

Modeling the Law of Motion of a Robotic Manipulator Using Polynomial Functions [†]

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Abstract

The increasing use of robotic manipulators in construction requires precise trajectory planning to reduce oscillations, dynamic loads, and positioning errors. Trajectory planning, understood as determining the manipulator's motion from an initial to a final configuration while satisfying specified kinematic and dynamic constraints, is a fundamental part of motion planning. Polynomial trajectories are widely applied because they ensure smoothness and allow specifying boundary conditions on velocity and acceleration. However, higher-degree polynomials introduce computational complexity. This study develops the analytical framework for seventh-degree polynomial trajectory generation for horizontal motion with smooth acceleration and deceleration, achieving prescribed maximum velocity and acceleration and identifying the conditions for including a uniform-motion segment.

Keywords: trajectory planning; uniform motion segment; kinematic constraints; dynamic loads reduction

1. Introduction

In recent decades, robotic technologies have been intensively implemented in construction. Numerous theoretical and experimental studies have been conducted by various researchers, resulting in the development of diverse concepts for automation and robotization of construction processes [1]. The increased degree of mechanization and automation in construction processes has led to the complexity of activities related to the design, adjustment, programming, and maintenance of equipment. Problems have arisen related to the difficulties in adapting to the working conditions of construction sites and the need to develop large-scale robotic machines while maintaining high accuracy in the operations performed [2]. This necessitates the definition and solution of fundamental research tasks that allow the synthesis of advanced designs for the construction of robotic manipulators with predetermined characteristics. One such fundamental task is to plan the movement of machines or their units to minimize dynamic structural loads and vibrations. Numerical studies in [3–6] showed that, by using laws with smooth increases and decreases in displacement, the dynamic behavior of machines can be significantly improved. Polynomials of a high degree are particularly suitable for motion planning, because they provide smoothness and allow the kinematic characteristics of the motion to be specified both at the beginning and the end of the motion [7–10], but the calculation of additional coefficients is complex.



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2. Problem Statement

In works [11,12] a mathematical model of a robotic construction manipulator with a Cartesian kinematic structure, a freely suspended load, and a fixed load were developed. The manipulator performs three types of rectilinear motions: longitudinal horizontal motion, transverse horizontal motion of the working tool, and vertical motion of the working tool. We considered the longitudinal horizontal motion of the manipulator because the largest displacements that occur during transportation operations are in this direction and involve high velocities and accelerations. This, combined with the considerable height of the manipulator and elasticity of its metal structure, results in significant oscillations. The effects of increasing machine speeds and accelerations are most pronounced during transitional phases such as startup and stopping. The primary effect is the excitation of vibrations in the structure, which can be mitigated by controlling the drive motor according to a law that ensures smooth starting and stopping. The motion law should satisfy various criteria [9], with particular importance placed on implementing a motion law that provides specified values of kinematic characteristics, such as velocity and acceleration, to enhance the functionality of the machine. In [13], an algorithm for trajectory planning based on a fifth-degree polynomial law with an intermediate segment of uniform motion was developed. A comparison was made between the two motion laws, where it was found that in a motion law with a uniform section, the duration of motion was shorter, and the preassigned maximum values of the speed and acceleration were achieved. The angle of oscillation of the vertical column is very small owing to the smooth acceleration and stopping. If a seventh-degree polynomial law is used for motion planning, which allows the inclusion of the derivative of acceleration, even smoother motion is ensured and the dynamic response of the machine is improved [11,13].

In the present work, the analytical foundations of manipulator trajectory planning according to a seventh-degree polynomial law are provided, which will provide an even smoother change in displacement. We set ourselves the goal of finding seventh-degree polynomials that describe the movement in stages of acceleration and stopping as well as determining the conditions under which a section with uniform motion can be included.

The displacement performed by the machine is represented by the diagram in Figure 1. To achieve the maximum performance of the machine, it is necessary to perform as much movement as possible at the maximum permissible speed, i.e., to include a section with uniform motion in the speed diagram. Half of the parameter T is used to accelerate to the preassigned speed during the interval from 0 to t_1 , followed by a period of uniform motion during the time interval from t_1 to t_2 . The other half of the parameter T is used to decelerate from the maximum speed to zero during the interval from t_2 to t_3 , where t_3 is the duration of the motion. The durations of the acceleration and deceleration periods are equal to $\frac{T}{2}$.

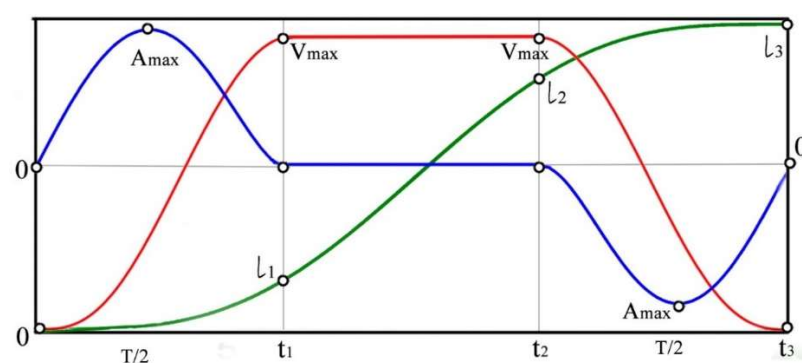


Figure 1. A trajectory with an intermediate section with uniform motion.

3. Main Statements and Proofs

3.1. Statement 1

In the set of all seventh-degree polynomials, $s(t) = \sum_{k=0}^7 c_k t^k$, there exists only one satisfying the following conditions:

$$\left| \begin{array}{l} s(0) = 0 \\ s'(0) = 0 \\ s''(0) = 0 \\ s'''(0) = 0 \end{array} \right. , \quad (1)$$

$$\left| \begin{array}{l} s(T) = l \\ s'(T) = 0 \\ s''(T) = 0 \\ s'''(T) = 0 \end{array} \right. , \quad (2)$$

where $l = \text{const} > 0$ and $T = \text{const} > 0$.

This polynomial is

$$s(t) = l \left(\frac{t}{T} \right)^4 \left(35 - 84 \left(\frac{t}{T} \right) + 70 \left(\frac{t}{T} \right)^2 - 20 \left(\frac{t}{T} \right)^3 \right), \quad (3)$$

and its derivatives up to the third order are

$$s'(t) = v(t) = 140 \frac{l}{T} \left(\frac{t}{T} \right)^3 \left(1 - \frac{t}{T} \right)^3, \quad (4)$$

$$s''(t) = a(t) = 420 \frac{l}{T^2} \left(\frac{t}{T} \right)^2 \left(1 - \frac{t}{T} \right)^2 \left(1 - 2 \frac{t}{T} \right), \quad (5)$$

$$s'''(t) = a'(t) = 840 \frac{l}{T^3} \left(\frac{t}{T} \right) \left(1 - \frac{t}{T} \right) \left(5 \left(\frac{t}{T} \right)^2 - 5 \left(\frac{t}{T} \right) + 1 \right). \quad (6)$$

Proof. Imposing the conditions (1) we obtain $c_0 = c_1 = c_2 = c_3 = 0$ and therefore

$$\begin{aligned} s(t) &= t^4 \sum_{k=4}^7 c_k t^{k-4}, \quad s'(t) = t^3 \sum_{k=4}^7 k c_k t^{k-4}, \\ s''(t) &= t^2 \sum_{k=4}^7 k(k-1) c_k t^{k-4} \text{ and } s'''(t) = t \sum_{k=4}^7 k(k-1)(k-2) c_k t^{k-4}. \end{aligned}$$

Imposing conditions (2) we obtain the system

$$\left| \begin{array}{l} \sum_{k=4}^7 c_k T^{k-4} = \frac{l}{T^4} \\ \sum_{k=4}^7 k c_k T^{k-4} = 0 \\ \sum_{k=4}^7 k(k-1) c_k T^{k-4} = 0 \\ \sum_{k=4}^7 k(k-1)(k-2) c_k T^{k-4} = 0 \end{array} \right.$$

which has only one solution: $c_4 = 35 \frac{l}{T^4}$, $c_5 = -84 \frac{l}{T^5}$, $c_6 = 70 \frac{l}{T^6}$, $c_7 = -20 \frac{l}{T^7}$.

We substitute the coefficients into the expression for $s'(t)$ and get

$$s'(t) = 140 \frac{l}{T} \left(\frac{t}{T} \right)^3 \left(1 - 3 \frac{t}{T} + 3 \left(\frac{t}{T} \right)^2 - \left(\frac{t}{T} \right)^3 \right) = 140 \frac{l}{T} \left(\frac{t}{T} \right)^3 \left(1 - \frac{t}{T} \right)^3$$

from where by differentiation, we obtain the final form of $s''(t)$ and $s'''(t)$. $\square$

3.2. Statement 2

Let $t \in [0; T]$ and $s(t)$, $v(t)$, $a(t)$ be the functions obtained in Statement 1. In such a case,

1. $s(t)$ is a strictly increasing function on the interval $[0; T]$

$$\begin{aligned} \min_{t \in [0; T]} s(t) &= s(0) = 0 \\ \max_{t \in [0; T]} s(t) &= s(T) = l' \end{aligned} \quad (7)$$

2. $v(t) > 0$ in the interval $(0; T)$ and

$$\begin{aligned} v_{\min} &= \min_{t \in [0; T]} v(t) = v(0) = v(T) = 0; \\ v_{\max} &= \max_{t \in [0; T]} v(t) = v\left(\frac{T}{2}\right) = \frac{35}{16} \frac{l}{T}. \end{aligned} \quad (8)$$

3. $a(t) > 0$ in the interval $\left(0; \frac{T}{2}\right)$, $a(t) < 0$ in the interval $\left(\frac{T}{2}; T\right)$,

$$\begin{aligned} a(0) &= a\left(\frac{T}{2}\right) = a(T) = 0 \\ a_{\max} &= \max_{t \in [0; T]} a(t) = a(\sigma_1 T) = \frac{84\sqrt{5}}{25} \frac{l}{T^2}; \\ a_{\min} &= \min_{t \in [0; T]} a(t) = a(\sigma_2 T) = -a_{\max} < 0. \end{aligned} \quad (9)$$

where $\sigma_1 = \frac{1}{2} \left(1 - \frac{\sqrt{5}}{5}\right) \in \left(0; \frac{1}{2}\right)$, $\sigma_2 = \frac{1}{2} \left(1 + \frac{\sqrt{5}}{5}\right) \in \left(\frac{1}{2}; 1\right)$.

Proof. From the expression (6) for $a'(t) = s'''(t)$ follows

$$\begin{aligned} s'''(t) &> 0, \quad 0 < t < \sigma_1 T, \\ s(\sigma_1 T) &= 0, \\ s'''(t) &< 0, \quad \sigma_1 T < t < \sigma_2 T, \\ s(\sigma_2 T) &= 0, \\ s'''(t) &> 0, \quad \sigma_2 T < t < T, \end{aligned}$$

where σ_1 and σ_2 are the roots of the polynomial $5\sigma^2 - 5\sigma + 1$. $\square$

Thus we obtain that the function $a(t) = s''(t)$ is strictly monotonically increasing in the intervals $[0; \sigma_1 T]$ and $[\sigma_2 T; T]$, and strictly monotonically decreasing in the interval $[\sigma_1 T; \sigma_2 T]$; therefore

$$\begin{aligned} a_{\max} &= \max_{t \in [0; T]} a(t) = a(\sigma_1 T) > 0; \\ a_{\min} &= \min_{t \in [0; T]} a(t) = a(\sigma_2 T) = -a_{\max}. \end{aligned}$$

From the expression (5) for $s''(t)$ follows

$$\begin{aligned} s''(t) &> 0, \quad 0 < t < \frac{T}{2}, \\ s''\left(\frac{T}{2}\right) &= 0, \\ s''(t) &< 0, \quad \frac{T}{2} < t < T \end{aligned}$$

and so, it follows that the function $v(t) = s'(t)$ is strictly monotonically increasing in the interval $\left[0; \frac{T}{2}\right]$ and strictly monotonically decreasing in the interval $\left[\frac{T}{2}; T\right]$; therefore

$$v_{\min} = \min_{t \in [0; T]} v(t) = v(0) = v(T) = 0.$$

$$v_{\max} = \max_{t \in [0; T]} v(t) = v\left(\frac{T}{2}\right) = \frac{35}{16} \frac{l}{T}.$$

From the expression (4) for $s'(t)$ follows $s'(t) > 0$, $0 < t < T$ and so we obtain that $s(t)$ is a strictly monotonically increasing function in the interval $[0; T]$.

As a direct consequence of (8) and (9), the following dependencies are obtained:

$$T = \frac{192\sqrt{5}}{125} \frac{v_{\max}}{a_{\max}} \quad (10)$$

$$l = \frac{3072\sqrt{5}}{4375} \frac{v_{\max}^2}{a_{\max}} \quad (11)$$

4. Obtained Results

The analysis of the obtained results indicates that three trajectory variants are possible depending on the values of the preset displacement and the calculated parameter l —a limit displacement, in which the set maximum values of speed and acceleration can be simultaneously achieved.

When the preset displacement is smaller than the limit displacement, it is not possible to concurrently achieve the pre-assigned values of the maximum velocity and acceleration. In most cases, the maximum acceleration is the determining factor for the kinematic characteristics of motion. From (11), the maximum possible velocity that satisfies the preassigned value of the maximum acceleration is determined

$$v_{\max} = \sqrt{\frac{la_{\max}}{k}} \quad \text{where} \quad k = \frac{3072\sqrt{5}}{4375}. \quad (12)$$

In the second variant, the preset displacement is equal to the limit displacement—the trajectory is entirely polynomial (3). The preset maximum velocity and acceleration can be reached simultaneously, but a segment of uniform motion cannot be included.

We are particularly interested in the third case, in which the preset displacement is greater than the limit displacement. It is possible to synthesize a trajectory with an intermediate section of uniform motion for preassigned values of the maximum velocity and acceleration.

Let $s(t)$, $v(t)$ and $a(t)$ be the functions defined by (3), (4) and (5), respectively.

Let $v_{\max}$ and $a_{\max}$ be predetermined maximum velocity and acceleration, respectively, and let the parameters T and l be chosen so that (10) and (11) are satisfied.

In the third variant we have

$$t_1 = \frac{T}{2}, \quad t_2 > t_1 \quad \text{and} \quad t_3 - t_2 = \frac{T}{2}. \quad (13)$$

From (13), it follows that

$$t_2 - t_1 = t_3 - T > 0. \quad (14)$$

By definition

$$v_1(t) = v(t), \quad 0 \leq t \leq t_1,$$

$$v_2(t) = v_{\max} = \text{const}, \quad t_1 \leq t \leq t_2,$$

$$v_3(t) = v(t - (t_2 - t_1)), \quad t_2 \leq t \leq t_3.$$

From the expression (4), it immediately follows that

$$v(T - t) = v(t) \text{ for each } t, \quad (15)$$

From (14) and (15) it follows that $v_3(t) = v(T - (t_3 - t)) = v(t_3 - t)$, i.e.,

$$v_3(t) = v(t_3 - t), \quad t_2 \leq t \leq t_3. \quad (16)$$

Let the function $\tilde{v}(t)$ describe the instantaneous velocity.

We define

$$\tilde{v}(t) = \begin{cases} v_1(t) = v(t), & 0 \leq t \leq t_1 \text{—the acceleration section;} \\ v_2(t) = v_{\max} = \text{const}, & t_1 \leq t \leq t_2 \text{—the uniform motion section;} \\ v_3(t) = v(t_3 - t) & t_2 \leq t \leq t_3 \text{—the deceleration section.} \end{cases} \quad (17)$$

We introduce the functions $s_1(t)$, $s_2(t)$ and $s_3(t)$ as follows:

$$s_1(t) = s(t); \quad 0 \leq t \leq t_1; \quad (18)$$

$$s_2(t) = l_1 + v_{\max}(t - t_1); \quad t_1 \leq t \leq t_2; \quad (19)$$

$$s_3(t) = l_2 + \int_{t_2}^t v_3(\tau) d\tau \quad t_2 \leq t \leq t_3, \quad (20)$$

where

$$l_1 = s_1(t_1) = s\left(\frac{T}{2}\right) = \frac{1}{2}l, \quad (21)$$

$$l_2 = l_1 + v_{\max}(t_2 - t_1). \quad (22)$$

Using (15) and (16) we obtain that

$$s_3(t) = l_1 + l_2 - s(t_3 - t), \quad t_2 \leq t \leq t_3. \quad (23)$$

If $l_3 = s(t_3)$, then by using (14), (21), (22) and (23) we have:

$$l_3 = l_1 + l_2 = l + v_{\max}(t_3 - T) > l$$

and therefore

$$t_3 = T + \frac{l_3 - l}{v_{\max}} > T.$$

We define the function $\tilde{s}(t)$ in the following way:

$$\tilde{s}(t) = \begin{cases} s_1(t), & 0 \leq t \leq t_1 \text{—the acceleration section;} \\ s_2(t), & t_1 < t < t_2 \text{—the uniform motion section;} \\ s_3(t), & t_2 \leq t \leq t_3 \text{—the deceleration section.} \end{cases}$$

It can be proven that $\tilde{s}(t)$ is a three-times continuously differentiable function in the interval $[0; t_3]$ (at all points t except $t = t_1$ and $t = t_2$, this function is infinitely differentiable).

Also, we have

$$\begin{cases} \tilde{s}'(t) = \tilde{v}(t), & 0 \leq t \leq t_3; \\ \tilde{s}(0) = 0. \end{cases}$$

Thus, the function $\tilde{s}(t)$ describes the displacement from the initial moment $t = 0$ to current moment t .

Let $\tilde{a}(t) = \tilde{v}'(t) = \tilde{s}''$, $0 \leq t \leq t_3$.

Then, by using (15), we obtain

$$\tilde{a} = \begin{cases} a_1(t) = a(t), & 0 \leq t \leq t_1; \\ a_2(t) = 0, & t_1 < t < t_2; \\ a_3(t) = -a(t_3 - t), & t_2 \leq t \leq t_3. \end{cases}$$

The function $\tilde{a}(t)$ describes the instantaneous acceleration.

5. Conclusions

The developed trajectory planning algorithm, based on seventh-degree polynomials, ensures smooth transitions through the three sections of motion: acceleration, uniform motion, and deceleration. Three possible trajectory variants for planning longitudinal horizontal motion are considered, depending on the length of displacement that needs to be covered and the achievement on the preassigned maximum speed and acceleration. When the preset displacement exceeds the calculated limit, the trajectory can consist of polynomial acceleration and deceleration phases with an intermediate segment of uniform motion. The resulting polynomials can be used for trajectory planning of the longitudinal motion of machines or their elements where smooth motion and precise positioning are required.

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References

1. Saidi, K.S.; Bock, T.; Georgoulas, C. Robotics in Construction. In *Springer Handbook of Robotics*; Siciliano, B., Khatib, O., Eds.; Springer: Cham, Switzerland, 2016; pp. 1493–1520.
2. Bulgakov, A.; Vorobyov, V. *Industrial Robots. Kinematics, Dynamics, Control and Management*; Solon Press: Moscow, Russia, 2012. (In Russian)
3. Abe, A. An effective trajectory planning method for simultaneously suppressing residual vibration and energy consumption of flexible structures. *Case Stud. Mech. Syst. Signal Process.* **2016**, *4*, 19–27. [[CrossRef](#)]
4. Qie, N.; Houa, W.-F.; He, J.-H. The fastest insight into the large amplitude vibration of a string. *Rep. Mech. Eng.* **2021**, *2*, 1–5. [[CrossRef](#)]
5. Siciliano, B.; Sciavicco, L.; Villani, L.; Oriolo, G. *Robotics: Modeling, Planning and Control*; Springer: Berlin/Heidelberg, Germany, 2009.
6. He, C.J.-H.; García, A. The simplest amplitude–period formula for non-conservative oscillators. *Rep. Mech. Eng.* **2021**, *2*, 143–148. [[CrossRef](#)]
7. Kalinkov, S.; Mitrev, R.; Ruzhekov, G. Simulation modelling and study of elevating transfer vehicle with controlled electric motor of the travel mechanism. *Mech. Mach.* **2009**, *82*, 71–78. (In Bulgarian)

8. Kalinkov, S. Research on Electronic Power Supply and Control Systems of Automated Storage and Retrieval Machines. Ph.D. Thesis, Technical University of Sofia, Sofia, Bulgaria, 2011. (In Bulgarian)
9. Mitrev, R. Development of the Theory for Designing, Modelling and Research of Construction and Lifting Manipulators. Doctoral Dissertation, Technical University of Sofia, Sofia, Bulgaria, 2019; pp. 290–301. (In Bulgarian)
10. Biagiotti, L.; Melchiorri, C. *Trajectory Planning of Automatic Machines and Robots*; Springer: Berlin/Heidelberg, Germany, 2008.
11. Ivanova, M. A Study of the Longitudinal Motion of a Robotic Construction Manipulator. Ph.D. Thesis, Technical University of Sofia, Sofia, Bulgaria, 2023. (In Bulgarian)
12. Panov, V.; Ivanova, M.; Mitrev, R. Mathematical modeling of the longitudinal motion of a robotic construction manipulator with a freely suspended payload. In *Proceedings of the American Institute of Physics Conference Series*; AIP Publishing: Melville, NY, USA, 2022; Volume 2557, p. 040003.
13. Mitrev, R.; Panov, V.; Ivanova, M. Trajectory planning of a robotic construction manipulator motion based on a fifth-degree polynomial with preassigned maximum velocity and acceleration. In *Proceedings of the American Institute of Physics Conference Series*; AIP Publishing: Melville, NY, USA, 2022; Volume 2557, p. 040004.

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