Entropy **2015**, *17*, 4028-4039; doi:10.3390/e17064028

OPEN ACCESS

entropy

ISSN 1099-4300

www.mdpi.com/journal/entropy

Article

Generalized Boundary Conditions for the Time-Fractional Advection Diffusion Equation

Yuriy Povstenko

Institute of Mathematics and Computer Science, Jan Długosz University in Częstochowa, Armii Krajowej 13/15, 42-200 Częstochowa, Poland; E-Mail: j.povstenko@ajd.czest.pl; Tel.: +048-343-612-269

Academic Editors: J. A. Tenreiro Machado and António M. Lopes

Received: 4 May 2015 / Accepted: 9 June 2015 / Published: 12 June 2015

Abstract: The different kinds of boundary conditions for standard and fractional diffusion and advection diffusion equations are analyzed. Near the interface between two phases there arises a transition region which state differs from the state of contacting media owing to the different material particle interaction conditions. Particular emphasis has been placed on the conditions of nonperfect diffusive contact for the time-fractional advection diffusion equation. When the reduced characteristics of the interfacial region are equal to zero, the conditions of perfect contact are obtained as a particular case.

Keywords: fractional calculus; non-Fickian diffusion; fractional advection diffusion equation; complex systems; nonperfect contact conditions

1. Introduction

In recent years considerable interest has been shown in fractional differential equations which describe important physical phenomena in amorphous, colloid, glassy and porous materials, in fractals and percolation clusters, comb structures, dielectric materials and semiconductors, biological systems, polymers, random and disordered media, geophysical and geological processes (see, for example, [1–7] and references therein). Fractional calculus also plays a significant part in studies of entropy. It should be emphasized that entropy is also used in the analysis of anomalous diffusion processes and fractional diffusion equations [8–18]. The entropy production rate for fractional diffusion processes was calculated in [8–10]. Well-posedness of the degenerate fractional convection-diffusion equation under the imposed entropy condition was investigated by Cifani and Jakobsen [11]. The spectral entropy for the case of fractional diffusion equation was calculated by Magin and Ingo [12,13]. The behavior of the relative entropy for anomalous diffusion was studied in [14,15]. The entropy approach to anomalous diffusion was used to analyze the magnetic resonance images in biological issues [16–18]. Fractional reaction-diffusion and advection-diffusion equations were studied by many authors (see [19–34], among many others). Several numerical methods were used to solve the problem: the implicit and explicit difference schemes [19–22], the Adomian decomposition method [23], the homotopy perturbation method [24] and homotopy analysis method [25], the collocation methods [26,27], the finite element method [28]. The finite volume spatial discretization using the matrix transfer technique and the time discretization were employed in [29]. Jiang and Lin [30] constructed the orthonormal basis in the reproducing kernel space and gave a Fourier series representation of the solution. The authors of [31] considered the time-fractional advection-diffusion equation and used the Laplace transform technique to obtain the corresponding time-independent inhomogeneous equation in the transform domain. Next they employ the boundary particle method to solve the transformed problem and implement the Stehfest numerical inverse Laplace transform. The space-time-fractional advection-diffusion equation with derivatives of the variable order was studied in [32]; the implicit Eulerian scheme and a Lagrangian solver were applied to solve this equation. The fractal and fractional derivatives were used in [33] to model anomalous diffusion; the Crank–Nicholson finite difference discretization scheme was utilized. We have cited the papers in which the bounded spatial domains were considered. In the majority of papers mentioned above the Dirichlet boundary condition was employed; the boundary conditions with the given normal derivative were prescribed in [29,31,34]. For the fractional advection-diffusion equation, the formulation of the proper boundary conditions describing the peculiarities of heat or mass exchange is a significant task.

Different kinds of boundary conditions for time-fractional diffusion equation were analyzed in [35–37]. Near the interface between two phases there arises a transition region which state differs from the state of contacting media owing to the different material particle interaction conditions. The transition region has its own physical, mechanical and chemical properties, and processes occurring in it differ from those occurring in the bulk. The properties of this region influence the course of processes such as diffusion, heat conduction, phase transition, corrosion, evolution of the defect structure, etc. Small thickness of the interface region allows us to consider it as a distinct two-dimensional phase and to formulate the corresponding two-dimensional equations for the interface. In this approach, mathematical description of processes occurring in the bulk phases consists in formulation and solution of some system of differential (or more complicated) equations with certain boundary conditions being the two-dimensional analogue of the corresponding three-dimensional equations [38].

2. Different Kinds of Boundary Conditions

2.1. The Classical Diffusion Equation

The standard theory of diffusion is based on the balance equation for mass:

$$\rho \frac{\partial c}{\partial t} = -\nabla \cdot \mathbf{J} \tag{1}$$

and the Fick law relating the matter flux $\mathbf{J}$ to the concentration gradient:

$$\mathbf{J} = -\kappa \nabla c, \tag{2}$$

where c is the concentration, ρ the mass density, κ the diffusive conductivity, respectively. From (1) and (2) we get the parabolic diffusion equation:

$$\rho \frac{\partial c}{\partial t} = \kappa \Delta c. \tag{3}$$

Generally speaking, we can consider the balance equation for the transported quantity and a phenomenological law which states the proportionality of the flux to the gradient of this quantity (the Fourier law, the Fick law, the Darcy law, *etc.*). Some of extensions of the laws mentioned above were formulated in terms of heat conduction, other in terms of diffusion or the flow of fluid in a porous medium. In what follows we will discuss diffusion, but in point of fact the consideration concerns various transport phenomena.

When the diffusion Equation (3) is studied in a bounded domain, the corresponding boundary conditions should be imposed. The boundary condition of the first kind (the Dirichlet boundary condition) specifies the concentration over the surface of a body:

$$c\big|_S = g(\mathbf{x}_S, t) \tag{4}$$

with $\mathbf{x}_S$ being a point at the surface *S*.

If the surface of a body is under the given matter flux $\mathbf{J}_e$ from the environment, then at the boundary we have:

$$\left(\mathbf{n} \cdot \mathbf{J} + \mathbf{n}_e \cdot \mathbf{J}_e\right)\big|_S = 0 \tag{5}$$

and, according to (2), the boundary condition of the second kind specifying the boundary value of the normal derivative of concentration (the Neumann boundary condition):

$$\kappa \frac{\partial c}{\partial n}\bigg|_S = g(\mathbf{x}_S, t), \tag{6}$$

where $g(\mathbf{x}_S, t) = \mathbf{n}_e \cdot \mathbf{J}_e(\mathbf{x}_S, t)$, $\mathbf{n}$ is the outer unit normal to the surface *S*.

The Newton condition of convective mass exchange between a body and the environment with the concentration c_e:

$$\mathbf{J} \cdot \mathbf{n}\big|_S = H\left(c\big|_S - c_e\right), \tag{7}$$

where *H* is the convective mass transfer coefficient, leads to a specification of a linear combination of the values of concentration and the values of the normal derivative of concentration at the boundary of a domain (the boundary condition of the third kind or the Robin boundary condition):

$$\left(\kappa \frac{\partial c}{\partial n} + Hc\right)\bigg|_S = g(\mathbf{x}_s, t). \tag{8}$$

where $g(\mathbf{x}_s, t) = Hc_e(\mathbf{x}_s, t)$.

When the surfaces of two bodies are in perfect diffusive contact, the concentrations at the contact surfaces are equal:

$$c_1\big|_S = c_2\big|_S \tag{9}$$

and the sum of the normal components of the matter fluxes should be equal to zero:

$$\left(\mathbf{n}_1 \cdot \mathbf{J}_1 + \mathbf{n}_2 \cdot \mathbf{J}_2\right)\Big|_S = 0, \tag{10}$$

which gives:

$$\kappa_1 \frac{\partial c_1}{\partial n}\bigg|_S = \kappa_2 \frac{\partial c_2}{\partial n}\bigg|_S, \tag{11}$$

where the subscripts 1 and 2 refer to the first and second body, respectively, and **n** is the common normal at the contact surface.

The boundary conditions presented above are well known and can be found in every textbook on diffusion or heat conduction. We have recalled them here to facilitate obtaining the proper boundary conditions for generalized equations.

2.2. The Standard Advection Diffusion Equation

The following constitutive equation for the matter flux (see, for example, [39]):

$$\mathbf{J} = -\kappa \nabla c + \rho\, \mathbf{v}\, c \tag{12}$$

in combination with the balance equation for mass (1) results in the standard advection diffusion equation:

$$\rho \frac{\partial c}{\partial t} = \kappa \Delta c - \rho\, \mathbf{v} \cdot \nabla c. \tag{13}$$

For the sake of simplicity we have restricted ourselves to the case $\rho = \text{const}$ and $\mathbf{v} = \text{const}$. Equation (13) can be interpreted in terms of diffusion or heat conduction with additional velocity field $\mathbf{v}$ as well as in terms of Brownian motion, transport processes in porous media, groundwater hydrology, *etc*. [39–44]. The specification of boundary conditions for the advection diffusion Equation (13) with taking into account the constitutive Equation (12) for the matter flux gives the following kinds of conditions. The Dirichlet boundary condition (4) with the given value of the concentration at the surface of a body remains unchanged. The prescribed boundary value of the matter flux yields:

$$\left(\kappa \frac{\partial c}{\partial n} - \rho\, v_{(n)} c\right)\bigg|_S = g(\mathbf{x}_S, t). \tag{14}$$

The convective mass exchange between a body and the environment provides:

$$\left[\kappa \frac{\partial c}{\partial n} + \left(H - \rho\, v_{(n)}\right) c\right]\bigg|_S = g(\mathbf{x}_S, t). \tag{15}$$

The boundary conditions of the perfect diffusive contact have the following form:

$$c_1\big|_S = c_2\big|_S, \tag{16}$$

$$\left(\kappa_1 \frac{\partial c_1}{\partial n} - \rho_1\, v_{(n)1} c_1\right)\bigg|_S = \left(\kappa_2 \frac{\partial c_2}{\partial n} - \rho_2\, v_{(n)2} c_2\right)\bigg|_S. \tag{17}$$

2.3. The Time-Fractional Diffusion-Wave Equation

The nonclassical theories in which the Fick law (2) or the constitutive Equation (12) as well as the standard diffusion Equation (3) and advection diffusion Equation (13) are replaced by more general equations, constantly attract the attention of the researchers.

The time-nonlocal dependence between the matter flux and the concentration gradient with the "long-tail" power kernel [45–49] can be interpreted in terms of fractional integrals and derivatives:

$$\mathbf{J} = -\kappa D_{RL}^{1-\alpha} \operatorname{grad} c, \qquad 0 < \alpha \le 2. \tag{18}$$

Here $D_{RL}^{\alpha} f(t)$ is the Riemann–Liouville fractional derivative of the order α (see [50–52]):

$$D_{RL}^{\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\, d\tau\right], \qquad m-1 < \alpha < m. \tag{19}$$

Following [4,51], we will not use a separate notation for the Riemann–Liouville fractional integral:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \qquad \alpha > 0; \tag{20}$$

the fractional integral of the order $\alpha > 0$ will be denoted as

$$I^{\alpha} f(t) = D_{RL}^{-\alpha} f(t), \qquad \alpha > 0. \tag{21}$$

The constitutive Equation (18) yields the time-fractional diffusion-wave equation:

$$\rho \frac{\partial^{\alpha} c}{\partial t^{\alpha}} = \kappa \Delta c, \qquad 0 < \alpha \le 2, \tag{22}$$

with the Caputo fractional derivative:

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m}\, d\tau, \qquad m-1 < \alpha < m. \tag{23}$$

The corresponding boundary conditions for the time-fractional diffusion-wave Equation (22) were fully analyzed in [35–37]. Here we recall them very briefly. Once again, the Dirichlet boundary condition (4) remains unaltered. The prescribed boundary value of the matter flux reads:

$$\kappa D_{RL}^{1-\alpha} \frac{\partial c}{\partial n}\bigg|_S = g(\mathbf{x}_S, t), \qquad 0 < \alpha \le 2. \tag{24}$$

The condition of convective mass exchange between a body and the environment is written as:

$$\left(\kappa D_{RL}^{1-\alpha} \frac{\partial c}{\partial n} + Hc\right)\bigg|_S = g(\mathbf{x}_S, t), \qquad 0 < \alpha \le 2. \tag{25}$$

The conditions of perfect diffusive contact are the following:

$$c_1\big|_S = c_2\big|_S, \tag{26}$$

$$\kappa_1 D_{RL}^{1-\alpha} \frac{\partial c_1}{\partial n}\bigg|_S = \kappa_2 D_{RL}^{1-\beta} \frac{\partial c_2}{\partial n}\bigg|_S, \qquad 0 < \alpha \le 2, \qquad 0 < \beta \le 2. \tag{27}$$

2.4. The Time-Fractional Advection Diffusion Equation

The time-nonlocal generalization of the constitutive Equation (12) for the matter flux with the "long-tail" power kernel [53]:

$$\mathbf{J} = D_{RL}^{1-\alpha}\left(-\kappa \nabla c + \rho \mathbf{v} c\right), \qquad 0 < \alpha \leq 2, \tag{28}$$

leads to the time-fractional advection diffusion equation:

$$\rho \frac{\partial^{\alpha} c}{\partial t^{\alpha}} = \kappa \Delta c - \rho \mathbf{v} \cdot \nabla c. \tag{29}$$

The boundary conditions for Equation (29) are the following. The Dirichlet condition (4) remains unaffected. The prescribed boundary value of the matter flux provides:

$$D_{RL}^{1-\alpha}\left(\kappa \frac{\partial c}{\partial n} - \rho v_{(n)} c\right)\bigg|_{S} = g(\mathbf{x}_S, t). \tag{30}$$

The convective mass exchanged between a medium and the environment yields:

$$\left[\kappa D_{RL}^{1-\alpha} \frac{\partial c}{\partial n} + \left(H - \rho v_{(n)} D_{RL}^{1-\alpha}\right) c\right]\bigg|_{S} = g(\mathbf{x}_S, t). \tag{31}$$

The boundary conditions of the perfect diffusive contact are of the form:

$$c_1\big|_S = c_2\big|_S, \tag{32}$$

$$D_{RL}^{1-\alpha}\left(\kappa_1 \frac{\partial c_1}{\partial n} - \rho_1 v_{(n)1} c_1\right)\bigg|_{S} = D_{RL}^{1-\beta}\left(\kappa_2 \frac{\partial c_2}{\partial n} - \rho_2 v_{(n)2} c_2\right)\bigg|_{S}. \tag{33}$$

3. Generalized Conditions of Nonperfect Contact

Consider a composite body consisting of three parts: the domain 1, the domain 2, and the intermediate domain designated by the index 0. Matter transport is described by the time-fractional advection diffusion equations appropriate to each domain:

$$\rho_1 \frac{\partial^{\alpha} c_1}{\partial t^{\alpha}} = \kappa_1 \Delta c_1 - \rho_1 \mathbf{v}_1 \cdot \nabla c_1, \tag{34}$$

$$\rho_2 \frac{\partial^{\beta} c_2}{\partial t^{\beta}} = \kappa_2 \Delta c_2 - \rho_2 \mathbf{v}_2 \cdot \nabla c_2, \tag{35}$$

$$\rho_0 \frac{\partial^{\gamma} c_0}{\partial t^{\gamma}} = \kappa_0 \Delta c_0 - \rho_0 \mathbf{v}_0 \cdot \nabla c_0. \tag{36}$$

At the boundary surfaces S_1 and S_2 between the intermediate domain and the corresponding body, the conditions of perfect diffusive contact are fulfilled:

$$c_1\big|_{S_1} = c_0\big|_{S_1}, \tag{37}$$

$$D_{RL}^{1-\alpha}\left(\kappa_1 \frac{\partial c_1}{\partial n} - \rho_1 v_{(n)1} c_1\right)\Bigg|_{S_1} = D_{RL}^{1-\gamma}\left(\kappa_0 \frac{\partial c_0}{\partial n} - \rho_0 v_{(n)0} c_0\right)\Bigg|_{S_1} \tag{38}$$

and:

$$c_2\big|_{S_2} = c_0\big|_{S_2}, \tag{39}$$

$$D_{RL}^{1-\beta}\left(\kappa_2 \frac{\partial c_2}{\partial n} - \rho_2 v_{(n)2} c_2\right)\Bigg|_{S_2} = D_{RL}^{1-\gamma}\left(\kappa_0 \frac{\partial c_0}{\partial n} - \rho_0 v_{(n)0} c_0\right)\Bigg|_{S_2}. \tag{40}$$

To investigate the transport processes in such a composite body in the general case is a very complicated problem. When the thickness $2h$ of the intermediate domain is small with respect to two other sizes and is constant, we can introduce the median surface Σ. Let R_1 and R_2 be the principal radii of curvature of the median surface. If $h/R_1 << 1$ and $h/R_2 << 1$, then a thin shell is obtained which allows us to reduce a three-dimensional problem in the intermediate layer to a two-dimensional one for the median surface. Thus we introduce the mixed coordinate system (ξ, η, z), where ξ and η are the curvilinear coordinates in the median surface and z is the normal coordinate ($-h \le z \le h$), see Figure 1.

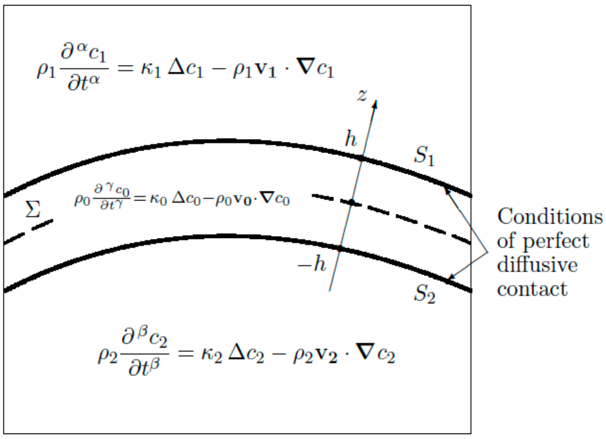

Figure 1. Thin intermediate layer between two media.

The time-fractional advection diffusion equation in the intermediate layer (36) is rewritten as:

$$\rho_0 \frac{\partial^\gamma c_0}{\partial t^\gamma} = \kappa_0 \Delta_\Sigma c_0 + \kappa_0 \frac{\partial^2 c_0}{\partial z^2} - \rho_0 \mathbf{v}_\Sigma \cdot \nabla_\Sigma c_0 - \rho_0 v_{(n)0} \frac{\partial c_0}{\partial z}, \tag{41}$$

where Δ_Σ is the surface Laplace operator, ∇_Σ denotes the surface del operator taking effect along a surface.

Next we average Equation (41) introducing the averaged characteristics of concentration:

$$\Theta_1(\xi, \eta, t) = \frac{1}{2h} \int_{-h}^{h} c_0(\xi, \eta, z, t)\, dz, \tag{42}$$

$$\Theta_2(\xi, \eta, t) = \frac{3}{2h^2} \int_{-h}^{h} c_0(\xi, \eta, z, t)\, z\, dz. \tag{43}$$

Introducing the averaged characteristics of concentration Θ_1 and Θ_2 is similar to introducing the stress resultants (forces and moments) in the theory of thin elastic shells. The interested reader is referred to the extended literature on this subject (see, for example, [54–57]). From Equation (41) we obtain:

$$\rho_0 \frac{\partial^\gamma \Theta_1}{\partial t^\gamma} = \kappa_0 \Delta_\Sigma \Theta_1 - \rho_0 \mathbf{v}_\Sigma \cdot \nabla_\Sigma \Theta_1 + \frac{\kappa_0}{2h}\frac{\partial c_0}{\partial z}\bigg|_{z=h} - \frac{\kappa_0}{2h}\frac{\partial c_0}{\partial z}\bigg|_{z=-h} - \frac{\rho_0 v_{(n)0} c_0}{2h}\bigg|_{z=h} + \frac{\rho_0 v_{(n)0} c_0}{2h}\bigg|_{z=-h} \tag{44}$$

$$\begin{aligned} \rho_0 \frac{\partial^\gamma \Theta_2}{\partial t^\gamma} &= \kappa_0 \Delta_\Sigma \Theta_2 - \rho_0 \mathbf{v}_\Sigma \cdot \nabla_\Sigma \Theta_2 + \frac{3\rho_0 v_{(n)0}}{h}\Theta_1 + \frac{3\kappa_0}{2h}\frac{\partial c_0}{\partial z}\bigg|_{z=h} + \frac{3\kappa_0}{2h}\frac{\partial c_0}{\partial z}\bigg|_{z=-h} \\ &\quad - \frac{3\kappa_0}{2h^2} c_0\big|_{z=h} + \frac{3\kappa_0}{2h^2} c_0\big|_{z=-h} - \frac{3\rho_0 v_{(n)0} c_0}{2h}\bigg|_{z=h} - \frac{3\rho_0 v_{(n)0} c_0}{2h}\bigg|_{z=-h} \end{aligned} \tag{45}$$

Furthermore, we use the conditions of perfect diffusive contact at the surfaces $z = h$ and $z = -h$ along with the assumption of linear dependence of concentration c_0 on the coordinate z and proceed to the limit $h \to 0$ keeping $\rho_\Sigma = 2h\rho_0$, $\kappa_\Sigma = 2h\kappa_0$, $R_\Sigma = 2h/\kappa_0$ and $v_{\{n\}\Sigma} = v_{(n)0}/(2h)$ constant. It should be noted that more general polynomial dependence of the concentration c_0 on the coordinate z or the operator method can also be used (see [58,59]). As a result we obtain the generalized boundary conditions at the median surface Σ (see Figure 2):

$$\begin{aligned} \rho_\Sigma \frac{\partial^\gamma (c_1 + c_2)}{\partial t^\gamma} &= \kappa_\Sigma \Delta_\Sigma (c_1 + c_2) - \rho_\Sigma \mathbf{v}_\Sigma \cdot \nabla_\Sigma (c_1 + c_2) \\ &\quad + 2\left[D_{RL}^{\gamma-\alpha}\left(\kappa_1 \frac{\partial c_1}{\partial n} - \rho_1 v_{(n)1} c_1 \right) - D_{RL}^{\gamma-\beta}\left(\kappa_2 \frac{\partial c_2}{\partial n} - \rho_2 v_{(n)2} c_2 \right) \right], \end{aligned} \tag{46}$$

$$\begin{aligned} \rho_\Sigma \frac{\partial^\gamma (c_1 - c_2)}{\partial t^\gamma} &= \kappa_\Sigma \Delta_\Sigma (c_1 - c_2) - \rho_\Sigma \mathbf{v}_\Sigma \cdot \nabla_\Sigma (c_1 - c_2) + 6\rho_\Sigma v_{(n)\Sigma} (c_1 + c_2) \\ &\quad + 6\left[D_{RL}^{\gamma-\alpha}\left(\kappa_1 \frac{\partial c_1}{\partial n} - \rho_1 v_{(n)1} c_1 \right) + D_{RL}^{\gamma-\beta}\left(\kappa_2 \frac{\partial c_2}{\partial n} - \rho_2 v_{(n)2} c_2 \right) \right] - \frac{12}{R_\Sigma}(c_1 - c_2). \end{aligned} \tag{47}$$

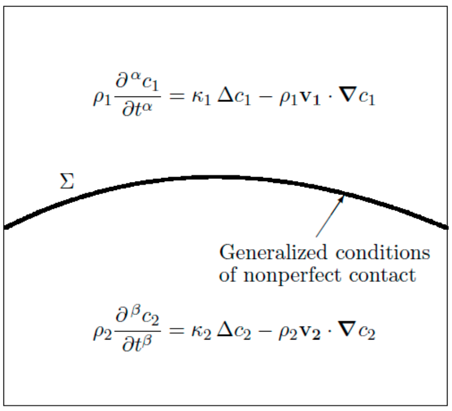

Figure 2. A contact surface Σ having its own physical characteristics.

When the drift terms are not considered, Equations (46) and (47) reduce to the following equations [7]:

$$\rho_\Sigma \frac{\partial^\gamma (c_1 + c_2)}{\partial t^\gamma} = \kappa_\Sigma \, \Delta_\Sigma \, (c_1 + c_2) + 2\left(\kappa_1 D_{RL}^{\gamma-\alpha} \frac{\partial c_1}{\partial n} - \kappa_2 D_{RL}^{\gamma-\beta} \frac{\partial c_2}{\partial n} \right), \tag{48}$$

$$\rho_\Sigma \frac{\partial^\gamma (c_1 - c_2)}{\partial t^\gamma} = \kappa_\Sigma \, \Delta_\Sigma \, (c_1 - c_2) + 6\left(\kappa_1 D_{RL}^{\gamma-\alpha} \frac{\partial c_1}{\partial n} + \kappa_2 D_{RL}^{\gamma-\beta} \frac{\partial c_2}{\partial n} \right) - \frac{12}{R_\Sigma} (c_1 - c_2). \tag{49}$$

For the classical diffusion equation ($\alpha = \beta = \gamma = 1$) we arrive at the result of Podstrigach [58,59]:

$$\rho_\Sigma \frac{\partial (c_1 + c_2)}{\partial t} = \kappa_\Sigma \, \Delta_\Sigma \, (c_1 + c_2) + 2\left(\kappa_1 \frac{\partial c_1}{\partial n} - \kappa_2 \frac{\partial c_2}{\partial n} \right), \tag{50}$$

$$\rho_\Sigma \frac{\partial (c_1 - c_2)}{\partial t} = \kappa_\Sigma \, \Delta_\Sigma \, (c_1 - c_2) + 6\left(\kappa_1 \frac{\partial c_1}{\partial n} + \kappa_2 \frac{\partial c_2}{\partial n} \right) - \frac{12}{R_\Sigma} (c_1 - c_2). \tag{51}$$

When the reduced diffusive characteristics of the median surface are equal to zero ($\rho_\Sigma = 0$, $\kappa_\Sigma = 0$, $R_\Sigma = 0$, $v_{(n)\Sigma} = 0$), from (46) and (47) we get the conditions of perfect diffusive contact (32) and (33) whereas from (50) and (51) we obtain the conditions of perfect contact (9) and (11).

4. Conclusions

We have analyzed different kinds of boundary conditions for the standard diffusion equation and advection diffusion equation as well for their fractional counterparts. It should be emphasized that due to the generalized constitutive equations for the matter flux the boundary conditions for the time-fractional diffusion equations have their particular traits in comparison with those for the standard ones. The proper physical boundary conditions should be formulated in terms of the matter flux, not in terms of the normal derivative of concentration alone. Specifying the boundary value of the matter flux in the case of the diffusion equation leads to the Neumann boundary condition, but in the case of the advection diffusion equation leads to the Robin boundary condition. The drift parameter **v** alters the convective mass transfer coefficient *H*. The transition region between two phases has been considered as a distinct phase having its own reduced characteristics (the reduced mass density ρ_Σ, the reduced diffusive conductivity κ_Σ, the reduced diffusive resistance R_Σ, and the reduced drift parameter $v_{(n)\Sigma}$ of the median surface), and the generalized conditions of nonperfect contact have been obtained.

Conflict of Interest

The author declares no conflict of interest.

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