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Link between Lie Group Statistical Mechanics and Thermodynamics of Continua

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Academic Editors: Frédéric Barbaresco and Frank Nielsen
Received: 29 April 2016; Accepted: 7 July 2016; Published: 12 July 2016

Abstract: In this work, we consider the value of the momentum map of the symplectic mechanics as an affine tensor called momentum tensor. From this point of view, we analyze the underlying geometric structure of the theories of Lie group statistical mechanics and relativistic thermodynamics of continua, formulated by Souriau independently of each other. We bridge the gap between them in the classical Galilean context. These geometric structures of the thermodynamics are rich and we think they might be a source of inspiration for the geometric theory of information based on the concept of entropy.

Keywords: Lie groups; symplectic geometry; affine tensors; continuum thermodynamics; statistical mechanics

1. Introduction

In [1], Souriau proposes to revisit mechanics, emphasizing its affine nature. It is this viewpoint that we will adopt here, starting from a generalization of the concept of momentum under the form of an affine object [2]. Our starting point is closely related to Souriau's approach on the basis of two key ideas: a new definition of momenta and the crucial part played by the affine group of $\mathbb{R}^n$. This group proposes an intentionally poor geometrical structure. Indeed, this choice is guided by the fact that it contains both Galileo and Poincaré groups [3,4], which allows the simultaneous involvement of the Galilean and relativistic mechanics. In the follow-up, we shall detail only the applications to classical mechanics and thermodynamics.

A class of tensors corresponds to each group. The components of these tensors are transformed according to the action of the considered group. The standard tensors discussed in the literature are those of the linear group of $\mathbb{R}^n$. We will call them linear tensors. A fruitful standpoint consists of considering the class of the affine tensors, corresponding to the affine group [2,5]. This viewpoint is closely related to symplectic mechanics [3,4,6] in the sense that the values of the momentum map *are just* the components of the momentum tensors.

The present paper is structured as follows. In Section 2, we present briefly the affine tensors, starting with the most simple ones: the *points* of an affine space which are 1-contravariant and the real affine functions on this space or *affine forms* which are 1-covariant. As a subgroup G of the affine group of $\mathbb{R}^n$ naturally acts onto the affine tensors by restriction to G of their transformation law, we define the corresponding G-tensor. In Section 3, we use this framework, defining the momentum as a mixed 1-covariant and 1-contravariant affine tensor. If G is a Lie group, we demonstrate the important fact that its transformation law *is nothing other* than the coadjoint representation of G in the dual $\mathfrak{g}^*$ of its Lie algebra. In Section 4, we recall classical tools of symplectic mechanics around the concept of symplectic action and a momentum map. An important result called the Kirillov–Kostant–Souriau theorem reveals the orbit symplectic structure. In Section 5, we recall shortly the main concepts of the Lie group statistical mechanics proposed by Souriau in [3,4], using geometric tools. In Section 6, we present briefly the cornerstone results of the Galilean version of a thermodynamics of continua compatible with general relativity proposed by Souriau in [7,8] independently of his statistical mechanics. In Section 7, we reveal the link between the previous relativistic thermodynamics of continua and Lie group statistical mechanics in the classical Galilean context, working in seven steps.

2. Affine Tensors

Points of an affine space. Let $A\mathcal{T}$ be an affine space associated to a linear space $\mathcal{T}$ of finite dimension n. By the choice of an affine frame f composed of a basis of $\mathcal{T}$ and an origin $\boldsymbol{a}_0$, we can associate to each point $\boldsymbol{a}$ a set of n (affine) components V^i gathered in the n-column $V \in \mathbb{R}^n$. For a change of affine frames, the transformation law for the components of a point reads:

$$V = C + P\,V' \,, \tag{1}$$

which is an affine representation of the affine group of $\mathbb{R}^n$ denoted $\mathbb{A}ff(n)$. It is clearly different from the usual transformation law of vectors $V = P\,V'$.

Affine forms. The affine maps $\boldsymbol{\Psi}$ from $A\mathcal{T}$ into $\mathbb{R}$ are called affine forms and their set is denoted $A^*\mathcal{T}$. In an affine frame, $\boldsymbol{\Psi}$ is represented by an affine function Ψ from $\mathbb{R}^n$ into $\mathbb{R}$. Hence, it holds:

$$\boldsymbol{\Psi}(\boldsymbol{a}) = \Psi(V) = \chi + \Phi\,V \,,$$

where $\chi = \Psi(0) = \boldsymbol{\Psi}(\boldsymbol{a}_0)$ and $\Phi = lin(\Psi)$ is a n-row. We call $\Phi_1, \Phi_2, \cdots, \Phi_n, \chi$ the components of $\boldsymbol{\Psi}$ or, equivalently, the couple of χ and the row Φ collecting the Φ_α. The set $A^*\mathcal{T}$ is a linear space of dimension $(n+1)$ called the vector dual of $A\mathcal{T}$. If we change the affine frame, the components of an affine form are modified according to the induced action of $\mathbb{A}ff(n)$, that leads to, taking into account (1):

$$\chi' = \chi - \Phi\,P^{-1}C, \qquad \Phi' = \Phi\,P^{-1} \,, \tag{2}$$

which is a linear representation of $\mathbb{A}ff(n)$.

Affine tensors. We can generalize this construction and define an affine tensor as an object:

- that assigns a set of components to each affine frame f of an affine space $A\mathcal{T}$ of finite dimension n,
- with a transformation law, when changing of frames, which is an affine or a linear representation of $\mathbb{A}ff(n)$.

With this definition, the affine tensors are a natural generalization of the classical tensors that we shall call linear tensors, these last ones being trivial affine tensors for which the affine transformation $a = (C, P)$ acts through its linear part $P = lin(a)$. An affine tensor can be constructed as a map which is affine or linear with respect to each of its arguments. Similar to the linear tensors, the affine ones can be classified in three families: covariants, contravariant and mixed. The most simple affine tensors are the points which are 1-contravariant and the affine forms which are 1-covariant but we can construct more complex ones having a strong physical meaning: the *torsors* (proposed in [5]), the *co-torsors* and the *momenta* extensively detailled in [2]. For more details on the affine dual space, affine tensor product, affine wedge product and affine tangent bundles, the reader interested in this topic is referred to the so-called AV-differential geometry [9].

G-tensors. A subgroup G of $\mathbb{A}ff(n)$ naturally acts onto the affine tensors by restriction to G of their transformation law. Let F_G be a set of affine frames of which G is a transformation group. The elements of F_G are called G-frames. A G-tensor is an object:

- that assigns a set of components to each G-frame f,
- with a transformation law, when changing of frames, which is an affine or a linear representation of G.

For instance, if G is the group of Euclidean transformations, we recover the classical Euclidean tensors. Hence, each G-tensor can be identified with an orbit of G within the space of the tensor components.

3. Momentum as Affine Tensor

Let $\mathcal{M}$ be a differential manifold of dimension n and G a Lie subgroup of $\mathbb{A}ff(n)$. In the applications to physics, $\mathcal{M}$ will be for us typically the space-time and G a subgroup of $\mathbb{A}ff(n)$ with a physical meaning in the framework of classical mechanics (Galileo's group) or relativity (Poincaré's group). The points of the space-time $\mathcal{M}$ are events of which the coordinate X^0 is the time t and $X^i = x^i$ for i running from 1 to 3 gives the position.

The tangent space to $\mathcal{M}$ at the point $\boldsymbol{X}$ equipped with a structure of affine space is called the affine tangent space and is denoted $AT_{\boldsymbol{X}}\mathcal{M}$. Its elements are called tangent points at $\boldsymbol{X}$. The set of affine forms on the affine tangent space is denoted $A^*T_{\boldsymbol{X}}\mathcal{M}$. We call momentum a bilinear map $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} : T_{\boldsymbol{X}}\mathcal{M} \times A^*T_{\boldsymbol{X}}\mathcal{M} \to \mathbb{R} : (\overrightarrow{\boldsymbol{V}}, \boldsymbol{\Psi}) \mapsto \boldsymbol{\mu}(\overrightarrow{\boldsymbol{V}}, \boldsymbol{\Psi})$$

It is a mixed 1-covariant and 1-contravariant affine tensor. Taking into account the bilinearity, it is represented in an affine frame f by:

$$\boldsymbol{\mu}(\overrightarrow{\boldsymbol{V}}, \boldsymbol{\Psi}) = (\chi\, K_\beta + \Phi_\alpha L^\alpha_\beta)\, V^\beta$$

where K_β and L^α_β are the components of $\boldsymbol{\mu}$ in the affine frame f or, equivalently, the couple $\mu = (K, L)$ of the row K collecting the K_β and the $n \times n$ matrix L of elements L^α_β. Owing to (2), the transformation law is given by the induced action of $\mathbb{A}ff(n)$:

$$K' = K\, P^{-1}, \qquad L' = (P\, L + C\, K)\, P^{-1} \tag{3}$$

If the action is restricted to the subgroup G, the momentum $\boldsymbol{\mu}$ is a G-tensor.

On the other hand, have a look to the Lie algebra $\mathfrak{g}$ of G, that is the set of infinitesimal generators $Z = da = (dC, dP)$ with $a \in G$. Let us identify the space of the momentum components $\mu = (K, L)$ to the dual $\mathfrak{g}^*$ of the Lie algebra thanks to the dual pairing:

$$\mu\, Z = \mu\, da = (K, L)\,(dC, dP) = K\, dC + Tr(L\, dP) \tag{4}$$

We know that the group acts on its Lie algebra by the adjoint representation:

$$Ad(a) : \mathfrak{g} \to \mathfrak{g} : Z' \mapsto Z = Ad(a)\, Z' = a\, Z'\, a^{-1}\,.$$

As G is a group of affine transformations, any infinitesimal generator Z is represented by:

$$\tilde{Z} = d\tilde{P} = d \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ dC & dP \end{pmatrix}\,.$$

Then $\tilde{Z} = \tilde{P}\, \tilde{Z}'\, \tilde{P}^{-1}$ leads to:

$$dC = P\,(dC' - dP'\, P^{-1} C), \qquad dP = P\, dP'\, P^{-1}\,. \tag{5}$$

This adjoint representation induces the coadjoint representation of G in $\mathfrak{g}^*$ defined by:

$$(Ad^*(a)\, \mu')\, Z = \mu'\, (Ad(a^{-1})\, Z)\,.$$

Taking into account (4), one finds that the coadjoint representation:

$$Ad^*(a) : \mathfrak{g}^* \to \mathfrak{g}^* : \mu' \mapsto \mu = Ad^*(a)\,\mu'$$

is given by:

$$K = K'\,P^{-1}, \qquad L = (P\,L' + C\,K')\,P^{-1}\,.$$

It is noteworthy to observe that the transformation law (3) of momenta *is nothing other* than the coadjoint representation!

However, this mathematical construction is not relevant for all considered physical applications and we need to extend it by considering a map θ from G into $\mathfrak{g}^*$ and a generalized transformation law:

$$\mu = a \cdot \mu' = Ad^*(a)\,\mu' + \theta(a)\,, \tag{6}$$

where θ eventually depends on an invariant of the orbit. It is an affine representation of G in $\mathfrak{g}^*$ (because we wish the momentum to be an affine tensor) provided:

$$\forall a, b \in G, \qquad \theta(ab) = \theta(a) + Ad^*(a)\,\theta(b) \tag{7}$$

Remark 1. *This action induces a structure of affine space on the set of momentum tensors. Let $\pi : \mathcal{F} \to \mathcal{M}$ be a G-principal bundle of affine frames with the free action $(a, f) \mapsto f' = a \cdot f$ on each fiber. Then we can build the associated G-principal bundle:*

$$\hat{\pi} : \mathfrak{g}^* \times \mathcal{F} \to (\mathfrak{g}^* \times \mathcal{F})/G : (\mu, f) \mapsto \boldsymbol{\mu} = orb(\mu, f)$$

for the free action:

$$(a, (\mu, f)) \mapsto (\mu', f') = a \cdot (\mu, f) = (a \cdot \mu, a \cdot f)$$

where the action on $\mathfrak{g}^*$ is (6). Clearly, the orbit $\boldsymbol{\mu} = orb(\mu, f)$ can be identified to the momentum G-tensor $\boldsymbol{\mu}$ of components μ in the G-frame f.

4. Symplectic Action and Momentum Map

Let $(\mathcal{N}, \omega)$ be a symplectic manifold [3,4,6,10]. A Lie group G smoothly left acting on $\mathcal{N}$ and preserving the symplectic form ω is said to be symplectic.The interior product of a vector $\overrightarrow{V}$ and a p-form ω is denoted $\iota(\overrightarrow{V})\,\omega$. A map $\psi : \mathcal{N} \to \mathfrak{g}^*$ such that:

$$\forall \eta \in \mathcal{N}, \quad \forall Z \in \mathfrak{g}, \qquad \iota(Z \cdot \eta)\,\omega = -d(\psi(\eta)Z)\,,$$

is called a momentum map of G. It is the quantity involved in Noether's theorem that claims ψ is constant on each leaf of $\mathcal{N}$. In [3] (Theorem 11.17, p. 109, or its English translation [4]), Souriau proved there exists a smooth map θ from G into $\mathfrak{g}^*$:

$$\theta(a) = \psi(a \cdot \eta) - Ad^*(a)\,\psi(\eta)\,, \tag{8}$$

which is a symplectic cocycle, that is a map $\theta : G \to \mathfrak{g}$ verifying the identity (7) and such that $(D\theta)(e)$ is a 2-form. An important result, called the Kirillov–Kostant–Souriau theorem, reveals the orbit symplectic structure [3] (Theorem 11.34, Pages 116–118). Let G be a Lie group and an orbit of the coadjoint representation $orb\,(\mu) \subset \mathfrak{g}^*$. Then the orbit $orb\,(\mu)$ is a symplectic manifold, G is a symplectic group and any $\mu \in \mathfrak{g}^*$ is its own momentum.

Remark 2. *Replacing η by $a^{-1} \cdot \eta$ in (8), this formula reads:*

$$\psi(\eta) = Ad^*(a)\,\psi'(\eta) + \theta(a)\,,$$

where $\psi \mapsto \psi' = a \cdot \psi$ is the induced action of the one of G on $\mathcal{N}$. It is worth observing it is just (6) with $\mu = \psi(\eta)$ and $\mu' = \psi'(\eta)$. In this sense, the values of the momentum map are just the components of the momentum G-tensors defined in the previous Section.

Remark 3. *We saw at Remark of Section 3 that the momentum G-tensor $\boldsymbol{\mu}$ is identified to the orbit $\boldsymbol{\mu} = orb(\mu, f)$ and, disregarding the frames for simplification, we can identify $\boldsymbol{\mu}$ to the component orbit $orb(\mu)$.*

5. Lie Group Statistical Mechanics

In order to discover the underlined geometric structure of the statistical mechanics, we are interested in the affine maps $\boldsymbol{\Theta}$ on the affine space of momentum tensors. In an affine frame, $\boldsymbol{\Theta}$ is represented by an affine function Θ from $\mathfrak{g}^*$ into $\mathbb{R}$:

$$\boldsymbol{\Theta}(\boldsymbol{\mu}) = \Theta(\mu) = z + \mu\, Z\,,$$

where $z = \Theta(0) = \boldsymbol{\Theta}(\boldsymbol{\mu}_0)$ and $Z = lin(\Theta) \in \mathfrak{g}$ are the affine components of $\boldsymbol{\Theta}$. If the components of the momentum tensors are modified according to (6), the change of affine components of $\boldsymbol{\Theta}$ is given by the induced action:

$$z = z' - \theta(a)\, Ad(a)\, Z', \qquad Z = Ad(a)\, Z'\,. \tag{9}$$

Then $\boldsymbol{\Theta}$ is a G-tensors. In [3,4], Souriau proposed a statistical mechanics model using geometric tools. Let $d\lambda$ be a measure on $\boldsymbol{\mu} = orb\,(\mu)$ and a Gibbs probability measure $p\, d\lambda$ with:

$$p = e^{-\Theta(\mu)} = e^{-(z+\mu\, Z)}\,.$$

The normalization condition $\int_{orb(\mu)} p\, d\lambda = 1$ links the components of $\boldsymbol{\Theta}$, allowing to express z in terms of Z:

$$z(Z) = \ln \int_{orb(\mu)} e^{-\mu\, Z}\, d\lambda\,. \tag{10}$$

The corresponding entropy and mean momenta are:

$$\begin{aligned} s(Z) &= -\int_{orb(\mu)} p\, \ln p\, d\lambda = z + M\, Z, \\ M(Z) &= \int_{orb(\mu)} \mu\, p\, d\lambda = -\frac{\partial z}{\partial Z}\,, \end{aligned} \tag{11}$$

satisfying the same transformation law as the one (6) of μ. Hence M are the components of a momentum tensor $\boldsymbol{M}$ which can be identified to the orbit $orb(M)$, that defines a map $\boldsymbol{\mu} \mapsto \boldsymbol{M}$, i.e., a correspondance between two orbits. This construction is formal and, for reasons of integrability, the integrals will be performed only on a subset of the orbit according to an heuristic way explained latter on.

People generally consider that the definition of the entropy is relevant for applications insofar as the number of particles in the system is very huge. For instance, the number of atoms contained in one mole is Avogadro's number equal to 6×10^{23}. It is worth noting that Vallée and Lerintiu proposed a generalization of the ideal gas law based on convex analysis and a definition of entropy which does not require the classical approximations (Stirling's Formula) [11].

6. Relativistic Thermodynamics of Continua

Independently of his statistical mechanics, Souriau proposed in [7,8] a thermodynamics of continua compatible with general relativity. Following in his footsteps, one can quote the works by Iglesias [12] and Vallée [13]. In his Ph.D thesis, Vallée studied the invariant form of constitutive laws in the context of special relativity where the gravitation effects are neglected. In [14], the author and Vallée proposed a Galilean version of this theory of which we recall the cornerstone results. For more details, the reader is referred to [2].

Galileo's group $\mathbb{GAL}$ is a subgroup of the affine group $\mathbb{A}ff(4)$, collecting the Galilean transformations, that is the affine transformations $dX' \mapsto dX = P\,dX' + C$ of $\mathbb{R}^4$ such that:

$$C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \tag{12}$$

where $u \in \mathbb{R}^3$ is a Galilean boost, $R \in \mathbb{SO}(3)$ is a rotation, $k \in \mathbb{R}^3$ is a spatial translation and $\tau_0 \in \mathbb{R}$ is a clock change. Hence, Galileo's group is a Lie group of dimension 10. The $\mathbb{GAL}$-tensors will also be called *Galilean tensors*.

$\mathcal{M}$ is the space-time equipped with a symmetric $\mathbb{GAL}$-connection ∇ representing the gravitation, the matter and its evolution is characterized by a line bundle $\pi_0 : \mathcal{M} \mapsto \mathcal{M}_0$. The trajectory of the particle $\boldsymbol{X}_0 \in \mathcal{M}_0$ is the corresponding fiber $\pi_0^{-1}(\boldsymbol{X}_0)$. In local charts, $\boldsymbol{X}_0$ is represented by $s' \in \mathbb{R}^3$ and its position x at time t is given by a map:

$$x = \varphi(t, s'). \tag{13}$$

The 4-velocity:

$$\overrightarrow{\boldsymbol{U}} = \frac{\overrightarrow{d\boldsymbol{X}}}{dt},$$

is the tangent vector to the fiber parameterized by the time. In a local chart, it is represented by:

$$U = \begin{pmatrix} 1 \\ v \end{pmatrix}, \tag{14}$$

where v is the usual velocity. Conversely, φ can be obtained as the flow of the 4-velocity.

β being the reciprocal temperature, that is $1\,/\,k_B T$ where k_B is Boltzmann's constant and T the absolute temperature, there are five basic tensor fields defined on the space-time $\mathcal{M}$:

- the 4-flux of mass $\overrightarrow{\boldsymbol{N}} = \rho\, \overrightarrow{\boldsymbol{U}}$ where ρ is the density,
- the 4-flux of entropy $\overrightarrow{\boldsymbol{S}} = \rho\, s\, \overrightarrow{\boldsymbol{U}} = s\, \overrightarrow{\boldsymbol{N}}$ where s is the specific entropy,
- Planck's temperature vector $\overrightarrow{\boldsymbol{W}} = \beta\, \overrightarrow{\boldsymbol{U}}$,
- its gradient $\mathbf{f} = \nabla \overrightarrow{\boldsymbol{W}}$ called friction tensor,
- the momentum tensor of a continuum $\boldsymbol{T}$, a linear map from $T_{\boldsymbol{X}}\mathcal{M}$ into itself.

In local charts, they are respectively represented by two 4-columns N, W and two 4×4 matrices f and T. Then we proved in [14] the following result characterizing the reversible processes:

Theorem 1. *If Planck's potential ζ smoothly depends on s', W and $F = \partial x / \partial s'$ through right Cauchy strains:*

$$\mathcal{C} = F^T F, \tag{15}$$

then:

$$T = U\,\Pi + \begin{pmatrix} 0 & 0 \\ -\sigma v & \sigma \end{pmatrix} \tag{16}$$

with

$$\Pi = -\rho\, \frac{\partial \zeta}{\partial W}, \qquad \sigma = -\frac{2\rho}{\beta}\, F\, \frac{\partial \zeta}{\partial \mathcal{C}}\, F^T, \tag{17}$$

represents the momentum tensor of the continuum and is such that:

$$(\nabla \zeta)\, N = -Tr\,(T\, f),$$

Combining this result with the geometric version of the first principle of thermodynamics:

$$\boldsymbol{Div}\, \boldsymbol{T} = \boldsymbol{0}, \qquad \boldsymbol{Div}\, \overrightarrow{\boldsymbol{N}} = \boldsymbol{0},\,, \tag{18}$$

In [7,8], Souriau claimed that the 4-flux of entropy is given by:

$$\overrightarrow{\boldsymbol{S}} = \boldsymbol{T}\,\overrightarrow{\boldsymbol{W}} + \zeta\,\overrightarrow{\boldsymbol{N}}\,, \tag{19}$$

and proved it is divergence free. Moreover the specific entropy s is an integral of the motion [2].

Let us introduce now the 5-temperature $\hat{\overrightarrow{\boldsymbol{W}}}$ represented by the 5-column:

$$\hat{W} = \begin{pmatrix} W \\ \zeta \end{pmatrix}, \tag{20}$$

and the tensor $\hat{\boldsymbol{T}}$ represented by the 4×5 matrix

$$\hat{T} = \begin{pmatrix} T & N \end{pmatrix} \tag{21}$$

which allows gathering Equation (18) in the more compact form

$$Div\,\hat{T} = 0$$

and representing (19) in the more compact form:

$$S = \hat{T}\,\hat{W}\,,$$

local expression of the contracted product of $\hat{\mathbf{T}}$ and $\hat{\vec{\boldsymbol{W}}}$:

$$\vec{\boldsymbol{S}} = \hat{\mathbf{T}} \cdot \hat{\vec{\boldsymbol{W}}}\,, \tag{22}$$

It is the cornerstone equation of Souriau's theory. In this form, it can be seen as a geometrization of Clausius' definition of the entropy as state function of a system:

$$\mathcal{S} = \frac{Q}{\theta}\,, \tag{23}$$

where Q is a the amount of heat absorbed in an isothermal process. Scalar quantities are replaced by analogous tensorial ones: $\mathcal{S}$ by its 4-flux $\vec{\mathbf{S}}$, Q by $\hat{\mathbf{T}}$ and $\beta = 1\ /\ \theta$ by its 5-flux $\hat{\vec{\mathbf{W}}}$. Replacing (19) by (22) is not a purely formal manipulation but it takes a strong meaning when considering Bargmann's group $\mathbb{B}$ [15], a central extension of Galileo's one [16], set of the affine transformations $d\hat{X}' \mapsto d\hat{X} = \hat{P}\,d\hat{X}' + \hat{C}$ of $\mathbb{R}^5$ such that.

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 \\ u & R & 0 \\ \frac{1}{2}\parallel u \parallel^2 & u^T R & 1 \end{pmatrix}. \tag{24}$$

The $\mathbb{B}$-tensors are called Bargmannian tensors. From this viewpoint, the 5-column (20) represents a Bargmannian vector $\hat{\vec{\boldsymbol{W}}}$ of transformation law:

$$\hat{W} = \hat{P}\,\hat{W}'\,, \tag{25}$$

and the 4×5 matrix (21) represents a Bargmannian 1-covariant and 1-contravariant tensor $\hat{\boldsymbol{T}}$ of transformation law:

$$\hat{T} = P\,\hat{T}'\,\hat{P}^{-1}\,.$$

7. Planck's Potential of a Continuum

Now, let us reveal the link between the previous relativistic thermodynamics of continua and Lie group statistical mechanics in the classical Galilean context and, to simplify, in absence of gravitation. In other words, how to deduce $\boldsymbol{T}$ from $\boldsymbol{M}$ and ζ from z? We work in seven steps:

- *Step 1: defining the orbit.* To begin with, we consider the momentum as an Galilean tensor, i.e., its components ar modified only by the action of Galilean transformations. In order to calculate the integral (10), the orbit is parameterized thanks to a momentum map. Calculating the infinitesimal generators $Z = (dC, dP)$ by differentiation of (12):

$$dC = \begin{pmatrix} d\tau_0 \\ dk \end{pmatrix}, \qquad dP = \begin{pmatrix} 0 & 0 \\ du & j(d\varpi) \end{pmatrix},$$

where $j(d\varpi)\, v = d\varpi \times v$, the dual pairing (4) reads:

$$\mu\, Z = l \cdot d\varpi - q \cdot du + p \cdot dk - e\, d\tau_0\,. \tag{26}$$

The most general form of the action (6) itemizes in:

$$p = R\, p' + m\, u, \qquad q = R\,(q' - \tau_0\, p') + m\,(k - \tau_0\, u)\,, \tag{27}$$

$$l = R\, l' - u \times (R\, q') + k \times (R\, p') + m\, k \times u\,, \tag{28}$$

$$e = e' + u \cdot (R\, p') + \frac{1}{2}\, m \parallel u \parallel^2\,. \tag{29}$$

where the orbit invariant m occuring in the symplectic cocycle θ is physically interpreted as the particle mass. In [3] (Theorem 11.34, p. 151), the cocycle of Galileo's group is derived from an explicit form of the symplectic form. An alternative method to obtain it using only the Lie group structure is proposed in [2] (Theorem 16.3, p. 329 and Theorem 17.4, p. 374).

Taking into account (3), the transformation law (6) of the Galilean momentum tensor $\boldsymbol{\mu}$ reads:

$$K = K'\, P^{-1} + K_m(C, P), \qquad L = (P\, L' + C\, K')\, P^{-1} + L_m(C, P)\,, \tag{30}$$

where K_m and L_m are the components of θ. In particular, one has:

$$K_m(C, P) = m\left(-\frac{1}{2} \parallel u \parallel^2, u^T\right)\,. \tag{31}$$

- *Step 2: representing the orbit by equations.* To obtain them, we have to determine a functional basis. The first step is to calculate their number. We start determining the isotropy group of μ. The analysis will be restricted to massive particles: $m \neq 0$. The components p, q, l, e being given, we have to solve the following system:

$$p = R\, p + m\, u\,, \tag{32}$$

$$q = R\, q - \tau_0 (R\, p + m\, u) + m\, k\,, \tag{33}$$

$$l = R\, l - u \times (R\, q) + k \times (R\, p) + m\, k \times u\,, \tag{34}$$

$$u \cdot (R\, p) + \frac{1}{2}\, m \parallel u \parallel^2 = 0\,, \tag{35}$$

with respect to τ_0, k, R, u. Owing to (32), the boost u can be expressed with respect to the rotation R by:

$$u = \frac{1}{m}\,(p - R\, p)\,, \tag{36}$$

that allows us to satisfy automatically (35). Next, owing to (32), Equation (33) can be simplified as follows:

$$q = R\,q - \tau_0 p + m\,k\,,$$

that allows to determine the spatial translation k with respect to R and the clock change τ_0:

$$k = \frac{1}{m}\,(q - R\,q + \tau_0 p). \tag{37}$$

Finally, because of (32), Equation (34) is simplified as follows:

$$l = R\,l - u \times (R\,q) + k \times p\,.$$

Substituting (37) into the last relation gives:

$$l = R\,l - u \times (R\,q) + \frac{1}{m}\,q \times p - \frac{1}{m}\,(R\,q) \times p\,.$$

Owing to (32) and the definition of the spin angular momentum l_0

$$l_0 = l - q \times p\ /\ m\,,$$

leads to:

$$l_0 = R\,l_0\,. \tag{38}$$

These quantity being given, we have to determine the rotations satisfying the previous relation. It turns out that two cases must be considered.

- *Generic orbits : massive particle with spin or rigid body.* If l_0 does not vanish, the solutions of (38) are the rotations of an arbitrary angle ϑ about the axis l_0. We know by (36) and (37) that u and k are determined in a unique manner with respect to R and τ_0. The isotropy group of μ can be parameterised by ϑ and τ_0. It is a Lie group of dimension 2. The dimension of the orbit of μ is $10 - 2 = 8$. The maximum number of independent invariant functions is $10 - 8 = 2$. A possible functional basis is composed of:

$$s_0 = \parallel l_0 \parallel\,, \tag{39}$$

$$e_0 = e - \frac{1}{2\,m}\ \parallel p \parallel^2\,, \tag{40}$$

  of which the values are constant on the orbit which represents a massive particle with spin or a rigid body (seen from a long way off).
- *Singular orbits : spinless massive particle.* In the particular case $l_0 = 0$, all the rotations of $\mathbb{SO}(3)$ satisfy (38), then the isotropy group is of dimension 4. By similar reasoning to the case of non vanishing l_0, we conclude that dimension of the orbit is 6 and the number of invariant functions is 4. A possible functional basis is composed of e_0 and the three null components of l_0.

For the orbits with $m = 0$, the reader is referred to [6] (pp. 440, 441).

To physically interpret the components of the momentum, let consider a coordinate system X' in which a particle is at rest and characterized by the components $p' = 0$, $q' = 0$, $l' = l_0$ and $e' = e_0$ of the momentum tensor. Let us consider another coordinate system $X = PX' + C$ with a Galilean boost v and a translation of the origin at $k = x_0$ (hence $\tau_0 = 0$ and $R = 1_{\mathbb{R}^3}$), providing the trajectory equation:

$$x = x_0 + v\,t\,, \tag{41}$$

of the particle moving in uniform straight motion at velocity v. Owing (27) and (28), we can determine the new components of the torsor in X:

$$p = m\,v, \quad q = m\,x_0, \quad l = l_0 + q \times v, \quad e = e_0 + \frac{m}{2} \parallel v \parallel^2 , \tag{42}$$

The third relation of (42) is the classical *transport law of the angular momentum*. In fact, it is a particular case of the general transformation laws (28) when considering only a Galilean boost. The transformation law reveals the physical meaning of the momentum tensor components:

- The quantity p, proportional to the mass and to the velocity, is the *linear momentum*.
- The quantity q, proportional to the mass and to the initial position, provides the trajectory equation. It is called *passage* because indicating the particle is passing through x_0 at time $t = 0$.
- The quantity l splits into two terms. The second one, $q \times v \;= x \times m\,v \;= x \times p$, is the *orbital angular momentum*. The first one, $l_0 = l - \; q \times p \;/\; m$, is the *spin angular momentum*. Their sum, l, is the *angular momentum*.

- *Step 3: parameterizing the orbit.* If the particle has an internal structure, introducing the moment of inertia matrix $\mathcal{J}$ and the spin ϖ, we have, according to König's theorem:

$$l_0 = \mathcal{J}\,\varpi, \qquad e_0 = \frac{1}{2}\,\varpi \cdot (\mathcal{J}\,\varpi)\;.$$

Hence each orbit defines a particle of mass m, spin s_0, inertia $\mathcal{J}$ and can be parameterized by 8 coordinates, the 3 components of q, the 3 components of p and the 2 components of the unit vector n defining the spin direction, thanks to the momentum map $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathfrak{g}^* : (q, p, n) \mapsto \mu = \psi(q, p, n)$ such that:

$$l = \frac{1}{m}\,q \times p + s_0 n, \qquad e = \frac{1}{2\,m} \parallel p \parallel^2 + \frac{s_0^2}{2}\,n \cdot (\mathcal{J}^{-1}\,n)\;.$$

The corresponding measure is $d\lambda \;=\; d^3q\,d^3p\,d^2n$. For simplicity, we consider further only a singular orbit of dimension 6 representing a spinless particle of mass m, which corresponds to the particular case $l_0 = 0$ then $n = 0$. It can be parameterized by 6 coordinates, the 3 components of q and the 3 components of p thanks to the map:

$$\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{g}^* : (q, p) \mapsto \mu = \psi(q, p)\;,$$

such that:

$$l = \frac{1}{m}\,q \times p, \qquad e = \frac{1}{2\,m} \parallel p \parallel^2 \;. \tag{43}$$

- *Step 4: modelling the deformation.* Statistical mechanics is essentially based on a set of discrete particles and, in essence, incompatible with continuum mechanics. Thus, according to usual arguments, the passage from the statistical mechanics to continuum mechanics is obtained by equivalence between the set of N particles (in huge number) and a box of finite volume V occupied by them, large with respect to the particle size but so small with respect to the continuous medium that it can be considered as infinitesimal. Let us consider N identical particles contained in V, large with respect to the particles but representing the volume element of the continuum thermodynamics. The motion of the matter being characterized by (13), let us consider the change of coordinate

$$t = t', \qquad x = \varphi(t', s')\;.$$

The jacobean matrix reads:

$$\frac{\partial X}{\partial X'} = P = \begin{pmatrix} 1 & 0 \\ v & F \end{pmatrix} . \tag{44}$$

From then on, the momentum is considered as an affine tensor, i.e., its components are modified by the action of any affine transformation.

Besides, we suppose that the box of initial volume V_0 is at rest in the considered coordinate system ($v = 0$) and the deformation gradient F is uniform in the box, then:

$$dx = F\, ds' .$$

According to (3), the linear momentum is transformed according to:

$$p = F^{-T} p' . \tag{45}$$

For a particle initially at position x, the passage is given by (42):

$$q = m\, x .$$

The measure becomes

$$d\lambda = m^3 d^3x\, d^3p\, d^2n = m^3 d^3s'\, d^3p'\, d^2n .$$

For reasons that will be justified at Step 5, we consider the infinitesimal generator:

$$Z = (-W, 0) .$$

As the box is at rest in the considered coordinate system, the velocity is null and, owing to (14):

$$W = \beta\, U = \begin{pmatrix} \beta \\ 0 \end{pmatrix} . \tag{46}$$

Hence the dual pairing (26) is reduced to:

$$\mu\, Z = \beta\, e ,$$

and, owing to (43), (45) and (15), for a spinless massive particle:

$$\mu\, Z = \frac{\beta}{2\,m} \parallel p \parallel^2 = \frac{\beta}{2\,m} \parallel F^{-T} p' \parallel^2 = \frac{\beta}{2\,m}\, p'^T C^{-1} p' .$$

For reasons of integrability as explained in Section 6, *it is usual to replace the orbit by the subset* $V_0 \times \mathbb{R}^3 \times \mathbb{S}^2 \subsetneq orb(\mu)$. It is worth remarking that, unlike the orbit, this set is not preserved by the action but the integrals in (10) and (11) are invariant. Equation (10) gives for a particle:

$$z = \ln(m^3 I_0 I_1 I_2) ,$$

where:

$$I_0 = \int_{V_0} d^3s' = V_0 ,$$

$$I_1 = \int_{\mathbb{R}^3} e^{-\frac{\beta}{2\,m}\, p'^T C^{-1} p'}\, d^3p' ,$$

$$I_2 = \int_{\mathbb{S}^2} d^2n = 4\,\pi .$$

Finally:

$$z = \frac{1}{2}\,\ln(\det(\mathcal{C})) - \frac{3}{2}\,\ln\beta + C^{te}\,, \tag{47}$$

where the value of the constant is not relevant in the sequel since it does not depend on W and F (through $\mathcal{C}$). It is worth remarking that, unlike $orb(\mu)$, the subset $V_0 \times \mathbb{R}^3 \times \mathbb{S}^2$ is not preserved by the action and depends on the arbitrary choice of V_0. Nevertheless, z—then s and M—depends on V_0 only through $\ln(V_0)$ which is absorbed in the constant and has no influence on the derivatives (17).

As pointed out by Barbaresco [17], there is a puzzling analogy between the integral occuring in (10) and Koszul–Vinberg characteristic function [18,19]:

$$\psi_\Omega(Z) = \int_{\Omega^*} e^{-\mu\,Z}\,d\lambda\,,$$

where Ω is a sharp open convex cone and Ω^* is the set of linear strictly positive forms on $\bar{\Omega} - \{0\}$. Considering Galileo's group, it is worth remarking that the cone of future directed timelike vectors (i.e., such that $\beta > 0$) [20] is preserved by linear Galilean transformations. The momentum orbits are contained in Ω^* but the integral does not converge on the orbits or on Ω^*.

- *Step 5: identification.* It is based on the following result.

Theorem 2. *The transformation law of the temperature vector* $\hat{\vec{W}}$ *is the same as the one of affine maps* $\mathbf{\Theta}$ *on the affine space of momentum tensors through the identification:*

$$Z = (-W, 0), \qquad z = m\,\zeta\,,$$

Proof. First of all, let us verify that the form $Z = (-W, 0)$ does not depend on the choice of the affine frame. Indeed, starting from $Z' = (-W', 0)$ and applying the adjoint representation (5) with $dC' = -W'$ and $dP' = 0$, we find that $dC = -W$ and $dP = 0$ with:

$$W = P\,W'\,.$$

Besides, using the notations of (30), Equation (9) gives:

$$z = z' - \theta(a)\,Ad(a)\,Z' = z' + K_m P\,W'\,.$$

On the other hand, let $\hat{W}$ be the 5-column (20) representing the temperature vector:

$$\hat{W} = \begin{pmatrix} W \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix}.$$

Taking into account (12) and (31), it is easy to verify that its transformation law (25) with the linear Bargmannian transformation (24) can be recast as:

$$\begin{pmatrix} W \\ \zeta \end{pmatrix} = \begin{pmatrix} P & 0 \\ F_1 P & 1 \end{pmatrix} \begin{pmatrix} W' \\ \zeta' \end{pmatrix},$$

which is the transformation law of the affine map $\mathbf{\Theta}$ provided $z = m\,\zeta$, that achieves the proof. □

- *Step 6: boost method.* For the box at rest in the coordinate system X, the temperature 4-vector is given by (46):

$$W = \begin{pmatrix} \beta \\ 0 \end{pmatrix}.$$

A new coordinate system $\bar{X}$ in which the box has the velocity v can be deduced from $X = P\,\bar{X} + C$ by applying a boost $u = -v$ (hence $k = 0$, $\tau_0 = 0$ and $R = 1_{\mathbb{R}^3}$). The transformation law of vectors gives the new components

$$\bar{W} = \begin{pmatrix} \beta \\ \beta\, v \end{pmatrix},$$

and (9) leads to:

$$\bar{z} = z + \frac{m\,\beta}{2} \parallel v \parallel^2 = z + \frac{m}{2\,\beta} \parallel w \parallel^2 \ .$$

Taking into account (47) and leaving out the bars:

$$z = \frac{1}{2}\ln(\det(\mathcal{C})) - \frac{3}{2}\ln\beta + \frac{m}{2\,\beta} \parallel w \parallel^2 + C^{te}\,. \tag{48}$$

It is clear from (11) that s is Legendre conjugate of $-z$, then, introducing the internal energy (which is nothing other than the Galilean invariant (40)):

$$e_{int} = e - \frac{1}{2\,m} \parallel p \parallel^2,$$

the entropy is:

$$s = \frac{3}{2}\ln e_{int} + \frac{1}{2}\ln(\det(\mathcal{C})) + C^{te},$$

and, by $Z = \partial s \,/\, \partial M$, we derive the corresponding momenta:

$$\beta = \frac{\partial s}{\partial e} = \frac{3}{2\,e_{int}}, \qquad w = -grad_{\,p}\, s = \frac{3}{2\,e_{int}}\,\frac{p}{m}\,.$$

As Equation (47), Equation (48) and the expressions of s, β and w are not affected by the arbitrary choice of V_0.

- *Step 7: link between z and ζ.* As z is an extensive quantity, its value for N identical particles is $z_N = N\,z$. Planck's potential ζ being a specific quantity, we claim that:

$$\zeta = \frac{z_N}{N\,m} = \frac{z}{m} = \frac{1}{2\,m}\ln(\det(\mathcal{C})) - \frac{3}{2\,m}\ln\beta + \frac{1}{2\,\beta} \parallel w \parallel^2 + C^{te}\,.$$

By (16) and (17), we obtain the linear 4-momentum $\Pi = (\mathcal{H}, -p^T)$ and Cauchy's stresses:

$$\mathcal{H} = \rho\left(\frac{3}{2}\frac{k_B T}{m} + \frac{1}{2} \parallel v \parallel^2\right), \qquad p = \rho v, \qquad \sigma = -q\,1_{\mathbb{R}^3}\,,$$

where, by the expression of the pressure, we recover the *ideal gas law*:

$$q = \frac{\rho}{m}\,k_B T = \frac{N}{V}\,k_B T\,.$$

The first principle of thermodynamics (18) reads:

$$\frac{\partial \mathcal{H}}{\partial t} + div\ (\mathcal{H} v - \sigma v) = 0, \qquad \rho\,\frac{dv}{dt} = -grad\,q, \qquad \frac{\partial \rho}{\partial t} + div\,(\rho\, v) = 0\,.$$

We recognize the balance of energy, linear momentum and mass.

Remark 4. *The Hessian matrix I of $-z$, considered as function of W through Z, is positive definite [3]. It is Fisher metric of the Information Geometry. For the expression (48), it is easy to verify it:*

$$-\delta M\,\delta Z = \frac{1}{\beta}\left(e_{int}(\delta\beta)^2 + m\,\parallel \delta w - \frac{\delta\beta}{m}\,p \parallel^2\right) > 0\,,$$

for any non vanishing δZ taking into account $\beta > 0, e_{int} > 0$ and $m > 0$. On this basis, we can construct a thermodynamic length of a path $t \mapsto X(t)$ [21]:

$$\mathcal{L} = \int_{t_0}^{t_1} \sqrt{(\delta W(t))^T I(t)\,\delta W(t)}dt\,,$$

where $\delta W(t)$ is the perturbation of the temperature vector, tangent to the space-time at $X(t)$. We can also define a related quantity, Jensen–Shannon divergence of the path:

$$\mathcal{J} = (t_1 - t_0)\int_{t_0}^{t_1} (\delta W(t))^T I(t)\,\delta W(t)dt\,.$$

8. Conclusions

The above approach is not limited to classical mechanics but can be used as guiding ideas to tackle the relativistic mechanics. Beyond the strict application to physics, it can be taken as source of inspiration to broach other topics such as the science of information from the viewpoint of differential geometry and Lie groups. We hope to have modestly contributed to this aim.

Conflicts of Interest: The authors declare no conflict of interest.

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