period t [p.u.]
$v_{i,t}$	Squared voltage magnitude at node i in period t [p.u. ²]
$l_{ij,t}$	Squared current magnitude of the distribution line flowing between nodes i and j in period t [p.u. ²]
$P_{ij,t}/Q_{ij,t}$	Active/reactive power flowing from node i to j in period t [p.u.]
p_{imb}	Absolute value of the imbalance between required and procured flexibility (active power) in period t [p.u.]
z_t^+/z_t^-	Auxiliary variables representing the increase/decrease rate from the predicted available PV generation power in period t
$z_t'^+/z_t'^-$	Binary variables used to linearize bilinear terms related to z_t^+ and z_t^- . $z_t'^+, z_t'^- \in \{0, 1\}$
C^{bt}	Battery operation cost [JPY]
C^{pv}	PV curtailment cost [JPY]
C^{fl}	FL control cost [JPY]
C^{penal}	Imbalance penalty cost for flexibility requirement [JPY]
C^{loss}	Distribution line loss cost [JPY]
C^{total}	Total net cost [JPY]
$C^{total'}$	Total net cost excluding distribution line loss cost [JPY]
Constants	
P_t^{req}	Flexible requirement in period t [p.u.]
R_{ij}/X_{ij}	Resistance/reactance of the distribution line between nodes i and j [p.u.]
P_t^{db}	Active power of base load at node i in period t [p.u.]
Q_t^{db}	Reactive power of base load at node i in period t [p.u.]
$\hat{P}_{i,t}^{pv}$	Predicted value of available PV generation power at node i in period t [p.u.]
η^{ch}/η^{dch}	Charging/discharging efficiency of the battery $\eta^{ch}, \eta^{dch} \in [0.0, 1.0]$
λ_t^{price}	Electricity price in period t [JPY/p.u.h]
θ	Power factor angle corresponding to the minimum power factor of the inverter
Γ	Uncertainty budget
p_t^{bl}	Active power flowing into the TSO-DSO connection node in period t (baseline) [p.u.]
P_t^{req}	Flexibility required from the external network in period t [p.u.]
c_t^{bt}	Cost required to control batteries per unit of energy in period t [JPY/p.u.h]
$c_t^{pv,cut}$	Cost required to curtail PV generation per unit of energy in period t [JPY/p.u.h]
c^{pfl+}/c^{pfl-}	Cost required to decrease/increase the FL load (active power) per unit of energy [JPY/p.u.h]
c^{qfl+}/c^{qfl-}	Cost required to increase the FL load (leading/lagging reactive power) per unit of energy [JPY/p.u.h]
c_t^{penal}	Penalty cost for flexibility imbalance per unit of energy in period t [JPY/p.u.h]
v_t^*	Squared sending-end voltage reference value at the TSO-DSO connection node in period t [p.u. ²]
τ	Time interval, $\tau = 24/T$ [hour]
$\square/\square$	Upper/lower bounds of the variable $\square$

Appendix A. Distribution System Model Used in Case Study

The IEEE 33-bus test system used in the numerical experiments uses data from the original model [28] for loads, line impedances, base voltage, and base power, and is additionally integrated with DERs and flexible loads. Table A1 shows the installation locations and specifications of the DERs, and Table A2 presents the common parameters.

Note that nodes without DER installations are omitted from Table A1. The capacity of the flexible load is uniformly set to 15% of the base load at each node. Furthermore, the upper and lower bounds of the confidence intervals for the predicted available PV generation were determined using a data-driven approach [25] based on available PV generation data under clear weather conditions.

Table A1. Specifications of the PV units and batteries integrated into the standard IEEE 33-bus test system [28]. Nodes omitted from this table contain neither PV nor battery installations.

Node	PV Capacity [kW]	Battery	
		Capacity [kWh]	Inverter Capacity [kVA]
2	–	150	50
4	240	–	–
6	120	–	–
7	–	300	100
9	120	–	–
11	90	–	–
12	–	90	30
15	120	–	–
16	120	–	–
17	–	90	30
19	180	–	–
21	180	–	–
22	–	135	45
24	840	–	–
26	120	–	–
27	–	90	30
31	300	–	–
32	420	315	105
33	120	90	30

Table A2. Technical and economic parameters used in the case studies. Note that while the optimization problem is formulated using per-unit (p.u.) values, the economic parameters are presented here in physical units (JPY/kWh and JPY/kvarh) for intuitive understanding. These values are appropriately scaled by the base power $S^{\text{base}} = 4369 \text{ kVA}$ in the actual computation.

Parameter	Symbol	Value
Parameters		
Flexible load capacity limit	$\bar{P}_{i,t}^{\text{fl}+}, \bar{P}_{i,t}^{\text{fl}-}, \bar{Q}_{i,t}^{\text{fl}+}, \bar{Q}_{i,t}^{\text{fl}-}$	15% of load capacity
PCS power factor	$\cos \theta$	0.98
Battery charging/discharging efficiency	$\eta^{\text{ch}}, \eta^{\text{dch}}$	0.98
Initial SOE	$E_{i,1}^{\text{bt}}$	$0.5 \bar{E}_i^{\text{bt}}$
Minimum SOE limit	$\underline{E}_i^{\text{bt}}$	0.0 p.u.h.
Substation sending-end squared voltage	$v_{0,t}$	1.00 p.u. ²
Cost parameters		
Grid electricity price	λ_t^{price}	20.0 JPY/kWh
PV curtailment penalty	$c_t^{\text{pv, cut}}$	20.0 JPY/kWh
Flexible load (active) cost	$c_t^{\text{pfl}+}, c_t^{\text{pfl}-}$	30.0 JPY/kWh
Flexible load (reactive) cost	$c_t^{\text{qfl}+}, c_t^{\text{qfl}-}$	1.0 JPY/kvarh
Battery operation cost	c_t^{bt}	5.0 JPY/kWh
Imbalance penalty	c_t^{penal}	100.0 JPY/kWh

Appendix B. Flexibility Requirement

Entities that provide flexibility, such as DSOs, procure flexibility to meet the requirements of the external network [3,4,13]. The required amount of flexibility has quantitative upper and lower bounds that depend on the capacity and number of individual DERs, and the deliverable range is constrained by the permissible limits of voltage and current within the distribution system. Therefore, when providing flexibility from DERs within a distribution system, it is crucial to procure the flexibility from the DERs while satisfying the network constraints of the distribution system [3,4,7,13,29]. In this paper, we determine the power exchanged at the TSO-DSO connection node without DER control by solving Equation (A1) to maximize PV utilization.

$$\underset{\mathbf{x}}{\text{maximize}} \quad \sum_{t \in \mathcal{T}} P_t^{\text{PV}}, \quad (\text{A1a})$$

$$\text{subject to} \quad (2), (4), (5), \quad (\text{A1b})$$

$$z_t^+ = z_t^- = 0, \quad \forall t \in \mathcal{T}. \quad (\text{A1c})$$

Because Equation (A1) includes the non-convex equality constraints (5f), it is solved using an optimization solver capable of handling non-convex constraints (e.g., Gurobi Optimizer ver. 13.0).

The active power at the TSO-DSO connection node based on the optimal solution $\mathbf{x}^{\text{opt}}$ of Equation (A1) is defined as the baseline, denoted as $P_t^{\text{bl}}(\mathbf{x}^{\text{opt}}) = \sum_{j \in \sigma(0)} P_{0j,t}(\mathbf{x}^{\text{opt}})$. The DSO procures flexibility from DERs within the distribution system to meet the flexibility requirement P_t^{req} imposed by the external network. That is, letting $\mathbf{x}^*$ be the distribution system state during flexibility provision, the cluster of DERs is controlled such that the active power at the interconnection point satisfies Equation (A2) below.

$$P_t^{\text{req}} = P_t^{\text{sub}}(\mathbf{x}^*) - P_t^{\text{bl}}(\mathbf{x}^{\text{opt}}), \quad \forall t \in \mathcal{T}. \quad (\text{A2})$$

Appendix C. Polyhedral Approximation

Equation (7), which represents the SOC-relaxed DistFlow equations, generally takes the form $P_{ij,t}^2 + Q_{ij,t}^2 \leq v_{i,t} l_{ij,t}$. This can be equivalently expressed as $(2P_{ij,t})^2 + (2Q_{ij,t})^2 + (v_{i,t} - l_{ij,t})^2 \leq (v_{i,t} + l_{ij,t})^2$ or, using the Euclidean norm, $\|2P_{ij,t}, 2Q_{ij,t}, (v_{i,t} - l_{ij,t})\|_2 \leq v_{i,t} + l_{ij,t}$. The linearization method for this constraint proceeds as follows. First, by introducing a non-negative auxiliary variable $W_{ij,t} \geq 0$, the initial SOC constraint is decomposed into the following two standard SOC constraints, yielding Equation (A3):

$$\|2P_{ij,t}, 2Q_{ij,t}\|_2 \leq W_{ij,t}, \quad \forall j \in \mathcal{N} \setminus \{0\}, i = \pi(j), \forall t \in \mathcal{T}, \quad (\text{A3a})$$

$$\|W_{ij,t}, (v_{i,t} - l_{ij,t})\|_2 \leq v_{i,t} + l_{ij,t}, \quad \forall j \in \mathcal{N} \setminus \{0\}, i = \pi(j), \forall t \in \mathcal{T}. \quad (\text{A3b})$$

Subsequently, Equation (A4) is derived by applying the polyhedral approximation technique [27] to linearize Equation (A3). Here, R is a positive integer denoting the number of iterations that dictates the approximation accuracy, while $\delta_{ij,t}^{pq,r}, \tau_{ij,t}^{pq,r}$ and $\delta_{ij,t}^{vl,r}, \tau_{ij,t}^{vl,r}$ ($r = 0, 1, \dots, R$) act as auxiliary variables introduced specifically for this approximation process.

$$2P_{ij,t} + \delta_{ij,t}^{pq,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4a})$$

$$-2P_{ij,t} + \delta_{ij,t}^{pq,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4b})$$

$$2Q_{ij,t} + \tau_{ij,t}^{pq,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4c})$$

$$-2Q_{ij,t} + \tau_{ij,t}^{pq,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4d})$$

$$\delta_{ij,t}^{pq,r} - \cos\left(\frac{\pi}{2^{r+1}}\right)\delta_{ij,t}^{pq,r-1} - \sin\left(\frac{\pi}{2^{r+1}}\right)\tau_{ij,t}^{pq,r-1} = 0, \quad r = 1, 2, \dots, R, \\ \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4e})$$

$$\sin\left(\frac{\pi}{2^{r+1}}\right)\delta_{ij,t}^{pq,r-1} - \cos\left(\frac{\pi}{2^{r+1}}\right)\tau_{ij,t}^{pq,r-1} + \tau_{ij,t}^{pq,r} \geq 0, \quad r = 1, 2, \dots, R, \\ \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4f})$$

$$-\sin\left(\frac{\pi}{2^{r+1}}\right)\delta_{ij,t}^{pq,r-1} + \cos\left(\frac{\pi}{2^{r+1}}\right)\tau_{ij,t}^{pq,r-1} + \tau_{ij,t}^{pq,r} \geq 0, \quad r = 1, 2, \dots, R, \\ \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4g})$$

$$-\delta_{ij,t}^{pq,R} + W_{ij,t} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4h})$$

$$-\tau_{ij,t}^{pq,R} + \tan\left(\frac{\pi}{2^{R+1}}\right)\delta_{ij,t}^{pq,R} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4i})$$

$$W_{ij,t} + \delta_{ij,t}^{vl,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4j})$$

$$-W_{ij,t} + \delta_{ij,t}^{vl,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4k})$$

$$(v_{i,t} - l_{ij,t}) + \tau_{ij,t}^{vl,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4l})$$

$$-(v_{i,t} - l_{ij,t}) + \tau_{ij,t}^{vl,0} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4m})$$

$$\delta_{ij,t}^{vl,r} - \cos\left(\frac{\pi}{2^{r+1}}\right)\delta_{ij,t}^{vl,r-1} - \sin\left(\frac{\pi}{2^{r+1}}\right)\tau_{ij,t}^{vl,r-1} = 0, \quad r = 1, 2, \dots, R, \\ \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4n})$$

$$\sin\left(\frac{\pi}{2^{r+1}}\right)\delta_{ij,t}^{vl,r-1} - \cos\left(\frac{\pi}{2^{r+1}}\right)\tau_{ij,t}^{vl,r-1} + \tau_{ij,t}^{vl,r} \geq 0, \quad r = 1, 2, \dots, R, \\ \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4o})$$

$$-\sin\left(\frac{\pi}{2^{r+1}}\right)\delta_{ij,t}^{vl,r-1} + \cos\left(\frac{\pi}{2^{r+1}}\right)\tau_{ij,t}^{vl,r-1} + \tau_{ij,t}^{vl,r} \geq 0, \quad r = 1, 2, \dots, R, \\ \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4p})$$

$$-\delta_{ij,t}^{vl,R} + v_{i,t} + l_{ij,t} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T}, \quad (\text{A4q})$$

$$-\tau_{ij,t}^{vl,R} + \tan\left(\frac{\pi}{2^{R+1}}\right)\delta_{ij,t}^{vl,R} \geq 0, \quad \forall j \in \mathcal{N} \setminus \{0\}, \quad i = \pi(j), \quad \forall t \in \mathcal{T} \quad (\text{A4r})$$

Appendix D. Proof of Reformulation

Proposition A1. Formulation (17) and Reformulation (18) are equivalent, yielding the same optimal objective function value $R(\mathbf{y}^*)$ for any given $\mathbf{y}^*$.

Proof. We show component-wise that the variable vectors Φ^+, Φ^- in Formulation (17) and ξ^+, ξ^- in Reformulation (18) form identical feasible regions and take the same values for any admissible set of binary variables z'^+, z'^- .

Focusing on an arbitrary i -th component, constraint (17j) ($z_i'^+ + z_i'^- \leq 1$) restricts the possible values of the binary variable pair ($z_i'^+, z_i'^-$) to three cases: (1,0), (0,1), and (0,0). We compare the constraints of both formulations for each case.

Case A: ($z_i'^+, z_i'^-$) = (1,0)

From the constraints in Formulation (17), the following holds:

$$0 \leq \Phi_i^+ \leq M_i, \quad \zeta_i \leq \Phi_i^+ \leq \zeta_i \quad \Rightarrow \quad \Phi_i^+ = \zeta_i \quad (\text{A5a})$$

$$0 \leq \Phi_i^- \leq 0, \quad \zeta_i - M_i \leq \Phi_i^- \leq \zeta_i \quad \Rightarrow \quad \Phi_i^- = 0 \quad (\text{and } 0 \leq \zeta_i \leq M_i) \quad (\text{A5b})$$

On the other hand, from the constraints in Formulation (18), the following holds:

$$0 \leq \zeta_i^+ \leq M_i, \quad 0 \leq \zeta_i^- \leq 0 \quad \Rightarrow \quad \zeta_i^- = 0 \quad (\text{A6a})$$

$$\zeta_i \geq \zeta_i^+ + \zeta_i^- \quad \Rightarrow \quad \zeta_i \geq \zeta_i^+ \quad (\text{A6b})$$

$$\zeta_i - \zeta_i^+ - \zeta_i^- \leq M_i(1 - 1 - 0) \quad \Rightarrow \quad \zeta_i \leq \zeta_i^+ \quad (\text{A6c})$$

From these, we obtain $\zeta_i^+ = \zeta_i$ and $\zeta_i^- = 0$ (and $0 \leq \zeta_i \leq M_i$). Therefore, in Case A, $\Phi_i^+ = \zeta_i^+$ and $\Phi_i^- = \zeta_i^-$ hold.

Case B: $(z_i'^+, z_i'^-) = (0, 1)$

Since it has a symmetric structure to Case A, the following is derived through a similar expansion:

- Formulation (17): $\Phi_i^+ = 0, \Phi_i^- = \zeta_i$;
- Formulation (18): $\zeta_i^+ = 0, \zeta_i^- = \zeta_i$.

Thus, $\Phi_i^+ = \zeta_i^+$ and $\Phi_i^- = \zeta_i^-$ hold.

Case C: $(z_i'^+, z_i'^-) = (0, 0)$

From the constraints in Formulation (17),

$$0 \leq \Phi_i^+ \leq 0 \quad \Rightarrow \quad \Phi_i^+ = 0 \quad (\text{A7a})$$

$$\zeta_i - M_i \leq \Phi_i^+ \leq \zeta_i \quad \Rightarrow \quad \zeta_i - M_i \leq 0 \leq \zeta_i \quad (\text{A7b})$$

Through a similar expansion, we have $\Phi_i^- = 0$. On the other hand, from the constraints in Formulation (18),

$$0 \leq \zeta_i^+ \leq 0 \quad \Rightarrow \quad \zeta_i^+ = 0 \quad (\text{A8a})$$

$$0 \leq \zeta_i^- \leq 0 \quad \Rightarrow \quad \zeta_i^- = 0 \quad (\text{A8b})$$

$$\zeta_i \geq 0 + 0 \quad \Rightarrow \quad \zeta_i \geq 0 \quad (\text{A8c})$$

$$\zeta_i - 0 - 0 \leq M_i(1 - 0 - 0) \quad \Rightarrow \quad \zeta_i \leq M_i \quad (\text{A8d})$$

From these, $\zeta_i^+ = 0$ and $\zeta_i^- = 0$ also hold in Formulation (18), and the constraint $0 \leq \zeta_i \leq M_i$ imposed on the main variable ζ_i is completely identical to that in Formulation (17).

From the above, $\Phi^+ = \zeta^+$ and $\Phi^- = \zeta^-$ hold in all cases, and the implicit bounds on the main variable ζ are perfectly identical. Since the objective function of Formulation (18) is obtained by replacing Φ^+ and Φ^- in the objective function of Formulation (17) with ζ^+ and ζ^- , respectively, their feasible regions and objective function spaces are strictly equivalent.

Therefore, Formulation (17) and Reformulation (18) are mathematically equivalent, yielding the same optimal solution and optimal value. $\square$

Appendix E. C&CG Algorithm

For convenience, we denote $R(\mathbf{y}) < +\infty$ if Equations (17a)–(17k) are feasible, and $R(\mathbf{y}) = +\infty$ if they are infeasible. The optimal value of the primal problem $S_P(\mathbf{y}, \hat{\mathbf{u}})$ takes either $R(\mathbf{y}) < +\infty$ or $R(\mathbf{y}) = +\infty$, depending on the feasibility of the dual problem. Using the C&CG algorithm [19], the upper bound UB and lower bound LB are iteratively updated. The computation terminates when the difference between these bounds becomes smaller

than a predefined threshold ϵ_{ccg} , at which point the optimal solution is obtained. Algorithm A1 presents the solution procedure using the C&CG algorithm [19]. In Algorithm A1, η is an auxiliary variable, and the bracketed superscript (m) on variables x and y denotes the iteration count in the C&CG algorithm.

Algorithm A1 C&CG algorithm for solving (15)

Require: Convergence tolerance $\epsilon_{ccg} > 0$

Ensure: A robust optimal solution: $(x^{*(m)}, y^{*(m)}, u^{*(m)})$

- 1: Initialization: Set $LB = -\infty$, $UB = +\infty$, $m = 1$, and $\mathcal{O} = \emptyset$
- 2: Set initial conditions: $\bar{z}_t^+ = \bar{z}_t^- = 0, \forall t \in \mathcal{T}$ and calculate the initial scenario u^{*1} . Update $\mathcal{O} \leftarrow \mathcal{O} \cup \{1\}$
- 3: **while** $UB - LB > \epsilon_{ccg}$ **do**
- 4: Let x_{ccg} be the set $x_{ccg} = \{x^{(1)}, x^{(2)}, \dots, x^{(m)}\}$. Solve the following master problem (A9):

$$\underset{x_{ccg}, y^{(m)}, \eta}{\text{minimize}} \quad \eta \quad (\text{A9a})$$

$$\text{subject to} \quad \eta \geq c^\top x^{(k)}, \quad \forall k \in \mathcal{O}, \quad (\text{A9b})$$

$$F_1 x_1^{(k)} + F_2 x_2^{(k)} + D y^{(m)} \geq d - E \hat{u}^{*(k)}, \quad \forall k \leq m, \quad (\text{A9c})$$

$$J_1 x_1^{(k)} + J_2 x_2^{(k)} + G y^{(m)} = e, \quad \forall k \leq m, \quad (\text{A9d})$$

$$x_1^{(k)} \geq 0, \quad x_2^{(k)} : \text{free}, \quad \forall k \in \mathcal{O}, \quad (\text{A9e})$$

$$y^{(m)} \in \mathcal{Y}^{1st}, \quad \eta \in \mathbb{R}. \quad (\text{A9f})$$

- Obtain the optimal solution $\{x^{*(1)}, x^{*(2)}, \dots, x^{*(m)}, y^{*(m)}, \eta^*\}$ and update $LB \leftarrow \eta^*$
 - 5: Solve the subproblem (18) where $y = y^{*(m)}$, and obtain the worst scenario (z'^+, z'^-) . Calculate $u^{*(m+1)}$ using (z'^+, z'^-) and optimal objective function $R(y^{*(m)})$.
 - 6: Update $UB \leftarrow \min\{UB, R(y^{*(m)})\}$
 - 7: **if** $UB - LB \leq \epsilon_{ccg}$ **then**
 - 8: Terminate the C&CG algorithm and output a robust optimal solution $(x^*, y^*, \hat{u}^*)$ where $x^* = x^{*(m)}$, $y^* = y^{*(m)}$ and $\hat{u}^* = \hat{u}^{*(m+1)}$.
 - 9: **else**
 - 10: Generate decision variables $x^{(m+1)}$.
 - 11: **end if**
 - 12: **if** $R(y^{*(m)}) < +\infty$ **then**
 - 13: Add $x^{(m+1)}$ and following constraints to master problem (A9): $\eta \geq c^\top x^{(m+1)}, F_1 x_1^{(m+1)} + F_2 x_2^{(m+1)} + D y^{(m+1)} \geq d - E \hat{u}^{*(m+1)}, J_1 x_1^{(m+1)} + J_2 x_2^{(m+1)} + G y^{(m+1)} = e$.
 - 14: Update $\mathcal{O} \leftarrow \mathcal{O} \cup (m+1)$ and $m \leftarrow (m+1)$.
 - 15: **else**
 - 16: Add $x^{(m+1)}$ and following constraints to master problem (A9): $F_1 x_1^{(m+1)} + F_2 x_2^{(m+1)} + D y^{(m+1)} \geq d - E \hat{u}^{*(m+1)}, J_1 x_1^{(m+1)} + J_2 x_2^{(m+1)} + G y^{(m+1)} = e$.
 - 17: Update $m \leftarrow (m+1)$.
 - 18: **end if**
 - 19: **end while**
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