

Geometrically Nonlinear Dynamics of Cracked Beams with Rotational Flexibility[†]

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Abstract

This study examines the geometrically nonlinear free vibration behaviour of a clamped–clamped Euler–Bernoulli beam weakened by an open transverse crack. The crack is modelled through an equivalent rotational flexibility formulation grounded in fracture mechanics, while geometric nonlinearity is incorporated by accounting for mid-plane stretching associated with large vibration amplitudes. A reduced-order formulation is established using a Galerkin approach with eigenfunctions that explicitly depend on the presence of the crack, leading to a nonlinear eigenvalue problem in which the response is amplitude-dependent. The primary aim is to clarify how crack severity and vibration amplitude jointly influence the distribution of nonlinear bending stresses along the beam. Unlike much of the existing literature, which predominantly emphasises frequency–amplitude interactions, this work adopts a stress-oriented framework and offers a detailed characterisation of the spatial variation in the normalised stress field. The results indicate that bending stresses increase markedly as vibration amplitude grows, with the most pronounced effects occurring near the clamped ends where curvature is highest. Furthermore, deeper cracks significantly intensify stress concentrations due to the associated local reduction in stiffness. Overall, the findings provide enhanced physical understanding of the nonlinear dynamic behaviour of cracked beam structures and establish a useful foundation for evaluating structural integrity under conditions of large-amplitude vibration.

Keywords: nonlinear vibration; clamped–clamped Euler–Bernoulli beam; open transverse crack; geometric nonlinearity



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1. Introduction

The dynamic behaviour of beam and plate structures has been extensively investigated, particularly in the context of nonlinear vibrations at large amplitudes. Early developments in this field established robust semi-analytical formulations for predicting nonlinear responses, among which the multimode approach has become a well-established framework for analysing the steady-state forced vibration of beams [1]. In the presence of structural defects such as open cracks, local stiffness reductions significantly alter the system's global dynamic characteristics, leading to noticeable changes in natural frequencies and mode shapes. This behaviour has been widely studied for cracked beams under various boundary conditions and configurations [2–5]. Geometric nonlinearity is particularly significant in

clamped–clamped beam configurations, where large transverse deflections induce mid-plane stretching. This mechanism introduces amplitude-dependent stiffness, resulting in inherently nonlinear dynamic behaviour that linear models cannot capture. Such effects have been reported in various structural systems, including composite and coupled-beam configurations [6–8]. More recently, research has expanded to more complex nonlinear structural systems involving distributed nonlinearities, structural coupling, and interactions between mechanical components. These investigations consistently demonstrate strong modal coupling and pronounced amplitude-dependent frequency responses in the nonlinear regime [9,10]. Concurrently, advanced numerical and semi-analytical approaches, including spectral element methods and nonlinear elastic foundation models, have been developed to address localised nonlinearities and complex boundary conditions with improved accuracy [11,12]. From a more focused perspective, the seminal contribution [1] remains a cornerstone in multimode-based nonlinear vibration analysis, providing a powerful semi-analytical framework for predicting the steady-state dynamic response of beams at large amplitudes. Building upon this methodological foundation [2], an established comprehensive formulation was developed for the geometrically nonlinear free vibration of clamped–clamped beams with an open-edge crack. Their approach introduced a rotational spring model to represent crack-induced local flexibility within a segmented Euler–Bernoulli beam formulation. It employed a Galerkin-based modal reduction that accounts for mid-plane stretching. This work clearly demonstrated the combined influence of geometric nonlinearity and local stiffness reduction on the nonlinear frequency–amplitude relationship and constitutes the primary methodological reference for the present study. The present work builds directly upon the framework [2] and extends it in the following specific directions, which collectively define the originality of the contribution: First, whereas [2] focused primarily on the nonlinear frequency ratios and their sensitivity to a limited set of crack parameters, the present study conducts a systematic, comprehensive parametric investigation across multiple levels of crack severity. This allows a more complete quantitative characterisation of the influence of damage intensity on the nonlinear dynamic response. Second, and most distinctively, the present work shifts the primary focus toward the nonlinear bending stress distribution along the beam length. Specifically, the amplitude-dependent evolution of the non-dimensional stress field $\sigma(x)$ derived from the nonlinear modal response is examined in detail for each crack configuration considered. The stress analysis is particularly emphasised in the vicinity of the clamped boundaries and the crack location, where both stress concentration and geometric nonlinearity simultaneously govern structural behaviour. This stress-centred perspective has not been systematically developed within the framework of [2] and constitutes a physically meaningful extension of the nonlinear cracked beam problem. Third, the non-dimensional formulation is applied consistently to yield results in a normalised, geometry-independent form, thereby facilitating direct comparison with existing linear and nonlinear reference data and broadening the applicability of the findings. The primary objective of this study is therefore to examine the coupled influence of crack severity and vibration amplitude on the nonlinear bending stress response of a clamped–clamped Euler–Bernoulli beam, with particular attention to regions of high curvature and stress concentration. The novelty of the work lies not in the mathematical framework itself, which is explicitly acknowledged as derived from [2], but in its application to a stress-focused nonlinear damage analysis that has not been previously reported in the literature. This perspective provides additional physically consistent information for damage assessment and structural integrity evaluation of beam-type structures operating under large-amplitude vibration conditions.

2. Model Formulation

2.1. Beam Configuration and Assumptions

A uniform Euler–Bernoulli beam of length L , width b , thickness h , cross-sectional area S , second moment of area I , bending rigidity EI , and mass density ρ is considered. The beam is clamped at both ends and contains an open transverse crack of depth a located at $x = x_0$, as shown in Figure 1. The depth ratio a/h is used to quantify the crack severity. Unless otherwise stated, dimensional quantities are expressed in SI units.

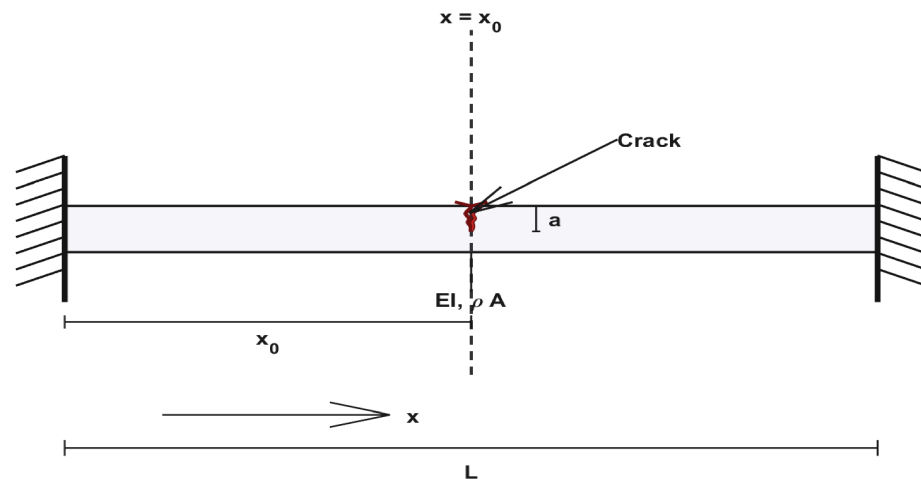


Figure 1. Clamped–clamped Euler–Bernoulli beam with an open transverse crack located at $x = x_0$.

2.2. Energetic Formulation and Physical Interpretation

The governing equations are derived from Hamilton’s principle:

$$\delta \int_{t_1}^{t_2} (V - T) dt = 0 \quad (1)$$

where T is the kinetic energy and V is the total potential energy.

For large-amplitude vibration, the strain energy is decomposed as:

$$V = V_b + V_c + V_a \quad (2)$$

where V_b is the classical bending energy, V_c is the additional local energy associated with the crack, and V_a is the nonlinear axial stretching energy induced by the end constraints.

The bending energy is written as:

$$V_b = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 W}{\partial x^2} \right)^2 dx \quad (3)$$

which measures the elastic energy stored by beam curvature.

The crack contribution is:

$$V_c = \frac{(EI)^2}{2K_f} \left[\frac{\partial^2 W}{\partial x^2} \Big|_{x=x_0} \right]^2 \quad (4)$$

where K_f is the equivalent rotational stiffness of the cracked section.

Physically, a lower value of K_f corresponds to a more compliant crack and therefore to a stronger local loss of bending stiffness.

The nonlinear stretching energy is:

$$V_a = \frac{ES}{8L} \left(\int_0^L \left(\frac{\partial W}{\partial x} \right)^2 dx \right)^2 \quad (5)$$

This term reflects the axial extension generated by large transverse deflections in a beam whose ends are restrained. It is the source of geometric nonlinearity in the present model.

The kinetic energy is given by

$$T = \frac{1}{2} \rho S \int_0^L \left(\frac{\partial W}{\partial t} \right)^2 dx \quad (6)$$

2.3. Crack Flexibility Model

The equivalent rotational flexibility C is introduced from fracture mechanics as

$$C = b \frac{\partial^2}{\partial M_f^2} \int_0^a \left(\frac{K_I^2}{E'} \right) d\alpha \quad (7)$$

$$K_I = \left(6 \frac{M_f}{b h^2} \right) \sqrt{\pi a} f\left(\frac{a}{h}\right) \quad (8)$$

$$f\left(\frac{a}{h}\right) = \sqrt{2h/(\pi a) \tan(\pi a/(2h))} \frac{[0.923 + 0.199 (1 - \sin(\frac{\pi a}{2h}))^4]}{\cos(\frac{\pi a}{2h})} \quad (9)$$

where M_f is the bending moment at the cracked section, K_I is the mode-I stress-intensity factor, $f(a/h)$ is the dimensionless geometry correction function, and is the effective elastic modulus consistent with the adopted plane-strain assumption. The associated rotational stiffness is

$$K_f = \frac{1}{C} \quad (10)$$

A dimensionless crack parameter is then defined by

$$\beta = \frac{EI}{K_f L} \quad (11)$$

The parameter β measures the relative influence of the crack on the global bending stiffness: the larger β , the more severe the stiffness reduction.

2.4. Linear Vibration Problem

The transverse motion of the beam is governed by

$$EI \frac{\partial^4 W}{\partial x^4} + \rho S \frac{\partial^2 W}{\partial t^2} = 0 \quad (12)$$

This equation is completed by clamped boundary conditions at $x = 0$ and $x = L$, together with continuity and compatibility conditions at the cracked section. In the rotational-spring representation, the transverse displacement, bending moment and shear force remain consistent across the crack location, whereas the slope exhibits a jump related to the local rotational flexibility.

Solving the characteristic equation $f(k) = 0$ yields the wave numbers k_n , and the corresponding natural frequencies are obtained as

$$\omega_n = k_n^2 \sqrt{\frac{EI}{\rho S}} \quad (13)$$

3. Nonlinear Reduced-Order Model

To capture the geometrically nonlinear dynamics, the transverse displacement is expanded over the linear eigenfunctions of the cracked beam:

$$W(x, t) = \sum_{i=1}^N a_i(t) \phi_i(x) \quad (14)$$

where $\phi_i(x)$ are the crack-dependent mode shapes and $a_i(t)$ are the generalised coordinates. This choice is physically relevant because the modal basis already embeds the effect of the crack at the linear level.

After the substitution of Equation (14) into the energetic expressions and application of a Galerkin projection, the nonlinear algebraic system becomes

$$([K] - \omega^2[M])A + \frac{3}{2}[B(A)]A = 0 \quad (15)$$

$$A = [a_1, a_2, \dots, a_N]^T. \quad (16)$$

In Equation (15), M is the generalised mass matrix, K is the linear stiffness matrix including the crack contribution, and $B(A)$ is the amplitude-dependent nonlinear stiffness operator arising from axial stretching. The first term represents the classical balance between inertia and bending stiffness, whereas the second term introduces the additional restoring effect associated with large-amplitude motion.

Using the non-dimensional quantities

$$x^* = \frac{x}{L}, \omega^{*2} = \frac{\rho S L^4}{EI} \omega^2 \quad (17)$$

and the dimensionless modal coefficients

$$k^*_{ij} = \int_0^1 \left(\frac{\partial^2 \phi^*_{*i}}{\partial x^{*2}} \right) \left(\frac{\partial^2 \phi^*_{*j}}{\partial x^{*2}} \right) dx^* + \beta \left(\frac{\partial^2 \phi^*_{*i}}{\partial x^{*2}} \right) \Big|_{\{x^*=0\}} \left(\frac{\partial^2 \phi^*_{*j}}{\partial x^{*2}} \right) \Big|_{\{x^*=0\}} \quad (18)$$

$$b^*_{ijkl} = \alpha \int_0^1 \left(\frac{\partial \phi^*_{*i}}{\partial x^*} \right) \left(\frac{\partial \phi^*_{*j}}{\partial x^*} \right) dx^* \int_0^1 \left(\frac{\partial \phi^*_{*k}}{\partial x^*} \right) \left(\frac{\partial \phi^*_{*l}}{\partial x^*} \right) dx^* \quad (19)$$

$$m^*_{ij} = \int_0^1 \phi^*_{*i} \phi^*_{*j} dx^* \quad (20)$$

Introducing these non-dimensional quantities, the nonlinear eigenproblem becomes

$$([K^*] - \omega^{*2}[M^*])A + \frac{3}{2}[B^*(A)]A = 0 \quad (21)$$

Equation (17) shows that the modal frequency depends on the response amplitude. Mechanically, this reflects the fact that the structure becomes stiffer as the transverse deformation increases.

4. Results and Discussion

Figures 2 and 3 present the non-dimensional bending-stress distribution along the clamped–clamped cracked beam for several vibration amplitudes and for two crack-depth ratios, $a/h = 0.1$ and $a/h = 0.6$, respectively.

$$\sigma(x, t) = -E \frac{h}{2} \frac{\partial^2 W(x, t)}{\partial x^2}, \sigma^*(x^*) = -\frac{\partial^2 \phi^*(x^*)}{\partial x^{*2}} \quad (22)$$

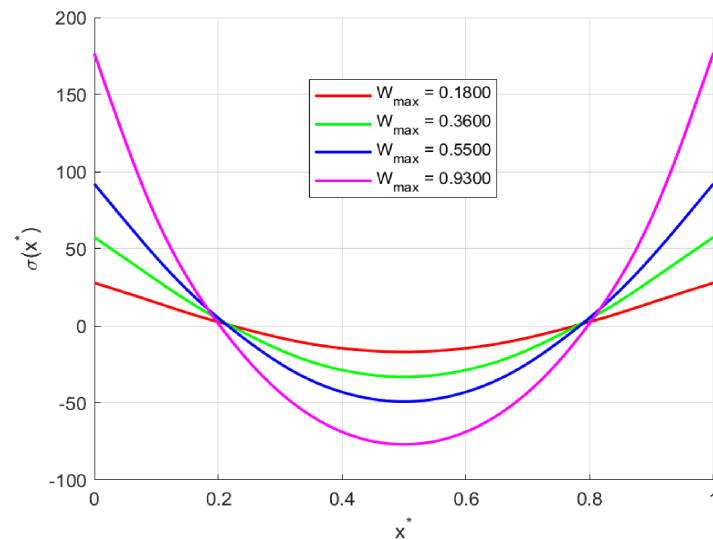


Figure 2. Non-dimensional bending-stress distribution $\sigma(x^*)$ in the first mode for different vibration amplitudes, with crack-depth ratio $a/h = 0.1$.

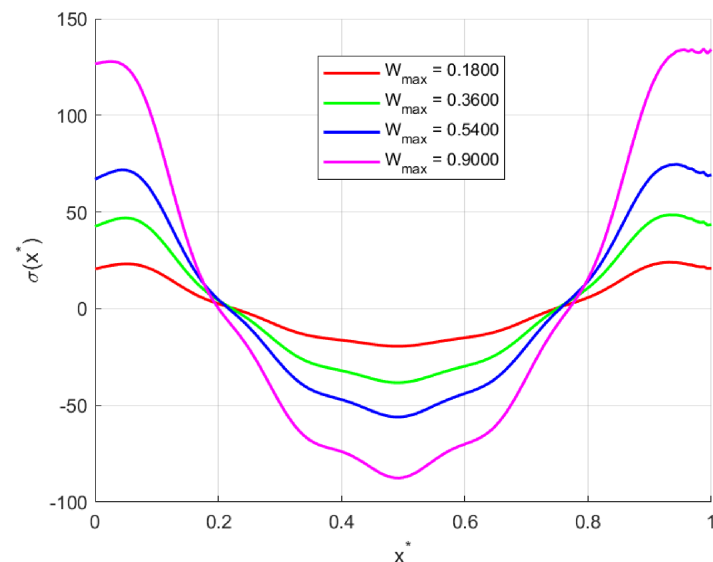


Figure 3. Non-dimensional bending-stress distribution $\sigma(x^*)$ in the first mode for different vibration amplitudes, with crack-depth ratio $a/h = 0.6$.

For both crack depths, the stress level increases markedly with vibration amplitude. This trend is physically consistent with the nonlinear formulation, since larger deflections produce larger curvature and stronger stretching effects. The highest stress values are observed near the clamped ends, where the boundary conditions impose high bending curvature. A second important observation is the influence of crack severity. When the crack-depth ratio increases from $a/h = 0.1$ to $a/h = 0.6$, the stress response becomes significantly higher. Mechanically, a deeper crack introduces a stronger local flexibility, which modifies the curvature field and amplifies the stress concentration in the structure. The results, therefore, highlight a coupled effect: vibration amplitude governs the level of nonlinear response, while crack severity controls the intensity of the stiffness reduction and the redistribution of bending effects. From a structural-integrity standpoint, this coupling implies that neglecting geometric nonlinearity would underestimate the peak bending stresses near the clamped ends—particularly for deep cracks—and hence the associated fatigue and failure risk.

Although only stress distributions are reported here, the formulation also provides access to amplitude-dependent modal characteristics through the nonlinear eigenproblem of Equation (17).

Figure 4 illustrates the first eleven mode shapes of a clamped–clamped beam with a transverse crack located at ($x = 0.5$). As the modal parameter (k) increases, the number of oscillations grows, reflecting higher-frequency modes. A noticeable change in slope appears at the crack position for all modes, consistent with a local rotational flexibility that preserves displacement continuity but introduces a rotation discontinuity. For intermediate modes, the deformation tends to concentrate near the crack, indicating a local reduction in stiffness and partial mode localisation. At higher modes, irregular oscillations appear, likely due to numerical sensitivity and reduced solution stability at large wave numbers. Overall, the results clearly show that the crack modifies both the global modal shapes and induces local deformation effects.

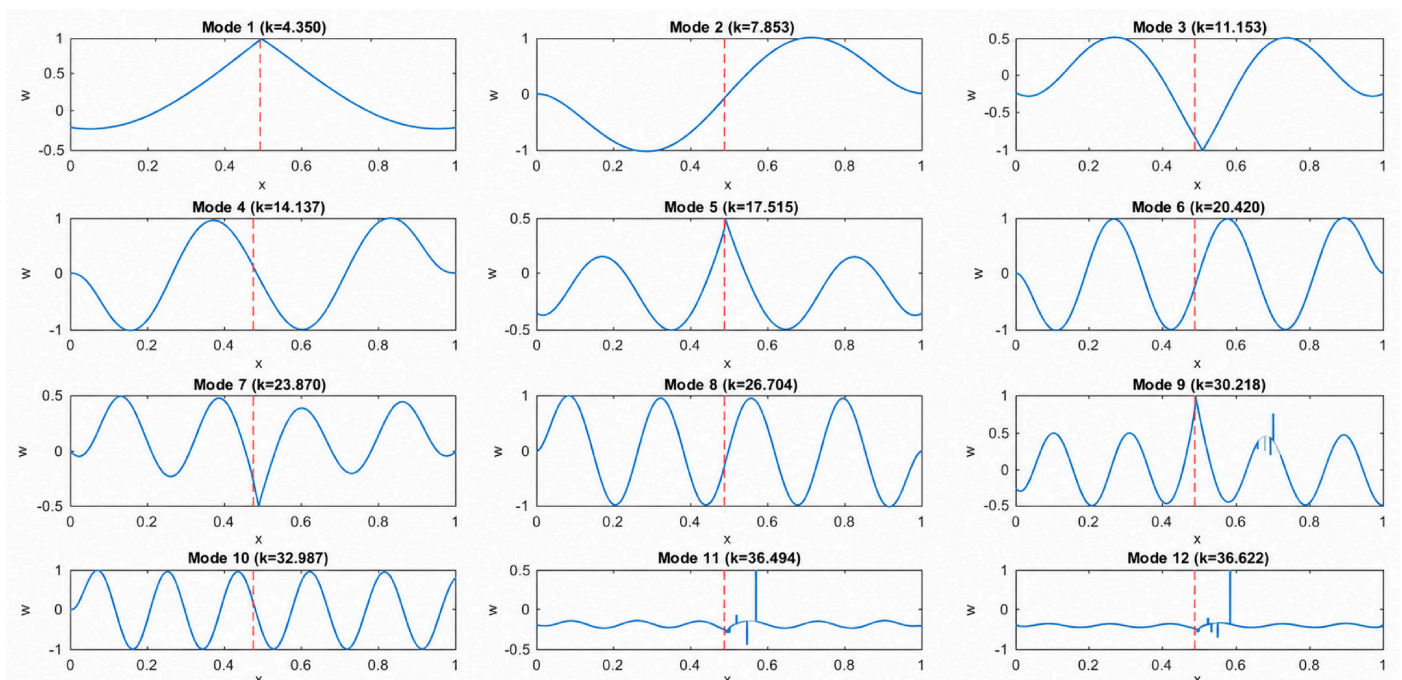


Figure 4. First eleven mode shapes of a cracked clamped–clamped beam; the dashed line indicates the crack position at $x = 0.5$.

Figures 5 and 6 illustrate that the nonlinear frequency response reveals a clear amplitude-dependent behaviour, where the ratio $\frac{\omega_n^*}{\omega_L^*}$ consistently increases with the normalised maximum deflection w_{max}^* , indicating a typical hardening nonlinearity driven by geometric effects. For the shallow crack configuration ($a/h = 0.1$), this increase remains gradual and closely aligns with previously published results [2], thereby confirming the accuracy and reliability of the present formulation. In contrast, the deep crack case ($a/h = 0.6$) exhibits a more pronounced response: at low amplitudes, the frequency ratio is reduced due to the loss of stiffness induced by the crack, whereas at higher amplitudes, a significant increase is observed as geometric stiffening becomes dominant. This behaviour reflects the interplay between damage-induced flexibility and nonlinear stiffening mechanisms, highlighting the necessity of incorporating nonlinear effects to accurately describe the dynamic response of cracked structures.

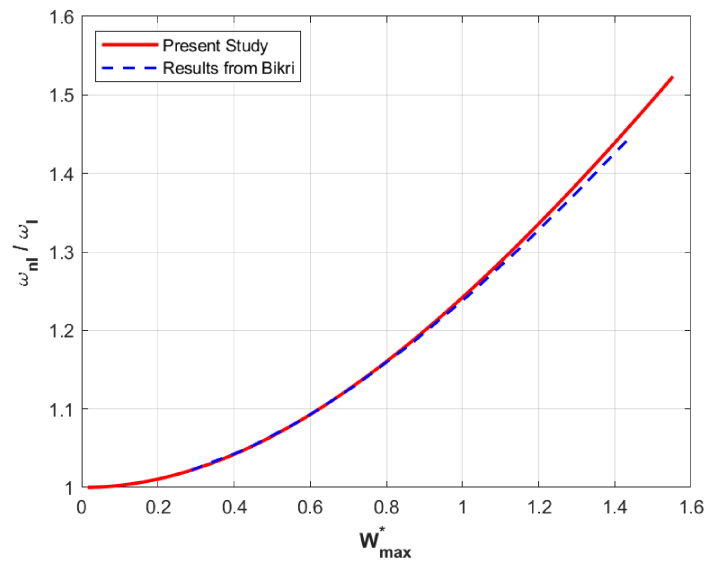


Figure 5. Comparison of nonlinear frequency ratio with reference results for a shallow crack ($a/h = 0.1$) mode 1.

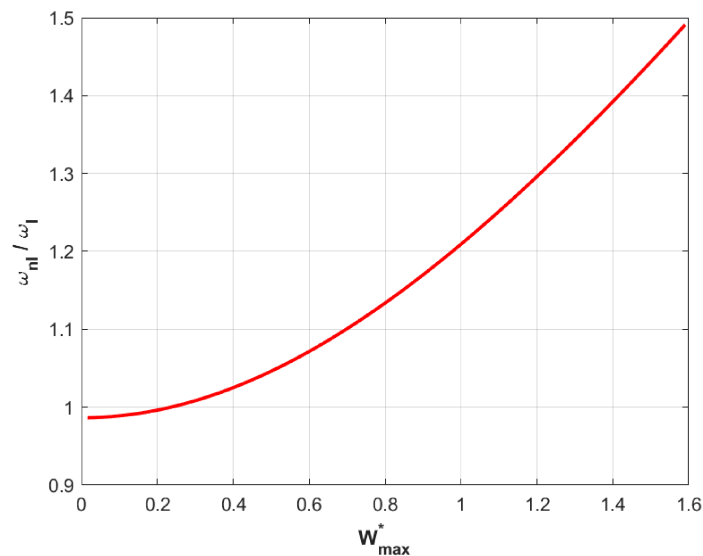


Figure 6. Nonlinear frequency ratio evolution for a deep crack ($a/h = 0.6$) mode 1.

5. Conclusions

A reduced-order formulation has been developed for the geometrically nonlinear free vibration of a clamped–clamped beam containing an open transverse crack. The model combines a fracture-mechanics-based representation of crack flexibility with a modal Galerkin reduction that accounts for axial stretching.

From a physical standpoint, the crack acts as a local compliance that reduces bending resistance, whereas large-amplitude motion activates an additional stretching-induced restoring force. The reported results for the first mode show that both vibration amplitude and crack depth strongly influence the non-dimensional bending-stress distribution, especially near the clamped boundaries. The proposed framework is therefore suitable for the nonlinear dynamic assessment of damaged beam-like structures and can serve as a basis for further analyses of amplitude-dependent frequencies and backbone curves. The present study is nevertheless limited to a single open (non-breathing) transverse crack, to the first vibration mode and to Euler–Bernoulli kinematics; extensions to multiple or breath-

ing cracks, higher modes, Timoshenko shear deformation and experimental validation constitute natural directions for future work.

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Nomenclature

Symbol	Description
L	beam length
b	beam width
h	beam thickness
S	cross-sectional area
I	second moment of area
E	Young's modulus
ν	Poisson's ratio
ρ	mass density
$W(x,t)$	transverse displacement
a	crack depth
x_0	crack location
a/h	crack-depth ratio
C	crack flexibility
K_f	equivalent rotational stiffness
β	dimensionless crack parameter
$\phi_i(x)$	linear mode shape of the cracked beam
$a_i(t)$	generalised modal coordinate
M	generalised mass matrix
K	linear stiffness matrix
$B(A)$	nonlinear stiffness operator
ω_n	linear natural frequency
ω^*	dimensionless nonlinear frequency

References

1. Azrar, L.; Benamar, R.; White, R.G. A semi-analytical approach to the non-linear dynamic response problem of beams at large vibration amplitudes, Part II: Multimode approach to the steady state forced periodic response. *J. Sound Vib.* **2002**, *255*, 1–41. [\[CrossRef\]](#)
2. El Bikri, K.; Benamar, R.; Bennouna, M.M. Geometrically non-linear free vibrations of clamped–clamped beams with an edge crack. *Comput. Struct.* **2006**, *84*, 485–502. [\[CrossRef\]](#)

3. Fakhreddine, H.; Adri, A.; Chajdi, M.; Benamar, R. Large amplitude forced vibrations of restrained beams resting on elastic point supports. *Eng. Trans.* **2021**, *69*, 225–242. [[CrossRef](#)]
4. El Khouddar, Y.; Adri, A.; Said, R.; Benamar, R. Geometrically non-linear forced vibrations of Euler–Bernoulli laminated composite beams. *J. Phys. Conf. Ser.* **2021**, *1896*, 012016. [[CrossRef](#)]
5. Moukhliiss, A.; Rahmouni, A.; Bouksour, O.; Benamar, R. Using the discrete model for the processing of natural vibrations, tapered beams made of AFG materials carrying masses at different spots. *Mater. Today Proc.* **2022**, *52*, 21–28. [[CrossRef](#)]
6. Hassa, M.; Adri, A.; El Khouddar, Y.; Outassafte, O.; Hantati, I.E.; Echouai, E.K.; Echouai, B.; Benamar, R. A multimode approach to nonlinear geometrically free and forced vibration analysis of two beams interconnected by several double spring-mass. *ZAMM—J. Appl. Math. Mech.* **2025**, *105*, e70268. [[CrossRef](#)]
7. Hassa, M.; Adri, A.; El Khouddar, Y.; Outassafte, O.; El Hantati, I.; Echouai, B.; Benamar, R. Nonlinear geometrically free vibration analysis of two beams coupled by double spring-mass. *ZAMM—J. Appl. Math. Mech.* **2025**, *105*, e70026. [[CrossRef](#)]
8. Janati, M.; Ikhharrazne, L.; Lamine, M. Analysis of Beam Transverse Vibration with Elastic Cubic Nonlinearity. *Constr. Technol. Archit.* **2026**, *21*, 65–78. [[CrossRef](#)]
9. Zhang, J.; Chen, H. Voltage-induced beating vibration of a dielectric elastomer membrane. *Nonlinear Dyn.* **2020**, *100*, 2547–2563. [[CrossRef](#)]
10. Wang, C.; Yao, G. Nonlinear vibration analysis of thin plate with periodically distributed tunable-stiffness nonlinear oscillators in subsonic flow. *J. Vib. Control* **2024**, *31*, 2312–2327. [[CrossRef](#)]
11. Stojanović, V. Geometrically nonlinear vibrations of beams supported by a nonlinear elastic foundation with variable discontinuity. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *28*, 66–80. [[CrossRef](#)]
12. Sepehry, N.; Ehsani, M.; Asadi, S.; Shamshirsaz, M.; Bakhtiari Nejad, F. Fourier spectral element for simulation of vibro-acoustic modulation caused by contact nonlinearity in the beam. *Thin-Walled Struct.* **2022**, *177*, 109112. [[CrossRef](#)]

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