

Integration of XFEM and XIGA for Stress Concentration Analysis of Holes and Cracks in Isotropic and Functionally Graded Plates [†]

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[†] Presented at the 3rd International Electronic Conference on Machines and Applications, 12–14 May 2026;
Available online: <https://sciforum.net/event/IECMA2026>.

Abstract

In recent decades, numerical methods have become indispensable tools for solving complex problems in science and engineering. Among these, the finite element method (FEM) is widely recognized as a powerful computational approach. However, traditional FEM has significant limitations when modeling discontinuities such as cracks, holes, or material interfaces, particularly in functionally graded materials (FGMs). To address these challenges, this study proposes an advanced framework that integrates the Extended Finite Element Method (XFEM) with Isogeometric Analysis (XIGA), referred to as XFEM–XIGA, to simulate stress concentration factors (SCFs) around circular holes in both isotropic and FGM plates. The proposed methodology employs the level-set method to represent discontinuous boundaries and incorporates appropriate enrichment functions into the displacement field, allowing accurate modeling of stress concentrations without the need for remeshing. MATLAB codes were developed to implement this integration, providing a flexible computational platform for practical engineering applications. The performance of the proposed XFEM–XIGA approach was evaluated using several benchmark problems, including isotropic plates with circular holes near material boundaries and FGMs subjected to uniaxial tensile loading. The results obtained from the XFEM–XIGA model were compared with analytical solutions, standard FEM results, and available experimental data. For isotropic plates, the XFEM–XIGA model achieved a stress concentration error of 1.71%. For FGM plates with cracks or circular holes, the error was 2.55% compared with exact solutions. These findings demonstrate the robustness and accuracy of the integrated method in handling complex geometries and heterogeneous material properties. Overall, this study shows that the combination of XFEM and XIGA offers an efficient and reliable tool for analyzing stress concentration factors in FGM structures. The proposed approach provides improved modeling capabilities for industrial components where stress concentrations at material boundaries are critical to structural integrity and performance.



Academic Editor: Kai Cheng

Published: 2 July 2026

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Keywords: XFEM; XIGA method; holes; FGM; SIFs; level-set method

1. Introduction

Structural components in engineering are often designed with holes, either intentionally to reduce weight and cost or due to operating conditions that create discontinuities in the geometry. The presence of holes, inclusions, or material interfaces typically induces localized high-stress regions known as stress concentrations. These stress concentrations

significantly influence the strength and durability of the structure. Therefore, a thorough understanding of stress concentration phenomena is crucial for preventing premature failure and ensuring structural safety. Traditional analytical methods for determining stress concentration factors (SCFs) are generally limited to simple geometries and loading conditions. As engineering projects become increasingly complex, more advanced numerical approaches are required to accurately capture the mechanical behavior of structures containing discontinuities and heterogeneous materials [1–5].

Over the past few decades, the Finite Element Method (FEM) has emerged as a powerful and widely used tool for simulating stress and deformation in engineering problems. However, conventional FEM requires complex meshing and repeated re-meshing when modeling discontinuities such as cracks, pores, or holes. This process significantly increases computational costs and can reduce accuracy. To overcome these limitations, the Extended Finite Element Method (XFEM) was developed. XFEM incorporates enrichment functions into the displacement field to represent discontinuous behavior independently of the mesh [6,7]. The level-set method further enhances XFEM by implicitly describing voids and interfaces, thereby allowing arbitrary placement of holes or cracks within the mesh without remeshing. Despite its effectiveness, XFEM alone may encounter difficulties in maintaining exact geometry representation and achieving higher-order continuity, especially when modeling functionally graded materials (FGMs) or complex curved boundaries [8–14].

To address these challenges, Isogeometric Analysis (IGA) was introduced [15]. IGA employs Non-Uniform Rational B-Splines (NURBS) and related spline functions to represent both geometry and field variables with CAD-level precision. The integration of IGA with XFEM [15,16], commonly referred to as the eXtended Isogeometric Analysis (XIGA) method, provides a unified framework that combines the geometric accuracy of IGA with the discontinuity-handling capabilities of XFEM. XIGA enables smooth and precise representation of curved boundaries, such as circular holes in isotropic or FGM plates, while efficiently capturing stress concentration effects without the need for remeshing [17].

This study applies the XIGA method, implemented in MATLAB R2019b codes with level-set enrichment, to simulate the stress concentration factors (SCFs) of isotropic and functionally graded plates with circular holes near material boundaries. The proposed framework is validated through comprehensive comparisons with analytical solutions, conventional FEM results, and available experimental data, demonstrating its robustness and effectiveness in solving complex structural mechanics problems.

2. Methodology

2.1. XIGA Method for Modeling Holes

The proposed XFEM–XIGA framework combines the exact geometric representation capability of Isogeometric Analysis (IGA) with the discontinuity enrichment strategy of XFEM. In the present work, the computational domain is described by NURBS basis functions defined from control points, knot vectors, and associated weights. The same NURBS basis functions are employed to approximate the displacement field. The circular hole is represented implicitly by a level-set function [2,13], while discontinuities are incorporated through XFEM enrichment functions. Consequently, the displacement approximation consists of the standard NURBS interpolation together with the Heaviside enrichment across the hole boundary and branch enrichment functions near crack tips. This unified formulation enables exact CAD geometry representation while preserving the mesh-independent enrichment capability of XFEM, forming the proposed XFEM–XIGA approach.

In the XIGA framework, both the geometry of the structure and discontinuities such as holes are represented using Non-Uniform Rational B-Splines (NURBS) basis functions [15].

This approach provides an exact description of curved boundaries without relying on traditional mesh refinement.

The hole is implicitly described by a level-set function, which serves as an enrichment tool to introduce discontinuities into the displacement field.

Let Γ denote the boundary of the hole within the plate. For any point x inside the domain, the projection point x_Γ on Γ is found such that the Euclidean distance $\|x - x_\Gamma\|$ is minimized. The signed distance function can then be expressed as:

$$\phi(x) = \pm \min_{x_\Gamma} \|x - x_\Gamma\| \quad (1)$$

where the positive sign applies when x is outside the hole boundary, and the negative sign applies when x lies inside the hole. Thus, the position of the hole is fully captured by the level-set function $\phi(x)$. Specifically, $\phi(x) < 0$ indicates a point inside the circular hole, while $\phi(x) > 0$ denotes a point outside (see Figure 1).

For a circular hole with center x_c and radius r , the signed distance function can be explicitly expressed as:

$$\phi(x) = \|x - x_c\| - r \quad (2)$$

In XIGA, the level-set function is interpolated within NURBS elements as:

$$\phi(x) = \sum_{I=1}^n R_I(x) \phi_I \quad (3)$$

where $R_I(x)$ are the NURBS basis functions associated with control points, and ϕ_I are the nodal level-set values. This formulation ensures exact geometry representation and smooth higher-order continuity, which are particularly advantageous when modeling circular and curved holes in isotropic or FGM plates.

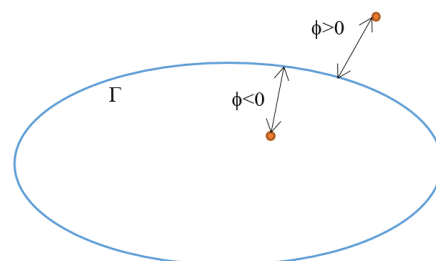


Figure 1. Signed distance function considering the sign convention.

2.2. Enrichment Functions in XFEM for Fracture Mechanics

The Extended Finite Element Method (XFEM) extends the standard finite element approximation space by introducing enrichment functions that can capture discontinuities and singularities without requiring remeshing [16].

For nodes located outside the hole, the Heaviside function takes the value $H(x) = 1$, while for nodes inside the hole, $H(x) = 0$. This discontinuous enrichment allows the displacement field to represent jumps across the hole boundary.

The branch enrichment functions near the crack tip are defined as:

$$\psi(x)|_{j=1}^4 = \left(\sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2} \sin \theta \right) \quad (4)$$

where r is the radial distance from the crack tip and θ is the polar angle relative to the crack axis (see Figure 2). These functions enrich the displacement approximation and capture the singular stress field near cracks or voids.

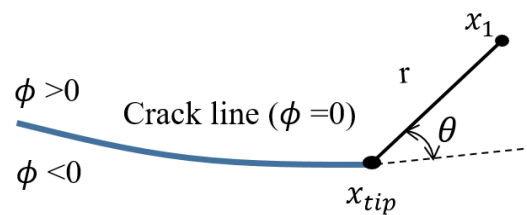


Figure 2. A schematic representation of distance r and angle θ at the crack tip.

Accordingly, the displacement field in XFEM is approximated as:

$$u(x) = \sum_{I \in W} R_I(x) u_I + \sum_{J \in H} R_J(x) H(x) a_j + \sum_{K \in T} R_K(x) \sum_{j=1}^4 F_j(x) b_{Kj} \quad (5)$$

where $R_I(x)$ are NURBS basis functions (in the XIGA framework), u_I are the nodal displacements, $H(x)$ is the Heaviside function across the hole boundary, and $F_j(x)$ are the branch functions describing the near-tip singularity. Sets W , H , and T represent the standard nodes, nodes cut by the discontinuity, and nodes around the crack tip, respectively (Figure 3).

In enriched elements, numerical integration is a critical step. Each element intersected by the zero level-set function is subdivided into subdomains via Delaunay triangulation. Gaussian quadrature is then performed on each sub-triangle, where second- or third-order integration is generally sufficient for accuracy. Nodes lying completely inside the void are excluded from the system of equations, effectively reducing computational cost while maintaining solution accuracy.

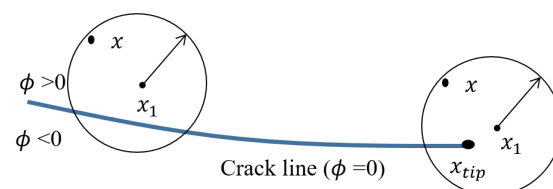


Figure 3. Diagram of domain definition for the interpolation.

2.3. Numerical Implementation

The computational domain was discretized using quadratic NURBS basis functions ($p = q = 2$). A structured control-point mesh was employed throughout the analysis, while the knot vectors were generated using the standard open-knot configuration. Gaussian quadrature with 3×3 integration points was used for regular elements, whereas enriched elements intersected by the level-set interface were subdivided into triangular subcells and integrated separately. Control points whose support was intersected by the zero level-set were enriched using the Heaviside function, while control points surrounding the crack tip were additionally enriched by the four crack-tip branch functions. The circular hole was represented implicitly through the signed distance level-set function without remeshing. Essential boundary conditions were imposed directly on the corresponding control points, and the external tensile load was applied as a uniform traction. A mesh refinement study was also conducted, showing that the numerical results converged rapidly and remained stable with increasing control-point density.

3. Numerical Results

3.1. Isotropic Plate with a Concentric Circular Hole

Consider a thin isotropic plate containing a concentric circular hole subjected to uniaxial tensile loading. The numerical analyses were performed under plane stress conditions, which are appropriate for the thin plate considered in this study. The geometric and material parameters of the problem are as follows (see Figure 4): $E = 210,000 \text{ N/mm}^2$; $\nu = 0.3$; $l = 200 \text{ mm}$; $a = 10 \text{ mm}$; $w = 100 \text{ mm} = 2b$; $t = 10 \text{ mm}$, $\sigma_0 = 100 \text{ N/mm}^2$

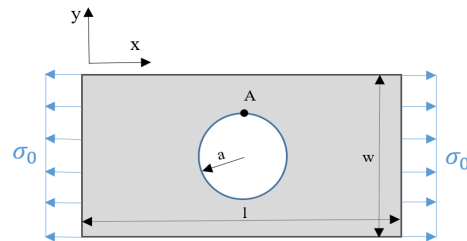


Figure 4. Circular hole in a thin plate subjected to uniaxial uniform tensile load.

According to analytical theory, the maximum stress occurs at point A. The theoretical stress concentration factor K_t is given by:

$$(\sigma_\theta)_{max} = 3\sigma_0 = 300 \text{ N/mm}^2$$

Stress concentration factor at A:

$$K_t = \frac{\sigma_{max}}{\sigma_0} = 3$$

XFEM-XIGA calculation results (Figure 5):

$$K_{tXFEM-XIGA} = \frac{\sigma_{max}}{\sigma_0} = \frac{295.9982}{100} = 2.96$$

Error:

$$K_{tXFEM-XIGA} = \frac{K_t - K_{tXFEM-XIGA}}{K_t} \cdot 100\% = 4\%$$

```
102.0332 POST PROCESSING
102.0335 Deformed configuration
108.3684 Stress computation
477.9351 Stress Intensity Factors - SIFs
```

```
>> max(max(stress(:,2)))
ans =
285.9982
```

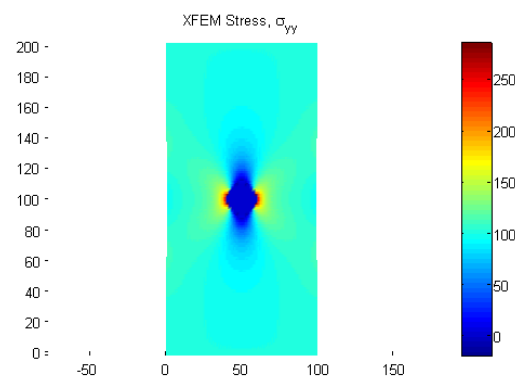


Figure 5. Stress distribution obtained using the proposed XFEM-XIGA method.

3.2. Isotropic FGM Plate with a Central Crack

An isotropic rectangular functionally graded material (FGM) plate with an edge crack was considered. The crack length $a = 0.5$ cm, plate width $L = 2$ cm, and plate length $D = 10$ cm. The problem was analyzed under plane strain conditions (see Figures 6–9).

Poisson's ratio was assumed to be constant at $\nu = 0.3$, while the elastic modulus (E) varied exponentially along the (x)-direction according to the following expression:

$$E(x) = E_0 e^{\beta x}$$

where $E_0 = 10^3$ and $\beta = 2$, $\sigma = 10^6$ N/m²

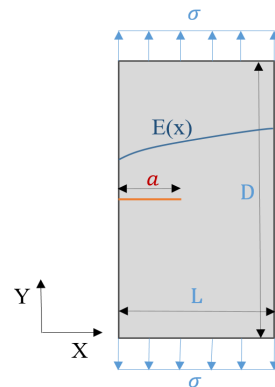


Figure 6. An isotropic FGM plate with an edge crack and material properties varying in the (x)-direction.

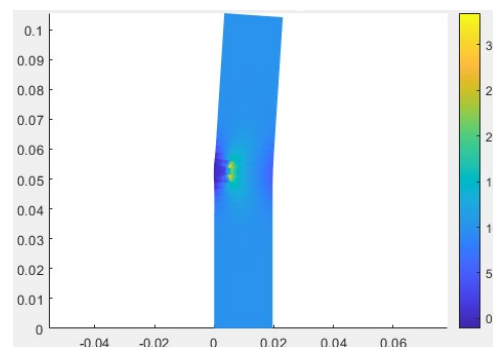


Figure 7. Displacement distribution obtained using the proposed XFEM–XIGA method.

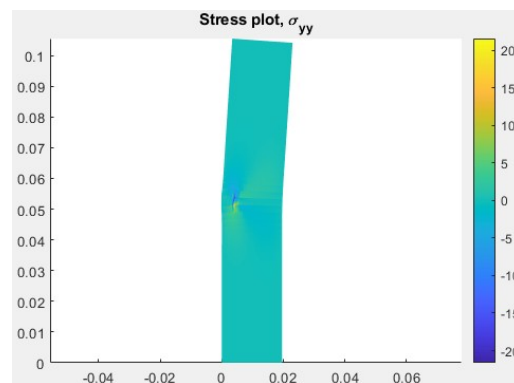


Figure 8. Von Mises stress distribution obtained using the proposed XFEM–XIGA method.

Computing the exact SIFs:

Crack geometric correction factor: $\alpha = 1.12 - 0.23\left(\frac{a}{W}\right) + 10.55\left(\frac{a}{W}\right)^2 - 21.71\left(\frac{a}{W}\right)^3 + 30.38\left(\frac{a}{W}\right)^4 = 1.5013$;

$$K_{exact} = \alpha \cdot \sigma \cdot \sqrt{\pi \cdot a} = 5.9503 \text{ (MPa} \cdot \sqrt{\text{mm}}) \quad (6)$$

The stress intensity factors K_I of the FGM plate for different crack lengths obtained from the XFEM-XIGA are presented in Figure 10. The percentage errors of the XFEM-XIGA results relative to the exact solutions are also shown in Figure 10 and summarized in Table 1. The results indicate that the XFEM-XIGA provides good agreement with the exact method.

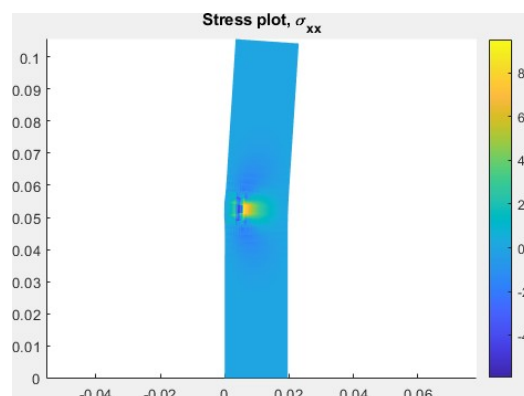


Figure 9. Principal stress distribution obtained using the proposed XFEM–XIGA method.

Table 1. Comparing SIF results among the exact method, MATLAB codes, XFEM-XIGA, from Figure 10, Equation (6).

Methods	Values K_I (MPa·√mm)	% Error
Exact	5.9503	-
XFEM	5.8003	2.52%

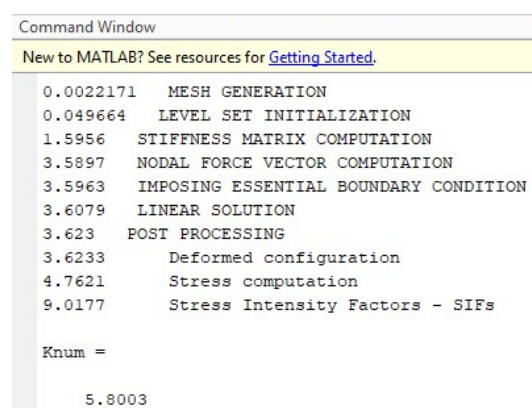
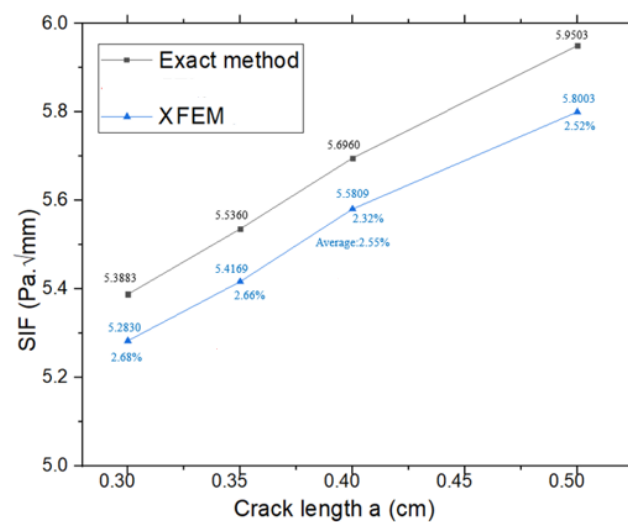


Figure 10. Stress intensity factor obtained using the proposed XFEM–XIGA method.

The relationship between the stress intensity factor K_I , crack length (a), and the corresponding percentage error is illustrated in Figure 11 and Table 2. As expected, the stress intensity factor increases with increasing crack length. This trend is consistent with fracture mechanics theory for FGM plates. A comparison of four different crack lengths shows that the XFEM-XIGA yields accurate predictions, with errors remaining within an acceptable range compared to the exact solutions.

Table 2. Relationship among SIFs, a% error, and crack length a .

Crack length a (cm)	0.3	0.35	0.4	0.5
Exact. ($\text{MPa}\cdot\sqrt{\text{mm}}$)	5.3785	5.5261	5.6861	5.9402
XFEM – XIGA ($\text{MPa}\cdot\sqrt{\text{mm}}$)	5.2910	5.4256	5.5818	5.8102
Error (%)	$Error_{FEM} = 5.99$	$Error_{FEM} = 5.89$	$Error_{FEM} = 5.96$	$Error_{FEM} = 5.82$
	$Error_{XFEM-XIGA} = 3.99$	$Error_{XFEM-XIGA} = 4.05$	$Error_{XFEM-XIGA} = 3.92$	$Error_{XFEM-XIGA} = 3.62$
	$Error_{(XFEM-XIGA)average} = 3.94$			

**Figure 11.** Relationship between the stress intensity factor K_I , crack length (a), and percentage error obtained from the XFEM-XIGA.

4. Discussion

The results obtained from the numerical analysis demonstrate the accuracy and efficiency of the XFEM-XIGA in modeling stress concentration and fracture parameters in both isotropic and functionally graded material (FGM) plates. For the isotropic plate with a circular hole, the XFEM-XIGA results showed only a small deviation from the exact analytical solution, with the error kept within an acceptable range. This confirms that the XFEM can accurately capture local stress concentration phenomena without requiring excessive mesh refinement around discontinuities.

For the FGM plate with a central crack, the displacement and stress distributions predicted by the XFEM-XIGA were consistent with theoretical expectations. The stress intensity factors (SIFs) obtained from the XFEM-XIGA agreed closely with the exact solution, with a relative error of only 3.62% for a crack length of 0.5 cm. Furthermore, the relationship between crack length and SIFs exhibited a clear increasing trend, which is in good agreement with fracture mechanics theory. The comparative analysis across multiple crack lengths further confirms that the XFEM-XIGA can accurately model crack behavior in heterogeneous materials with spatially varying elastic modulus.

Overall, the XFEM-XIGA provided stable and reliable numerical solutions while avoiding the complexity of remeshing around discontinuities. These advantages make it superior to the conventional FEM for analyzing problems involving holes, cracks, and material gradation.

5. Conclusions

This study investigated the application of the XFEM-XIGA for analyzing isotropic and functionally graded material (FGM) plates containing holes and cracks. The main findings can be summarized as follows:

- For isotropic plates with circular holes, the XFEM-XIGA successfully captured the maximum stress concentration, yielding results very close to the exact analytical solution.
- For FGM plates with central cracks, the XFEM-XIGA provided accurate predictions of displacement fields, stress distributions, and stress intensity factors (SIFs). The relative error compared with exact solutions was small (less than 3%), confirming the reliability of the method.
- The relationship between SIFs and crack length showed a consistent increasing trend, demonstrating that the XFEM can effectively describe fracture mechanics behavior in heterogeneous materials.
- The XFEM-XIGA eliminated the need for remeshing and offered high computational efficiency, making it a highly suitable tool for analyzing both fracture and stress concentration problems in isotropic and graded materials.

In conclusion, the XFEM-XIGA has proven to be a robust and accurate numerical method for simulating fracture and stress concentration phenomena in isotropic and FGM plates. It holds great potential for engineering applications involving complex geometries and heterogeneous materials.

Although the proposed XFEM-XIGA framework demonstrates good accuracy and robustness for the benchmark problems considered in this study, several limitations should be acknowledged. The present formulation is limited to two-dimensional analyses with relatively simple FGM distributions and crack configurations. More complex material gradation, multiple interacting cracks, curved crack propagation, and fully three-dimensional fracture problems have not yet been investigated. These topics constitute important directions for future research. Nevertheless, the integration of XFEM and IGA provides a flexible computational framework that can be extended to these more challenging applications.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The author thanks DNC for their support.

Conflicts of Interest: The author declares no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

IGA	Isogeometric Analysis
XFEM	Extended Finite Element Method
NURBS	Non-Uniform Rational B-Splines
SCFs	Stress concentration factors

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