

Article

Fractional Calculus Application in Modelling Non-Linear Sloshing

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Abstract

Many models exist to describe the nature of sloshing, an intricate phenomena, including the linear mechanical model and the Duffing oscillator. This paper attempts to provide an alternative model that describes the maximum slosh wave height (MSWH) of different sloshing liquids, to more accurately predict the nuanced phenomena as well as to better understand the inherent nature of sloshing. We explore three different models, the classical linear mechanical model, the non-linear Duffing model and finally the novel fractional visco-elastic model. The latter illustrates promising results after being validated via experimentation. The paper leads to the conclusion that sloshing of liquids, particularly low-viscosity fluids, is better described to have a spectrum of fractional-order visco-elastic forces at near and post resonant excitation frequencies rather than independent viscous and elastic forces as assumed by the linear and non-linear mechanical models.

Keywords: sloshing; fractional calculus; visco-elastic model; linear mechanical model; non-linear mechanical model; duffing oscillator

1. Introduction

Sloshing is a naturally occurring phenomenon that takes place when a container, partially filled with a liquid, experiences horizontal or vertical excitations. This can be observed in situations, including liquid rocket fuel tanks during lift off or tanker lorries when accelerating, to name a few. The effect of sloshing may appear to be minimal, but in worst cases, when the frequency of periodic excitations matches the natural frequency of the sloshing system, it can lead to disasters due to resonance, such as, losing control over tanker lorries when applying brakes. Hence, it is important to find these frequencies to ensure the working frequency of the system is below the sloshing natural frequency to avoid such incidents. For example, the liquid fuels on NASAs space vehicle Saturn V have a fundamental slosh frequency of 0.3–0.4 Hz [1]. Considering a safety factor, the vehicle was designed to work at a frequency of 0.16 Hz, thus being able to safely lift off without resonant sloshing [1].

The maximum slosh wave height, hereinafter MSWH, of the liquid also pertains significance, particularly in the context of designing tanks to carry liquids safely, ensuring vehicle stability, predicting overflows or spillovers and in the design of baffles or anti-slosh devices. Presently, the most popular methods used by engineers to compute the MSWH include computational fluid dynamics (CFD) or smoothed particle hydrodynamics (SPH), two resource- and time-intensive tasks, a major setback for engineers. Hence, there is the need for a fast, simple and precise model to predict MSWH and thereby reduce design time and cost.



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During the rapid development of rockets, with increased caution and focus on stability, the necessity to develop simple models to predict sloshing had arisen. Previous models based on fluid dynamics, derived from Navier-Stokes equations required excessive computational power at the time, emphasising the need for simpler models. Authors, including Abramson et al. [1] and Vikas Sharma et al. [2], have endeavoured in depth into studies of sloshing equivalent mechanical models, including multi-degree of freedom models, in the context of rocketry. However, there still remain opportunities to study the use of equivalent non-linear mechanical models, due to the inherent non-linear nature of sloshing.

Adding on, authors such as Y. Kim and Makoto et al., have used models from fluid dynamics to numerically simulate sloshing in 2D and 3D, both in the context of cargo in freight [3,4]. Kim additionally investigated the effect of load impact on sloshing, which results in forced excitation [3], and Makoto et al. particularly examined how LNG carrier tank's length-to-breadth ratio impacted sloshing. In the 2011 paper, Rafiee et al. investigated liquid sloshing using numerical methods, including SPH, and verified them experimentally in a very similar setup to the one presented in this paper [5]. While the results from both papers generally show agreement, their simulations may be viewed as resource-intensive.

Many have also explored the reliability of stochastic models in sloshing, including Fillon et al. and Cetin et al. [6,7]. However, the caveat of these approaches is that they require a lot of data before being statistically fitted. For example, Fillon et al. required 480 h of sloshing data to construct their model [6].

Recently, the use of fractional calculus in modelling natural phenomena, arising particularly in engineering disciplines, has gained a lot of interest, especially because it seems to explain many anomalies arising in traditional models. For example, Riewe was among the first to apply fractional calculus to model non-conservative forces like friction, which appeared to improve predictions over the Lagrangian and Hamiltonian mechanics [8]. In liquid dynamics, a spring-mass-damper system is often used as a model for sloshing. P. Stinga, in his survey of the applications of fractional calculus, argues that not all fluids return to their original shape nor deform indefinitely, as assumed by the elastic and viscous terms in the classical models [9]. This makes fractional models promising, because they can predict behaviour of materials with memory [9]. As opposed to the traditional spring-mass-damper system the concept of a spring-pot was introduced and tested by many authors, including Pirrota et al., in the context of FEM analysis [10]. Authors like R.C. Koeller were the first to propose the use of fractional calculus to model visco-elastic behaviour [11], and authors like F.C. Meral et al. [12] were among the first to verify such models experimentally. It is, thus, clear that while classical mechanical models use ordinary differential equations to govern the laws of motion, there is no need to restrict oneself with integer-order calculus if nature is intrinsically fractional. This means that one could replace the time-derivative in evolution equations with fractional derivatives, as done by Riewe [8]. Previously, authors like Matteo et al. [13] have modelled U-tube Tuned Liquid Column Damper (TLCD) using fractional calculus, as existing integer-order differential equations do not accurately predict liquid motion [13]. However, there still exists room for the application of fractional models to more general container geometries. Other related works, such as the one by Spanos and Evangelatos, have pointed out that changing the degree of the fractional derivatives in the fractional differential equation can alter the resonant frequency and level of damping [14], despite resonance and damping being linear concepts, which is due to stress-relaxation causing non-linearities in viscoelastic materials. This gives potential to modelling frequency-response of various liquids, particularly near-resonant behaviour, with fractional calculus.

While addressing sloshing and oscillatory behaviour, many authors have tried novel mathematical models and approaches to model this behaviour, including Juan et al. [15]

who used generalised local derivatives. Our model is another, yet novel approach using fractional calculus.

In this paper, a comprehensive study of linear, non-linear and fractional mechanical models, with which MSWH can be calculated or computed, will be assessed and validated with experimental results. The domain of the paper will be restricted to a horizontally excited vertical cylindrical tank, although similar methods can be used for various orientations and geometries. The different methods will be compared and contrasted with one another to find an ideal method to predict the MSWH at particular frequencies. We first derive the solutions to the linear, non-linear mechanical, and finally fractional models. Then, we present experimental data (setup similar to [3,5]) across five liquids with different viscosities and proceed to compare the three models with each other.

2. Linear Mechanical Model

We may assume sloshing exhibits a pattern analogous to that of a single degree of freedom spring-mass-damper (SMD) system, a system forming a mechanical model often used to emulate a vibrating system. Sloshing is similar to a SMD system as both approximately exhibit similar harmonic response to forced oscillations, resonance phenomena, damping effects and phase lag between oscillations and response [1,2]. Figure 1 illustrates how a sloshing liquid can be modelled by a SMD system.

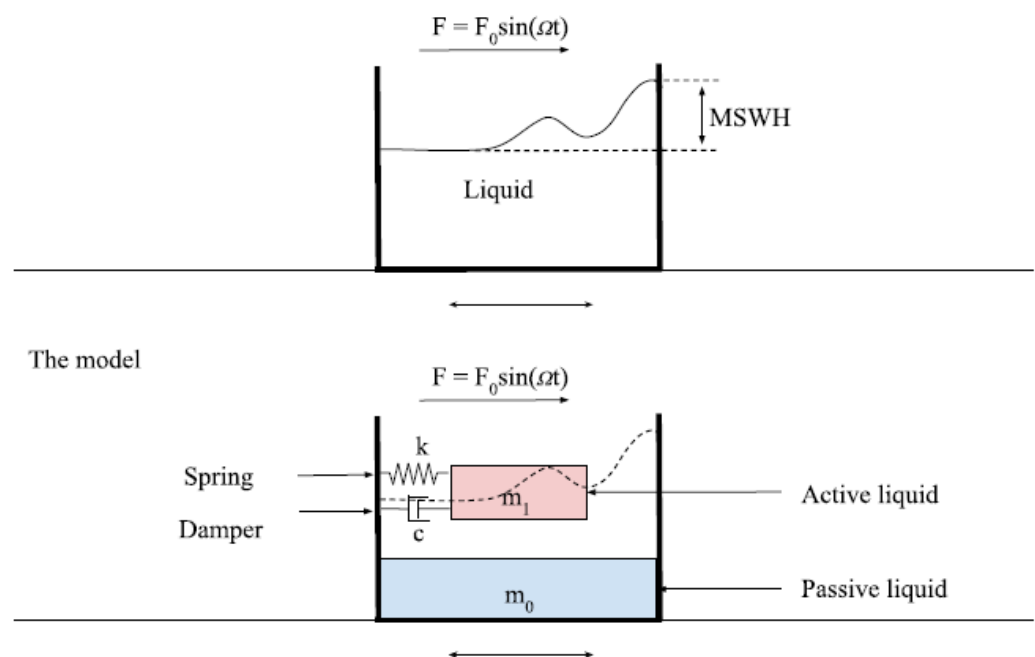


Figure 1. Sloshing equivalent linear 1-DoF SMD model.

As the container oscillates from left to right by some forcing function $F = F_0 \sin(\Omega t)$, the liquid begins to form peaks, whose maximum is the MSWH. The model attempts to capture this behaviour by splitting the liquid into two masses: the passive mass m_0 that remains static during sloshing, and the active mass m_1 that shifts positions, forming the peaks. By meticulously adjusting the spring stiffness and damping coefficient, the model can be used to predict the MSWH of the liquid at particular frequencies.

By applying Newtons Second Law to Figure 1, the equation of motion can be found.

$$m_1 \ddot{x} + c \dot{x} + kx = F$$

The expression can be simplified by using the natural frequency of the sloshing system $\omega = \sqrt{k/m_1}$ and the damping ratio $\zeta = \frac{c}{2\omega m_1}$.

$$\ddot{x} + 2\omega\zeta\dot{x} + \omega^2x = \frac{\omega^2 F_0}{k} \sin(\Omega t) \quad (1)$$

For the scope of this paper, we assume the sloshing liquid is an underdamped system, wherein $\zeta < 1$ and when let to oscillate freely, the liquid oscillates at its natural frequency and comes to a stop gradually. Only for a minority of certain liquids with very high viscosity, the system may behave as an overdamped system, where $\zeta > 1$. Nonetheless, for an underdamped system, the steady state solution to (1) as a function of time is,

$$x_{ss}(t) = \frac{F_0 \Gamma}{k} \sin(\Omega t - \phi)$$

where Γ is the magnification factor and ϕ is the phase lag, given as [16],

$$\Gamma = \frac{1}{\sqrt{(1 - \bar{\Omega}^2)^2 + (2\zeta\bar{\Omega})^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta\bar{\Omega}}{1 - \bar{\Omega}^2}\right)$$

Maximising $x_{ss}(t)$ will give the MSWH.

$$x_{max} = \frac{F_0 \Gamma}{k} = \frac{F_0}{k\sqrt{(1 - \bar{\Omega}^2)^2 + (2\zeta\bar{\Omega})^2}} \quad (2)$$

Note, the frequency ratio is given by $\bar{\Omega} = \frac{\Omega}{\omega}$. This equation can be validated using experimentation.

3. Non-Linear Mechanical Model

Sloshing is inherently a non-linear phenomena. The non-linearities arise due to liquid compressions, change in centre of gravity and many more [17,18]. As a result, the use of a non-linear mechanical model may be better suited to simulate a sloshing system. The non-linear model is simply a perturbed version of the equation of motion (1), by a non-linear term βx^3 . The Duffing model does not include quadratic and higher even-degree terms to ensure that the magnitude of the restoring force is the same at both extremes. The β coefficient determines whether the liquid under exam shows hardness (spring stiffness increases for larger displacements when $\beta > 0$) or softness (spring stiffness decreases for larger displacement when $\beta < 0$). A schematic is illustrated in Figure 2.

$$\ddot{x} + 2\omega\zeta\dot{x} + \omega^2x + \beta x^3 = \frac{\omega^2 F_0}{k} \sin(\Omega t)$$

This form, otherwise known as the Duffing oscillator, can be solved using the harmonic balance method. It has a general steady-state solution of the following approximation [19].

$$x_{ss}(t) \approx A \sin(\Omega t + \phi) \quad (3)$$

where ϕ is the phase lag and A is the solution to the following amplitude expression [19],

$$\left[\left(\Omega^2 - \omega^2 + \frac{3}{4}\beta A^2 \right)^2 + (2\omega\Omega\zeta)^2 \right] A^2 = \left(\frac{\omega^2 F_0}{k} \right)^2 \quad (4)$$

The MSWH is thus given by maximising (3), i.e.,

$$x_{max} = A$$

which can be found by solving (4). For A is the solution to a polynomial function, simple numerical methods, such as Brent's method or Newton-Raphson's method, can be used to find the roots.

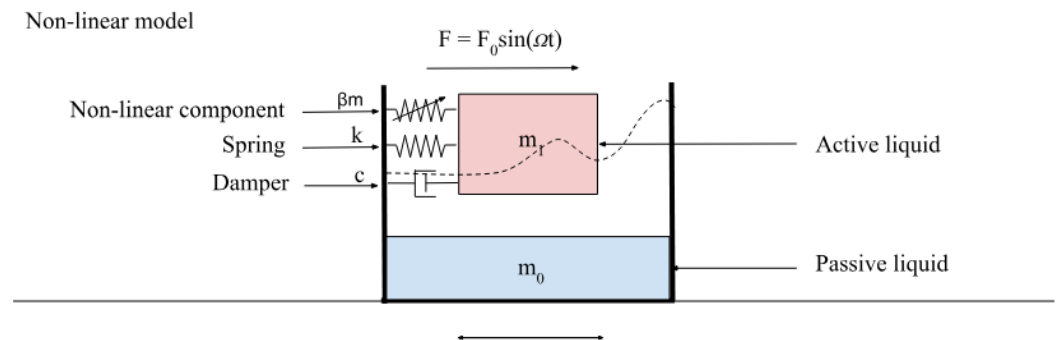


Figure 2. Sloshing equivalent non-linear mechanical model.

4. Fractional Model

The fractional derivative has many definitions, including the most well-known Riemann-Liouville, and Caputo derivatives, that has many applications in physics and engineering. However, since we are working with periodic functions (as the vertical displacement of the mass is periodic under periodic excitation), it is quite natural that the best suitable fractional derivative for this model should be based on differentiation on the Fourier space. The following is the definition of the fractional derivative, which works well with the Fourier transform [20].

Definition 1 (Fractional Derivative via Fourier Transform). *For all real-valued functions $f \in C^{[\alpha]}(\mathbb{R})$,*

$$\mathcal{F}\{D_z^\alpha f\}(u) = (iu)^\alpha \mathcal{F}\{f\}(u)$$

where $\alpha \in \mathbb{R}$ [20–23].

These definitions were used by many author's when solving differential equations. In his 2020 paper, Bagarello specifically analyses this Fourier definition of the fractional derivative on functions in $\mathcal{S}(\mathbb{R})$ and its applications to quantum mechanics.

We are particularly interested in the fractional derivative of elementary trigonometric functions. Formally speaking, the Fourier transform is only defined for functions in $\mathcal{S}(\mathbb{R})$, $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, however, sin and cos can be viewed as tempered distributions, so their Fourier transforms involve the Dirac delta function. With the Fourier transform definition of the fractional derivative [22,24],

$$D_z^\alpha [\sin(\lambda z)] = \lambda^\alpha \sin\left(\lambda z + \frac{\pi\alpha}{2}\right)$$

$$D_z^\alpha [\cos(\lambda z)] = \lambda^\alpha \cos\left(\lambda z + \frac{\pi\alpha}{2}\right)$$

Previously, the non-linear mechanical model modelled the liquid as a perfect SMD system with non-linearities. However, this is often an oversimplification of nature, for which we shall consider the cases of liquids exhibiting both viscous and elastic properties. There are altogether many ways one could use fractional derivatives to model this be-

haviour, but provided is the specific model that most closely resembles the experimental results obtained.

Fractional Visco-Elastic Model

The model to be considered assumes the liquid behaves in a hybrid manner; it shows semi-elastic and semi-viscous properties, along with some extent of memory [11]. While we normally use the null-th time-derivative to represent a spring, and the first time-derivative to represent a damper, we use the fractional derivative to represent a spring-pot. We choose some $0 < \alpha < 1$, and use the fractional α th order derivative to model visco-elastic behaviour.

$$\ddot{x} + \gamma D_t^\alpha [x] = \frac{\omega^2 F_0}{k} \sin(\Omega t) \quad (5)$$

Here we let $\gamma(\zeta, \omega)$ be an unknown arbitrary function, wherein we substitute $\gamma D_t^\alpha [x]$ for $2\zeta\omega\dot{x} + \omega^2 x$, which represents the viscous and elastic terms. We now assume the solution takes form $x = a \sin(\Omega t) + b \cos(\Omega t)$, since we know that x is a periodic function. Solving Equation (5), and collecting the sin and cos terms, we get the system of equations

$$\begin{cases} \frac{\omega^2 F_0}{k} = -a\Omega^2 + \gamma a\Omega^\alpha \cos \frac{\pi\alpha}{2} - \gamma b\Omega^\alpha \sin \frac{\pi\alpha}{2} \\ 0 = -b\Omega^2 + \gamma a\Omega^\alpha \sin \frac{\pi\alpha}{2} + \gamma b\Omega^\alpha \cos \frac{\pi\alpha}{2} \end{cases} \quad (6)$$

The model has been solved by fitting the data for a and b , by converting the above into a system of linear equation,

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} -\Omega^2 + \gamma\Omega^\alpha \cos\left(\frac{\pi\alpha}{2}\right) & -\gamma\Omega^\alpha \sin\left(\frac{\pi\alpha}{2}\right) \\ \gamma\Omega^\alpha \sin\left(\frac{\pi\alpha}{2}\right) & -\Omega^2 + \gamma\Omega^\alpha \cos\left(\frac{\pi\alpha}{2}\right) \end{bmatrix} = \begin{bmatrix} \frac{\omega^2 F_0}{k} \\ 0 \end{bmatrix}$$

The solution to this model provides the fractional mechanical model for MSWH.

5. Experimental Validation

5.1. Slosh Rig Setup

An upright cylinder was setup on a horizontal shake table, as illustrated in Figure 3. This particular geometry, a tall upright canister, was chosen to ensure the MSWH can be captured, also serving as a point of comparison with [2]. The frequency of the shake table was varied and the corresponding MSWH was measured using video footage and frame-by-frame analysis.

5.2. Experimental Parameters

Table 1 lists the parameters of the experiment and the setup. The liquids used in the experiment include water, sunflower oil, rapeseed oil, engine oil and honey.

Table 1. Parameters of the experiment.

Parameters	Values
Height of liquid h	50 mm
Radius of container R	44 mm
Amplitude of oscillation/crank radius X_0	45 mm
Motor torque τ	0.2 Nm
Force of oscillations $F_0 = \tau / X_0$	4.44 N

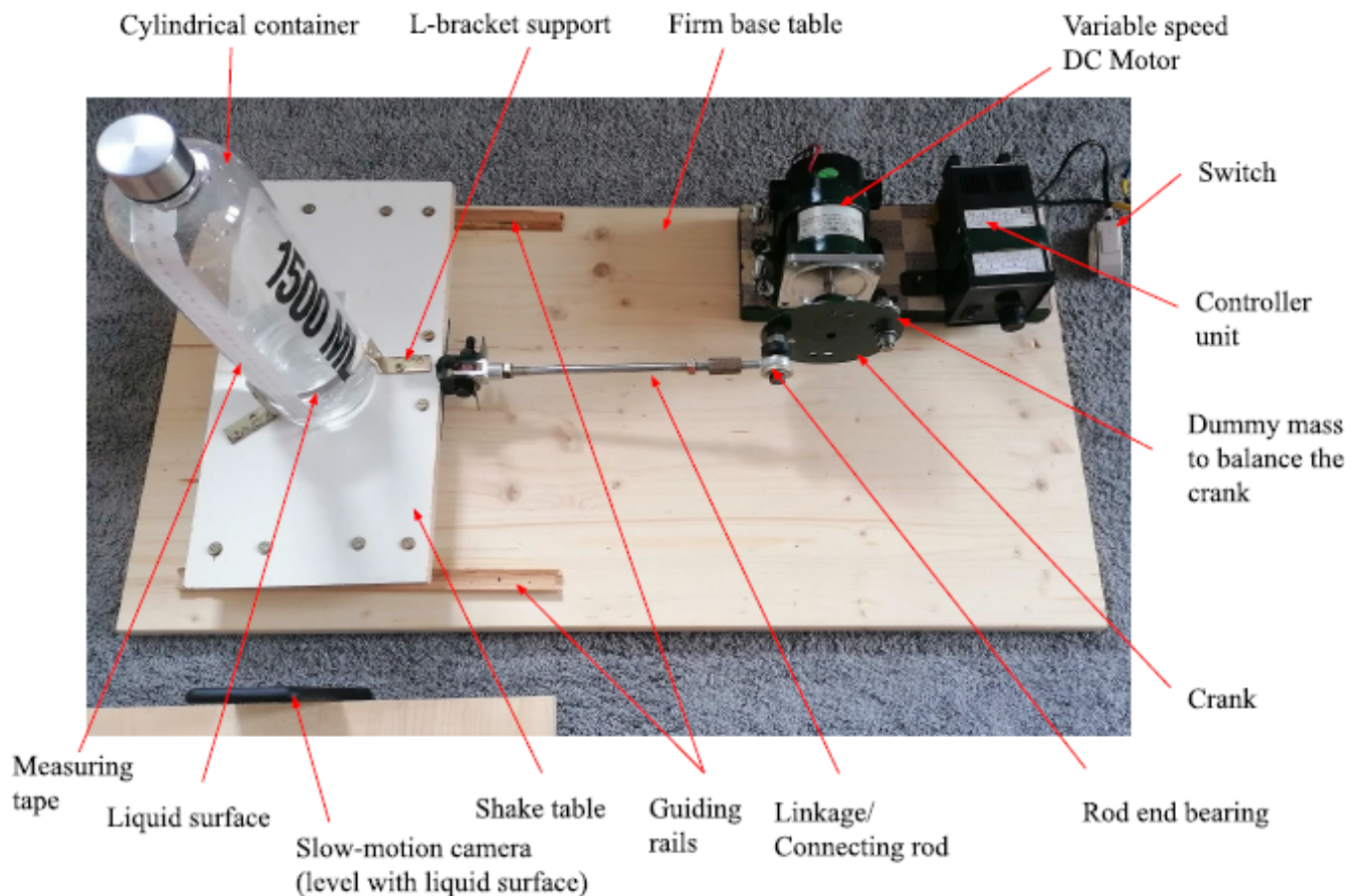
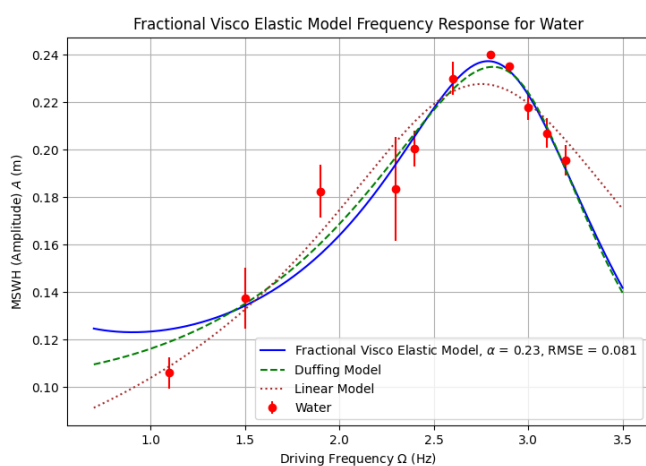


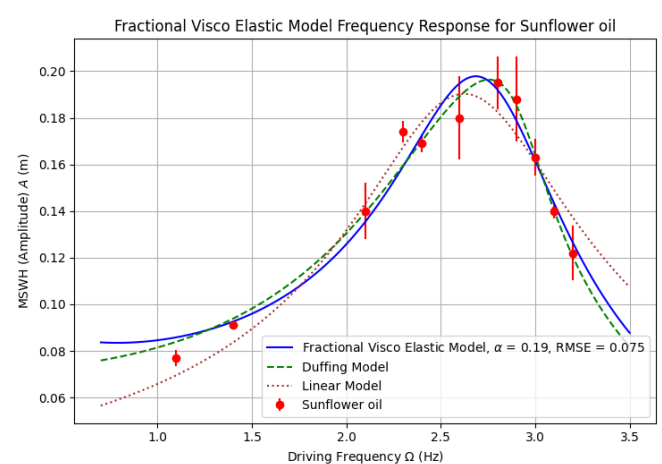
Figure 3. Slosh rig experimental setup.

5.3. Results

Figure 4 displays the frequency-response curves for the various liquids. The model coefficients were fitted using SciPy optimise. The error bars illustrated are corresponding uncertainties propagated for each measurement uncertainty.



(a) Water



(b) Sunflower oil

Figure 4. Cont.

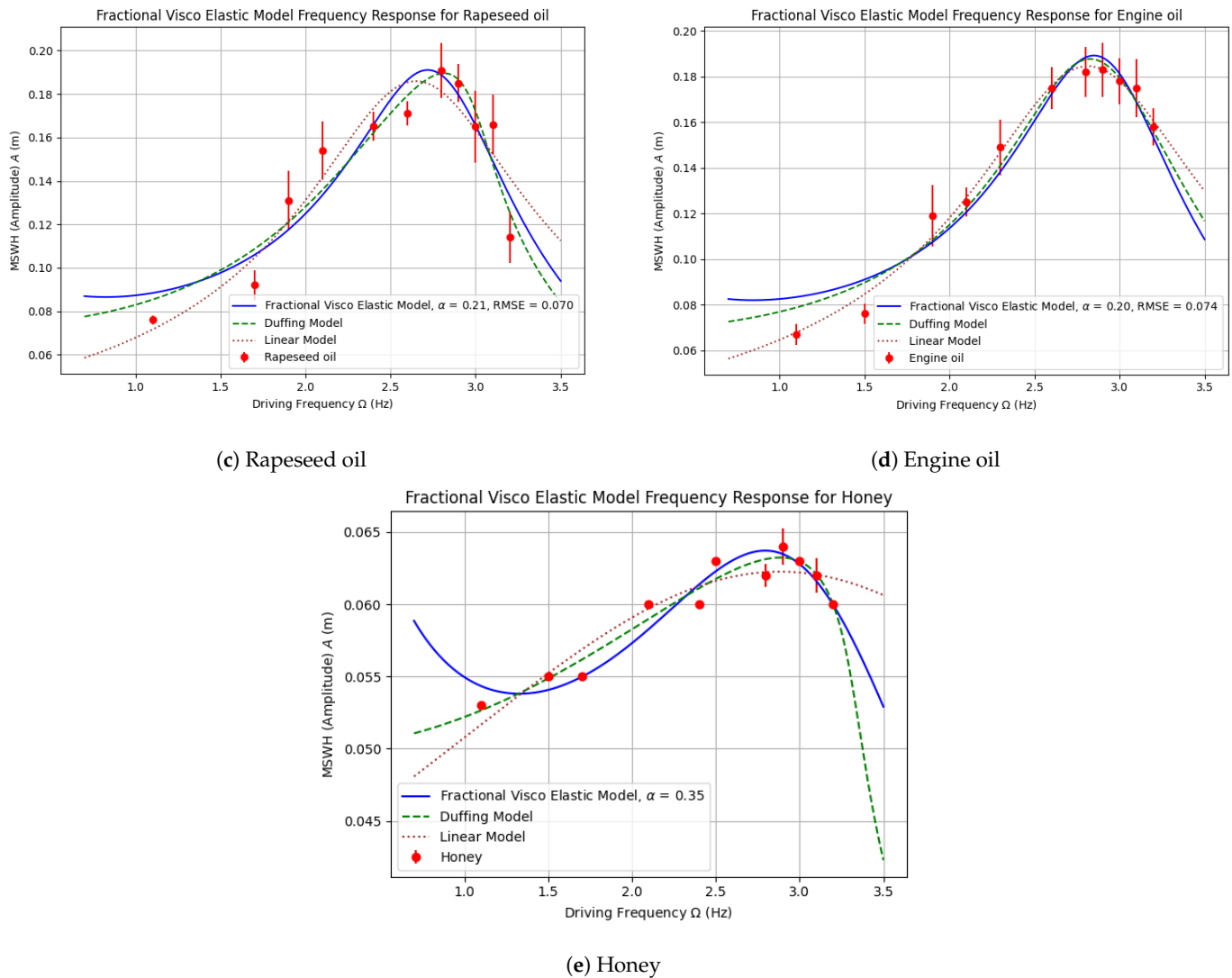


Figure 4. Frequency-response curves comparing fractional visco-elastic model, Duffing model and 1-DOF SMD System.

6. Conclusions

In this paper, a shake table attached to a motor was designed to convert angular motion into linear harmonic motion that drives the sloshing in an upright cylindrical container. We strive to determine the maximum slosh wave height (MSWH) by observing and making measurements utilising camera footage. Various models were fitted, including the linear, non-linear (Duffing) and fractional mechanical models.

The results presented in this paper show clearly that each model is successful in its own ways, and that neither can be classified as superior to one another. Regarding the fractional visco-elastic model, the order of the fractional derivative for the visco-elastic fluids should be in the range 0.05–0.35, as suggested by Koeller [11]. This matches the results presented in this paper, ranging from 0.19–0.35.

Expectedly, the Duffing model outperforms the linear model in most liquids (all but perhaps engine oil). This is because the linear model is a simplification of the Duffing oscillator when $\beta = 0$.

Overall, both the Duffing and fractional models elicit greatest accuracy at resonance and post-resonance frequencies. The Duffing model takes into account of non-linearity in the restoring force (controlled by β) and acknowledges that stiffness and damping varies with amplitude, making this model most accurate when MSWH is highest (i.e., most variation in

amplitude). Since the fractional model does not explicitly address this issue, it is evident that the Duffing oscillator is superior at near-resonant frequencies. For water, sunflower oil, engine oil and honey, the fractional visco-elastic model presents a nearly perfect fit for post-resonance frequencies, making it generally a useful model to predict post-resonance MSWH. However, the fractional visco-elastic model tends to diverge away from experimental results at the lower sub-resonance frequencies. A possible explanation of this phenomenon could be linked to the observation of Grunfest, who claims that viscous heating, due to liquid flow and increased contact at the molecular level, leads to deviations in Newtonian behaviour [25]. At post-resonant frequencies, when the driving frequency is higher, the effects of viscous heating are more probable, making the fractional model a better fit.

Nevertheless, the behaviour of all the liquids near and post resonant driving frequencies is very close to the fractional model. A possible extension would be to explore how well the fractional model can predict the liquid response at frequencies higher than resonance. Furthermore, one could explore what factors affect the fractional order, for example, the container's aspect ratio, or changing the overall geometry of the container.

Furthermore, the results presented in this paper reveal the inherent non-linear but also variable and volatile nature of a sloshing system. The sloshing liquid cannot solely be characterised by linear or non-linear damping, as depending on the viscosity of the liquid under exam, there is a possibility of viscoelastic effects. This can be verified by the empirical equation given by Abrahamson et al. [26].

$$\zeta = 0.79\nu^{1/2}R^{-3/4}g^{-1/4}$$

where ν is the kinematic viscosity. From this equation, it can be recognised that the fractional term of the fractional model is impacted by the kinematic viscosity of the liquid. For that particular reason, it is also observed that the fractional order α is not fixed for all liquids and the RMSE (root mean square error) is typically very low. Furthermore, this hints at the continuous nature of sloshing liquids in terms of viscous and elastic forces.

For example, water, which has a relatively low dynamic viscosity (see Table 2), is best described using the fractional visco-elastic model, whereas sunflower oil, which has greater viscosities, is better fit by the non-linear Duffing oscillator for the frequency-response. These observations empirically identify the variable nature of sloshing and its dependence on viscosity.

Table 2. Properties of the different liquids (at $T = 20\text{ }^{\circ}\text{C}$) [27–29].

	Water	Sunflower Oil	Rapeseed Oil	Engine Oil	Honey
Dynamic viscosity (mPa · s)	1.0016	48.8	59.2	>200	>2000
Density (kg/m ³)	997	920	915	860	1430

Some anomalies found in the fractional visco-elastic model include the unanticipated high MSWH at lower frequencies. Possible future opportunities for research include the potential of a non-linear fractional model.

In conclusion, the fractional visco-elastic model serves as an important model that attempts to mimic the convoluted nature of sloshing. Showing very promising results for certain liquids at particular frequencies, it indicates the inherent fractional nature of damping and its dependence on the viscosity of the sloshing liquid, as well as the continuous nature between viscous and elastic behaviour of sloshing liquids.

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