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Explicit Formula of Koszul–Vinberg Characteristic Functions for a Wide Class of Regular Convex Cones

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Abstract: The Koszul–Vinberg characteristic function plays a fundamental role in the theory of convex cones. We give an explicit description of the function and related integral formulas for a new class of convex cones, including homogeneous cones and cones associated with chordal (decomposable) graphs appearing in statistics. Furthermore, we discuss an application to maximum likelihood estimation for a certain exponential family over a cone of this class.

Keywords: convex cone; homogeneous cone; graphical model; Koszul–Vinberg characteristic function

1. Introduction

Let Ω be an open convex cone in a vector space $\mathcal{Z}$. The cone Ω is said to be regular if Ω contains no straight line, which is equivalent to the condition $\overline{\Omega} \cap (-\overline{\Omega}) = \{0\}$. In this paper, we always assume that a convex cone is open and regular. The dual cone Ω^* with respect to an inner product $(\cdot|\cdot)$ on $\mathcal{Z}$ is defined by:

$$\Omega^* := \{\, \xi \in \mathcal{Z}\,;\, (x|\xi) > 0 \ (\forall x \in \overline{\Omega} \setminus \{0\})\,\}.$$

Then, Ω^* is again a regular open convex cone, and we have $(\Omega^*)^* = \Omega$. The Koszul–Vinberg characteristic function $\varphi_\Omega : \Omega \to \mathbb{R}_{>0}$ defined by:

$$\varphi_\Omega(x) := \int_{\Omega^*} e^{-(x|\xi)}\, d\xi \qquad (x \in \Omega)$$

plays a fundamental role in the theory of regular convex cones [1–4].

In particular, φ_Ω is an important function in the theory of convex programming [5], and it has also been studied recently in connection with thermodynamics [6,7]. There are several (not many) classes of cones for which an explicit formula of the Koszul–Vinberg characteristic function is known. Among them, the class of homogeneous cones [8–10] and the class of cones associated with chordal graphs [11] are particularly fruitful research objects. In this paper, we present a wide class of cones, including both of them, and give an explicit expression of the Koszul–Vinberg characteristic function (Section 3). Moreover, we get integral formulas involving the characteristic functions and the so-called generalized power functions, which are expressed as some product of powers of principal minors of real symmetric matrices (Section 4). After investigating the multiplicative Legendre transform of generalized power functions in Section 5, we study a maximum likelihood estimator for a Wishart-type natural exponential family constructed from the integral formula (Section 6).

A regular open convex cone $\Omega \subset \mathcal{Z}$ is said to be homogeneous if the linear automorphism group $GL(\Omega) := \{\, \alpha \in GL(\mathcal{Z})\,;\, \alpha\Omega = \Omega\,\}$ acts on Ω transitively. The cone $\mathcal{P}_n$ of positive definite $n \times n$ real symmetric matrices is a typical example of homogeneous cones. It is known [12–16] that every homogeneous cone is linearly isomorphic to a cone $\mathcal{P}_n \cap \mathcal{Z}$ with an appropriate subspace $\mathcal{Z}$ of the vector space $\mathrm{Sym}(n, \mathbb{R})$ of all $n \times n$ real symmetric matrices, where $\mathcal{Z}$ admits a specific block decomposition. Based on such results, our matrix realization method [15,17,18] has been developed for the purpose of the efficient study of homogeneous cones. In this paper, we present a generalization of matrix realization dealing with a wide class of convex cones, which turns out to include cones associated with chordal graphs. Actually, it was an enigma for the author that some formulas in [11,19] for the chordal graph resemble the formulas in [8,17] for homogeneous cones so much, and the mystery is now solved by the unified method in this paper to get the formulas. Furthermore, the techniques and ideas in the theory of homogeneous cones, such as Riesz distributions [8,20,21] and homogeneous Hessian metrics [4,18,22], will be applied to various cones to obtain new results in our future research.

Here, we fix some notation used in this paper. We denote by $\mathrm{Mat}(p, q, \mathbb{R})$ the vector space of $p \times q$ real matrices. For a matrix A, we write ${}^{\mathrm{t}}A$ for the transpose of A. The identity matrix of size p is denoted by I_p.

2. New Cones $\mathcal{P}_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{V}}^*$

2.1. Setting

We fix a partition $n = n_1 + n_2 + \cdots + n_r$ of a positive integer n. Let $\mathcal{V} = \{\mathcal{V}_{lk}\}_{1 \le k < l \le r}$ be a system of vector spaces $\mathcal{V}_{lk} \subset \mathrm{Mat}(n_l, n_k, \mathbb{R})$ satisfying

(V1) $A \in \mathcal{V}_{lk} \Rightarrow A\,{}^{\mathrm{t}}A \in \mathbb{R}I_{n_l} \quad (1 \le k < l \le r)$,
(V2) $A \in \mathcal{V}_{lj},\ B \in \mathcal{V}_{kj} \Rightarrow A\,{}^{\mathrm{t}}B \in \mathcal{V}_{lk} \quad (1 \le j < k < l \le r)$.

The integer r is called the rank of the system $\mathcal{V}$. We denote by n_{lk} the dimension of $\mathcal{V}_{lk}$. Note that some n_{lk} can be zero. Let $\mathcal{Z}_{\mathcal{V}}$ be the space of real symmetric matrices $x \in \mathrm{Sym}(n, \mathbb{R})$ of the form:

$$x = \begin{pmatrix} X_{11} & {}^{\mathrm{t}}X_{21} & \dots & {}^{\mathrm{t}}X_{r1} \\ X_{21} & X_{22} & & {}^{\mathrm{t}}X_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & \dots & X_{rr} \end{pmatrix} \qquad \begin{pmatrix} X_{kk} = x_{kk} I_{n_k},\ x_{kk} \in \mathbb{R},\ k = 1, \dots, r \\ X_{lk} \in \mathcal{V}_{lk},\ 1 \le k < l \le r \end{pmatrix}, \tag{1}$$

and $\mathcal{P}_{\mathcal{V}}$ the subset of $\mathcal{Z}_{\mathcal{V}}$ consisting of positive definite matrices. Then, $\mathcal{P}_{\mathcal{V}}$ is a regular open convex cone in $\mathcal{Z}_{\mathcal{V}}$.

Example 1. *Let $r = 3$, and set $\mathcal{V}_{21} := \left\{ \begin{pmatrix} a & 0 \end{pmatrix} ; a \in \mathbb{R} \right\}$, $\mathcal{V}_{31} := \left\{ \begin{pmatrix} 0 & a \end{pmatrix} ; a \in \mathbb{R} \right\}$, and $\mathcal{V}_{32} := \mathbb{R}$. Then, $\mathcal{Z}_{\mathcal{V}}$ is the space of symmetric matrices x of the form:*

$$x = \begin{pmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_1 & 0 & x_5 \\ x_4 & 0 & x_2 & x_6 \\ 0 & x_5 & x_6 & x_3 \end{pmatrix}. \tag{2}$$

We shall see later that the cone $\mathcal{P}_{\mathcal{V}} = \mathcal{Z}_{\mathcal{V}} \cap \mathcal{P}_4$ is not homogeneous in this case, but admits various integral formulas, as well as explicit expression of the Koszul–Vinberg characteristic function.

2.2. Inductive Description of $\mathcal{P}_{\mathcal{V}}$

If the system $\mathcal{V} = \{\mathcal{V}_{lk}\}_{1 \le k < l \le r}$ satisfies (V1) and (V2), any subsystem $\mathcal{V}_{\mathcal{I}} := \{\mathcal{V}_{lk}\}_{k,l \in \mathcal{I}}$ with $\mathcal{I} \subset \{1, \dots, r\}$ also satisfies the same conditions. In particular, the cone corresponding to the subsystem

$\{\mathcal{V}_{lk}\}_{2\le k<l\le r}$ will play an important role in this paper. Let us define $\mathcal{V}' := \{\mathcal{V}'_{lk}\}_{1\le k<l\le r-1}$ by $\mathcal{V}'_{lk} := \mathcal{V}_{l+1,k+1}$. Then, $\mathcal{V}'$ is a system of rank $r-1$. Any $x \in \mathcal{Z}_{\mathcal{V}}$ is written as:

$$x = \begin{pmatrix} x_{11} I_{n_1} & {}^{\mathrm{t}}U \\ U & x' \end{pmatrix} \qquad (x_{11} \in \mathbb{R},\, U \in \mathcal{W},\, x' \in \mathcal{Z}_{\mathcal{V}'}), \tag{3}$$

where:

$$\mathcal{W} := \left\{ U = \begin{pmatrix} X_{21} \\ \vdots \\ X_{r1} \end{pmatrix} ;\, X_{l1} \in \mathcal{V}_{l1} \ (1 < l \le r) \right\}. \tag{4}$$

If $x_{11} \neq 0$, then we have:

$$\begin{pmatrix} x_{11} I_{n_1} & {}^{\mathrm{t}}U \\ U & x' \end{pmatrix} = \begin{pmatrix} I_{n_1} & \\ x_{11}^{-1} U & I_{n-n_1} \end{pmatrix} \begin{pmatrix} x_{11} I_{n_1} & \\ & x' - x_{11}^{-1} U {}^{\mathrm{t}}U \end{pmatrix} \begin{pmatrix} I_{n_1} & x_{11}^{-1}{}^{\mathrm{t}}U \\ & I_{n-n_1} \end{pmatrix}. \tag{5}$$

Note that $U{}^{\mathrm{t}}U$ belongs to $\mathcal{Z}_{\mathcal{V}'}$ thanks to (V1) and (V2). Thus, we deduce the following lemma immediately from (5).

Lemma 1. (i) *Let $x \in \mathcal{Z}_{\mathcal{V}}$ as in (3). Then, $x \in \mathcal{P}_{\mathcal{V}}$ if and only if $x_{11} > 0$ and $x' - x_{11}^{-1} U{}^{\mathrm{t}}U \in \mathcal{P}_{\mathcal{V}'}$.*
(ii) *For $x \in \mathcal{P}_{\mathcal{V}}$, there exist unique $\tilde{U} \in \mathcal{W}$ and $\tilde{x} \in \mathcal{P}_{\mathcal{V}'}$ for which:*

$$\begin{aligned} x &= \begin{pmatrix} I_{n_1} & \\ \tilde{U} & I_{n-n_1} \end{pmatrix} \begin{pmatrix} x_{11} I_{n_1} & \\ & \tilde{x}' \end{pmatrix} \begin{pmatrix} I_{n_1} & {}^{\mathrm{t}}\tilde{U} \\ & I_{n-n_1} \end{pmatrix} \\ &= \begin{pmatrix} x_{11} I_{n_1} & x_{11}{}^{\mathrm{t}}\tilde{U} \\ x_{11}\tilde{U} & \tilde{x}' + x_{11}\tilde{U}{}^{\mathrm{t}}\tilde{U} \end{pmatrix}. \end{aligned} \tag{6}$$

(iii) *The closure $\overline{\mathcal{P}_{\mathcal{V}}}$ of the cone $\mathcal{P}_{\mathcal{V}}$ is described as:*

$$\overline{\mathcal{P}_{\mathcal{V}}} := \left\{ \begin{pmatrix} x_{11} I_{n_1} & x_{11}{}^{\mathrm{t}}\tilde{U} \\ x_{11}\tilde{U} & \tilde{x}' + x_{11}\tilde{U}{}^{\mathrm{t}}\tilde{U} \end{pmatrix} ;\, x_{11} \ge 0,\ \tilde{U} \in \mathcal{W},\ \tilde{x}' \in \overline{\mathcal{P}_{\mathcal{V}'}} \right\}.$$

2.3. The Dual Cone $\mathcal{P}_{\mathcal{V}}^*$

We define an inner product on the space $\mathcal{V}_{lk}$ by $(A|B)_{\mathcal{V}_{lk}} := n_l^{-1} \mathrm{tr}\, A{}^{\mathrm{t}}B$ for $A, B \in \mathcal{V}_{lk}$. Then, we see from (V1) that:

$$A\,{}^{\mathrm{t}}B + B\,{}^{\mathrm{t}}A = 2(A|B)_{\mathcal{V}_{lk}} I_{n_l}.$$

Gathering these inner products $(\cdot|\cdot)_{\mathcal{V}_{lk}}$, we introduce the standard inner product on the space $\mathcal{Z}_{\mathcal{V}}$ defined by:

$$(x|x') := \sum_{k=1}^{r} x_{kk} x'_{kk} + 2 \sum_{1\le k<l\le r} (X_{lk}|X'_{lk})_{\mathcal{V}_{lk}} \tag{7}$$

for $x, x' \in \mathcal{Z}_{\mathcal{V}}$ of the form (1). When $n_1 = n_2 = \cdots = n_r = 1$ (and only in this case), the standard inner product above equals the trace inner product $\mathrm{tr}\,(xx')$.

Let $\widetilde{\mathcal{W}}_k$ $(k = 1, \ldots, r)$ be the vector space of $W \in \mathrm{Mat}(n, n_k, \mathbb{R})$ of the form:

$$W = \begin{pmatrix} 0_{n_1+\cdots+n_{k-1}, n_k} \\ X_{kk} \\ X_{k+1,k} \\ \vdots \\ X_{rk} \end{pmatrix} \qquad (X_{kk} = x_{kk} I_{n_k},\ x_{kk} \in \mathbb{R},\ X_{lk} \in \mathcal{V}_{lk},\ l > k).$$

Clearly, the space $\widetilde{\mathcal{W}}_k$ is isomorphic to $\mathbb{R} \oplus \sum_{l>k} \mathcal{V}_{lk}$, which implies $\dim \widetilde{\mathcal{W}}_k = 1 + q_k$ with $q_k := \sum_{l>k} n_{lk}$. Gathering orthogonal bases of $\mathcal{V}_{lk}$'s, we take a basis of $\widetilde{\mathcal{W}}_k$, so that we have an isomorphism $\widetilde{\mathcal{W}}_k \ni W \mapsto w = \mathrm{vect}(W) \in \mathbb{R}^{1+q_k}$, where the first component w_1 of w is assumed to be x_{kk}. Let us introduce a linear map $\phi_k : \mathcal{Z}_{\mathcal{V}} \to \mathrm{Sym}(1 + q_k, \mathbb{R})$ defined in such a way that:

$$(W^{\mathrm{t}}W|\xi) = {}^{\mathrm{t}}w\phi_k(\xi)w \qquad (\xi \in \mathcal{Z}_{\mathcal{V}},\ W \in \widetilde{\mathcal{W}}_k,\ w = \mathrm{vect}(W) \in \mathbb{R}^{1+q_k}). \tag{8}$$

It is easy to see that $\phi_r(\xi) = \xi_{rr}$ for $\xi \in \mathcal{Z}_{\mathcal{V}}$.

Theorem 1. *The dual cone $\mathcal{P}_{\mathcal{V}}^* \subset \mathcal{Z}_{\mathcal{V}}$ of $\mathcal{P}_{\mathcal{V}}$ with respect to the standard inner product is described as:*

$$\begin{aligned} \mathcal{P}_{\mathcal{V}}^* &= \{\, \xi \in \mathcal{Z}_{\mathcal{V}}\,;\, \phi_k(\xi) \text{ is positive definite for all } k = 1, \ldots, r \,\} \\ &= \{\, \xi \in \mathcal{Z}_{\mathcal{V}}\,;\, \det \phi_k(\xi) > 0 \text{ for all } k = 1, \ldots, r \,\}. \end{aligned} \tag{9}$$

Proof. We shall prove the statement by induction on the rank r. When $r = 1$, we have $\phi_1(\xi) = \xi_{11}$ and $\xi = \xi_{11} I_{n_1}$. Thus, (9) holds in this case.

Let us assume that (9) holds when the rank is smaller than r. In particular, the statement holds for $\mathcal{P}_{\mathcal{V}'}^* \subset \mathcal{Z}_{\mathcal{V}'}$, that is,

$$\begin{aligned} \mathcal{P}_{\mathcal{V}'}^* &= \{\, \xi' \in \mathcal{Z}_{\mathcal{V}'}\,;\, \phi'_k(\xi') \text{ is positive definite for all } k = 1, \ldots, r-1 \,\} \\ &= \{\, \xi' \in \mathcal{Z}_{\mathcal{V}'}\,;\, \det \phi'_k(\xi') > 0 \text{ for all } k = 1, \ldots, r-1 \,\}, \end{aligned}$$

where ϕ'_k is defined similarly to (8) for $\mathcal{Z}_{\mathcal{V}'}$. On the other hand, if:

$$\xi = \begin{pmatrix} \xi_{11} I_{n_1} & {}^{\mathrm{t}}V \\ V & \xi' \end{pmatrix} \qquad (\xi_{11} \in \mathbb{R},\ V \in \mathcal{W},\ \xi' \in \mathcal{Z}_{\mathcal{V}'}), \tag{10}$$

we observe that:

$$\phi_k(\xi) = \phi'_{k-1}(\xi') \qquad (k = 2, \ldots, r).$$

Therefore, in order to prove (9) for $\mathcal{P}_{\mathcal{V}}^*$ of rank r, it suffices to show that:

$$\begin{aligned} \mathcal{P}_{\mathcal{V}}^* &= \{\, \xi \in \mathcal{Z}_{\mathcal{V}}\,;\, \xi' \in \mathcal{P}_{\mathcal{V}'}^* \text{ and } \phi_1(\xi) \text{ is positive definite} \,\} \\ &= \{\, \xi \in \mathcal{Z}_{\mathcal{V}}\,;\, \xi' \in \mathcal{P}_{\mathcal{V}'}^* \text{ and } \det \phi_1(\xi) > 0 \,\}. \end{aligned} \tag{11}$$

If $q_1 = 0$, then any element $\xi \in \mathcal{Z}_{\mathcal{V}}$ is of the form:

$$\xi = \begin{pmatrix} \xi_{11} I_{n_1} & \\ & \xi' \end{pmatrix},$$

which belongs to $\mathcal{P}_{\mathcal{V}}$ if and only if $\xi' \in \mathcal{P}_{\mathcal{V}'}$ and $\phi_1(\xi) = \xi_{11} > 0$, so that (11) holds.

Assume $q_1 > 0$. Keeping in mind that $\widetilde{\mathcal{W}}_1 \simeq \mathbb{R} \oplus \mathcal{W}$ and $\mathcal{W} \simeq \mathbb{R}^{q_1}$ by (4), we have for $\xi \in \mathcal{Z}_{\mathcal{V}}$ as in (10),

$$\phi_1(\xi) = \begin{pmatrix} \xi_{11} & {}^{\mathrm{t}}v \\ v & \psi(\xi') \end{pmatrix} \in \mathrm{Sym}(1+q_1, \mathbb{R}), \tag{12}$$

where $v = \mathrm{vect}(V) \in \mathbb{R}^{q_i}$ and $\psi : \mathcal{Z}_{\mathcal{V}'} \to \mathrm{Sym}(q_i, \mathbb{R})$ is defined in such a way that:

$$(U^{\mathrm{t}}U|\xi') = {}^{\mathrm{t}}u\psi(\xi')u \quad (\xi' \in \mathcal{Z}_{\mathcal{V}'},\, U \in \mathcal{W},\, u = \mathrm{vect}(U) \in \mathbb{R}^{q_1}). \tag{13}$$

On the other hand, for $x \in \mathcal{Z}_{\mathcal{V}}$ as in (6), we have:

$$(x|\xi) = x_{11}\xi_{11} + 2x_{11}{}^{\mathrm{t}}\tilde{u}v + x_{11}{}^{\mathrm{t}}\tilde{u}\psi(\xi')\tilde{u} + (\tilde{x}'|\xi). \tag{14}$$

Owing to Lemma 1 (iii), the element $\xi \in \mathcal{Z}_{\mathcal{V}}$ belongs to $\mathcal{P}_{\mathcal{V}}^*$ if and only if the right-hand side is strictly positive for all $x_{11} \geq 0$, $\tilde{U} \in \mathcal{W}$ and $\tilde{x}' \in \overline{\mathcal{P}_{\mathcal{V}'}}$ with $(x_{11}, \tilde{x}') \neq (0,0)$. Assume $\xi \in \mathcal{P}_{\mathcal{V}}^*$. Considering the case $x_{11} = 0$, we have $(\tilde{x}'|\xi') > 0$ for all $\tilde{x}' \in \overline{\mathcal{P}_{\mathcal{V}'}} \backslash \{0\}$, which means that $\xi' \in \mathcal{P}_{\mathcal{V}'}^*$. Then, the quantity in (13) is strictly positive for non-zero U because $U^{\mathrm{t}}U$ belongs to $\overline{\mathcal{P}_{\mathcal{V}}} \setminus \{0\}$. Thus, $\psi(\xi')$ is positive definite, and (14) is rewritten as:

$$(x|\xi) = x_{11}(\xi_{11} - {}^{\mathrm{t}}v\psi(\xi')^{-1}v) + x_{11}{}^{\mathrm{t}}(\tilde{u} + \psi(\xi')^{-1}v)\psi(\xi')(\tilde{u} + \psi(\xi')^{-1}v) + (\tilde{x}'|\xi'). \tag{15}$$

Therefore, we obtain:

$$\mathcal{P}_{\mathcal{V}}^* = \left\{ \xi \in \mathcal{Z}_{\mathcal{V}}\,;\, \xi' \in \mathcal{P}_{\mathcal{V}'}^* \text{ and } \xi_{11} - {}^{\mathrm{t}}v\psi(\xi')^{-1}v > 0 \right\}. \tag{16}$$

On the other hand, we see from (12) that:

$$\phi_1(\xi) = \begin{pmatrix} 1 & {}^{\mathrm{t}}v\psi(\xi')^{-1} \\ & I_{q_1} \end{pmatrix} \begin{pmatrix} \xi_{11} - {}^{\mathrm{t}}v\psi(\xi')^{-1}v & \\ & \psi(\xi') \end{pmatrix} \begin{pmatrix} 1 & \\ \psi(\xi')^{-1}v & I_{q_1} \end{pmatrix}. \tag{17}$$

Hence, we deduce (11) from (16) and (17). $\square$

We note that, if $q_1 > 0$, the $(1,1)$-component of the inverse matrix $\phi_1(\xi)^{-1}$ is given by:

$$(\phi_1(\xi)^{-1})_{11} = (\xi_{11} - {}^{\mathrm{t}}v\psi(\xi')^{-1}v)^{-1} \tag{18}$$

thanks to (17).

3. Koszul–Vinberg Characteristic Function of $\mathcal{P}_{\mathcal{V}}^*$

We denote by $\varphi_{\mathcal{V}}$ the Koszul–Vinberg characteristic function of $\mathcal{P}_{\mathcal{V}}^*$. In this section, we give an explicit formula of $\varphi_{\mathcal{V}}$.

Recall that the linear map $\psi : \mathcal{Z}_{\mathcal{V}'} \to \mathrm{Sym}(q_1, \mathbb{R})$ plays an important role in the proof of Theorem 1. We shall introduce similar linear maps $\psi_k : \mathcal{Z}_{\mathcal{V}} \to \mathrm{Sym}(q_k, \mathbb{R})$ for k such that $q_k > 0$. Let $\mathcal{W}_k$ be the subspace of $\widetilde{\mathcal{W}}_k$ consisting of $W \in \widetilde{\mathcal{W}}_k$ for which $w_1 = x_{kk} = 0$. Then, clearly, $\mathcal{W}_k \simeq \sum_{l>k}^{\oplus} \mathcal{V}_{lk}$ and $\dim \mathcal{W}_k = q_k$. If $q_k > 0$, using the same orthogonal basis of $\mathcal{V}_{lk}$ as in the previous section, we have the isomorphism $\mathcal{W}_k \ni W \mapsto w = \mathrm{vect}(W) \in \mathbb{R}^{q_k}$. Similarly to (8), we define ψ_k by:

$$(W^{\mathrm{t}}W|\xi) = {}^{\mathrm{t}}w\psi_k(\xi)w \qquad (\xi \in \mathcal{Z}_{\mathcal{V}},\ W \in \mathcal{W}_k,\, w = \mathrm{vect}(W) \in \mathbb{R}^{q_k}). \tag{19}$$

Then, we have:

$$\phi_k(\xi) = \begin{pmatrix} \xi_{kk} & {}^{\mathrm{t}}v_k \\ v_k & \psi_k(\xi) \end{pmatrix} \qquad (\xi \in \mathcal{Z}_{\mathcal{V}}), \tag{20}$$

where $v_k \in \mathbb{R}^{q_k}$ is a vector corresponding to the $\mathcal{W}_k$-component of ξ. If $\xi \in \mathcal{P}_{\mathcal{V}}^*$, we see from (19) that $\psi_k(\xi)$ is positive definite. In this case, we have:

$$\phi_k(\xi) = \begin{pmatrix} 1 & {}^{\mathrm{t}}v_k\psi_k(\xi)^{-1} \\ & I_{q_k} \end{pmatrix} \begin{pmatrix} \xi_{kk} - {}^{\mathrm{t}}v_k\psi_k(\xi)^{-1}v_k & \\ & \psi_k(\xi) \end{pmatrix} \begin{pmatrix} 1 & \\ \psi_k(\xi)^{-1}v_k & I_{q_k} \end{pmatrix}, \tag{21}$$

so that we get a generalization of (18), that is,

$$(\phi_k(\xi)^{-1})_{11} = (\xi_{kk} - {}^{\mathrm{t}}v_k\psi_k(\xi)^{-1}v_k)^{-1}. \tag{22}$$

On the other hand, if $q_k = 0$, then $\phi_k(\xi)^{-1} = \xi_{kk}^{-1}$.

We remark that $\psi_1(\xi) = \psi(\xi')$, and that some part of the argument above is parallel to the proof of Theorem 1.

Theorem 2. *The Koszul–Vinberg characteristic function $\varphi_{\mathcal{V}}$ of $\mathcal{P}_{\mathcal{V}}^*$ is given by the following formula:*

$$\varphi_{\mathcal{V}}(\xi) = C_{\mathcal{V}} \prod_{k=1}^{r} (\phi_k(\xi)^{-1})_{11}^{1+q_k/2} \prod_{q_k>0} (\det \psi_k(\xi))^{-1/2} \qquad (\xi \in \mathcal{P}_{\mathcal{V}}^*), \tag{23}$$

where $C_{\mathcal{V}} := (2\pi)^{(N-r)/2} \prod_{k=1}^{r} \Gamma(1 + \frac{q_k}{2})$ and $N := \dim \mathcal{Z}_{\mathcal{V}}$.

Proof. We shall show the statement by induction on the rank as in the proof of Theorem 1. Then, it suffices to show that:

$$\varphi_{\mathcal{V}}(\xi) = (2\pi)^{q_1/2}\Gamma(1 + \frac{q_1}{2})(\phi_1(\xi)^{-1})_{11}^{1+q_1/2}(\det \psi_1(\xi))^{-\mathrm{sgn}(q_1)/2}\varphi_{\mathcal{V}'}(\xi') \tag{24}$$

for $\xi \in \mathcal{P}_{\mathcal{V}}^*$ as in (10), where $(\det \psi_1(\xi))^{-\mathrm{sgn}(q_1)/2}$ is interpreted as:

$$(\det \psi_1(\xi))^{-\mathrm{sgn}(q_1)/2} := \begin{cases} 1 & (q_1 = 0), \\ (\det \psi_1(\xi))^{-1/2} & (q_1 > 0). \end{cases}$$

When $q_1 = 0$, we have:

$$\begin{aligned} \varphi_{\mathcal{V}}(\xi) &= \int_0^\infty \int_{\mathcal{P}_{\mathcal{V}'}} e^{-x_{11}\xi_{11}} e^{-(x'|\xi')} dx_{11}\, dx' \\ &= \xi_{11}^{-1}\varphi_{\mathcal{V}'}(\xi'), \end{aligned}$$

which means (24).

When $q_1 > 0$, the Euclidean measure dx equals $2^{q_1/2}x_{11}^{q_1}\, dx_{11} d\tilde{u} d\tilde{x}'$ by the change of variables in (6). Indeed, the coefficient $2^{q_1/2}$ comes from the normalization of the inner product on $\mathcal{W} \simeq \mathbb{R}^{q_1}$ regarded as a subspace of $\mathcal{Z}_{\mathcal{V}}$. Then, we have by (15):

$$\begin{aligned} \varphi_{\mathcal{V}}(\xi) = \int_0^\infty \int_{\mathbb{R}^{q_1}} \int_{\mathcal{P}_{\mathcal{V}'}} & e^{-x_{11}(\xi_{11} - {}^{\mathrm{t}}v\psi(\xi')^{-1}v)} e^{-x_{11}{}^{\mathrm{t}}(\tilde{u}+\psi(\xi')^{-1}v)\psi(\xi')(\tilde{u}+\psi(\xi')^{-1}v)} e^{-(\tilde{x}'|\xi')} \\ & \times 2^{q_1/2}x_{11}^{q_1}\, dx_{11} d\tilde{u} d\tilde{x}'. \end{aligned}$$

By the Gaussian integral formula, we have:

$$\int_{\mathbb{R}^{q_1}} e^{-x_{11}{}^{\mathrm{t}}(\tilde{u}+\psi(\xi')^{-1}v)\psi(\xi')(\tilde{u}+\psi(\xi')^{-1}v)} d\tilde{u} = \pi^{q_1/2}x_{11}^{-q_1/2}(\det \psi(\xi'))^{-1/2}.$$

Therefore, we get:

$$\begin{aligned}\varphi_{\mathcal{V}}(\xi) &= (2\pi)^{q_1/2}(\det\psi(\xi'))^{-1/2}\int_0^\infty e^{-x_{11}(\xi_{11}-{}^t v\psi(\xi')^{-1}v)}x_{11}^{q_1/2}\,dx_{11}\int_{\mathcal{P}_{\mathcal{V}'}} e^{-(\tilde{x}'|\xi')}\,d\tilde{x}' \\ &= (2\pi)^{q_1/2}(\det\psi_1(\xi))^{-1/2}\Gamma(1+\frac{q_1}{2})(\xi_{11}-{}^t v\psi(\xi')^{-1}v)^{-1-q_k/2}\varphi_{\mathcal{V}'}(\xi'),\end{aligned}$$

which together with (18) leads us to (24). □

Example 2. *Let $\mathcal{V} = \{\mathcal{V}_{lk}\}_{1\le k<l\le 3}$ be as in Example 1. For:*

$$\xi = \begin{pmatrix} \xi_1 & 0 & \xi_4 & 0 \\ 0 & \xi_1 & 0 & \xi_5 \\ \xi_4 & 0 & \xi_2 & \xi_6 \\ 0 & \xi_5 & \xi_6 & \xi_3 \end{pmatrix} \in \mathcal{Z}_{\mathcal{V}}, \tag{25}$$

we have:

$$\phi_1(\xi) = \begin{pmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{pmatrix}, \quad \phi_2(\xi) = \begin{pmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{pmatrix}, \quad \phi_3(\xi) = \xi_3,$$

$$\psi_1(\xi) = \begin{pmatrix} \xi_2 & 0 \\ 0 & \xi_3 \end{pmatrix}, \quad \psi_2(\xi) = \xi_3.$$

The cone $\mathcal{P}^*_{\mathcal{V}}$ is described as:

$$\mathcal{P}^*_{\mathcal{V}} = \left\{ \xi \in \mathcal{Z}_{\mathcal{V}}\,;\, \begin{vmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{vmatrix} > 0,\ \begin{vmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{vmatrix} > 0,\ \xi_3 > 0 \right\},$$

and its Koszul–Vinberg characteristic function $\varphi_{\mathcal{V}}$ is expressed as:

$$\begin{aligned}\varphi_{\mathcal{V}}(\xi) &= C_{\mathcal{V}} \left\{ \begin{vmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{vmatrix} / (\xi_2\xi_3) \right\}^{-2} \left\{ \begin{vmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{vmatrix} / \xi_3 \right\}^{-3/2} \xi_3^{-1}\cdot(\xi_2\xi_3)^{-1/2}(\xi_3)^{-1/2} \\ &= C_{\mathcal{V}} \begin{vmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{vmatrix}^{-2} \begin{vmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{vmatrix}^{-3/2} \xi_2^{3/2}\xi_3^{3/2},\end{aligned}$$

where $C_{\mathcal{V}} = (2\pi)^{3/2}\Gamma(2)\Gamma(3/2)\Gamma(1) = \sqrt{2}\pi^2$.

Suppose that the cone $\mathcal{P}_{\mathcal{V}}$ is homogeneous. Then, $\mathcal{P}^*_{\mathcal{V}}$, as well as $\mathcal{P}_{\mathcal{V}}$, is a homogeneous cone of rank 3, so that the Koszul–Vinberg characteristic function of $\mathcal{P}^*_{\mathcal{V}}$ has at most three irreducible factors (see [8]). However, we have seen that there are four irreducible factors in the function $\varphi_{\mathcal{V}}$. Therefore, we conclude that neither $\mathcal{P}_{\mathcal{V}}$, nor $\mathcal{P}^*_{\mathcal{V}}$ is homogeneous.

4. Γ-Type Integral Formulas

For an $n \times n$ matrix $A = (A_{ij})$ and $1 \le m \le n$, we denote by $A^{[m]}$ the upper-left $m \times m$ submatrix $(A_{ij})_{i,j \le m}$ of A. Put $M_k := \sum_{i=1}^{k} n_k \ \ (k = 1, \dots, r)$. For $\underline{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, we define functions $\Delta_{\underline{s}}^{\mathcal{V}}$ on $\mathcal{P}_{\mathcal{V}}$ and $\delta_{\underline{s}}^{\mathcal{V}}$ on $\mathcal{P}_{\mathcal{V}}^*$ respectively by:

$$\begin{aligned} \Delta_{\underline{s}}^{\mathcal{V}}(x) &:= (\det x^{[M_1]})^{s_1/n_1} \prod_{k=2}^{r} \Big(\frac{\det x^{[M_k]}}{\det x^{[M_{k-1}]}} \Big)^{s_k/n_k} \\ &= (\det x)^{s_r/n_r} \prod_{k=1}^{r-1} (\det x^{[M_k]})^{s_k/n_k - s_{k-1}/n_{k-1}} \qquad (x \in \mathcal{P}_{\mathcal{V}}), \end{aligned} \tag{26}$$

$$\begin{aligned} \delta_{\underline{s}}^{\mathcal{V}}(\xi) &:= \prod_{k=1}^{r} (\phi_k(\xi)^{-1})_{11}^{-s_k} \\ &= \prod_{q_k=0} \xi_{kk}^{s_k} \prod_{q_k>0} (\xi_{kk} - {}^{\mathrm{t}}v_k \psi_k(\xi)^{-1} v_k)^{s_k} \qquad (\xi \in \mathcal{P}_{\mathcal{V}}^*). \end{aligned} \tag{27}$$

Recall (22) for the second equality of (27).

For $\underline{a} = (a_1, \dots, a_r) \in \mathbb{R}_{>0}^r$, let $D_{\underline{a}}$ denote the diagonal matrix defined by:

$$D_{\underline{a}} := \begin{pmatrix} a_1 I_{n_1} & & & \\ & a_2 I_{n_2} & & \\ & & \ddots & \\ & & & a_r I_{n_r} \end{pmatrix} \in GL(n, \mathbb{R}).$$

Then, the linear map $\mathcal{Z}_{\mathcal{V}} \ni x \mapsto D_a x D_a \in \mathcal{Z}_{\mathcal{V}}$ preserves both $\mathcal{P}_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{V}}^*$, and we have:

$$\Delta_{\underline{s}}^{\mathcal{V}}(D_a x D_a) = (\prod_{k=1}^{r} a_k^{2s_k}) \Delta_{\underline{s}}^{\mathcal{V}}(x) \qquad (x \in \mathcal{P}_{\mathcal{V}}), \tag{28}$$

$$\delta_{\underline{s}}^{\mathcal{V}}(D_a \xi D_a) = (\prod_{k=1}^{r} a_k^{2s_k}) \delta_{\underline{s}}^{\mathcal{V}}(\xi) \qquad (\xi \in \mathcal{P}_{\mathcal{V}}). \tag{29}$$

Assume $q_1 > 0$. For $B \in \mathcal{W}$, we denote by τ_B the linear transform on $\mathcal{Z}_{\mathcal{V}}$ given by:

$$\begin{aligned} \tau_B x &:= \begin{pmatrix} I_{n_1} & \\ B & I_{n-n_1} \end{pmatrix} \begin{pmatrix} x_{11} I_{n_1} & {}^{\mathrm{t}}U \\ U & x' \end{pmatrix} \begin{pmatrix} I_{n_1} & {}^{\mathrm{t}}B \\ & I_{n-n_1} \end{pmatrix} \\ &= \begin{pmatrix} x_{11} I_{n_1} & {}^{\mathrm{t}}U + x_{11} {}^{\mathrm{t}}B \\ U + x_{11} B & x' + U {}^{\mathrm{t}}B + B {}^{\mathrm{t}}U + x_{11} B {}^{\mathrm{t}}B \end{pmatrix}, \end{aligned}$$

where $x \in \mathcal{Z}_{\mathcal{V}}$ is as in (3). Indeed, since:

$$U {}^{\mathrm{t}}B + B {}^{\mathrm{t}}U = (U + B) {}^{\mathrm{t}}(U + B) - U {}^{\mathrm{t}}U - B {}^{\mathrm{t}}B \in \mathcal{Z}_{\mathcal{V}'},$$

the matrix $\tau_B x$ belongs to $\mathcal{Z}_{\mathcal{V}}$. Clearly, τ_B preserves $\mathcal{P}_{\mathcal{V}}$, and we have:

$$\Delta_{\underline{s}}^{\mathcal{V}}(\tau_B x) = \Delta_{\underline{s}}^{\mathcal{V}}(x) \qquad (x \in \mathcal{P}_{\mathcal{V}}). \tag{30}$$

The formula (5) is rewritten as:

$$\tau_{-x_{11}^{-1} U}(x) = \begin{pmatrix} x_{11} I_{n_1} & \\ & x' - x_{11}^{-1} U {}^{\mathrm{t}}U \end{pmatrix},$$

which together with (30) tells us that:

$$\Delta_{\underline{s}}^{\mathcal{V}}(x) = x_{11}^{s_1}\Delta_{\underline{s}'}^{\mathcal{V}'}(x' - x_{11}^{-1}U^{\mathrm{t}}U), \tag{31}$$

where $\underline{s}' := (s_2, \ldots, s_r) \in \mathbb{C}^{r-1}$.

Let us consider the adjoint map $\tau_B^* : \mathcal{Z}_{\mathcal{V}} \to \mathcal{Z}_{\mathcal{V}}$ of τ_B with respect to the standard inner product. Let $b \in \mathbb{R}^{q_1}$ be the vector corresponding to $B \in \mathcal{W}$. For $x \in \mathcal{Z}_{\mathcal{V}}$ and $\xi \in \mathcal{Z}_{\mathcal{V}}$ as in (3) and (10), respectively, we observe that:

$$\begin{aligned}(\tau_B x|\xi) &= x_{11}\xi_{11} + 2^{\mathrm{t}}(u + x_{11}b)v + (x' + U^{\mathrm{t}}B + B^{\mathrm{t}}U + x_{11}B^{\mathrm{t}}B|\xi') \\ &= x_{11}(\xi_{11} + 2^{\mathrm{t}}bv + {}^{\mathrm{t}}b\psi(\xi')b) + 2^{\mathrm{t}}u(v + \psi(\xi')b) + (x'|\xi').\end{aligned}$$

Thus, if we write:

$$\iota(\xi_{11}, v, \xi') := \begin{pmatrix} \xi_{11}I_{n_1} & {}^{\mathrm{t}}V \\ V & \xi' \end{pmatrix},$$

we have:

$$\tau_B^*\iota(\xi_{11}, v, \xi') = \iota(\xi_{11} + 2^{\mathrm{t}}bv + {}^{\mathrm{t}}b\psi(\xi')b, v + \psi(\xi')b, \xi'). \tag{32}$$

Furthermore, we see from (12) that $\phi_1(\tau_B^*\iota(\xi_{11}, v, \xi'))$ equals:

$$\begin{pmatrix} \xi_{11} + 2^{\mathrm{t}}bv + {}^{\mathrm{t}}b\psi(\xi')b & {}^{\mathrm{t}}v + {}^{\mathrm{t}}b\psi(\xi') \\ v + \psi(\xi')b & \psi(\xi') \end{pmatrix} = \begin{pmatrix} 1 & {}^{\mathrm{t}}b \\ & I_{q_1} \end{pmatrix} \begin{pmatrix} \xi_{11} & {}^{\mathrm{t}}v \\ v & \psi(\xi') \end{pmatrix} \begin{pmatrix} 1 & \\ b & I_{q_1} \end{pmatrix},$$

so that we get for $\xi = \iota(\xi_{11}, v, \xi')$:

$$\phi_1(\tau_B^*\xi) = \begin{pmatrix} 1 & {}^{\mathrm{t}}b \\ & I_{q_1} \end{pmatrix} \phi_1(\xi) \begin{pmatrix} 1 & \\ b & I_{q_1} \end{pmatrix}.$$

Therefore:

$$(\phi_1(\tau_B^*\xi)^{-1})_{11} = (\phi_1(\xi)^{-1})_{11}.$$

On the other hand, we have for $\xi = \iota(\xi_{11}, v, \xi') \in \mathcal{P}_{\mathcal{V}}^*$:

$$\delta_{\underline{s}}^{\mathcal{V}}(\xi) = (\phi_1(\xi)^{-1})_{11}^{-s_1}\,\delta_{\underline{s}'}^{\mathcal{V}'}(\xi'). \tag{33}$$

Thus, we conclude that:

$$\delta_{\underline{s}}^{\mathcal{V}}(\tau_B^*\xi) = \delta_{\underline{s}}^{\mathcal{V}}(\xi). \tag{34}$$

Theorem 3. *When $\Re s_k > -1 - q_k/2$ for $k = 1, \ldots, r$, one has:*

$$\int_{\mathcal{P}_{\mathcal{V}}} e^{-(x|\xi)}\Delta_{\underline{s}}^{\mathcal{V}}(x)\,dx = C_{\mathcal{V}}^{-1}\gamma_{\mathcal{V}}(\underline{s})\,\delta_{-\underline{s}}^{\mathcal{V}}(\xi)\varphi_{\mathcal{V}}(\xi), \tag{35}$$

where $\gamma_{\mathcal{V}}(\underline{s}) := (2\pi)^{(N-r)/2}\prod_{k=1}^{r}\Gamma(s_k + 1 + \frac{q_k}{2})$.

Proof. Recalling Theorem 2, we rewrite the right-hand side of (35) as:

$$(2\pi)^{(N-r)/2}\prod_{k=1}^{r}\Gamma(s_k + 1 + \frac{q_k}{2})\prod_{k=1}^{r}(\phi_k(\xi)^{-1})_{11}^{s_k+1+q_k/2}\prod_{q_k>0}(\det\psi_k(\xi))^{-1/2},$$

which is similar to the right-hand side of (23). Thus, the proof is parallel to Theorem 2. Namely, by induction on the rank, it suffices to show that:

$$\begin{aligned}&\int_{\mathcal{P}_{\mathcal{V}}} e^{-(x|\xi)}\Delta_{\underline{s}}^{\mathcal{V}}(x)\,dx\\&=(2\pi)^{q_1/2}\Gamma(s_1+1+\frac{q_1}{2})(\phi_1(\xi)^{-1})_{11}^{s_1+1+q_1/2}(\det\psi_1(\xi))^{-\mathrm{sgn}(q_1)/2}\\&\quad\times\int_{\mathcal{P}_{\mathcal{V}'}} e^{-(x'|\xi)}\Delta_{\underline{s}'}^{\mathcal{V}'}(x')\,dx'\end{aligned}\tag{36}$$

thanks to (33).

When $q_1=0$, we have $(x|\xi)=x_{11}\xi_{11}+(x'|\xi')$ and $\Delta_{\underline{s}}^{\mathcal{V}}(x)=x_{11}^{s_1}\Delta_{\underline{s}}^{\mathcal{V}'}(x')$. Thus:

$$\int_{\mathcal{P}_{\mathcal{V}}} e^{-(x|\xi)}\Delta_{\underline{s}}^{\mathcal{V}}(x)\,dx=\int_0^\infty e^{-x_{11}\xi_{11}}x_{11}^{s_1}\,dx_{11}\times\int_{\mathcal{P}_{\mathcal{V}'}} e^{-(x'|\xi)}\Delta_{\underline{s}'}^{\mathcal{V}'}(x')\,dx'.$$

Since $\int_0^\infty e^{-x_{11}\xi_{11}}x_{11}^{s_1}\,dx_{11}=\Gamma(s_1+1)\xi_{11}^{-s_1-1}$, we get (36).

When $q_1>1$, we use the change of variable (6). Since $\tilde{x}'=x'-x_{11}^{-1}U^{\mathrm{t}}U$, we have $\Delta_{\underline{s}}^{\mathcal{V}}(x)=x_{11}^{s_1}\Delta_{\underline{s}'}^{\mathcal{V}'}(\tilde{x}')$ by (31). Therefore, by the same Gaussian integral formula as in the proof of Theorem 2, the integral $\int_{\mathcal{P}_{\mathcal{V}}} e^{-(x|\xi)}\Delta_{\underline{s}}^{\mathcal{V}}(x)\,dx$ equals:

$$\begin{aligned}&\int_0^\infty\int_{\mathcal{W}}\int_{\mathcal{P}_{\mathcal{V}'}} e^{-x_{11}(\xi_{11}-{}^{\mathrm{t}}v\psi(\xi')^{-1}v)}e^{-x_{11}{}^{\mathrm{t}}(\tilde{u}+\psi(\xi')^{-1}v)\psi(\xi')(\tilde{u}+\psi(\xi')^{-1}v)}e^{-(\tilde{x}'|\xi')}x_{11}^{s_1}\Delta_{\underline{s}'}^{\mathcal{V}'}(\tilde{x}')\\&\qquad\qquad\times 2^{q_1/2}x_{11}^{q_1}\,dx_{11}d\tilde{u}d\tilde{x}'\\&=(2\pi)^{q_1/2}(\det\psi(\xi))^{-1/2}\int_0^\infty e^{-x_{11}(\xi_{11}-{}^{\mathrm{t}}v\psi(\xi')^{-1}v)}x_{11}^{s_1+q_1/2}\,dx_{11}\\&\quad\times\int_{\mathcal{P}_{\mathcal{V}'}} e^{-(\tilde{x}'|\xi')}\Delta_{\underline{s}'}^{\mathcal{V}'}(\tilde{x}')\,d\tilde{x}'\\&=(2\pi)^{q_1/2}(\det\psi(\xi))^{-1/2}\Gamma(s_k+1+\frac{q_1}{2})(\xi_{11}-{}^{\mathrm{t}}v\psi(\xi')^{-1}v)^{-s_k-1-q_k/2}\\&\quad\times\int_{\mathcal{P}_{\mathcal{V}'}} e^{-(\tilde{x}'|\xi')}\Delta_{\underline{s}'}^{\mathcal{V}'}(\tilde{x}')\,d\tilde{x}'.\end{aligned}$$

Hence, we get (36) by (18). □

We shall obtain an integral formula over $\mathcal{P}_{\mathcal{V}}^*$ as follows.

Theorem 4. *When $\Re s_k>q_k/2$ for $k=1,\ldots,r$, one has:*

$$\int_{\mathcal{P}_{\mathcal{V}}^*} e^{-(x|\xi)}\delta_{\underline{s}}^{\mathcal{V}}(\xi)\,\varphi_{\mathcal{V}}(\xi)\,d\xi=C_{\mathcal{V}}\Gamma_{\mathcal{V}}(\underline{s})\Delta_{-\underline{s}}^{\mathcal{V}}(x)\qquad(x\in\mathcal{P}_{\mathcal{V}}),\tag{37}$$

where $\Gamma_{\mathcal{V}}(\underline{s}):=(2\pi)^{(N-r)/2}\prod_{k=1}^r\Gamma(s_k-q_k/2)$.

Proof. Using (24), (31) and (33), we rewrite (37) as:

$$\begin{aligned}&\int_{\mathcal{P}_{\mathcal{V}}^*} e^{-(x|\xi)}(\phi_1(\xi)^{-1})_{11}^{-s_1+1+q_1/2}(\det\psi_1(\xi))^{-\mathrm{sgn}(q_1)/2}\delta_{\underline{s}'}^{\mathcal{V}'}(\xi')\,\varphi_{\mathcal{V}'}(\xi')\,d\xi\\&=C_{\mathcal{V}'}(2\pi)^{q_1/2}\Gamma(s_1-q_1/2)\Gamma_{\mathcal{V}'}(\underline{s}')\,x_{11}^{-s_1}\Delta_{-\underline{s}'}^{\mathcal{V}'}(\tilde{x}'),\end{aligned}\tag{38}$$

where:

$$\tilde{x}':=\begin{cases}x' & (q_1=0),\\ x'-x_{11}^{-1}U^{\mathrm{t}}U & (q_1>0).\end{cases}$$

Therefore, by induction on the rank, it suffices to show that the left-hand side of (38) equals:

$$(2\pi)^{q_1/2}\Gamma(s_1-q_1/2)x_{11}^{-s_1}\int_{\mathcal{P}^*_{\mathcal{V}'}} e^{-(\tilde{x}'|\xi')}\delta^{\mathcal{V}'}_{\underline{s}'}(\xi')\,\varphi_{\mathcal{V}'}(\xi')\,d\xi'. \tag{39}$$

When $q_1=0$, since $d\xi=d\xi_{11}d\xi'$, the left-hand side of (38) equals:

$$\int_0^\infty e^{-x_{11}\xi_{11}}\xi_{11}^{s_1-1}\,d\xi_{11}\int_{\mathcal{P}^*_{\mathcal{V}'}} e^{-(x'|\xi')}\delta^{\mathcal{V}'}_{\underline{s}'}(\xi')\,\varphi_{\mathcal{V}'}(\xi')\,d\xi',$$

which coincides with (39) in this case.

Assume $q_1>0$. Keeping (16) and (18) in mind, we put $\tilde{\xi}_{11}:=\xi_{11}-{}^{\mathrm{t}}v\psi(\xi')^{-1}v=(\phi_1(\xi)^{-1})_{11}^{-1}>0$. By the change of variables $\xi=\iota(\tilde{\xi}_{11}+{}^{\mathrm{t}}v\psi(\xi')^{-1}v,\ v,\ \xi')$, we have $d\xi=2^{q_1/2}\,d\tilde{\xi}_{11}dvd\xi'$. On the other hand, we observe:

$$\begin{aligned}(x|\xi)&=x_{11}(\tilde{\xi}_{11}+{}^{\mathrm{t}}v\psi(\xi')^{-1}v)+2{}^{\mathrm{t}}uv+(x'|\xi')\\&=x_{11}\tilde{\xi}_{11}+x_{11}{}^{\mathrm{t}}(v+x_{11}^{-1}\psi(\xi')u)\psi(\xi')^{-1}(v+x_{11}^{-1}\psi(\xi')u)+(x-x_{11}^{-1}U{}^{\mathrm{t}}U|\xi').\end{aligned}$$

Thus, the left-hand side of (39) equals:

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^{q_1}}\int_{\mathcal{P}^*_{\mathcal{V}'}} & e^{-x_{11}\tilde{\xi}_{11}}e^{-x_{11}{}^{\mathrm{t}}(v+x_{11}^{-1}\psi(\xi')u)\psi(\xi')^{-1}(v+x_{11}^{-1}\psi(\xi')u)}e^{-(x-x_{11}^{-1}U{}^{\mathrm{t}}U|\xi')}\\&\times\tilde{\xi}_{11}^{s_1-1-q_1/2}(\det\psi(\xi'))^{-1/2}\delta^{\mathcal{V}'}_{\underline{s}'}(\xi')\,\varphi_{\mathcal{V}'}(\xi')\,2^{q_1/2}\,d\tilde{\xi}_{11}dvd\xi'.\end{aligned} \tag{40}$$

By the Gaussian integral formula, we have:

$$\int_{\mathbb{R}^{q_1}} e^{-x_{11}{}^{\mathrm{t}}(v+x_{11}^{-1}\psi(\xi')u)\psi(\xi')^{-1}(v+x_{11}^{-1}\psi(\xi')u)}\,dv=\pi^{q_1/2}x_{11}^{-q_1/2}(\det\psi(\xi'))^{1/2},$$

so that (40) equals:

$$(2\pi)^{q_1/2}x_{11}^{-q_1/2}\int_0^\infty e^{-x_{11}\tilde{\xi}_{11}}\tilde{\xi}_{11}^{s_1-1-q_1/2}\,d\tilde{\xi}_{11}\int_{\mathcal{P}^*_{\mathcal{V}'}} e^{-(x-x_{11}^{-1}U{}^{\mathrm{t}}U|\xi')}\delta^{\mathcal{V}'}_{\underline{s}'}(\xi')\,\varphi_{\mathcal{V}'}(\xi')d\xi',$$

which coincides with (39) because: $\int_0^\infty e^{-x_{11}\tilde{\xi}_{11}}\tilde{\xi}_{11}^{s_1-1-q_1/2}\,d\tilde{\xi}_{11}=\Gamma(s_1-q_1/2)x_{11}^{-s_1+q_1/2}$. $\square$

Example 3. *Let $\mathcal{Z}_{\mathcal{V}}$ be as in Example 1, and let $x\in\mathcal{P}_{\mathcal{V}}$ and $\xi\in\mathcal{P}^*_{\mathcal{V}}$ be as in (2) and (25), respectively. Then, we have for $\underline{s}=(s_1,s_2,s_3)\in\mathbb{C}^3$,*

$$\begin{aligned}\Delta^{\mathcal{V}}_{\underline{s}}(x)&=(x_{11}^2)^{s_1/2-s_2}\begin{vmatrix}x_1&0&x_4\\0&x_1&0\\x_4&0&x_2\end{vmatrix}^{s_2-s_3}(\det x)^{s_3}\\&=x_{11}^{s_1-s_2-s_3}\begin{vmatrix}x_1&x_4\\x_4&x_2\end{vmatrix}^{s_2-s_3}(\det x)^{s_3},\end{aligned}$$

and:

$$\begin{aligned}\delta^{\mathcal{V}}_{\underline{s}}(\xi)&=(\xi_1-\frac{\xi_4^2}{\xi_2}-\frac{\xi_5^2}{\xi_3})^{s_1}(\xi_2-\frac{\xi_6^2}{\xi_3})^{s_2}\xi_3^{s_3}\\&=\begin{vmatrix}\xi_1&\xi_4&\xi_5\\\xi_4&\xi_2&0\\\xi_5&0&\xi_3\end{vmatrix}^{s_1}\begin{vmatrix}\xi_2&\xi_6\\\xi_6&\xi_3\end{vmatrix}^{s_2}\xi_2^{-s_1}\xi_3^{s_3-s_1-s_2}.\end{aligned}$$

When $\Re s_1 > -2$, $\Re s_2 > -3/2$ and $\Re s_3 > -1$, the integral formula (35) holds with:

$$\gamma_{\mathcal{V}}(\underline{s}) = (2\pi)^{3/2}\Gamma(s_1+2)\Gamma(s_2+3/2)\Gamma(s_3+1).$$

Furthermore, when $\Re s_1 > 1$, $\Re s_2 > 1/2$ and $\Re s_3 > 0$, the integral formula (37) holds with:

$$\Gamma_{\mathcal{V}}(\underline{s}) = (2\pi)^{3/2}\Gamma(s_1-1)\Gamma(s_2-1/2)\Gamma(s_3).$$

5. Multiplicative Legendre Transform of Generalized Power Functions

For $\underline{s} \in \mathbb{R}^r_{>0}$, we see that $\log \Delta_{-\underline{s}}$ is a strictly convex function on the cone $\mathcal{P}_{\mathcal{V}}$. In fact, $\Delta_{-\underline{s}}$ is defined naturally on $\mathcal{P}_n$ as a product of powers of principal minors, and it is well known that such $\log \Delta_{-\underline{s}}$ is strictly convex on the whole $\mathcal{P}_n$. In this section, we shall show that $\log \Delta^{\mathcal{V}}_{-\underline{s}}$ and $\log \delta^{\mathcal{V}}_{-\underline{s}}$ are related by the Fenchel–Legendre transform.

For $x \in \mathcal{P}_{\mathcal{V}}$, we denote by $\mathcal{I}^{\mathcal{V}}_{\underline{s}}(x)$ the minus gradient $-\nabla \log \Delta_{-\underline{s}}(x)$ at x with respect to the inner product. Namely, $\mathcal{I}^{\mathcal{V}}_{\underline{s}}(x)$ is an element of $\mathcal{Z}_{\mathcal{V}}$ for which:

$$(\mathcal{I}^{\mathcal{V}}_{\underline{s}}(x)|y) = -\left(\frac{d}{dt}\right)_{t=0} \log \Delta_{-\underline{s}}(x+ty) \qquad (y \in \mathcal{Z}_{\mathcal{V}}).$$

Similarly, $\mathcal{J}^{\mathcal{V}}_{\underline{s}}(\xi) := -\nabla \log \delta_{-\underline{s}}(\xi)$ is defined for $\xi \in \mathcal{P}^*_{\mathcal{V}}$. If $q_1 > 0$, then for any $B \in \mathcal{W}$, we have:

$$\mathcal{I}^{\mathcal{V}}_{\underline{s}} \circ \tau_B = \tau^*_B \circ \mathcal{I}^{\mathcal{V}}_{\underline{s}}, \tag{41}$$
$$\mathcal{J}^{\mathcal{V}}_{\underline{s}} \circ \tau^*_B = \tau_B \circ \mathcal{J}^{\mathcal{V}}_{\underline{s}} \tag{42}$$

owing to (30) and (34), respectively.

Theorem 5. *For any $\underline{s} \in \mathbb{R}^r_{>0}$, the map $\mathcal{I}^{\mathcal{V}}_{\underline{s}} : \mathcal{P}_{\mathcal{V}} \to \mathcal{Z}_{\mathcal{V}}$ gives a diffeomorphism from $\mathcal{P}_{\mathcal{V}}$ onto $\mathcal{P}^*_{\mathcal{V}}$, and $\mathcal{J}^{\mathcal{V}}_{\underline{s}}$ gives the inverse map.*

Proof. We shall prove the statement by induction on the rank. When $r = 1$, we have $\mathcal{I}^{\mathcal{V}}_{\underline{s}}(x_{11}I_{n_1}) = \frac{s_1}{x_{11}}I_{n_1} = \mathcal{J}^{\mathcal{V}}_{\underline{s}}(x_{11}I_{n_1})$ for $x_{11} > 0$. Thus, the statement is true in this case.

When $r > 1$, assume that the statement holds for the system of rank $r-1$. Let $\mathcal{Z}^0_{\mathcal{V}}$ be the subspace of $\mathcal{Z}_{\mathcal{V}}$ defined by:

$$\mathcal{Z}^0_{\mathcal{V}} := \left\{ \begin{pmatrix} x_{11}I_{n_1} & 0 \\ 0 & x' \end{pmatrix} ; x_{11} \in \mathbb{R},\ x' \in \mathcal{Z}_{\mathcal{V}'} \right\}.$$

By direct computation with (31) and (33), we have:

$$\mathcal{I}^{\mathcal{V}}_{\underline{s}} \begin{pmatrix} x_{11}I_{n_1} & 0 \\ 0 & x' \end{pmatrix} = \begin{pmatrix} \frac{s_1}{x_{11}}I_{n_1} & 0 \\ 0 & \mathcal{I}^{\mathcal{V}'}_{\underline{s}'}(x') \end{pmatrix}, \tag{43}$$
$$\mathcal{J}^{\mathcal{V}}_{\underline{s}} \begin{pmatrix} \xi_{11}I_{n_1} & 0 \\ 0 & \xi' \end{pmatrix} = \begin{pmatrix} \frac{s_1}{\xi_{11}}I_{n_1} & 0 \\ 0 & \mathcal{J}^{\mathcal{V}'}_{\underline{s}'}(\xi') \end{pmatrix} \tag{44}$$

for $x_{11}, \xi_{11} > 0$, $x' \in \mathcal{P}_{\mathcal{V}'}$ and $\xi' \in \mathcal{P}^*_{\mathcal{V}'}$. By the induction hypothesis, we see that $\mathcal{I}^{\mathcal{V}}_{\underline{s}} : \mathcal{P}_{\mathcal{V}} \cap \mathcal{Z}^0_{\mathcal{V}} \to \mathcal{P}^*_{\mathcal{V}} \cap \mathcal{Z}^0_{\mathcal{V}}$ is bijective with the inverse map $\mathcal{J}^{\mathcal{V}}_{\underline{s}} : \mathcal{P}^*_{\mathcal{V}} \cap \mathcal{Z}^0_{\mathcal{V}} \to \mathcal{P}_{\mathcal{V}} \cap \mathcal{Z}^0_{\mathcal{V}}$.

If $q_1 = 0$, the statement holds because $\mathcal{Z}_{\mathcal{V}} = \mathcal{Z}^0_{\mathcal{V}}$. Assume $q_1 > 0$. Lemma 1 (ii) tells us that, for $x \in \mathcal{P}_{\mathcal{V}}$, there exist unique $x^0 \in \mathcal{Z}^0_{\mathcal{V}} \cap \mathcal{P}_{\mathcal{V}}$ and $B \in \mathcal{W}$ for which $x = \tau_B x^0$. Similarly, we see from (32) that, for $\xi \in \mathcal{P}^*_{\mathcal{V}}$, there exist unique $\xi^0 \in \mathcal{Z}^0_{\mathcal{V}} \cap \mathcal{P}^*_{\mathcal{V}}$ and $C \in \mathcal{W}$ for which $\xi = \tau^*_C \xi^0$. Therefore, we deduce from (41) and (42) that $\mathcal{I}^{\mathcal{V}}_{\underline{s}} : \mathcal{P}_{\mathcal{V}} \mapsto \mathcal{P}^*_{\mathcal{V}}$ is a bijection with $\mathcal{J}^{\mathcal{V}}_{\underline{s}}$ the inverse map. □

Proposition 1. *Let $\underline{s} \in \mathbb{R}^r_{>0}$. For $\xi \in \mathcal{P}^*_{\mathcal{V}}$, one has:*

$$\Delta_{-\underline{s}}(\mathcal{J}_{\underline{s}}(\xi))^{-1} = (\prod_{k=1}^{r} s_k^{s_k})\delta_{-\underline{s}}(\xi). \tag{45}$$

Proof. We prove the statement by induction on the rank. When $r = 1$, the equality (45) is verified directly. Indeed, the left-hand side of (45) is computed as $(\frac{s_1}{\xi_{11}})^{s_1} = s_1^{s_1}\xi_{11}^{-s_1}$.

When $r > 1$, assume that (45) holds for a system of rank $r - 1$. We deduce from (31), (33), (43), (44) and the induction hypothesis that (45) holds for $\xi \in \mathcal{P}^*_{\mathcal{V}} \cap \mathcal{Z}^0_{\mathcal{V}}$. Therefore, (45) holds for all $\xi \in \mathcal{P}^*_{\mathcal{V}}$ by (30), (34) and (42). □

In general, for a non-zero function f, the function $\frac{1}{f\circ(\nabla \log f)^{-1}}$ is called the multiplicative Legendre transform of f. Thanks to Theorem 5 and Proposition 1, we see that the multiplicative Legendre transform of $\Delta_{-\underline{s}}(x)$ is equal to $\delta_{-\underline{s}}(-\xi)$ on $-\mathcal{P}^*_{\mathcal{V}}$ up to constant multiple. As a corollary, we arrive at the following result.

Theorem 6. *The Fenchel–Legendre transform of the convex function $\log \Delta_{-\underline{s}}$ on $\mathcal{P}_{\mathcal{V}}$ is equal to the function $\log \delta_{-\underline{s}}(-\xi)$ of $\xi \in -\mathcal{P}^*$ up to constant addition.*

6. Application to Statistics and Optimization

Take $\underline{s} \in \mathbb{R}^r$ for which $s_k > q_k/2$ $(k = 1, \dots, r)$. We define a measure $\rho^{\mathcal{V}}_{\underline{s}}$ on $\mathcal{P}^*_{\mathcal{V}}$ by:

$$\rho^{\mathcal{V}}_{\underline{s}}(d\xi) := C_{\mathcal{V}}^{-1}\Gamma_{\mathcal{V}}(\underline{s})^{-1}\delta^{\mathcal{V}}_{\underline{s}}(\xi)\varphi_{\mathcal{V}}(\xi)\,d\xi \qquad (\xi \in \mathcal{P}^*_{\mathcal{V}}). \tag{46}$$

Theorem 4 states that:

$$\int_{\mathcal{P}^*_{\mathcal{V}}} e^{-(x|\xi)}\rho^{\mathcal{V}}_{\underline{s}}(d\xi) = \Delta^{\mathcal{V}}_{-\underline{s}}(x) \qquad (x \in \mathcal{P}_{\mathcal{V}}).$$

Then, we obtain the natural exponential family generated by $\rho^{\mathcal{V}}_{\underline{s}}$, that is a family $\{\mu^{\mathcal{V}}_{\underline{s},x}\}_{x\in\mathcal{P}_{\mathcal{V}}}$ of probability measures on $\mathcal{P}^*_{\mathcal{V}}$ given by:

$$\mu^{\mathcal{V}}_{\underline{s},x}(d\xi) := \Delta^{\mathcal{V}}_{\underline{s}}(x)e^{-(x|\xi)}\rho^{\mathcal{V}}_{\underline{s}}(d\xi).$$

In particular, when $\underline{s} = (n_1\alpha, n_2\alpha, \dots, n_r\alpha)$ for sufficiently large α, we have $\mu^{\mathcal{V}}_{\underline{s},x}(d\xi) = (\det x)^{\alpha}e^{-(x|\xi)}\rho^{\mathcal{V}}_{\underline{s}}(d\xi)$. We call $\mu^{\mathcal{V}}_{\underline{s},x}$ the Wishart distributions on $\mathcal{P}^*_{\mathcal{V}}$ in general.

From a sample $\xi_0 \in \mathcal{P}^*_{\mathcal{V}}$, let us estimate the parameter $x \in \mathcal{P}_{\mathcal{V}}$ in such a way that the likelihood function $\Delta^{\mathcal{V}}_{\underline{s}}(x)e^{-(x|\xi)}$ attains its maximum at the estimator x_0. Then, we have the likelihood equation $\xi_0 = \mathcal{I}^{\mathcal{V}}_{\underline{s}}(x_0)$, whereas Theorem 5 gives a unique solution by $x_0 = \mathcal{J}^{\mathcal{V}}_{\underline{s}}(\xi_0)$.

The same argument leads us to the following result in semidefinite programming. For a fixed $\xi_0 \in \mathcal{P}^*_{\mathcal{V}}$ and $\alpha > 0$, a unique solution x_0 of the minimization problem of $(x|\xi_0) - \alpha \log\det x$ subject to $x \in \mathcal{P}_{\mathcal{V}} = \mathcal{Z}_{\mathcal{V}} \cap \mathcal{P}_n$ is given by $x_0 = \mathcal{J}^{\mathcal{V}}_{\underline{s}}(\xi_0)$, where $\underline{s} = (n_1\alpha, \dots, n_r\alpha)$. Note that $\mathcal{J}^{\mathcal{V}}_{\underline{s}}$ is a rational map because $\delta^{\mathcal{V}}_{\underline{s}}$ is a product of powers of rational functions.

7. Special Cases

7.1. Matrix Realization of Homogeneous Cones

Let us assume that the system $\mathcal{V} = \{\mathcal{V}_{lk}\}_{1\le k<l\le r}$ satisfies not only the conditions (V1) and (V2), but also the following:

(V3) $A \in \mathcal{V}_{lk}$, $B \in \mathcal{V}_{kj} \Rightarrow AB \in \mathcal{V}_{lj}$ $\quad (1 \le j < k < l \le r)$.

Then, the set $H_{\mathcal{V}}$ of lower triangular matrices T of the form:

$$T = \begin{pmatrix} T_{11} & & & \\ T_{21} & T_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & \dots & T_{rr} \end{pmatrix}$$

becomes a linear Lie group, and $H_{\mathcal{V}}$ acts on the space $\mathcal{Z}_{\mathcal{V}}$ by $\rho(T)x := Tx\,{}^{\mathrm{t}}T$ $(T \in H_{\mathcal{V}},\ x \in \mathcal{Z}_{\mathcal{V}})$. The group $H_{\mathcal{V}}$ acts on the cone $\mathcal{P}_{\mathcal{V}}$ simply transitively by this action ρ, so that $\mathcal{P}_{\mathcal{V}}$ is a homogeneous cone. Moreover, it is shown in [15] that every homogeneous cone is linearly isomorphic to such $\mathcal{P}_{\mathcal{V}}$ (see also [18]).

Let $\mathcal{V}^0 = \{\mathcal{V}^0_{lk}\}_{1\le k<l\le 3}$ be the system given by $\mathcal{V}^0_{21} = \{0\}$ and $\mathcal{V}^0_{lk} = \mathbb{R}$ $((l,k) \neq (2,1))$. Then:

$$\mathcal{Z}_{\mathcal{V}^0} = \left\{ \begin{pmatrix} x_1 & 0 & x_4 \\ 0 & x_2 & x_5 \\ x_4 & x_5 & x_3 \end{pmatrix} ; x_1, \dots, x_5 \in \mathbb{R} \right\},$$

and $\mathcal{P}_{\mathcal{V}^0} := \mathcal{Z}_{\mathcal{V}^0} \cap \mathcal{P}_3$ is homogeneous because (V1)–(V3) are satisfied in this case. On the other hand, let $\mathcal{V}^1 = \{\mathcal{V}^1_{lk}\}_{1\le k<l\le 3}$ be the system given by $\mathcal{V}^1_{31} = \{0\}$ and $\mathcal{V}^1_{lk} = \mathbb{R}$ $((l,k) \neq (3,1))$. Then:

$$\mathcal{Z}_{\mathcal{V}^1} = \left\{ \begin{pmatrix} x_1 & x_4 & 0 \\ x_4 & x_2 & x_5 \\ 0 & x_5 & x_3 \end{pmatrix} ; x_1, \dots, x_5 \in \mathbb{R} \right\}.$$

Note that $\mathcal{V}^1$ satisfies only (V1) and (V2), but $\mathcal{P}_{\mathcal{V}^1}$ is homogeneous because $\mathcal{P}_{\mathcal{V}^1}$ is isomorphic to the homogeneous cone $\mathcal{P}_{\mathcal{V}^0}$ via the map:

$$\mathcal{P}_{\mathcal{V}^1} \ni \begin{pmatrix} x_1 & x_4 & 0 \\ x_4 & x_2 & x_5 \\ 0 & x_5 & x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_4 & 0 \\ x_4 & x_2 & x_5 \\ 0 & x_5 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} x_1 & 0 & x_4 \\ 0 & x_3 & x_5 \\ x_4 & x_5 & x_2 \end{pmatrix} \in \mathcal{P}_{\mathcal{V}^0}.$$

This example tells us that our matrix realization of a convex cone is not unique and that the condition (V3) is merely a sufficient condition for the homogeneity of the cone.

Many ideas in this work are inspired by the theory of homogeneous cones. The notion of generalized power functions, as well as the Γ-type integral formulas are due to Gindikin [8] (see also [23]). The Wishart distributions for homogeneous cones are studied in [17,21,24,25].

7.2. Cones Associated with Chordal Graphs

If $n_1 = n_2 = \cdots = n_r = 1$, then $\mathcal{V}_{lk}$ equals either $\mathbb{R}$ or $\{0\}$. In this case, $\mathcal{Z}_{\mathcal{V}}$ is the space of symmetric matrices with prescribed zero components. Such a space is described by using an undirected graph in the graphical model theory.

Let us recall some notion in the graph theory. Let G be a graph and V_G the set of vertices of G. We assume that G has no multiple edge, that is, for any two vertices $i, j \in V_G$, either there is one edge connecting them or there is no edge between them. These relations of the vertices i and j are denoted by $i \sim j$ and $i \not\sim j$, respectively. Assume further that G has no loop, which means that $i \not\sim i$ for $i \in V_G$. We define the edge set $E_G \subset V_G \times V_G$ by:

$$E_G := \{\, (i,j) \in V_G \times V_G\,;\, i \sim j \,\}.$$

Since V_G and E_G have all of the information of G, the graph G is often identified with the pair (V_G, E_G). For a non-empty subset V' of V_G, put $E' := E_G \cap (V' \times V')$. The graph $G' := (V', E')$ is called an induced subgraph of G. The graph G is said to be chordal or decomposable if G contains no cycle of length greater than three as an induced subgraph, and said to be A_4-free if G contains no A_4 graph $\bullet - \bullet - \bullet - \bullet$ as an induced subgraph. Let $\preceq$ be a total order on the vertex set V_G, and for $i \in V_G$, put $V_G^{[i]} := \{ j \in V_G ; i \sim j \text{ and } i \preceq j \} \subset V_G$. Then, $\preceq$ is said to be an eliminating order on the graph G if the induced subgraph with the vertex set $V_G^{[i]}$ is complete for each $i \in V_G$. It is known that there exists an eliminating order on G if and only if the graph G is chordal.

Let us identify the vertex set V_G with $\{1, 2, \ldots, r\}$. Let $\mathcal{Z}_G$ be the space of symmetric matrices $x = (x_{ij}) \in \mathrm{Sym}(r, \mathbb{R})$, such that, if $i \neq j$ and $i \not\sim j$, then $x_{ij} = 0$. Define $\mathcal{P}_G := \mathcal{Z}_G \cap \mathcal{P}_r$. We can show ([11] (Theorem 2.2), [26]) that the cone $\mathcal{P}_G$ is homogeneous if and only if the graph G is chordal and A_4-free. On the other hand, it is known in the graphical model theory as well as the sparse matrix linear algebra that even though $\mathcal{P}_G$ is not homogeneous, various formulas still hold for $\mathcal{P}_G$ if G is chordal.

The cone $\mathcal{P}_G$ is expressed as $\mathcal{P}_{\mathcal{V}}$ with $n_1 = n_2 = \cdots = n_r = 1$ and:

$$\mathcal{V}_{ji} = \begin{cases} \mathbb{R} & (j \sim i), \\ \{0\} & (j \not\sim i). \end{cases}$$

Then, the condition (V2) means exactly that the order $\leq$ is an eliminating order on G. Therefore, any cone $\mathcal{P}_G$ with chordal G can be treated as $\mathcal{P}_{\mathcal{V}}$ in our framework. Most of the integral formulas for $\mathcal{P}_G$ in [11,27] can be deduced from Theorems 3 and 4, while the Wishart distribution is a central object in the theory of graphical model. In [28], the analysis for generalized power functions associated with all eliminating orders is discussed for a specific graph $A_n : \bullet - \bullet - \cdots - \bullet$ by direct computations.

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