

Article

A Geared Five-Bar Linkage for the Walker Gait Trainer: Synthesis, Kinematic Analysis, and Device Integration

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Abstract

Stroke is a major cause of lower-limb paresis, and clinical practice has long shown that early, repetitive gait training can accelerate motor recovery. End-effector gait trainers mounted on a wheeled walker provide a portable and low-cost alternative to bulky treadmill-based exoskeletons. This paper presents a geared five-bar linkage as the trajectory-generating mechanism of a Walker Gait Trainer (WGT), a single-actuator rehabilitation device for over-ground use. The two cranks of the five-bar linkage are linked by a gear train made up of two identical spur gears and an intermediate idler. This arrangement reduces the dimensional synthesis problem from four defect constraints, Grashof, order, and two circuit constraints, to a single order constraint, because branch and circuit defects are removed by design when both cranks are compelled to rotate continuously. Dimensional synthesis is formulated as a path-generation problem based on seven precision points obtained from normative gait data, and the mechanism dimensions are found through a systematic design-by-analysis search carried out interactively in the GIM kinematic simulation environment, using the closed-form kinematic model developed in this study. The final mechanism is then reconstructed in GIM as a kinematic cross-check of the closed-form model. The geared five-bar linkage reproduces the typical teardrop ankle path of healthy gait with one fewer link than conventional six-bar designs, while also adding the gear ratio as an extra parameter for shaping the trajectory, which is not available in six-bar topologies. The paper also presents the full device integration, in which the input crank is powered by a single speed-controlled actuator mounted on the walker frame, together with a three-dimensional CAD assembly model.

Keywords: gait rehabilitation; walker gait trainer; geared five-bar linkage; path generation; dimensional synthesis



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1. Introduction

Stroke is among the leading causes of acquired motor impairment in the lower limbs worldwide. With population ageing, the incidence of stroke and the demand for rehabilitation continue to rise, and demographic forecasts suggest a substantial increase in the number of older stroke survivors over the next decade. A frequent and disabling outcome of stroke is hemiparesis, in which one side of the body, including the lower limb, has only partial motor function. The resulting gait impairment limits mobility and independence and is repeatedly reported by patients as a major rehabilitation issue [1,2].

Clinical evidence shows that gait function is recovered most effectively when rehabilitation starts in the acute or sub-acute phase and includes frequent, repeated practice of a physiologically correct walking pattern [1]. Conventional manual therapy, in which one or more therapists physically guide the patient's limb through the gait cycle, is effective but requires considerable labor, varies across sessions, and is constrained by therapist fatigue and availability. These drawbacks have encouraged the use of robotic and mechanically assisted rehabilitation devices, which can provide controlled, repetitive movement without the same limitations [3,4].

Many existing rehabilitation devices, however, create their own constraints. Treadmill-based exoskeletons are usually bulky, costly, restricted to specialist clinics, and reliant on a body-weight support system. Portable exoskeletons, by contrast, require crutches for balance. There is, therefore, a clear clinical need for a device that is lightweight, portable, affordable, easy to put on and take off, and suitable for home use without constant therapist supervision, while still guiding the paretic limb through a natural gait pattern.

Gait rehabilitation devices may be grouped into three categories with progressively lower mechanical complexity: treadmill-based exoskeletons, over-ground end-effector devices, and walker-based devices.

Treadmill-based systems such as the Lokomat [5], the G-EO system [6], and Loko-Help [7] use powered orthoses attached to the patient's legs to move the limbs through the gait cycle, while a body-weight support system helps maintain balance. These devices provide highly controlled training and are particularly suitable for patients with severe impairments, including those with spinal-cord injury. Their main drawbacks are high cost, bulk, the need for a body-weight support system, and limited use in specialist clinics. As a result, they are poorly suited to home settings or to small clinics.

End-effector devices guide the patient's foot along a predefined path by means of a foot plate, without an exoskeletal frame running along the limb. Reported examples in the literature include linkage-based gait trainers, adjustable elliptical exercisers, and pediatric trainers [3,8,9]. End-effector devices have been shown to produce walking-speed gains comparable to, or better than, those of exoskeleton-based devices, while being mechanically much simpler. In particular, several end-effector trainers use five-bar linkages, which support the five-bar topology as a practical foundation for rehabilitation trajectory generation [10,11].

Walker-based devices integrate a trajectory-generating linkage into a conventional wheeled walker, which at the same time offers postural support and a passive power source through its rolling wheels. Recent studies have reported walker-mounted gait trainers based on six-bar linkages in the recent literature [12]; for instance, Singh et al. implemented a Stephenson III six-bar linkage driven passively by a belt-pulley system connected to the walker wheel [13]. Such studies demonstrate the feasibility of the walker-mounted, end-effector approach. The present work proposes a separate design for this class of device, based on a geared five-bar linkage, with the goal of lowering mechanical and synthesis complexity while maintaining direct control over the generated trajectory.

More recently, the scope of walker-assisted rehabilitation has extended beyond linkage-based trainers. Loureiro et al. investigated the gait kinematics of smart walker-assisted locomotion and provided experimental reference kinematics for walker-assisted gait [14]. Li et al. proposed a single device that incorporates a multi-function mechanism for mobility support, sit-to-stand transfer, and walking assistance [15]. Table 1 compares representative single-degree-of-freedom gait-training mechanisms in terms of topology, defect constraints, reported trajectory accuracy, and actuation to quantitatively position the proposed design. The six-bar mechanisms of Singh et al. [13], Haghjoo et al. [12], and Li et al. [4] all require simultaneous treatment of Grashof, order, and circuit constraints during synthesis. The

proposed geared five-bar offers two structural advantages over these six-bar solutions: one fewer moving link and the elimination of the branch and circuit constraints (Section 3.4). For balance, its principal disadvantages are the added mass, cost, backlash, and guarding requirements of the gear train, and the need for lubrication and periodic maintenance of the meshes; the sensitivity of the generated trajectory to gear backlash is discussed in Section 5.5.

Table 1. Quantitative comparison of representative single-degree-of-freedom gait-training mechanisms and the proposed device.

Device/Ref.	Mechanism	Links/Joints	Defect Constraints	Reported Accuracy (7-pt Benchmark)	Actuation
Singh et al. [13]	Stephenson III six-bar	6/7 revolute	4 (Grashof, order, 2 circuit)	RMS 42.56 mm; sum 104.12 mm	Passive belt-pulley from wheel
Haghjoo et al. [12]	Single-DOF six-bar (movable frame)	6/7 revolute	4 (Grashof, order, 2 circuit)	Not reported on this benchmark	Motorized
Li et al. [4]	Six-bar (full-limb gait)	6/7 revolute	4 (Grashof, order, 2 circuit)	Not reported on this benchmark	Motorized
Proposed WGT	Geared five-bar	5/5 revolute + 1 gear pair	1 (order only)	RMS 16.07 mm; sum 87.06 mm	Single speed-controlled actuator

Four-bar linkages are the most basic closed-chain path generators. They are compact and mechanically reliable, but they can produce only a limited range of coupler-curve shapes, which confines them mainly to single-joint tasks such as knee guidance. Six-bar linkages, including the Watt and Stephenson families, generate much more varied coupler curves and have been used for full-limb gait trajectories as well as finger rehabilitation [4,16]. Their drawback is the difficulty of defect management: a six-bar path generator must be synthesized under Grashof, order, and circuit constraints, often across more than one constituent loop, which makes the optimization problem more complicated [16,17].

A geared five-bar linkage is a five-bar mechanism in which the two cranks are coupled by a gear pair, reducing the inherently two-degree-of-freedom five-bar linkage to a single degree of freedom. Geared five-bar mechanisms are well established in mechanism theory, with coupler-curve atlases available since the classical work of Zhang, Norton, and Hammonds [18], and have been used, for instance, with non-circular gears for exact path generation [19–21]. The gear ratio introduces an additional shape-control parameter that is not available in a plain five-bar or in a six-bar linkage. Despite this attractive property, the geared five-bar linkage has not previously been applied to a walker-mounted gait rehabilitation device; this is the principal novelty addressed by the present work.

Dimensional synthesis for path generation is a non-linear, multimodal optimization problem. A wide range of algorithms has been used, including genetic algorithms, particle swarm optimization, Differential Evolution, and parameter-free metaheuristics such as Teaching–Learning-Based Optimization (TLBO), Jaya, and the Rao algorithms [16,22]. While these automated methods are suitable for fully automatic synthesis, the present work instead adopts a design-by-analysis approach: the closed-form kinematic model developed in Section 3 is implemented in the GIM kinematic-simulation environment [23,24], and the mechanism dimensions are determined through a systematic, model-guided interactive search against the seven precision points. This approach makes use of the structural defect

elimination established in Section 3.4, which removes branch and circuit defects in advance and thereby keeps the feasible design space well behaved during manual exploration.

Four distinct gaps emerge from the literature review:

- i. The Stephenson III six-bar linkages used in current walker-based gait trainers must satisfy four defect constraints at the same time (Grashof, order, and circuit constraints on two loops), which makes the synthesis optimization problem considerably more difficult.
- ii. Six-bar linkage synthesis does not offer a mechanism-level parameter, similar to a gear ratio, that can control the coupler-curve shape independently of the link lengths.
- iii. No previous study has applied a geared five-bar linkage to a walker-mounted gait rehabilitation device, even though this topology is well established in general mechanism theory.
- iv. Existing walker-based six-bar devices still leave substantial room for better trajectory accuracy on the same set of normative-gait precision points, which supports the need for a mechanically simpler but more controllable alternative.

Building on the limitations identified in the previous section, this paper presents a mechanically simpler and more controllable alternative to the Stephenson III six-bar mechanisms that are often used in walker-based gait rehabilitation systems. The central idea is to employ a geared five-bar linkage as the trajectory-generating mechanism of a Walker Gait Trainer. This arrangement combines the compact structure of a five-bar linkage with the extra control offered by a gear-coupling parameter. Consequently, the coupler-curve shape can be modified not only by changing the link dimensions, but also by adjusting the gear ratio imposed between the two input cranks. This introduces an additional mechanism-level design variable that is not available in conventional six-bar linkage synthesis.

The contributions of this paper, all of which are supported by the kinematic modeling, dimensional synthesis, simulation results, and CAD integration presented in Sections 3–6, are summarized as follows:

- i. A geared five-bar linkage is proposed and dimensionally synthesized as the trajectory-generating mechanism of a Walker Gait Trainer, the first use of this mechanism topology in walker-based gait rehabilitation.
- ii. The general synthesis formulation includes the gear ratio as a trajectory-shaping design variable with no counterpart in six-bar topologies; in the present implementation, a unit gear ratio ($N = 1$, identical gears) was retained for manufacturing simplicity, and the mechanism reproduces the teardrop ankle trajectory with one fewer link than a conventional six-bar linkage, while non-unit gear ratios are left for future optimization.
- iii. Branch and circuit defects are shown to be structurally removed by the gear coupling, which reduces the synthesis problem from four constraints to only one order constraint.
- iv. The mechanism is synthesized through a model-guided design-by-analysis procedure using seven normative-gait precision points in the GIM kinematic-simulation environment, based on the closed-form kinematic model derived in this paper.
- v. A full device integration is presented, including a single-actuator drive of the input crank, a gear housing, and a three-dimensional CAD assembly model.

2. Clinical Design Specifications and Desired Ankle Trajectory

2.1. Biomechanics of Human Gait

2.1.1. Gait Cycle Phases

Human walking is a repetitive cyclic motion. A gait cycle is typically divided into a stance phase, which accounts for about 60% of the cycle and during which the foot remains

in contact with the ground, and a swing phase, which makes up the remaining 40% and during which the foot is lifted and moved forward. The stance phase includes heel strike, loading response, mid-stance, terminal stance, and pre-swing; the swing phase includes initial swing, mid-swing, and terminal swing. During one full cycle, the ankle joint follows a distinct asymmetric closed trajectory, commonly described as a teardrop shape, in the sagittal plane [2].

2.1.2. Sagittal-Plane Kinematic Model

Walking is primarily characterized by motion in the sagittal plane. The lower limb is represented as a planar two-link (2R) kinematic chain, with single-axis revolute joints at the hip and knee, as shown in Figure 1. In the sagittal-plane model, the hip joint is assumed fixed. The model controls the trajectory of the ankle point in the sagittal plane; foot orientation and plantar-/dorsiflexion are constrained by the foot plate of the Walker Gait Trainer and are therefore not modeled explicitly, which is a limitation to be addressed in future work. With the hip angle and knee angle known, the ankle position is determined using forward kinematics:

$$A_{ex} = h_x + L_f \cos(3\pi/2 + \theta_{hip}) + L_s \cos(3\pi/2 + \theta_{hip} - \theta_{knee}) \quad (1)$$

$$A_{ey} = h_y + L_f \sin(3\pi/2 + \theta_{hip}) + L_s \sin(3\pi/2 + \theta_{hip} - \theta_{knee}) \quad (2)$$

where $A_e = (A_{ex}, A_{ey})$ is the ankle position, $H_p = (h_x, h_y)$ is the fixed hip position, L_f is the femur length, L_s is the shank length, and θ_{hip} and θ_{knee} are the hip and knee angles obtained from normative gait data [2].

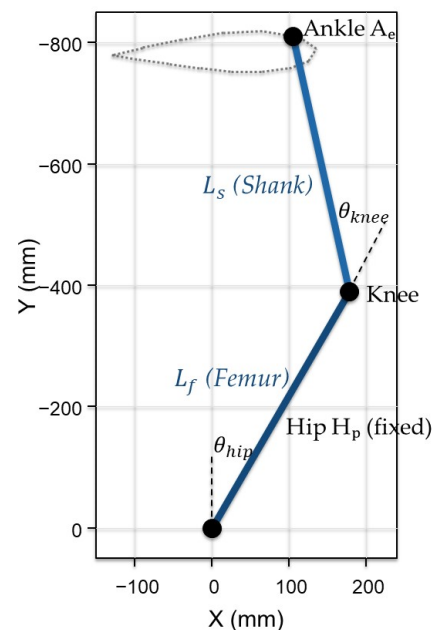


Figure 1. Sagittal-plane 2R model used to define the desired ankle trajectory from the hip and knee angles.

2.2. Anthropometric Data

To enable a direct and fair comparison with prior walker-based devices, the anthropometric parameters are taken to correspond to a representative adult subject of height $H = 1.75$ m. Standard anthropometric relations give the segment lengths as follows [2,13].

Femur length: $L_f = 0.245 H = 0.245 \times 1750 \approx 429$ mm.

Shank length: $L_s = 0.246 H = 0.246 \times 1750 \approx 431$ mm.

Using a fixed, well-defined subject anthropometry ensures that any difference in trajectory accuracy between candidate mechanisms is attributable to the mechanism and the synthesis method, and not to a change in the target curve.

2.3. Desired Ankle Trajectory

The desired ankle trajectory is obtained by inserting the normative hip and knee angle profiles for one full gait cycle into the forward-kinematics equations, Equations (1) and (2). The closed path produced in the (X , Y) plane serves as the synthesis target for the mechanism.

Trajectory Features

This desired ankle path is marked by several key features: a horizontal extent, which corresponds to stride length, of about 700 mm; a vertical rise at peak swing of about 100 mm; and an asymmetric closed-loop form with a comparatively flat lower section, representing forward foot motion during stance, and a curved upper arc, representing heel-off, swing, and descent.

The aim of the mechanism synthesis is to reproduce both the size and the general form of this curve.

2.4. Precision Points Selection

Seven precision points are chosen on the desired ankle trajectory to define the synthesis target. The first and last points of the gait cycle are included, and the remaining points are placed at biomechanically important gait events while maintaining the smoothness of the curve. These seven precision points are given in Table 2; they match the precision-point set used by Singh et al. [13], ensuring that the trajectory-accuracy comparison in Section 5 is based on a common and clear reference.

Table 2. Desired ankle-joint precision points.

Point	Index	A_{ex} (mm)	A_{ey} (mm)	Gait Phase
P1	1st	1226.2	606.07	Heel strike (stance start)
P2	5th	1075.9	565.99	Mid-stance loading
P3	15th	685.3	568.24	Opposite toe-off
P4	19th	560.9	631.11	Swing initiation (toe-off)
P5	21st	580.5	668.49	Early swing—maximum lift
P6	27th	1024.7	586.70	Terminal swing—descent
P7	31st	1252.3	617.04	Pre-heel strike

2.5. Walker Gait Trainer Design Specifications

The following requirements limit the mechanical design and define the range of the optimization variables in Section 4.3.

- Actuation: the mechanism is driven by a single speed-controlled actuator applied to the input crank, allowing the cadence of the guided gait cycle to be set and adjusted for the patient.
- Single degree of freedom: the input crank rotates continuously under actuator control, and one crank revolution generates one complete guided gait cycle while the patient advances with the walker.
- Frame envelope: the mechanism must fit within the lateral outline of the walker frame.
- Foot plate: a rigid foot plate is fixed at the coupler point and is intended to remain approximately horizontal, reducing uncontrolled plantar-/dorsiflexion during guided motion.

- v. Patient range: the device is intended for adult patients with heights between 1.65 and 1.85 m, and an adjustable foot-plate height is suggested for future work.

3. Geared Five-Bar Mechanism: Topology and Kinematics

3.1. Mechanism Topology

3.1.1. Link and Joint Inventory

The proposed mechanism is a planar five-bar linkage in which the two cranks are linked by a gear train [19]. As shown in the kinematic diagram in Figure 2, the mechanism includes five rigid links connected by five revolute joints, together with one gear constraint. The link and joint inventory are given below.

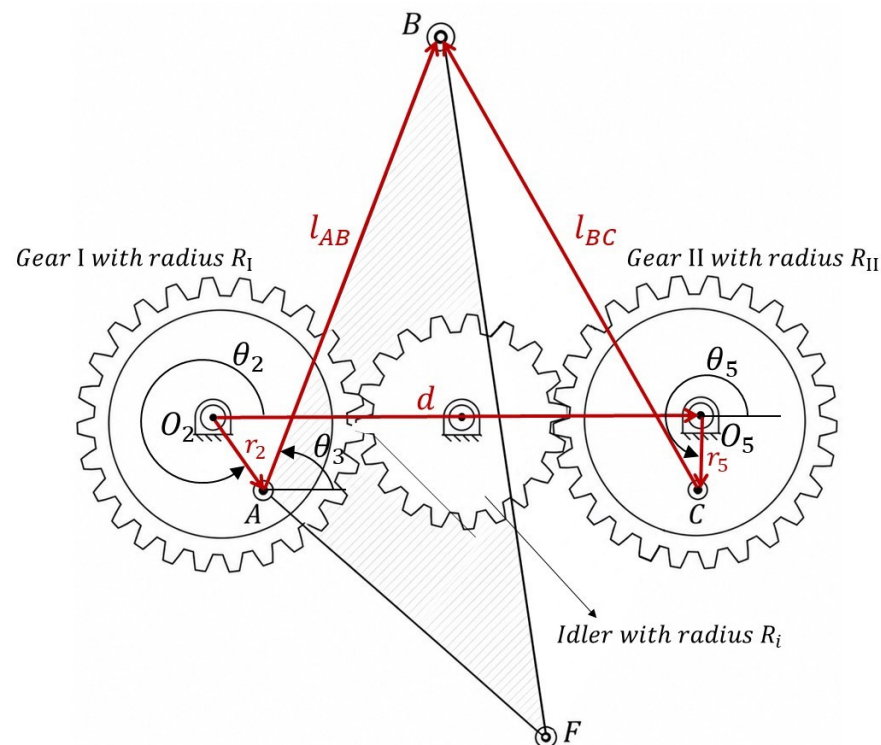


Figure 2. Kinematic diagram of the proposed geared five-bar mechanism, with labelled joints O_2 , A, B, C, O_5 and the foot-plate point F.

Link 1 (ground). The fixed walker frame, which carries the two ground pivots O_2 and O_5 at a constant distance d . Both pivots are installed on a lateral crossbar of the walker frame.

Link 2 (crank 1). A binary link of length $r_2 = O_2A$ that rotates fully about O_2 . Crank 1 is the input link; it is driven directly by a single speed-controlled actuator.

Link 3 (ternary coupler). A ternary link with vertices A, B, and F. Its three side lengths are l_{AB} , l_{AF} , and l_{BF} . Point F is the foot-plate attachment, serving as the end-effector that follows the ankle path.

Link 4 (coupler). A binary link of length l_{BC} joining vertex B of the ternary coupler to joint C on crank 2.

Link 5 (crank 2). A binary link of length $r_5 = O_5C$ that rotates fully about O_5 . Link 5 is kinematically linked to Link 2 by the gear train.

These five links create the closed kinematic loop O_2 -A-B-C- O_5 . The foot-plate point F is located on the ternary coupler and projects below the loop to ankle level. The gear train between O_2 and O_5 consists of two identical spur gears, Gear I, with pitch radius R_I , mounted coaxially with crank 1 (O_2A) about O_2 , and Gear II, with pitch radius R_{II} ,

mounted coaxially with crank 2 (O_5C) about O_5 , connected by a small intermediate idler gear with pitch radius R_i , as illustrated in Figure 2. The idler serves two purposes: it spans the center distance between the two ground pivots, which is usually larger than the sum of the two gear pitch radii, and it ensures an even number of external meshes so that the two cranks rotate in the same direction. Since the two end gears are identical, the speed ratio between the cranks is one.

3.1.2. Degree-of-Freedom Analysis

The mobility of the mechanism is established using the equivalent kinematic representation in which the three-gear train is reduced to one angular constraint between the two cranks. In this representation, the idler gear is not counted as an independent output link because it only transmits motion and does not add a task variable. The Grübler-Kutzbach criterion for planar mechanisms, whose foundations and known limitations are reviewed in the critical survey of Gogu [25], and whose result for the present gear-coupled topology is confirmed by alternative degree-of-freedom formulations based on constraint normal lines [26] and on the concept of spaces in structural analysis and synthesis [27], gives

$$M = 3(n - 1) - 2j_1 - j_2 \quad (3)$$

in which n is the number of links, j_1 the number of lower (revolute) pairs, and j_2 the number of higher (gear) pairs. For the proposed mechanism $n = 5$, $j_1 = 5$ (revolute pairs at O_2 , A, B, C and O_5) and $j_2 = 1$ (the gear pair coupling the two cranks). Substitution gives $M = 3(5 - 1) - 2(5) - 1 = 12 - 10 - 1 = 1$.

The mechanism therefore has one degree of freedom: once the input crank rotates, the configuration of all links and the position of the foot plate are fully fixed. This satisfies the design requirement of a device driven by a single actuator input. The basic five-bar linkage, if the gear pair were absent, would have two degrees of freedom ($M = 3 \times 4 - 2 \times 5 = 2$); the gear constraint eliminates one of them.

3.1.3. Gear Constraint

The gear train imposes a fixed relationship between the angular positions of the two cranks. Let θ_2 and θ_5 denote the angular positions of crank 1 and crank 2, measured from the positive X-axis. The gear constraint is

$$\theta_5 = N \theta_2 + \varphi_0 \quad (4)$$

where N is the gear ratio and φ_0 is the initial phase offset between the two cranks. The gear ratio is the inverse ratio of the tooth numbers of the two end gears, $N = Z_I/Z_{II}$. In the present configuration, the two end gears are identical, so $Z_I = Z_{II}$ and $N = 1$; both cranks therefore rotate with the same angular speed. The idler gear between the two end gears does not change the speed ratio: it only reverses the direction of rotation, and because it creates two external meshes, the two end gears rotate in the same direction. The phase offset φ_0 is a true design parameter, set by the relative angular mounting of the two cranks on their gear shafts, and it is obtained from the dimensional-synthesis procedure in Section 4.

Differentiating Equation (4) with respect to time gives the angular velocity and angular acceleration relations $\omega_5 = N\omega_2$ and $\alpha_5 = N\alpha_2$, which are used in the velocity and acceleration analysis of Section 3.3. Although the present device uses $N = 1$, the full kinematic and synthesis formulation below is written for a general gear ratio N , so that a future variable-ratio device can be developed within the same framework. The gear ratio is kept as a design variable in Section 4, specifically to treat it as a parameter for shaping the trajectory.

The gear train must also satisfy the meshing, or center-distance, condition. For the three-gear arrangement, the center distance between crank 1 and the idler is $R_I + R_i$, and that between the idler and crank 2 is $R_i + R_{II}$, where R_I , R_{II} , and R_i are the pitch radii of the two end gears and the idler. Since $R_I = R_{II}$, the idler lies on the line O_2O_5 , and the meshing condition determines its pitch radius and position once the ground-pivot spacing d is known. Thus, the gear geometry follows from the synthesized link geometry rather than serving as an independent set of unknowns.

3.2. Position Analysis

The position analysis finds the coordinates of all joints, as well as the foot plate F , for a given input crank angle θ_2 . Since this computation must be carried out repeatedly during the design search, an efficient closed-form geometric method is used instead of iterative root-finding.

3.2.1. Vector Loop-Closure Equation

The closed kinematic loop O_2 -A-B-C- O_5 is expressed as a vector loop-closure equation in the complex plane [28,29]. Traversing the loop from O_2 to O_5 by two distinct paths and equating them gives

$$r_2 e^{i\theta_2} + l_{AB} e^{i\theta_3} = d e^{i\alpha} + r_5 e^{i\theta_5} + l_{BC} e^{i\theta_4} \quad (5)$$

where r_2 and r_5 are the crank lengths, l_{AB} and l_{BC} the coupler-link lengths, d the ground-pivot distance O_2O_5 , α the inclination of the ground link, and θ_3 and θ_4 the unknown orientations of links AB and BC. Substituting the gear constraint, Equation (4), for θ_2 leaves Equation (5) as one complex equation, equivalently, two real scalar equations, in the two unknowns θ_3 and θ_4 .

Rather than solving the two transcendental scalar equations directly, the position problem is resolved geometrically. For a given θ_2 , the gear constraint fixes θ_5 , and the coordinates of joints A and C follow immediately from the crank geometry:

$$A_x = O_{2x} + r_2 \cos \theta_2 \quad A_y = O_{2y} + r_2 \sin \theta_2 \quad (6)$$

$$C_x = O_{5x} + r_5 \cos \theta_5 \quad C_y = O_{5y} + r_5 \sin \theta_5 \quad (7)$$

Joint B is then the common point of two circles: a circle of radius l_{AB} centered at A, and a circle of radius l_{BC} centered at C. Writing the vector from A to C as (e_x, e_y) with magnitude e , the two intersection points are

$$B = M \pm \mu \cdot \hat{n}_{AC} \quad (8)$$

where M is the point on line AC at distance $a = (l_{AB}^2 - l_{BC}^2 + e^2)/2e$ from A, the half-chord height is $\mu = \sqrt{(l_{AB}^2 - a^2)}$, and $\hat{n}_{AC}$ is the unit vector perpendicular to AC. The sign $\pm$ selects one of the two assembly branches; the branch is fixed at the start of synthesis and held constant for the whole crank rotation, which, as shown in Section 3.4, guarantees a defect-free circuit.

3.2.2. Coupler-Point Coordinates

Once joints A and B are known, the ternary coupler is fully located, since F is rigidly attached to it. The foot-plate point F is found as the intersection of a circle of radius l_{AF} centered at A with a circle of radius l_{BF} centered at B,

$$F = \text{circle}(A, l_{AF}) \cap \text{circle}(B, l_{BF}) \quad (9)$$

This intersection is evaluated by the same circle–circle construction as Equation (8), and again admits two branches; the branch placing F below the coupler loop, at ankle height, is selected. Equivalently, F may be written in terms of the orientation θ_3 of link AB and the fixed apex angle δ of the ternary triangle at vertex A,

$$F_x = A_x + l_{AF} \cos(\theta_3 + \delta) \quad F_y = A_y + l_{AF} \sin(\theta_3 + \delta) \quad (10)$$

where the apex angle δ is obtained from the three side lengths of the ternary triangle by the cosine rule, $\delta = \arccos[(l_{AB}^2 + l_{AF}^2 - l_{BF}^2) / (2 l_{AB} l_{AF})]$. As the input crank angle θ_2 sweeps from 0 to 2π , the point F traces a closed curve; this curve is the ankle trajectory generated by the mechanism, and its proximity to the desired trajectory of Section 2 is the objective of the synthesis.

3.3. Velocity and Acceleration Analysis

Velocity and acceleration analysis is required for two purposes: to verify that the foot-plate motion is smooth, which is a clinical requirement, and to provide the basis for a future dynamic analysis of the device.

3.3.1. Velocity Analysis

Differentiating the loop-closure equation, Equation (5), with respect to time, and using the gear relation $\omega_5 = N\omega_2$, yields a linear system in the two unknown coupler angular velocities ω_3 and ω_4 ,

$$[J] \begin{pmatrix} \omega_3, \omega_4 \end{pmatrix}^T = \omega_2 b \quad (11)$$

where the 2×2 Jacobian matrix $[J]$ collects the partial derivatives of the loop-closure equation with respect to θ_3 and θ_4 , and the vector b collects the known contributions of the two cranks. The system is solved for ω_3 and ω_4 at each crank position. The velocity of the foot plate is then obtained by differentiating the position expressions, Equation (10),

$$V_{Fx} = \dot{x}_A - l_{AF}\omega_3 \sin(\theta_3 + \delta) \quad V_{Fy} = \dot{y}_A + l_{AF}\omega_3 \cos(\theta_3 + \delta) \quad (12)$$

The magnitude of the foot-plate velocity, $|V_F| = \sqrt{V_{Fx}^2 + V_{Fy}^2}$, is reported in Section 5.4 as a function of the crank angle and is compared with the velocity profile expected from normative gait.

3.3.2. Acceleration Analysis

Differentiating the velocity equations once more with respect to time gives the acceleration analysis. The two unknown angular accelerations α_3 and α_4 are obtained from a linear system with the same Jacobian $[J]$ as Equation (11),

$$[J] \begin{pmatrix} \alpha_3, \alpha_4 \end{pmatrix}^T = c \quad (13)$$

in which the right-hand side c includes the known crank accelerations as well as the centripetal terms that are quadratic in the angular velocities. Reusing the Jacobian $[J]$ from the velocity stage keeps the acceleration analysis computationally inexpensive. The foot-plate acceleration is then obtained by differentiating Equation (12) once more. In this study, acceleration analysis is employed to verify that the foot-plate motion does not exhibit sudden changes that would be clinically undesirable; a dynamic force analysis based on these accelerations is proposed as future work.

3.4. Defect Analysis: Structural Elimination

This subsection presents the main theoretical result of the paper: the gear coupling eliminates, by structure, two of the three classical linkage defects, so that dimensional synthesis involves only one constraint. The three defects are discussed separately.

3.4.1. Grashof/Branch Defect

A branch, or Grashof, defect occurs when a mechanism cannot attain a required configuration unless it is taken apart, because a crank is unable to complete a full revolution and the motion stops at a dead-center position. In a four-bar or six-bar path generator, this is avoided by satisfying the Grashof condition on the link lengths, which is a standard constraint in conventional synthesis [16,17]. For geared five-bar chains specifically, formal mobility and rotatability criteria were established by Ting [30], systematic branch-identification procedures were developed by Dou and Ting [31], and the full rotatability and singularity of six-bar and geared five-bar linkages were characterized in [32]; the structural argument below is consistent with these criteria and is verified on the synthesized mechanism in Section 3.4.5. In the geared five-bar, both cranks are rigidly connected through gears to a common input that rotates continuously. As a result, each crank completes full revolutions by design, and neither one can stop at a dead-center position. No Grashof condition on the link lengths is needed, and the related constraint is therefore eliminated from the synthesis problem.

3.4.2. Circuit Defect

A circuit defect occurs when the desired precision points belong to different, disconnected circuits of the coupler curve, so the mechanism cannot move through all of them within a single assembly. In a Stephenson III six-bar, a circuit constraint must generally be imposed, often on more than one loop of the mechanism. In the geared five-bar, the gear constraint reduces the system to a strictly single-parameter mechanism: the input angle θ_2 alone determines the position of every joint through Equations (4)–(10). As θ_2 increases steadily through one revolution, the foot plate generates one, and only one, continuous closed curve. If the assembly branch selected in Equations (8) and (9) is kept fixed, which is done throughout both synthesis and operation, then no second circuit can appear, and a circuit defect cannot arise. The circuit constraints are therefore also removed from the synthesis problem.

3.4.3. Order Defect

An order defect arises when the mechanism visits the precision points in the wrong sequence. Unlike the branch and circuit defects, this defect is not removed by the gear coupling: the synthesized link lengths may still pass through the seven target points in a different order. Thus, the order defect remains as the only constraint in the synthesis problem. It is expressed by requiring that the input crank angles at successive precision points be strictly monotonic along the direction of crank rotation. With the crank angle unwrapped along the (clockwise) direction of rotation adopted here, that is, measured continuously without reduction to the interval $[0^\circ, 360^\circ)$, the constraint reads

$$\theta_2^{(j+1)} < \theta_2^{(j)}, \quad j = 1, 2, \dots, 6 \quad (14)$$

Equation (14) is the only defect constraint needed for the geared five-bar, compared with the four constraints required for a Stephenson III six-bar. In the synthesis result reported in Section 5, the crank rotates clockwise and the unwrapped crank angles at the seven precision points form the strictly decreasing sequence $311.54^\circ > 257.54^\circ > 188.54^\circ > 146.54^\circ > 116.54^\circ > 8.54^\circ > -51.46^\circ$, where the last value is the unwrapped equiv-

alent of the angle 308.54° reported for P7 in the synthesized mechanism (Section 5.1) ($308.54^\circ - 360^\circ = -51.46^\circ$); Equation (14) is therefore satisfied. If the angles were instead reduced to the interval $[0^\circ, 360^\circ)$, the sequence would appear non-monotonic at P7; the unwrapped representation removes this apparent inconsistency and is the form in which the order constraint is checked throughout this work.

3.4.4. Constraint Comparison

Table 3 compares the two mechanisms. The reduction from four defect constraints to a single one follows directly from the gear coupling structure. This offers two practical advantages for the synthesis. First, the allowable design region becomes much larger, which expands the set of admissible mechanisms that can be considered during the design search. Second, since only the order constraint needs to be checked, the interactive search in Section 4 depends on one feasibility condition instead of four, making the design space easier to examine manually.

Table 3. Comparison between a conventional Stephenson III six-bar and the proposed geared five-bar linkage.

Feature	Stephenson III Six-Bar	Geared Five-Bar (Proposed)
Number of links	6	5
Joints	7 revolute	5 revolute + 1 gear pair
Degrees of freedom	1	1
Grashof/branch constraint	Required	Structurally eliminated
Circuit constraint	Required (two loops)	Structurally eliminated
Order constraint	Required	Required
Total defect constraints	4	1
Crank rotation	Crank-rocker (Grashof-dependent)	Full rotation guaranteed
Trajectory-shaping parameter	None	Gear ratio N, phase φ_0

3.4.5. Quantitative Verification of Defect-Free Operation

To complement the structural argument of Sections 3.4.1–3.4.3 by a direct check on the synthesized mechanism, the circle-intersection construction of Equations (8) and (9) is checked over one full crank revolution using the synthesized mechanism parameters (Section 5.1). The assembly condition $|l_{AB} - l_{BC}| \leq e(\theta_2) \leq l_{AB} + l_{BC}$ remains valid over the whole revolution; therefore, the circle intersection of Equation (8) always has a real solution, and the half-chord height $\mu(\theta_2)$ never becomes zero. The coupler point of Equation (9) is also well defined at every crank angle. Hence, the mechanism does not involve a branch-switching configuration, which is in agreement with the structural argument of Section 3.4.1 and with the formal rotatability and singularity criteria for geared five-bar linkages and the intuitive geometric singularity measures presented in the authors' earlier kinematic studies [30–34]. Since only one assembly branch is retained (Section 3.4.2), the foot plate traces one continuous closed curve, as shown in Figure 3a. The only remaining requirement is the order constraint of Equation (14): The input crank angle θ_2 is plotted in Figure 3b at the seven precision points, unwrapped in the clockwise direction of rotation (so that P7 is plotted at $308.54^\circ - 360^\circ = -51.46^\circ$). The angles are seen to be a strictly decreasing sequence, and so the precision points are visited in the prescribed order. Thus, Figure 3 visually verifies the branch-, circuit- and order-defect-free operation structurally established on the synthesized mechanism in Section 3.4.

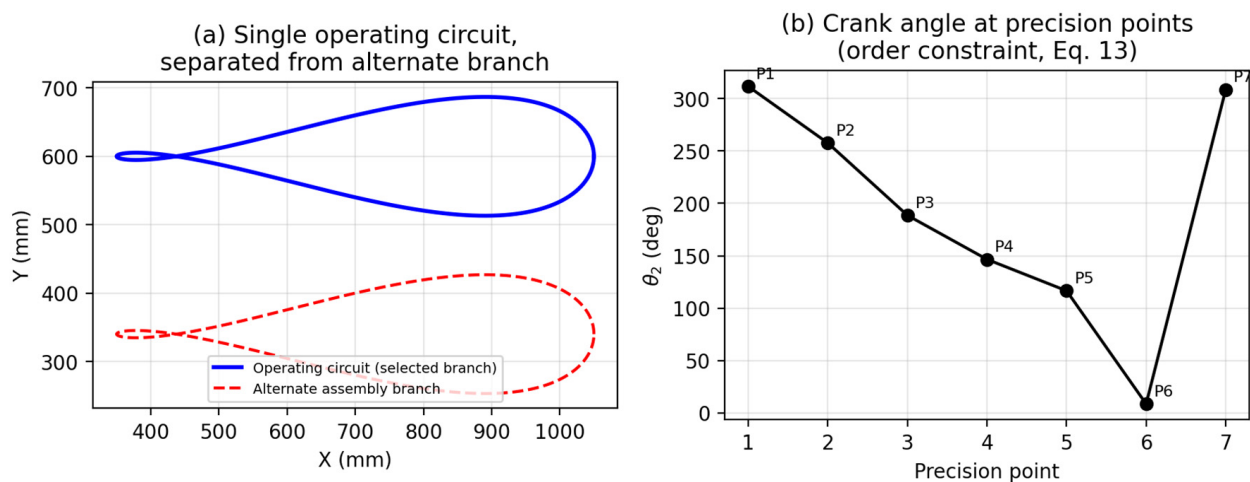


Figure 3. Defect-free operation of the synthesized mechanism: (a) continuous coupler curve of point F for the selected assembly branch; (b) unwrapped input-crank angles at the seven precision points.

3.5. Application-Specific Task-Space Feasibility

For the proposed Walker Gait Trainer, the clinically relevant task is not arbitrary positioning of the foot plate over a two-dimensional area, but the repeatable generation of one prescribed closed ankle trajectory. Thus, a conventional workspace map of the underlying two-degree-of-freedom five-bar, derived from removing the gear constraint, would describe a parent mechanism that is not physically operated in the final device and could be misleading as a performance metric. Workspace studies are useful for free five-bar manipulators [35]. For the present single-degree-of-freedom rehabilitation device, the relevant feasibility checks are: (i) the generated trajectory reaches the selected gait precision points; (ii) the retained assembly branch remains valid over the full crank revolution; (iii) the half-chord $\mu(\theta_2)$ does not approach zero, so no branch-switching singularity is encountered; and (iv) the precision points are visited in the prescribed gait order. These application-specific task-space conditions are verified directly in Section 3.4.5 and Figure 3. Consequently, the design is assessed through trajectory accuracy, order preservation, and singularity-free operation rather than by the area of an unconstrained two-degree-of-freedom workspace.

4. Dimensional Synthesis Formulation

4.1. Design Variables

The synthesis aims to determine the link geometry, gear parameters, and the set of input crank angles at which the foot plate passes through the seven precision points. These quantities are gathered in the design-variable vector

$$x = [O_{2x}, O_{2y}, O_{5x}, O_{5y}, r_2, r_5, l_{AB}, l_{BC}, l_{AF}, l_{BF}, N, \varphi_0, \theta_2^{(1)}, \dots, \theta_2^{(7)}] \quad (15)$$

This vector includes 19 design variables: twelve geometric parameters: four pivot coordinates, two crank lengths, two coupler-link lengths, two ternary-triangle side lengths, the gear ratio, and the phase offset, together with seven input crank angles, one for each precision point. The physical interpretation and bounds of each variable are listed in Table 4. The gear ratio and phase offset have no equivalent in the six-bar topology, and they provide separate control over the shape of the coupler curve.

Table 4. Design variables, bounds, and physical interpretation.

Variable	Description	Bounds	Remarks
O_{2x}, O_{2y}	Ground pivot 1 coordinates	[100, 1400] mm	Within frame envelope
O_{5x}, O_{5y}	Ground pivot 2 coordinates	[100, 1400] mm	Within frame envelope
r_2	Crank 1 length, O_2A	[30, 300] mm	Input crank
r_5	Crank 2 length, O_5C	[30, 300] mm	Geared crank
l_{AB}	Coupler link AB	[80, 600] mm	Free parameter
l_{BC}	Coupler link BC	[80, 700] mm	Free parameter
l_{AF}	Ternary side AF	[150, 900] mm	To ankle height
l_{BF}	Ternary side BF	[200, 1000] mm	To ankle height
N	Gear ratio, ω_5/ω_2	[0.5, 5.0]	$N = 1$ for identical gears
φ_0	Crank phase offset	[0, 2π] rad	Shaping parameter
$\theta_2^{(1)} \dots \theta_2^{(7)}$	Crank angles at precision points	[0, 2π] rad	Order constraint applies

4.2. Objective Function

The mechanism is required to generate a coupler curve that passes as closely as possible through the seven desired precision points. The primary objective is therefore the sum of the Euclidean distances between each desired precision point and the corresponding generated foot-plate point,

$$\text{minimise } f_1(x) = \sum_{i=1}^7 \|P_d(i) - P_g(i)\| \quad (16)$$

where $P_d(i)$ is the i -th desired precision point of Table 2 and $P_g(i)$ is the foot-plate point generated by the mechanism at the input crank angle θ_2 associated with that precision point. The generated point is obtained from the position analysis of Section 3.2.

A clinically motivated secondary objective penalizes non-uniformity of the foot-plate speed, since abrupt velocity changes are undesirable in rehabilitation. It is defined as the coefficient of variation of the foot-plate speed over one complete crank revolution,

$$f_2(x) = \sigma(|V_F|) / \mu(|V_F|) \quad (17)$$

where σ and μ denote the standard deviation and the mean of the foot-plate speed $|V_F|$. The two objectives are combined into a single weighted objective,

$$f(x) = f_1(x) + \lambda f_2(x) + P(x) \quad (18)$$

in which λ is a small weighting factor and $P(x)$ is the penalty term enforcing the order constraint. Setting $\lambda = 0$ recovers the pure path-generation objective, which is used for the primary accuracy comparison so that the comparison remains strictly like-for-like.

4.3. Constraints and Bounds

4.3.1. Order Constraint and Penalty Formulation

As established in Section 3.4, the only defect constraint required for the geared five-bar is the order constraint, Equation (14). The constrained problem is converted to an

unconstrained one by the exterior penalty method. A Boolean coefficient B flags any violation of the order constraint,

$$B = 0 \text{ if } \theta_2^{(j)} < \theta_2^{(j+1)} \text{ for all } j; \quad B = 1 \text{ otherwise} \quad (19)$$

and the penalty term added to the objective is

$$P(x) = B \cdot P_e, \quad P_e = 10^4 \quad (20)$$

where P_e is a large penalty constant that renders any order-violating candidate non-competitive. Only this single penalty term is needed; a six-bar synthesis, by contrast, requires four such terms, one each for the Grashof, order, and two circuit constraints. The simpler penalty structure is a direct consequence of the defect elimination proved in Section 3.4 and contributes to a smoother objective landscape.

4.3.2. Gear Meshing Condition

The gear train must engage properly in physical terms. In the identical-gear case with $N = 1$, the two outer gears have the same pitch radii $R_I = R_{II}$, and the pitch radius and placement of the idler are determined by the separation of the ground pivots through the center-distance condition in Section 3.1.3. As a result, the gear layout is determined after the fact from the synthesized pivot positions and crank lengths, and it is not included in the optimization as a separate constraint. For a general gear ratio $N \neq 1$, the tooth numbers Z_I and Z_{II} are chosen by rounding N to the nearest practical rational fraction after optimization, and the slight adjustment in N is then handled through a final re-optimization with N held fixed.

4.3.3. Variable Bounds

The bounds shown in Table 4 are selected to guarantee structural feasibility (that is, strictly positive link lengths), to keep the pivot locations inside the lateral outline of the walker frame, and to maintain the gear ratio within a range that is easy to manufacture. Since the geared five-bar needs only one defect constraint, a relatively broad search region can be examined without the optimizer being blocked by infeasible designs, which lowers the chance of ruling out promising solutions in advance.

4.4. Design Search Procedure

The formulation in Sections 4.1–4.3 specifies the design variables, the objective, and the single order constraint that every acceptable mechanism must meet. Instead of solving the problem with a fully automated optimizer, the mechanism dimensions in this work are obtained through a structured, model-based design-by-analysis search performed interactively in the GIM kinematic-analysis environment. This method is practical because of the structural defect removal established in Section 3.4: since branch and circuit defects are eliminated beforehand by the gear coupling, every candidate assembled in GIM with continuously rotating cranks is automatically free of these defects, and only the order of the precision points must be checked during the search.

The search is then performed as follows. First, the closed-form position analysis in Section 3.2 is used to generate an initial estimate of the link lengths, the ground-pivot locations, and the phase offset that aligns the foot-plate point near the target teardrop trajectory. This initial mechanism is then constructed in GIM, and its coupler curve is generated for one full crank revolution. The design variables in Table 4 are then varied iteratively: the ground-pivot positions, the crank and coupler lengths, the ternary-triangle dimensions, and the phase offset; after each variation, the resulting

curve is compared to the seven precision points and to the full desired curve in Section 2. Changes are kept when they reduce the precision-point error and the deviation from the full curve, and the order of the precision points along the generated curve is checked at every stage to ensure that the order constraint in Equation (14) remains satisfied. The two end gears are kept the same, so the gear ratio remains fixed at $N = 1$ throughout. The search ends when further manual adjustment no longer gives a meaningful improvement in trajectory error.

This design-by-analysis approach replaces the global assurance of an automated optimizer with direct physical understanding: each adjustment has an immediate, interpretable effect on the coupler curve that the designer can observe, which suits this low-dimensional, defect-free problem and a device that must also meet the practical mounting constraints in Section 2.5. The resulting mechanism is presented in Section 5.1, and its accuracy relative to the precision points and the desired curve is assessed in Section 5.3. The same GIM-based design-by-analysis strategy was previously applied by the authors to synthesize a planar six-bar crank-driven leg mechanism for walking robots [36], confirming its suitability for this class of low-dimensional path-generation problem.

To position the adopted design-by-analysis procedure with respect to alternative synthesis approaches, two representative alternatives are considered. Population-based global optimizers such as differential evolution are directly applicable to the seven-point path-generation problem of Equations (15)–(20) under the same variable bounds and the single order constraint of Equation (14); because the gear coupling removes the branch and circuit defects beforehand (Section 3.4), the feasible space presented to such an optimizer is unusually well behaved, and no defect-repair penalties are required. Exact synthesis methods, such as the cognate-based route recently formalized by Soriano-Heras et al. [37], are attractive for low precision-point counts but do not extend directly to the seven-point geared five-bar problem considered here. The approaches are therefore complementary. The design-by-analysis route was retained because it preserves full physical interpretability of every design adjustment, enforces the mounting constraints of Section 2.5 directly, and exploits the structurally defect-free search space established in Section 3.4; a systematic quantitative benchmarking of the retained design against a population-based global optimizer on the same problem is identified as future work.

4.5. Kinematic Cross-Verification in GIM

The final mechanism was reconstructed and animated in GIM as a kinematic cross-check of the closed-form model of Section 3. Because the design search also used GIM interactively, the cross-check should be understood as a consistency check between the analytical equations, the final geometric parameters, and the implemented kinematic model, rather than as an experimental validation. The link lengths, pivot locations, and phase offset of the final mechanism are given in Table 5, and the foot-plate point is animated through one complete revolution to generate the coupler curve. This curve is then compared point by point with the desired precision points in Table 2 and with the complete desired ankle curve in Section 2; the findings are reported in Section 5. The match confirms that the kinematic model in Section 3, the precision-point targets in Section 2, and the implemented mechanism are mutually consistent.

Table 5. Synthesized geared five-bar mechanism parameters.

Parameter	Symbol	Value
Crank 1 length	$r_2 = O_2A$	120 mm
Coupler link AB	l_{AB}	484.44 mm
Coupler link BC	l_{BC}	477.41 mm
Crank 2 length	$r_5 = O_5C$	95.11 mm
Ternary side AF	l_{AF}	702.65 mm
Ternary side BF	l_{BF}	1102 mm
Ground pivot 1	$O_2 = (O_{2x}, O_{2y})$	(710, 1260) mm
Ground pivot 2	$O_5 = (O_{5x}, O_{5y})$	(1392, 1260) mm
Gear ratio	N	1 (identical end gears)
Phase offset	φ_0	57°

5. Results and Discussion

5.1. Synthesized Mechanism Parameters

The design search procedure described in Section 4.4 was applied to the seven precision points listed in Table 2. The resulting geared five-bar mechanism is specified by the link lengths, ground-pivot positions, gear ratio, and phase offset reported in Table 5; the corresponding input crank angles at the seven precision points are presented in Table 6. The synthesized mechanism is shown to scale in Figure 4, together with the generated coupler curve and gear train.

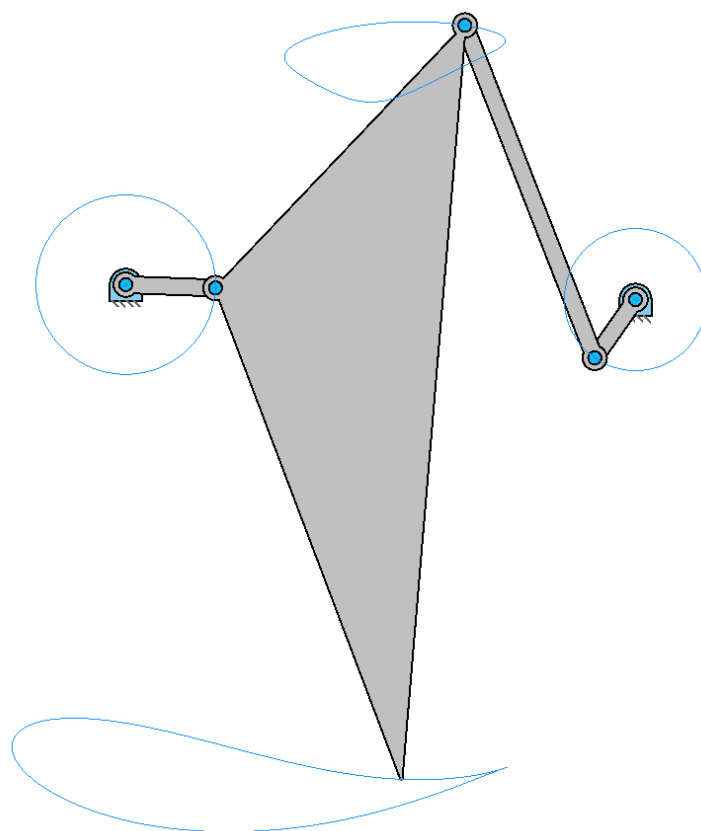
**Figure 4.** Synthesized geared five-bar mechanism, with the generated ankle trajectory and the gear train.

Table 6. Input crank angles at the seven precision points.

Precision Point	Crank Angle θ_2 (rad)	Crank Angle θ_2 (deg)
P1	5.437	311.54
P2	4.495	257.54
P3	3.291	188.54
P4	2.558	146.54
P5	2.034	116.54
P6	0.149	8.54
P7	5.385	308.54

Note: the angles are reported reduced to the interval $[0^\circ, 360^\circ)$. For the order-constraint verification of Equation (14), the angles are unwrapped along the clockwise direction of rotation, so that P7 corresponds to $308.54^\circ - 360^\circ = -51.46^\circ$ ($5.385 - 2\pi = -0.898$ rad) and the sequence P1–P7 is strictly decreasing.

5.2. GIM Kinematic Cross-Verification

The synthesized mechanism was reconstructed in GIM, as described in Section 4.5, as a kinematic cross-check of the closed-form model of Section 3. Figure 5 presents the ankle path generated by the GIM model together with the target path derived in Section 2. The strong match between the GIM path and the target curve confirms the validity of the kinematic formulation in Section 3. It must be emphasized that the deviations reported in Section 5.3 are synthesis residuals, that is, differences between the trajectory of the synthesized mechanism and the normative gait target, and not numerical errors of the model: the closed-form formulation of Section 3 and the GIM reconstruction agree to numerical round-off at all seven precision points, since both implement the same rigid-body kinematics. The synthesis residual itself has three identifiable sources: the finite number of precision points (seven) representing a continuous gait curve; the sharp curvature reversal of the target near the heel-strike tip, where P7 is located, and the largest single error occurs; and the deliberate restriction to a unit gear ratio ($N = 1$), which withholds one trajectory-shaping parameter for reasons of manufacturing simplicity.

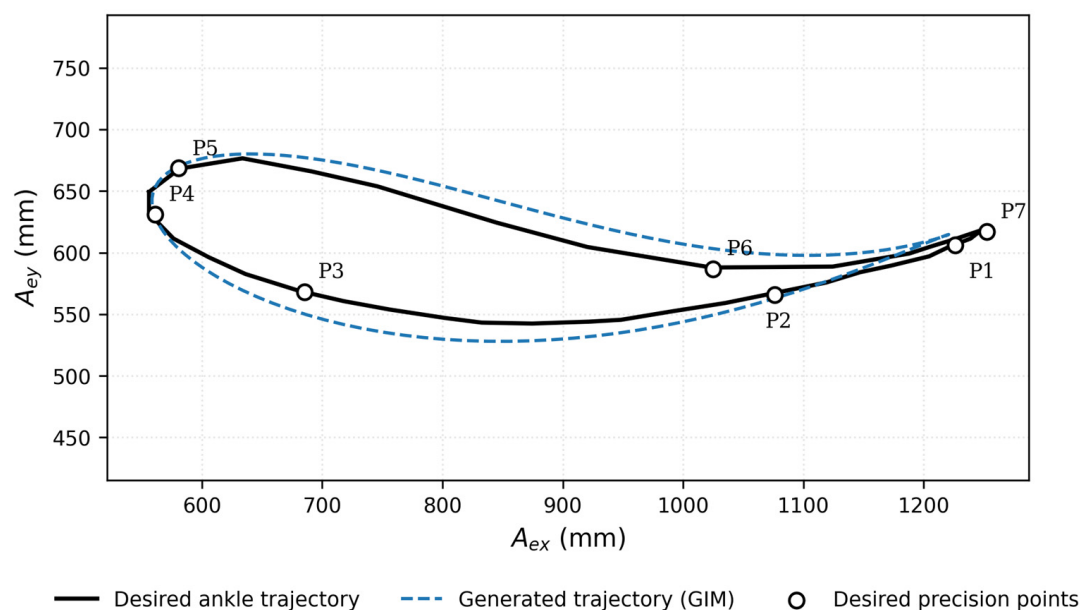


Figure 5. Desired ankle trajectory and the trajectory generated by the GIM model of the synthesized mechanism.

5.3. Trajectory Accuracy

5.3.1. Precision-Point Errors

Table 7 presents, for each of the seven precision points, the target coordinates, the coordinates produced by the synthesized mechanism, and the Euclidean error between the two sets of coordinates. The table footer reports the sum of errors and the root-mean-square (RMS) error. The synthesis yields a total error of 87.06 mm and an RMS error of 16.07 mm; the greatest single error is 31.70 mm at P7 (pre-heel-strike), and three of the seven points are located within 5 mm of the targets. For comparison, the walker-based six-bar device of Singh et al. reported a total error of 104.115 mm and an RMS error of 42.56 mm for the same precision-point set [13]. The proposed geared five-bar thus reduces the total point error by 16% and the RMS error by 62% on this benchmark.

Table 7. Precision-point errors of the synthesized geared five-bar mechanism.

Point	Desired A_{ex}/A_{ey} (mm)	Generated A_{ex}/A_{ey} (mm)	Euclidean Error (mm)
P1	1226.2/606.07	1220.49/614.55	10.23
P2	1075.9/565.99	1077.29/563.26	3.06
P3	685.3/568.24	687.10/549.56	18.77
P4	560.9/631.11	561.42/628.26	2.90
P5	580.5/668.49	581.85/670.59	2.50
P6	1024.7/586.70	1033.95/602.04	17.91
P7	1252.3/617.04	1220.68/614.70	31.70
Sum of errors			87.06
RMS error			16.07

5.3.2. Full-Trajectory Shape Comparison

Precision-point error alone does not fully describe a path generator: a curve may pass exactly through the prescribed points while still departing from the target elsewhere. For this reason, the deviation of the entire generated curve from the full desired ankle curve is also reported. This whole-curve deviation, calculated as the RMS of the shortest distance from each generated point to the desired curve, is a stricter and more representative indicator of trajectory fidelity.

The whole-curve deviation (Figure 6) between the trajectory produced by the synthesized mechanism (verified in GIM) and the desired ankle curve of Section 2, computed as the shortest distance from each generated point to the closed piecewise-linear desired curve, reaches a maximum of 19.11 mm, with a mean of 9.15 mm and an RMS value of 10.74 mm across 121 samples evenly distributed over one crank revolution. The median of this distribution is 8.76 mm. The whole-curve RMS deviation (10.74 mm) is somewhat lower than the precision-point RMS reported in Section 5.3.1 (16.07 mm), showing that the synthesized mechanism tracks the desired curve more closely on average than the precision-point results alone imply; the precision-point error is mainly affected by a small number of outlier points, especially P7, while the rest of the curve remains close to the target. The deviation pattern in Figure 6 reveals two clear peaks: a stance-phase peak in the interval $\theta_2 = 0$ to 60° , associated with the lower boundary of the coupler curve between precision points P6 and P1, and a swing-phase peak near $\theta_2 = 197^\circ$, associated with the upper boundary of the curve between precision points P3 and P5. Both peaks stay below 20 mm.

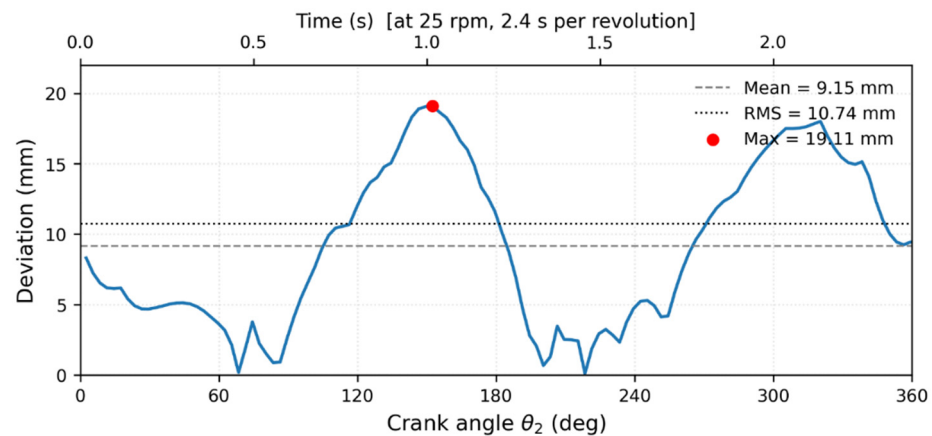


Figure 6. Whole-curve deviation of the generated ankle trajectory from the desired ankle curve as a function of the input crank angle θ_2 .

5.4. Velocity and Smoothness Analysis

Using the velocity analysis of Section 3.3, the foot-plate velocity was assessed over one full crank revolution with the cranks operated at a constant input speed of 25 rpm, corresponding to a cycle time of 2.4 s per stride. Figure 7 shows the signed tangential foot-plate velocity, taken as positive along the direction in which the coupler curve is traversed, as a function of the input crank angle; the term speed is reserved for its magnitude. The profile is smooth and continuous throughout the cycle: the velocity increases to an initial maximum of about 0.79 m/s near the beginning of the cycle, drops to a local minimum of about 0.24 m/s in mid-cycle, rises again to a peak of 0.91 m/s during the swing phase, and then declines steadily, changing sign near the heel-strike configuration as the foot-plate reverses direction to about -0.73 m/s. The mean foot-plate speed is 0.585 m/s, which equals the trajectory perimeter of 1404 mm divided by the 2.4 s cycle time. This mean speed places the device in the slow, controlled range suitable for early gait rehabilitation, below the 0.6 m/s threshold often linked with limited household ambulation, while still covering the full physiological range of foot motion. The overall trend, slower motion during foot contact and faster motion during swing, matches the expected normative gait foot-speed pattern, and the resulting speed range is consistent with the walker-assisted gait kinematics measured experimentally by Loureiro et al. [14].

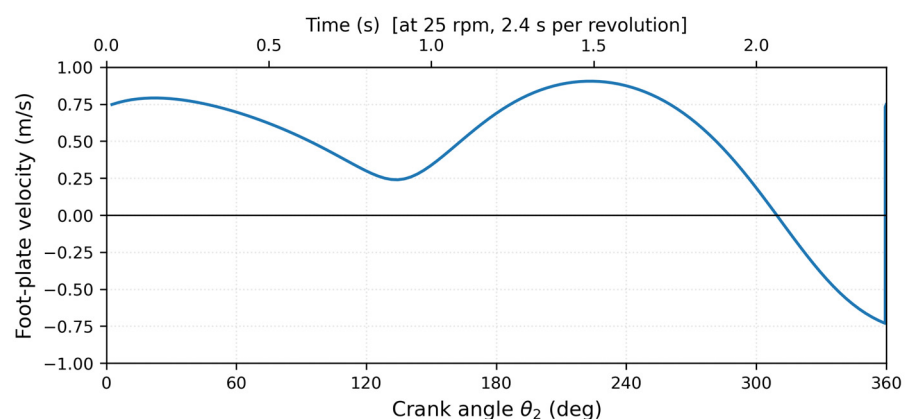


Figure 7. Signed tangential foot-plate velocity of point F over one complete crank revolution at a constant input speed of 25 rpm (2.4 s per cycle).

The foot-plate velocity profile in Figure 7 is smooth and has a single peak in each half of the cycle, reaching a maximum speed of 0.91 m/s and showing a gradual change of sign near heel strike; there are no abrupt jumps or sharp reversals within the cycle. This

indicates that the accelerations remain finite, which is consistent with the clinical need for a gentle, jerk-free guided path. The peak speed of 0.91 m/s and mean of 0.585 m/s at 25 rpm remain within the range tolerated by rehabilitation patients; because the input crank is driven by a single speed-controlled actuator, the cadence and hence the foot-plate speed can be set directly and adjusted for the patient, so the 25 rpm value should be read as a representative operating point that can be reduced for early-stage training and increased as the patient progresses. A complete dynamic force study based on the acceleration field in Section 3.3.2 is left for future work.

5.5. Sensitivity of Trajectory Accuracy to Gear Backlash

Because the two cranks are coupled through a physical gear train, angular play (backlash) in the meshes perturbs the gear constraint of Equation (4), which may be written as $\theta_5 = N\theta_2 + \varphi_0 \pm \Delta$, where Δ is the total angular backlash reflected to the crank-2 shaft. Because the idler introduces two external meshes, the play of the two gear pairs accumulates. The first-order effect can be estimated from the crank-2 joint displacement $\delta_C \approx r_5\Delta$. With $r_5 = 95.11$ mm (Table 5), a reflected backlash of $\Delta = 0.5^\circ$ gives $\delta_C \approx 0.83$, and $\Delta = 1.0^\circ$ gives $\delta_C \approx 1.66$ mm. Even if the closed-chain transmission amplifies this local displacement by a conservative factor of two to three at the foot plate point F, the resulting trajectory perturbation remains on the order of a few millimeters, which is smaller than the precision-point RMS synthesis residual of 16.07 mm and the maximum whole-curve deviation of 19.11 mm reported in Section 5.3. This estimate is not a substitute for a full tolerance analysis, but it shows that realistic backlash is unlikely to dominate the trajectory error of the present design. In the physical prototype, the remaining effect can be reduced by using anti-backlash gears, preloaded meshes, or a calibration offset in the phase angle φ_0 . A complete experimental backlash and tolerance study is therefore identified as future work.

5.6. Gear Ratio as a Shaping Parameter

In principle, the gear ratio N is a real trajectory-shaping parameter: by differentiating the gear constraint in Equation (4), it can be seen that N changes the angular speed of the second crank relative to the first, which in turn modifies the set of coupler curves that can be reached in a way not available in the six-bar topology. In the mechanism presented here, matching end gears are adopted so that $N = 1$ and both cranks turn at the same rate; this choice keeps fabrication straightforward and preserves kinematic symmetry. A detailed examination of trajectory accuracy for a fixed non-unit gear ratio, making use of this extra degree of freedom, is identified as future work.

5.7. Discussion

The findings are discussed in relation to the contributions outlined at the end of the Introduction. The geared five-bar linkage reproduces the teardrop ankle path of normal gait with five links, one fewer than a standard Stephenson III six-bar. The defect analysis in Section 3.4 shows, and the synthesis confirms, that branch and circuit defects are removed by the structure itself, leaving only the order constraint active; this reduces the complexity of the design search and expands the feasible design space. The gear ratio and phase offset offer shape control that does not exist in six-bar topologies.

From a quantitative perspective, the synthesized mechanism passes through all seven precision points, with single-digit-millimeter errors at three of them (P2, 3.06 mm; P4, 2.90 mm; P5, 2.50 mm) and the largest error of 31.70 mm at the pre-heel-strike position (P7). The total point error is 87.06 mm and the RMS error is 16.07 mm, which corresponds to a 16% decrease in error sum and a 62% decrease in RMS error compared with the walker-mounted Stephenson III six-bar of Singh et al. using the same precision-point set [13]. The

clinically important positions of toe-off (P4) and maximum swing height (P5) are matched with single-digit-millimeter accuracy, supporting adequate foot clearance during swing. The whole-curve analysis in Section 5.3.2 (max 19.11 mm, mean 9.15 mm, RMS 10.74 mm) also shows that the full curve follows the intended path more closely than the worst precision-point errors alone imply; the precision-point error is driven mainly by a small number of outliers rather than by a consistent drift in the generated curve. The remaining error at P7 is located near the heel-strike tip of the coupler curve, where the trajectory has a sharp change in curvature; errors in this area affect the timing of foot contact but not the kinematic range of the cycle.

Several limitations should be noted. The model is planar and restricted to the sagittal plane, so out-of-plane motion is not captured. The actuator speed is adjustable, but the geometric trajectory is fixed once the mechanism is built and cannot be adapted to an individual patient within a session. The synthesis is based on a single representative anthropometric subject. Finally, the current validation is computational, that is, a synthesized mechanism cross-checked in the GIM kinematic simulation environment against the closed-form model, and a physical prototype with clinical evaluation remains future work. These limitations are discussed again in Section 7.2.

6. Walker Gait Trainer Device Integration and CAD Design

6.1. Mechanism Mounting on the Walker Frame

The mechanism is attached to the lateral side of a standard wheeled walker. The two ground pivots, O_2 and O_5 , are secured to a horizontal crossbar connected to the walker uprights at roughly knee height. The crossbar supports the gear train between the two pivots. The ternary coupler extends downward from the loop, and the foot-plate point F moves through the operating envelope at ankle height, where the rigid foot plate is mounted. The mechanism is positioned on the side of the patient's paretic limb, while the opposite limb moves freely.

6.2. Actuation

The input crank is powered by a single electric actuator, namely a speed-controlled geared motor, which is connected directly to the crank 1 shaft at O_2 . Since the gear coupling limits the mechanism to one degree of freedom, this single actuator completely defines the motion of all links and the trajectory of the foot plate. During steady operation, the actuator runs at a constant prescribed speed, generating one full gait cycle per crank revolution; its speed can be adjusted so that the cadence of the assisted gait cycle matches the patient's needs and can be increased gradually during rehabilitation. No belt-, pulley-, or wheel-driven transmission is needed, which reduces drivetrain complexity and separates the assisted cadence from the patient's forward walking speed.

6.3. CAD Assembly Model

A three-dimensional CAD assembly model of the integrated device is developed, shown in Figure 8. The model includes the walker frame, the crossbar carrying the two ground pivots, the gear pair and idler, the two cranks, the two coupler links, the ternary coupler with the foot plate, and the actuator connected to the input crank at O_2 . A motion study is performed over one complete crank revolution to verify that no link collides with the frame or the patient envelope, and that the foot plate remains clear of the ground throughout the stance phase of the cycle.

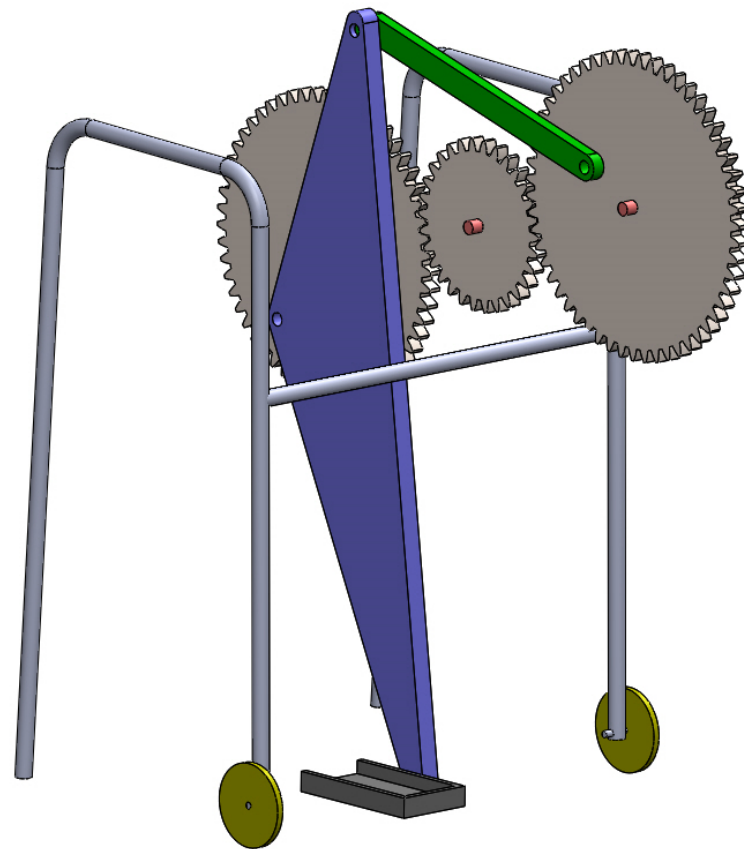


Figure 8. CAD assembly of the integrated Walker Gait Trainer.

6.4. Practical Embodiment and Implementation Considerations

To underline the technological feasibility of the proposed design, the main embodiment parameters are specified. With the ground-pivot spacing of Table 5, the two end gears are selected as identical standard spur gears of module $m = 2$ mm, and the idler pitch diameter follows from the center-distance condition of Section 3.1.3; all three gears are enclosed in a protective housing mounted on the crossbar, satisfying the safety requirements of a device operating next to a patient's limb. The links are sized as aluminum flat bars, and the cranks are keyed to the gear shafts, which run in sealed deep-groove ball bearings press-fitted into the crossbar housing; the estimated added mass of the complete mechanism, gears, and housing on one side of the walker is approximately 4–5 kg, compatible with a standard wheeled-walker frame. The actuator is a speed-controlled DC geared motor driving crank 1 at O_2 ; at the representative cadence of 25 rpm and the torque levels implied by light guided training, a motor in the 30–60 W class is sufficient, with the cadence continuously adjustable by the speed controller. The motion study of the CAD assembly (Section 6.3) verifies clearance between all links, the frame, and the patient envelope over the complete cycle. A benchtop prototype of the linkage for experimental verification of the generated trajectory, followed by a full device prototype and clinical evaluation, is planned as the immediate next stage of this research (Section 7.2).

6.5. Device-Level Summary

Table 8 summarizes the proposed device and contrasts it with a representative walker-based six-bar gait trainer [13]. The two devices share the walker-mounted, end-effector concept; the proposed device replaces the passive belt-pulley drive of the reference device with a single speed-controlled actuator on the input crank. They differ in the trajectory-generating mechanism: the proposed device replaces the six-bar linkage and its four defect

constraints with a five-bar linkage, a gear train, and a single order constraint, and gains the gear ratio as an additional shaping parameter at the cost of a gear train.

Table 8. Device-level summary and comparison with a representative walker-based six-bar gait trainer.

Aspect	Walker-Based Six-Bar Device	Proposed Walker Gait Trainer
Trajectory mechanism	Stephenson III six-bar	Geared five-bar
Links/joints	6 links, 7 revolute joints	5 links, 5 revolute joints + gear pair
Defect constraints	4	1 (order only)
Shaping parameter	None	Gear ratio and phase offset
Drive	Belt-pulley from wheel	Single speed-controlled actuator
Additional component	Extra coupler link	Gear train
RMS trajectory error	42.56 mm (reported)	See Section 5.3

7. Conclusions and Future Work

7.1. Conclusions

This paper proposed a geared five-bar linkage as the path-generating mechanism for a single-actuator Walker Gait Trainer intended for over-ground stroke rehabilitation. The two cranks of the five-bar linkage are connected through a gear train made up of two identical spur gears and an intermediate idler, which restricts the mechanism to one degree of freedom and enables actuation by a single speed-controlled motor applied to the input crank.

The main theoretical finding is that the gear connection removes two of the three traditional linkage defects at the structural level for the selected operating branch. Since both cranks are geared to a shared, continuously rotating input, the mechanism does not need a separate Grashof condition for the crank motion, and the fixed assembly branch prevents circuit changes during normal operation. The only remaining defect condition is the order constraint, which is satisfied by the unwrapped crank-angle sequence of the seven precision points.

The geared five-bar linkage reproduces the characteristic teardrop ankle path of healthy gait with only five links, one fewer than a standard six-bar design, and it adds the gear ratio and crank phase offset as trajectory-shaping variables that do not have direct equivalents in six-bar structures. A full device integration was also described, including the mechanism's attachment to the walker frame, the single-actuator drive of the input crank, and a three-dimensional CAD assembly model.

7.2. Limitations and Future Work

This study has several limitations that guide future research. The kinematic model is planar and restricted to the sagittal plane, so motion out of that plane is not captured. The actuator speed may be changed, but the geometric path is fixed once the mechanism is built; it cannot be tailored to a specific patient during a session. The synthesis relies on the anthropometry of a single representative subject. In addition, the validation presented here is computational and consists of cross-checking the synthesized mechanism between the closed-form kinematic model and the GIM kinematic-simulation environment; physical validation remains future work.

Future work will address these limitations in four directions. First, a physical prototype of the Walker Gait Trainer will be built and tested experimentally, followed by clinical assessment with stroke patients. Second, dynamic force analysis, based on the acceleration

analysis in Section 3.3, will be used to determine the actuator size and the structural design of the links and gear train, including a protective gear cover. Third, a variable-ratio or non-circular gear pair will be studied so that the trajectory-shaping potential offered by the gear ratio can be used more effectively for closer path matching. Fourth, an adjustable foot-plate height and a scalable linkage will be developed so that one device can fit patients of different statures.

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Data Availability Statement: The normative gait data used in this study are publicly available from van Criekinge et al. (2023), cited as Reference [2] in this manuscript. The kinematic model and synthesis parameters are fully reported in the paper. No other datasets were generated or analyzed during the current study.

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References

1. Sloom, L.H.; Baker, L.M.; Bae, J.; Porciuncula, F.; Clément, B.F.; Sivi, C.; Nuckols, R.W.; Baker, T.; Sloutsky, R.; Choe, D.K.; et al. Effects of a Soft Robotic Exosuit on the Quality and Speed of Overground Walking Depends on Walking Ability after Stroke. *J. Neuroeng. Rehabil.* **2023**, *20*, 113. [\[CrossRef\]](#) [\[PubMed\]](#)
2. Van Criekinge, T.; Saeys, W.; Truijen, S.; Vereeck, L.; Sloom, L.H.; Hallemans, A. A Full-Body Motion Capture Gait Dataset of 138 Able-Bodied Adults across the Life Span and 50 Stroke Survivors. *Sci. Data* **2023**, *10*, 852. [\[CrossRef\]](#) [\[PubMed\]](#)
3. Lee, J.; Li, L.; Shin, S.Y.; Deshpande, A.D.; Sulzer, J. Kinematic Comparison of Single Degree-of-Freedom Robotic Gait Trainers. *Mech. Mach. Theory* **2021**, *159*, 104258. [\[CrossRef\]](#)
4. Li, M.; Yan, J.; Zhao, H.; Ma, G.; Li, Y. Mechanically Assisted Neurorehabilitation: A Novel Six-Bar Linkage Mechanism for Gait Rehabilitation. *IEEE Trans. Neural Syst. Rehabil. Eng. Publ. IEEE Eng. Med. Biol. Soc.* **2021**, *29*, 985–992. [\[CrossRef\]](#) [\[PubMed\]](#)
5. Wu, L.; Xu, G.; Wu, Q. The Effect of the Lokomat[®] Robotic-Orthosis System on Lower Extremity Rehabilitation in Patients with Stroke: A Systematic Review and Meta-Analysis. *Front. Neurol.* **2023**, *14*, 1260652. [\[CrossRef\]](#) [\[PubMed\]](#)
6. Gandolfi, M.; Valè, N.; Dimitrova, E.; Zanolin, M.; Mattiuz, N.; Battistuzzi, E.; Beccari, M.; Geroin, C.; Picelli, A.; Waldner, A.; et al. Robot-Assisted Stair Climbing Training on Postural Control and Sensory Integration Processes in Chronic Post-Stroke Patients: A Randomized Controlled Clinical Trial. *Front. Neurosci.* **2019**, *13*, 1143. [\[CrossRef\]](#) [\[PubMed\]](#)
7. Sargsyan, S.; Arakelian, V.; Briot, S. Robotic Rehabilitation Devices of Human Extremities: Design Concepts and Functional Particularities. In *Engineering Systems Design and Analysis*; American Society of Mechanical Engineers: Nantes, France, 2012; Volume 3.
8. Mohan, D.S. Synthesis and Analysis of Jansen's Leg-Based Mechanism for Gait Rehabilitation. In *Mechanism and Machine Science: Select Proceedings of Asian MMS 2018*; Springer: Singapore, 2021; pp. 303–315. ISBN 978-981-15-4476-7.
9. Gazo-Hanna, E. A Review on Soft Planar Mechanisms: Modeling and Actuation for Robotic Applications. *Eng. Technol. Appl. Sci. Res.* **2026**, *16*, 31788–31794. [\[CrossRef\]](#)
10. Tsuge, B.Y.; Michael McCarthy, J. An Adjustable Single Degree-of-Freedom System to Guide Natural Walking Movement for Rehabilitation. *J. Med. Devices* **2016**, *10*, 044501. [\[CrossRef\]](#)
11. Yul Shin, S.; Deshpande, A.D.; Sulzer, J. Design of a Single Degree-of-Freedom, Adaptable Electromechanical Gait Trainer for People With Neurological Injury. *J. Mech. Robot.* **2018**, *10*, 044503. [\[CrossRef\]](#)
12. Haghighi, M.R.; Lee, H.; Afzal, M.R.; Eizad, A.; Yoon, J. Mech-Walker: A Novel Single-DOF Linkage Device With Movable Frame for Gait Rehabilitation. *IEEEASME Trans. Mechatron.* **2021**, *26*, 13–23. [\[CrossRef\]](#)
13. Singh, R.; Pathak, V.K.; Sharma, A.; Chakraborty, D.; Saxena, K.K.; Prakash, C.; Buddhi, D.; Salem, K. Caster Walker GAIT Trainer (CGT): A Robotic Assistive Device. *Robot. Auton. Syst.* **2023**, *159*, 104302. [\[CrossRef\]](#)

14. Loureiro, M.; Elias, A.; Machado, F.; Bezerra, M.; Zimerer, C.; Mello, R.; Frizera, A. Analysis of Gait Kinematics in Smart Walker-Assisted Locomotion in Immersive Virtual Reality Scenario. *Sensors* **2024**, *24*, 5534. [\[CrossRef\]](#) [\[PubMed\]](#)
15. Li, B.; Chen, X.; Liu, H.; Yuan, D.; Zhang, J.; Lu, S. Mechanism Design of a Novel Device to Facilitate Mobility, Sit-to-Stand Transfer Movement, and Walking Assistance. *Machines* **2025**, *13*, 134. [\[CrossRef\]](#)
16. Bulatović, R.R.; Đjordjević, S.R. Optimal Synthesis of a Path Generator Six-Bar Linkage. *J. Mech. Sci. Technol.* **2012**, *26*, 4027–4040. [\[CrossRef\]](#)
17. Nafees, K.; Mohammad, A. Dimensional Synthesis of Six-Bar Stephenson III Mechanism for 12 Precision Points Path Generation. *Int. J. Mech. Robot. Syst.* **2016**, *3*, 80. [\[CrossRef\]](#)
18. Zhang, C.; Norton, P.E.R.L.; Hammonds, T. Optimization of Parameters for Specified Path Generation Using an Atlas of Coupler Curves of Geared Five-Bar Linkages. *Mech. Mach. Theory* **1984**, *19*, 459–466. [\[CrossRef\]](#)
19. Lin, W.-Y. Optimum Path Synthesis of a Geared Five-Bar Mechanism. *Adv. Mech. Eng.* **2013**, *5*, 757935. [\[CrossRef\]](#)
20. Mather, S.; Erdman, A. Synthesis of Geared Planar Linkage Mechanisms through the Segmentation of Multiloop Mechanisms into Discrete Chains. *Machines* **2024**, *12*, 182. [\[CrossRef\]](#)
21. Mundo, D.; Gatti, G.; Dooner, D.B. Optimized Five-Bar Linkages with Non-Circular Gears for Exact Path Generation. *Mech. Mach. Theory* **2009**, *44*, 751–760. [\[CrossRef\]](#)
22. Sabaapour, M.R.; Yoon, J. A Novel Method for Optimal Path Synthesis of Mechanisms Based on Tracking Control of Shadow Robot. *Mech. Mach. Theory* **2019**, *131*, 218–233. [\[CrossRef\]](#)
23. Petuya, V.; Macho, E.; Altuzarra, O.; Pinto, C.; Hernandez, A. Educational Software Tools for the Kinematic Analysis of Mechanisms. *Comput. Appl. Eng. Educ.* **2014**, *22*, 72–86. [\[CrossRef\]](#)
24. Macho, E.; Urizar, M.; Petuya, V.; Hernández, A. Improving Skills in Mechanism and Machine Science Using GIM Software. *Appl. Sci.* **2021**, *11*, 7850. [\[CrossRef\]](#)
25. Gogu, G. Mobility of Mechanisms: A Critical Review. *Mech. Mach. Theory* **2005**, *40*, 1068–1097. [\[CrossRef\]](#)
26. Luo, G.; Wang, X.; Yan, X. DOF Calculation Method of Planar Mechanisms Based on Constraint Normal Line. *Math. Probl. Eng.* **2023**, *2023*, 3790060. [\[CrossRef\]](#)
27. Prikhodko, A.; Smelyagin, A. The Concept of Spaces in the Structural Analysis and Synthesis of Mechanisms and Machines. *Mech. Mach. Theory* **2025**, *205*, 105889. [\[CrossRef\]](#)
28. Gazo-Hanna, E.; Saber, A.; Amine, S. Design and Modeling of a Six-Bar Mechanism for Repetitive Tasks with Symmetrical End-Effector Motion. *Eng. Technol. Appl. Sci. Res.* **2024**, *14*, 16302–16310. [\[CrossRef\]](#)
29. Norton, R.L. *Design of Machinery: An Introduction to the Synthesis and Analysis of Mechanisms and Machines*; McGraw-Hill: Singapore, 2012; ISBN 978-0-07-352935-6.
30. Ting, K.-L. Mobility Criteria of Geared Five-Bar Linkages. *Mech. Mach. Theory* **1994**, *29*, 251–264. [\[CrossRef\]](#)
31. Dou, X.; Ting, K.-L. Branch Identification of Geared Five-Bar Chains. *J. Mech. Des.* **1996**, *118*, 384–389. [\[CrossRef\]](#)
32. Ting, K.-L.; Wang, J.; Xue, C.; Currie, K.R. Full Rotatability and Singularity of Six-Bar and Geared Five-Bar Linkages. *J. Mech. Robot.* **2010**, *2*, 011011. [\[CrossRef\]](#)
33. Amine, S.; Caro, S.; Wenger, P. Constraint and Singularity Analysis of the Exechon. *Appl. Mech. Mater.* **2012**, *162*, 141–150. [\[CrossRef\]](#)
34. Amine, S.; Gazo Hanna, E. Kinematic Analysis of HALF Parallel Robot. *J. Eng. Sci. Technol. Rev.* **2019**, *12*, 207–213. [\[CrossRef\]](#)
35. Erwin-Christian, L.; Valentin, C.; Tivadar, D.; Alexandru, O.; Elida-Gabriela, T.; Melania-Olivia, S. Optimal Synthesis of Five-Bar Linkage Based on Singularity-Free Workspaces with Predefined Shapes. *Robotics* **2024**, *13*, 173. [\[CrossRef\]](#)
36. Amine, S.; Prasad, B.; Saber, A.; Mokhiamar, O.; Gazo-Hanna, E. Design and Analysis of a Planar Six-Bar Crank-Driven Leg Mechanism for Walking Robots. *Appl. Sci.* **2024**, *14*, 8919. [\[CrossRef\]](#)
37. Soriano-Heras, E.; Pérez-Carrera, C.; Rubio, H. Mathematical Dimensional Synthesis of Four-Bar Linkages Based on Cognate Mechanisms. *Mathematics* **2025**, *13*, 11. [\[CrossRef\]](#)

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