Entropy **2006**, *8*, 134-142

entropy
ISSN 1099-4300
www.mdpi.org/entropy/

Full Paper

Calculations of System Aging through the Stochastic Entropy

Paolo Rocchi

IBM, via Shangai 53, 00144 Roma, Italy
Tel. +39 6 5966 5213; Fax +39 6 5966 5212 (paolorocchi@it.ibm.com)

Received: 12 July 2006 / Accepted: 27 July 2006 / Published: 11 August 2006

Abstract: The present research discusses four ‘physical’ models of system and calculates the reliability function during system’s aging and maturity on the basis of the system structure.

Keywords: System aging, Boltzmann-like entropy, reliability function, Weibull distribution, Gompertz distribution.

1. Introduction

Attempts to develop a fundamental quantitative theory of aging, mortality, and lifespan have deep historical roots. Studies upon aging started with the earliest statistical studies on human mortality and later embraced both biological and artificial systems. Endeavors to classify theories of aging have led to the two major classifications. We find ‘wear and tear theories’ that sustain aging is the effect of the sum effect of many kinds of environmental assaults (e.g. radiation, metal ions) and of internal deterioration (e.g. mechanical attrition, spin-off accumulation) [1]. At the other side ‘programmed aging theories’ hold aging is due to something inside an organism's control mechanisms that forces elderliness and deterioration [2], [3]. The latter school is particular popular in the biological realm, while the former is formally sustained by the *reliability theory*. The fact that the failure rate is an increasing function of time is the formal expression of an aging property of system in the course of its operations.

Reliability theory seems to be a great method to obtain a general theory of aging and degradation of technical and biological systems, and interesting efforts have been conducted to define accurate statistical models of aging [4]. The present work follows a different path in the sense we discuss ‘physical models’ of systems and not pure statistical models. In particular we derive the reliability function from those models using the *stochastic entropy*, a function different from Shannon's entropy and introduced in [5].

I briefly remind that the stochastic entropy $H(A_i)$ is function of the probability P_i of the state A_i

$$H(A_i) = H(P_i) = \log_e(P_i) \tag{1}$$

and quantifies the reversibility/ irreversibility of the state A_i of the discrete-state stochastic system S. Assuming S in A_i, it may be said in intuitive terms that when S often abandons this state, A_i is somewhat reversible and $H(A_i)$ is 'low'. When the system does not evolve from the state Ai, we say that the state A_i is irreversible, and the stochastic entropy $H(A_i)$ is 'high'.

By definition the entropy $H(A_i)$ is summable

$$H_i = H(A_i) = f(H_{i1}, H_{i2}, ...H_{in}) = \sum_{g=1}^{n} H_{ig} \tag{2}$$

where S has n substates and H_{ig} is the entropy of the generic substate (or component) g.

2. Aging interpretations

In the present work we consider the stochastic system is binary, namely S either works or is repaired, and S assumes either the *functioning state* A_f or the *recovery state* A_r

$$P_f = 1 - P_r \tag{3}$$

Current literature holds that when a system becomes old, the probability P_f slops down. This trend implies that the entropy $H_f = H(P_f)$ declines, namely the functioning state becomes reversible and the capability of working of S diminishes in the physical world. The stochastic entropy illustrates that aging consists both of increased failure risk and of low ability of working, and in such a way answer an open riddle. In fact physicians, sociologists and other researches in the biological realm understand aging as the worsening of the performance characteristics of a system (e.g. see biomarkers of aging [6]). At the other side mathematicians interpret aging as the dramatic increase of the hazard rate. These views seem irreconcilable; instead the notion of reversibility/ irreversibility unifies the 'failure interpretation' of aging and the 'performance interpretation'.

Authors agree that system reliability gradually decreases during system maturity and comes down in the last period of lifetime. Aging is the accelerated decay of a system because additional phenomena join to the maturity decaying and worsen the behavior of systems. Because of this pair of parallel phenomena, I formalize the decay of S during maturity, and later shall tackle the system aging.

3. Maturity

A wide assortment of natural and artificial components degenerates due to bare functioning. Prolonged attrition, oxidation processes and other secondary effects, caused solely by running, bring forth the progressive reduction of system components. We make clear that:

1) – The ability of working of the generic component *k* decreases at constant rate with the passage of time, as a result of processes inherent in the functional block.
2) – The more a component is bust and the more the overall system degenerates. The capability of good functioning of the entire system *S* slops down due to the continuous decay of the components.

Axiom 3.1 - We quantify the reduced capabilities of *S* by $H_f(t)$ which observes the following axioms:

1. The entropy of the generic component *k* in the state A_f decreases linearly in function of the time

$$H_{fk}(t) = -\lambda_k t \qquad \lambda_k > 0 \quad k=1,2..n \tag{4}$$

where λ_k is the time-constant of *k*.

2. $H_f(t) = f(H_{f1}, H_{f2}, ...H_{fn})$ is a monotonically increasing function of H_{fk} (k = 1,2,..,n).

Theorem 3.1 – The maturity function $H_f(t)$ that complies with axioms 1 and 2 is the following

$$H_f(t) = -\lambda t \tag{5}$$

Proof: The property (2) holds the stochastic entropy is summable

$$H_f(t) = \sum_{k=1}^{n} H_{fk} \tag{6}$$

By combining (4) and (6) we prove the theorem

$$H_f(t) = \sum_{k=1}^{n} H_{fk} = -\sum_{k=1}^{n} \lambda_k t = -\left(\sum_{k=1}^{n} \lambda_k\right) t = -\lambda t \tag{7}$$

Application 3.1 - From Eqn (5) and (1) may be easily derived that the reliability function $P_f(t)$ is exponential [7] during the system maturity

$$P_f(t) = \nu \exp\left(-\lambda t\right) \qquad \nu > 0 \tag{8}$$

In substance (5) yields that if the system components grow worse with a regular rule, then the system reliability decreases with the exponential law. This result is a classical issue in reliability theory which we have derived from the model (4). Some authors have gained the exponential law using Markov chains but the method of calculus has raised some problems which instead are not present here [7].

4. Aging

Current literature holds a component decays so heavily in the last period of the system lifetime that the component hastens close components to failure. A 'cascade effect' unites to the phenomenon calculated by (5) and accelerates the system death. This mechanism runs toward every direction, in fact a deteriorated piece damages the components all around and even others more distant in *S**. We calculate this mechanism of *S** by means of the *aging function* H_f^*, which has three special properties.

1) – The foregoing considerations imply that the entropy H_f^* of *S** increases in function of the entropy H_{fk} of the generic component *k*.
2) – A component is capable of stopping the entire system. If only one component gives away, the overall system may give away, and this means that if the entropy H_{fk} of the generic component *k* reaches the minimum, then the reliability entropy H_f^* of *S* reaches the minimum value.
3) – The aging mechanism begins when the system is no longer mature. Intuitively we may say that the reciprocal damaging of components can start only when H_f^* has reached a 'certain' level below zero, which is the maximum of H_f^*. We assume that aging starts when the reliability entropy is -1, in order to normalize the function H_f^*.

Axiom 4.1 - To recapitulate, we assume the following axioms.

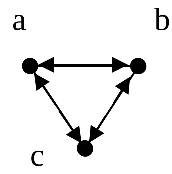
1. $H_f^* = f(H_{f1}, H_{f2}, ...H_{fn})$ is a monotonically increasing function of H_{fk} (k = 1,2,..,n).
2. If $H_{fk} = -\infty$, then $H_f^* = -\infty$
3. If $\left|H_{fk}\right| = 1$ for all the generic components *k*, then $\left|H_f^*\right| = 1$ (Axiom 3 is said 'normalization axiom').

Theorem 4.1 - The aging function that satisfies the three given axioms is

$$H_f^* = -\prod_{k=1}^{n} \left|H_{fk}\right| \tag{9}$$

Proof: Multiplication is a repeated addition of multipliers, thus axiom 1 is followed. It is sufficient a multiplier is infinite to obtain the product be infinite in (9). It is evident how the product $-|1\cdot 1\cdot 1\cdot 1\cdot \ldots \cdot 1| = -1$ proves the last axiom.

Remark 4.1 - The aging function (9) quantifies the worsening mechanism achieved by *n* components that systematically interact one with another. To exemplify the components a, b and c make the chains abc, acb, bac, bca, cab, and cba in *S** = {a,b,c,d}.

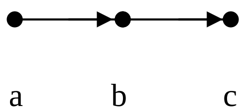


In short, *n*! chains carry on the overall aging process in the ideal *S** which is somewhat rare in the physical reality, thus we shall calculate two system structures that are more realistic.

Remark 4.2 - For the sake of exhaustiveness, we highlight that the so-called ‘cascade effect’ does not work in some systems. For example, the probability of failure show special trends during aging of electronic circuits in fact the components do not damage one another in accordance to the mechanism above discussed.

5. Aging of linear systems

Let $S^{\wedge}$ has a linear structure, thus the spoiled component *k* is capable of damaging only the component (*k* + 1).



A precise sequence of components accomplishes the aging process of the entire system.

Theorem 5.1 - The aging function for a linear system is the following

$$H_f^{\wedge} = -\frac{|H_{f1}\cdot H_{f2}\cdot H_{f3}\cdot \ldots \cdot H_{fn}|}{n!} \tag{10}$$

Proof: The aging function of the linear system may be easily derived from (9). In fact H_f^* calculates *n*! permutations without repetitions while a linear process consists of only one permutation. Supposing there are no special reasons that unbalance the system aging, we divide H_f^* by the factorial and obtain the aging function $H_F^{\wedge}$ for the linear chain

$$H_f^\wedge = -\frac{\prod_{k=1}^{n} \left| H_{fk} \right|}{n!} = -\frac{\left| H_{f1} \cdot H_{f2} \cdot H_{f3} \cdot \ldots \cdot H_{fn} \right|}{n!} \tag{11}$$

Application 5.1 : Eqn (4) is true during the system maturity and even during aging. We make explicit the entropies of the components in (11) to see the reduced capability of $S^\wedge$ in the time

$$H_f^\wedge(t) = -\frac{\left| t\lambda_{f1} \cdot t\lambda_{f2} \cdot t\lambda_{f3} \cdot \ldots \cdot t\lambda_{fn} \right|}{n!} \tag{12}$$

For the sake of simpleness we assume all the time-constants equal to λ_f, and we get

$$H_f^\wedge(t) = -\frac{\left| \lambda_f^n \cdot t^n \right|}{n!} = -\left(\frac{\lambda_f^n}{n!} \right) t^n = -\sigma t^n \tag{13}$$

Where σ is a constant depending on the system $S^\wedge$. From the logarithm function (1) we get the reliability function that follows the Weibull distribution

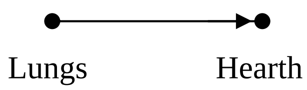
$$P_f = P_f(t) = \beta \cdot \exp(-\alpha t^n) \qquad \beta, \alpha > 0 \tag{14}$$

This theoretical result perfectly matches with current literature; in fact several artificial systems (e.g. vehicles, machines, industrial processes, automatic equipment) are linear and preferably follow the Weibull law [8].

6. Aging of compound systems

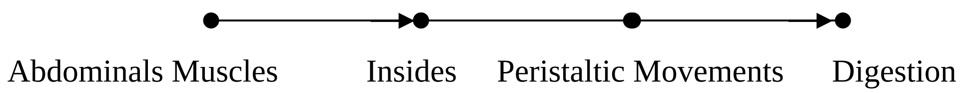
Biological systems are not linear and the interferences during aging are not systematic and follow distinguished rules. Heart, lungs, nerves, skin, bones, muscles, and stomach work together; they do not collapse but endanger according to selected relationships when they get old.

Example 6.1 - Lungs reduce the ventilation during aging and cause cardiac fatigue. There is a direct interference between this pair of components. Lungs do not affect other organs e.g. skeleton, spleen.

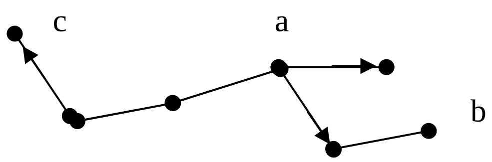


Example 6.2 - Abdominals muscles lose tone during aging and become flabby. As a consequence of this decaying, the insides are no longer contained and the peristaltic movements lose the appropriate

rhythms and the processes of digestion delays and become defective. An interference chain strikes four components



In conclusion a component makes cascade chains of different length. On principle we state the generic component *k* degenerates by-itself and even causes one-step chain, two-step chain, three-step chain, up-to *n*-step chain.



In other words, the degeneration chains stemming from *k* are unequal and range from one to *n* components.

Theorem 6.1 - The aging function for the complex system S^+ is the following

$$H_f^+ = -\sum_{k=1}^{n}\left(\frac{|H_{fk}|}{1!} + \frac{|H_{fk}H_{fh}|}{2!} + \frac{|H_{fk}H_{fl}H_{fd}|}{3!} + \ldots + \frac{|H_{fk}H_{fi}H_{fs}H_{fu}\ldots|}{n!}\right) \tag{16}$$

Proof: As first we calculate the aging function H_{fk}^+ for the generic component *k* which causes *n* chains of inference. We use (10) to calculate each chain and (2) to calculate the overall result

$$H_{fk}^+ = -\left(\frac{|H_{fk}|}{1!} + \frac{|H_{fk}H_{fh}|}{2!} + \frac{|H_{fk}H_{fl}H_{fd}|}{3!} + \ldots + \frac{|H_{fk}H_{fi}H_{fs}H_{fu}\ldots|}{n!}\right) \tag{17}$$

Eqn (17) is valid for every component of S^+, thus we calculate the aging function through summation and we prove the theorem is true

$$H_f^+ = \sum_{k=1}^{n} H_{fk}^+ = -\sum_{k=1}^{n}\left(\frac{|H_{fk}|}{1!} + \frac{|H_{fk}H_{fh}|}{2!} + \frac{|H_{fk}H_{fl}H_{fd}|}{3!} + \ldots + \frac{|H_{fk}H_{fi}H_{fs}H_{fu}\ldots|}{n!}\right) \tag{18}$$

Application 6.1 : Eqn (4) is valid for S^+, thus we make explicit the aging function in respect to time

$$H_f^+(t) = -\sum_{k=1}^{n}\left(\frac{\lambda_{fk}t}{1!} + \frac{\lambda_{fk}\lambda_{fh}t^2}{2!} + \frac{\lambda_{fk}\lambda_{fl}\lambda_{fd}t^3}{3!} + \frac{\lambda_{fk}\lambda_{fi}\lambda_{fs}\lambda_{fu}t^4}{4!} + ...\right) \tag{19}$$

For the sake of simpleness we put each product λ_{fk}, $\lambda_{fk}\lambda_{fh}$, $\lambda_{fk}\lambda_{fl}\lambda_{fd}$, $\lambda_{fk}\lambda_{fi}\lambda_{fs}\lambda_{fu}$ equals to λ_f. This assumption also fits with the experience because the speed toward fatal failure of a degeneration chain is indifferent to the number of the chain steps. Thus it is reasonable to take the constant-time of every mechanisms be equal and to write

$$\begin{aligned} H_f^+(t) &= -n\left(\frac{\lambda_f t}{1!} + \frac{\lambda_f t^2}{2!} + \frac{\lambda_f t^3}{3!} + \frac{\lambda_f t^4}{4!} + ... + \frac{\lambda_f t^n}{n!}\right) = \\ &= -n\lambda_f\left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ... + \frac{t^n}{n!}\right) \end{aligned} \tag{20}$$

The number of components n in a biological system is high, we conclude that (20) reasonably approximates the exponential series

$$H_f^+(t) = -n\lambda_f\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}\right) = -\zeta\exp(t) \qquad \zeta > 0 \tag{21}$$

Using (1) we obtain that the reliability function follows the exponential-exponential distribution

$$P_f = P_f(t) = \delta\exp[-\varepsilon(\exp)t] \qquad \delta, \varepsilon > 0 \tag{22}$$

This nice result conforms to current literature which claims that the reliability function or mortality force for biological beings complies with the Gompertz distribution [1]. The new of the present research is that the Gompertz distribution has been computed from a 'physical' model which does not raise the problems opposed by other theoretical models [7].

7. Conclusive Remarks

At present, most efforts revolve on the definition pure statistical models in reliability theory. Few theoretical researches try to derive the properties of the system from its very structure and the results appear questionable. The current study calculates the behavior of a system during its maturity and aging by means of four 'physical' models of system, in particular the last couple of models yield respectively the aging of artificial and biological systems.

The present work has derived the reliability function on the basis of those models, in other word it has passed from studying 'how' a system declines, to studying 'why' a system declines, and is worth of attention due to this special approach. I guess the present method will contribute to the maturation of the reliability theory from the statistical analysis of the events to the theoretical forecast of the events.

References

1. Gertsbakh, I. B. *Statistical Reliability Theory*, Marcel Dekker, 1988.
2. Arking, R. *Biology of Aging*, Oxford University Press, 2006 (3rd Ed).
3. Bengtson, V. L.; Schaie, K. W. (ed), *Handbook of Theories of Aging*, Springer Publishing, 1999.
4. Nikulin, M. Accelerated Life and Degradation Models in Reliability: State of Art, *Proc. Stochastic Models in Reliability, Safety, Security and Logistics* (SMRSSL'05), BeerSheva, 2005, pp.257-265.
5. Rocchi, P. Boltzmann-like Entropy in Reliability Theory. *Entropy* **2002**, *4*, 142-150.
6. Schulz, R. (ed.), Encyclopedia of Aging, Springer Publishing, 2006 (4th Ed), p.137.
7. Rocchi, P. The Stochastic Entropy and the Physical Behavior of the Systems, *Proc. Fifth Intl. Scientific School on Modelling and Analysis of Safety and Risk in Complex Systems,* St. Petersurg, 2005, pp. 71-77].
8. Rigdon, S. E.; Basu, A. P. *Statistical Methods for the Reliability of Repairable Systems*, Wiley, 2000.