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Existence of Solutions to a Nonlinear Parabolic Equation of Fourth-Order in Variable Exponent Spaces

Bo Liang [1,*], Xiting Peng [1] and Chengyuan Qu [2]

[1] School of Science, Dalian Jiaotong University, Dalian 116028, China; Pengxiting2030@163.com
[2] Department of Mathematics, Dalian Nationalities University, Dalian 116600, China; mathqcy@163.com
* Correspondence: lining@djtu.edu.cn; Tel.: +86-411-8410-6838

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Abstract: This paper is devoted to studying the existence and uniqueness of weak solutions for an initial boundary problem of a nonlinear fourth-order parabolic equation with variable exponent $v_t + \mathrm{div}(|\nabla\triangle v|^{p(x)-2}\nabla\triangle v) - |\triangle v|^{q(x)-2}\triangle v = g(x,v)$. By applying Leray-Schauder's fixed point theorem, the existence of weak solutions of the elliptic problem is given. Furthermore, the semi-discrete method yields the existence of weak solutions of the corresponding parabolic problem by constructing two approximate solutions.

Keywords: fourth-order parabolic equation; semi-discretization; variable exponent

1. Introduction

We mainly study the following fourth-order parabolic equations with variable exponents:

$$v_t + \mathrm{div}(|\nabla\triangle v|^{p(x)-2}\nabla\triangle v) - |\triangle v|^{q(x)-2}\triangle v = g(x,v), \qquad (x,t) \in \Omega_T, \tag{1}$$

$$v(x,t) = \triangle v(x,t) = 0, \qquad (x,t) \in \Gamma_T, \tag{2}$$

$$v(x,0) = v_0(x), \qquad x \in \Omega, \tag{3}$$

where Ω is an open, bounded domain in $\mathbb{R}^N$, $\partial\Omega \in C^1$. Define $\Omega_T = \Omega \times (0,T)$ and $\Gamma_T = \partial\Omega \times (0,T)$. If p is a constant (especially $p \equiv 2$ and $q \equiv 2$), the Equation (1) has the structure of the classical Cahn–Hilliard problem, which is often used to describe the evolution of a conserved concentration field during phase separation in physics. It is also related to the thin-film equation if $|\nabla\triangle v|^{p(x)-2}$ becomes v^p, which can analyze the motion of a very thin layer of viscous incompressible fluids along an include plane.

There have been some results related to the existence, uniqueness and properties of solutions to the fourth-order degenerate parabolic equations (see [1,2]). The paper [3] has studied the existence of the Cahn–Hilliard equation and the reader may refer to [4] to obtain its physical background. For the constant exponent case of (1), the paper [5] has given the existence and uniqueness of weak solutions. For the problems in variable exponent spaces, the papers [6–8] have studied the existence of some fourth-order parabolic equations with a variable exponent, and [9] has given the Fujita type conditions for fast diffusion equation.

For the research of the existence and long-time behavior of the fourth-order partial differential equations, the entropy functional method is often applied in order to obtain the necessary estimates and to show the entropy dissipation. The large time behavior of solutions of the thin film equation $u_t + (u^n u_{xxx})_x = 0$ was addressed in [10,11] by the entropy function method. For $0 < n < 3$, [12] proved the existence of (1) in the distributional sense and obtained the exponentially fast convergence in L^∞-norm via the entropy method of a regularized problem. We apply the idea of the entropy method to deal with the corresponding problems with variable exponents.

In this paper, we apply the Leray-Schauder's fixed point theorem to prove the existence of weak solutions of the corresponding elliptic problem of (1)–(3) in order to deal with the nonlinear source. Furthermore, the semi-discrete method yields the existence of weak solutions of the parabolic problem by constructing two approximate solutions. We will show the effect of the variable exponents and the second-order nonlinear diffusion to the degenerate parabolic Equation (1).

1.1. Preliminaries

We introduce some elementary concepts and lemmas related to the variable exponent spaces in this part.

Let $p(x) \geq 1$ be a continuous function in $\overline{\Omega}$ and we define the variable exponent space as follows:

$$\mathcal{L}^{p(x)}(\Omega) = \left\{ v(x) : v \text{ is measureable and } \mathbb{T}_{p(.)}(v) = \int_{\Omega} |v|^{p(x)} \mathrm{d}x < \infty \right\}$$

with the norm

$$\|v\|_{p(x)} = \inf \left\{ \gamma > 0 : \mathbb{T}_{p(.)}\left(\frac{v}{\gamma}\right) \leq 1 \right\}.$$

It is easy to check that the variable exponent space $\mathcal{L}^{p(x)}(\Omega)$ becomes the classical Lebesgue space $\mathcal{L}^{p}(\Omega)$ when $p(x)$ is a positive constant.

For convenience, we list some definitions and notations of the generalized Lebesgue–Sobolev space $\mathcal{W}^{k,p(x)}(\Omega)$:

$$\begin{aligned}
&\mathcal{W}^{k,p(x)}(\Omega) =: \left\{ v(x) \in \mathcal{L}^{p(x)}(\Omega) : D^{\alpha}v \in \mathcal{L}^{p(x)}(\Omega), |\alpha| \leq k \right\}, \\
&\|v\|_{\mathcal{W}^{k,p(x)}} =: \sum_{|\alpha| \leq k} |D^{\alpha}v|_{p(x)}, \\
&E_1 =: \{ v \in \mathcal{W}_0^{1,p(x)}(\Omega) \cap \mathcal{W}^{2,p(x)}(\Omega) \cap \mathcal{W}^{2,q(x)}(\Omega) | \triangle v \in \mathcal{W}_0^{1,p(x)}(\Omega) \}, \\
&E_2 =: \{ v \in \mathcal{H}_0^{1}(\Omega) \cap \mathcal{W}_0^{1,p(x)}(\Omega) \cap \mathcal{W}^{2,p(x)}(\Omega) \cap \mathcal{W}^{2,q(x)}(\Omega) | \triangle v \in \mathcal{W}_0^{1,p(x)}(\Omega) \}, \\
&\mathcal{L}^{p'(x)}(\Omega) \text{ denotes the dual space with } \frac{1}{p'(x)} + \frac{1}{p(x)} = 1.
\end{aligned}$$

Moreover, $\mathcal{W}_0^{k,p(x)}$ denotes the closure of $C_0^{\infty}(\Omega)$ in $\mathcal{W}^{k,p(x)}(\Omega)-$norm, $\mathcal{W}_0^{-1,p'(x)}(\Omega)$ denotes the dual space of $\mathcal{W}_0^{1,p(x)}(\Omega)$. For any positive continuous function $\theta(x)$, we define

$$\theta^{-} = \inf_{x \in \overline{\Omega}} \theta(x), \theta^{+} = \sup_{x \in \overline{\Omega}} \theta(x).$$

Throughout the paper, C and $C_i (i = 1, 2, 3, ...)$ denote the general positive constants independent of solutions and may change from line to line.

In the following, we list some known results for the variable exponent spaces (see [13,14]).

Lemma 1. *Letting $f \in \mathcal{L}^{p(x)}(\Omega)$, one has*

$$\begin{aligned}
&(1)\ \|f\|_{p(x)} < 1 (= 1; > 1) \Longleftrightarrow \mathbb{T}_{p(.)}(f) < 1 (= 1; > 1); \\
&(2)\ \|f\|_{p(x)} < 1 \Longrightarrow \|f\|_{p(x)}^{p^+} \leq \mathbb{T}_{p(.)}(f) \leq \|f\|_{p(x)}^{p^-}; \\
&\quad\ \|f\|_{p(x)} \geq 1 \Longrightarrow \|f\|_{p(x)}^{p^-} \leq \mathbb{T}_{p(.)}(f) \leq \|f\|_{p(x)}^{p^+}; \\
&(3) \|f\|_{p(x)} \to 0 \Longleftrightarrow \mathbb{T}_{p(.)}(f) \to 0; \|f\|_{p(x)} \to \infty \Longleftrightarrow \mathbb{T}_{p(.)}(f) \to \infty.
\end{aligned}$$

Lemma 2. *(Poincaré's inequality) Letting $f \in \mathcal{W}_0^{1,p(x)}(\Omega)$, there exists a positive constant C such that $\|f\|_{p(x)} \leq C\|\nabla f\|_{p(x)}$.*

Lemma 3. *(Hölder's inequality) Letting* $f \in \mathcal{L}^{p(x)}(\Omega)$ *and* $g \in \mathcal{L}^{p'(x)}(\Omega)$*, one has* $\left|\int_\Omega fg\mathrm{d}x\right| \leq \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|f\|_{p(x)}\|g\|_{p'(x)} \leq 2\|f\|_{p(x)}\|g\|_{p'(x)}$.

1.2. Results

In (1), we require that $p(x)$ and $q(x)$ are two continuous functions in $\overline{\Omega}$ and $p^-, q^- > 1$. Besides, the nonlinear source term $g(x,v) \in C^1(\Omega \times R)$ satisfies the growth condition:

$$|g(x,v)| \leq K|v|^{l(x)} + s(x), \qquad v \in (-\infty,+\infty), x \in \Omega, \tag{4}$$

where K is a positive constant, $l(x)$ is a continuous function in $\overline{\Omega}$ and $s(x) \in \mathcal{L}^{p'(x)}(\Omega)$. Furthermore, by letting $\pi(x) =: l(x)p'(x)$, we require that

$$\frac{\pi^+}{p^-} < 1, 0 \leq l(x) \leq \frac{Np(x)}{(N-p(x))p'(x)}. \tag{5}$$

The corresponding steady-state problem of (1)–(3) has the form:

$$\mathrm{div}(|\nabla\triangle v|^{p(x)-2}\nabla\triangle v) - |\triangle v|^{q(x)-2}\triangle v = g(x,v) \text{ in } \Omega, \tag{6}$$

$$v = \triangle v = 0 \text{ on } \partial\Omega. \tag{7}$$

The weak solution is defined in the following sense.

Definition 1. *A function* $v \in E_1$ *is said to be a weak solution of* (6) and (7) *provided that*

$$-\int_\Omega |\nabla\triangle v|^{p(x)-2}\nabla\triangle v\nabla\phi\mathrm{d}x - \int_\Omega |\triangle v|^{q(x)-2}\triangle v\phi\mathrm{d}x = \int_\Omega g(x,v)\phi\mathrm{d}x \tag{8}$$

for each $\phi \in \mathcal{W}_0^{1,p(x)}(\Omega) \cap \mathcal{L}^{q(x)}(\Omega)$ *and* $g(x,v(x)) \in \mathcal{L}^{p'(x)}(\Omega)$.

The following theorem gives the existence of solutions.

Theorem 1. *Let* $v_0 \in \mathcal{W}_0^{1,p(x)}(\Omega)$*. There exists at least a weak solution of* (6) and (7) *satisfying Definition 1.*

For the evolution equation case, we define the weak solution of (1)–(3) as following.

Definition 2. *A function* v *is said to be a weak solution of* (1)–(3) *provided that*

(i) $v \in C([0,T];\mathcal{L}^2(\Omega_T)) \cap \mathcal{L}^{p^-}(0,T;\mathcal{W}_0^{1,p(x)}(\Omega)) \cap \mathcal{L}^{q^-}(0,T;\mathcal{W}^{2,q(x)}(\Omega))$,
$\triangle v \in \mathcal{L}^{p'^-}(0,T;\mathcal{W}_0^{1,p(x)}(\Omega))$, $\frac{\partial v}{\partial t} \in \mathcal{L}^{p'^-}(0,T;\mathcal{W}^{-1,p(x)}(\Omega))$, $v(x,0) = v_0(x)$ *a.e. in* Ω;
(ii) *For any* $\phi \in \mathcal{L}^{p^+}(0,T;\mathcal{W}_0^{1,p(x)}(\Omega)) \cap \mathcal{L}^{q^+}(0,T;\mathcal{W}^{2,q(x)}(\Omega))$*, one has*

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, \phi \right\rangle \mathrm{d}t = \int_0^T \int_\Omega |\nabla\triangle v|^{p(x)-2}\nabla\triangle v\nabla\phi\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\triangle v|^{q(x)-2}\triangle v\phi\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega g(x,v)\phi\mathrm{d}x\mathrm{d}t.$$

The existence of solutions is the following theorem.

Theorem 2. *Let* $p'^- > 1$, $q'^- > 1$, $v_0 \in \mathcal{H}_0^1(\Omega) \cap \mathcal{W}_0^{1,p(x)}(\Omega)$ *and* $\triangle v_0 \in \mathcal{W}_0^{1,p(x)}(\Omega)$*. There exists at least a weak solution of (1)–(3).*

Moreover, the solution of (1)–(3) is unique when $g(x,v) = \mu(v(x,t) - b(x))$ *where* μ *is a constant and* $b(x) \in \mathcal{L}^{p'(x)}(\Omega)$.

This paper is organized as follows. In Section 2, we prove the existence and uniqueness of weak solution to the steady-state problem by using Leray-Schauder's fixed point theorem. In Section 3, we prove the existence of the solution to an evolution equation by applying the semi-discrete method with necessary uniform estimates.

2. Steady-State Problem

In order to apply the fixed point theorem, we consider a steady-state problem with the source $g(x)$:

$$\mathrm{div}(|\nabla\triangle v|^{p(x)-2}\nabla\triangle v) - |\triangle v|^{q(x)-2}\triangle v = g(x) \text{ in } \Omega, \tag{9}$$

$$v = \triangle v = 0 \text{ on } \partial\Omega. \tag{10}$$

By constructing an energy functional and obtaining its minimizer, we have the following existence of weak solutions.

Lemma 4. *Let $g \in \mathcal{L}^{p'(x)}(\Omega)$. There exists a unique weak solution $v \in E_1$ of* (9) and (10) *satisfying*

$$\begin{aligned}&-\int_\Omega |\nabla\triangle v|^{p(x)-2}\nabla\triangle v\nabla\triangle\phi \mathrm{d}x - \int_\Omega |\triangle v|^{q(x)-2}\triangle v\triangle\phi \mathrm{d}x\\ =&\int_\Omega g(x)\triangle\phi \mathrm{d}x\end{aligned} \tag{11}$$

for any $\phi \in E_1$.

Proof. Introduce a functional

$$F(v) = \int_\Omega \frac{1}{p(x)}|\nabla\triangle v|^{p(x)}\mathrm{d}x + \int_\Omega \frac{1}{q(x)}|\triangle v|^{q(x)}\mathrm{d}x - \int_\Omega g\triangle v\mathrm{d}x. \tag{12}$$

For the last term, Hölder's inequality, the Young inequality, the Sobolev embedding theorem (see [15]) and the $\mathcal{L}^p$-theory of the second-order elliptic equation (see [16]) gives

$$\begin{aligned}\left|\int_\Omega g\triangle v\mathrm{d}x\right| &\le \|g\|_{p'(x)}\|\triangle v\|_{p(x))}\\ &\le C\left(\frac{1}{\varepsilon}\right)\max\left\{\|g\|_{p'(x)}^{p'^-}, \|g\|_{p'(x)}^{p'^+}\right\} + \varepsilon\int_\Omega |\triangle v|^{p(x)}\mathrm{d}x\\ &\le C\max\left\{\|g\|_{p'(x)}^{p'^-}, \|g\|_{p'(x)}^{p'^+}\right\} + \frac{1}{2}\int_\Omega \frac{1}{p(x)}|\nabla\triangle v|^{p(x)}\mathrm{d}x.\end{aligned} \tag{13}$$

On the other hand, (12) implies

$$-C \le \inf_{\nu\in E_1} F(\nu) \le F(0) = 0.$$

Hence there exists a sequence $\{v_k\}_{k=1}^\infty \in E_1$ such that

$$F(v_k) \to \inf_{\nu\in E_1} F(\nu)(k\to\infty). \tag{14}$$

Equations (13) and (14) give

$$\|\nabla\triangle v_k\|_{p(x)} \le C, \|\triangle v_k\|_{q(x)} \le C,$$

which implies that $F(v_k)$ is bounded and thus Lemmas 1–3 yield

$$\begin{aligned}&\|v_k\|_{2,p(x)} \leq \|\triangle v_k\|_{p(x)} \leq C\|\nabla\triangle v_k\|_{p(x)}\\ &\leq C_1 \max\left\{\left(\int_\Omega |\nabla\triangle v_k|^{p(x)}\right)^{\frac{1}{p^-}}, \left(\int_\Omega |\nabla\triangle v_k|^{p(x)}\right)^{\frac{1}{p^+}}\right\} \leq C_2,\end{aligned}$$

and

$$\begin{aligned}&\|v_k\|_{2,q(x)} \leq \|\triangle v_k\|_{q(x)}\\ &\leq C_3 \max\left\{\left(\int_\Omega |\triangle v_k|^{q(x)}\right)^{\frac{1}{q^-}}, \left(\int_\Omega |\triangle v_k|^{q(x)}\right)^{\frac{1}{q^+}}\right\} \leq C_4.\end{aligned}$$

It shows that v_k belongs to the space $\mathcal{W}_0^{1,p(x)} \bigcap \mathcal{W}^{2,p(x)} \bigcap \mathcal{W}^{2,q(x)}$ uniformly, and then there exists a function $v \in E_1$ such that

$$\begin{aligned}v_k &\rightharpoonup v \text{ weakly in } \mathcal{W}_0^{1,p(x)}(\Omega) \cap \mathcal{W}^{2,p(x)}(\Omega) \cap \mathcal{W}^{2,q(x)}(\Omega),\\ \triangle v_k &\rightharpoonup \triangle v \text{ weakly in } \mathcal{W}_0^{1,p(x)}(\Omega) \cap \mathcal{L}^{q(x)}(\Omega).\end{aligned}$$

Furthermore, since $F(v)$ is weakly lower semi-continuous on E_1, we have

$$\inf_{\nu \in E_1} F(\nu) \leq F(v) \leq \lim_{k\to\infty} \inf F(v_k) = \inf_{\nu\in E_1} F(\nu),$$

i.e., v is a minimizer of $F(\cdot)$ and $F(v) = \inf_{\nu\in E_1} F(\nu)$. It guarantees that v is a weak solution of (9) and (10).

The uniqueness is obvious and we omit the details. □

Now, we consider the problem (6) and (7) with the nonlinear source $g(x, v)$.

Lemma 5. *Letting $v(x) \in E_1$ be a weak solution of (6) and (7), one has $\|v\|_{E_1} \leq C$.*

Proof. Multiplying (8) by v gives

$$\begin{aligned}&\int_\Omega |\nabla\triangle v|^{p(x)}\mathrm{d}x + \int_\Omega |\triangle v|^{q(x)}\mathrm{d}x\\ =& -\int_\Omega g(x,v)\triangle v\mathrm{d}x \leq 2\|g\|_{p'(x)}\|\triangle v\|_{p(x)}\\ \leq& C\|g\|_{p'(x)}\|\nabla\triangle v\|_{p(x)}\\ \leq& \frac{1}{4}\int_\Omega |\nabla\triangle v|^{p(x)}\mathrm{d}x + C\max\{\|g\|_{p'(x)}^{p'^-}, \|g\|_{p'(x)}^{p'^+}\}.\end{aligned} \tag{15}$$

By Lemmas 2 and 3 and $\mathcal{L}^p$-estimate (see [16]), we conclude that

$$\begin{aligned}\int_\Omega |g(x,v)|^{p'(x)}\mathrm{d}x &\leq C\int_\Omega (K|v|^{l(x)} + s(x))^{p'(x)}\mathrm{d}x\\ &\leq CM^{p'^+}\int_\Omega |v|^{l(x)p'(x)}\mathrm{d}x + C\int_\Omega |s(x)|^{p'(x)}\mathrm{d}x + C\\ &\leq C\|v\|_{\mathcal{W}^{1,p(x)}}^{l(x)p'(x)} + C \leq C\|\triangle v\|_{p(x)}^{l(x)p'(x)} + C\\ &\leq C\|\nabla\triangle v\|_{p(x)}^{l(x)p'(x)} + C,\end{aligned} \tag{16}$$

and thus

$$\max\{\|g\|_{p'(x)}^{p'^-}, \|g\|_{p'(x)}^{p'^+}\} \leq C\max\{\|\nabla\triangle v\|_{p(x)}^{\pi^+}, \|\nabla\triangle v\|_{p(x)}^{\pi^-}\} + C$$
$$\leq \frac{1}{4}\int_\Omega |\nabla\triangle v|^{p(x)}\mathrm{d}x + C. \tag{17}$$

Equations (15)–(17) yield

$$\int_\Omega |\nabla\triangle v|^{p(x)}\mathrm{d}x + \int_\Omega |\triangle v|^{q(x)}\mathrm{d}x \leq C. \tag{18}$$

It completes the proof of Lemma 5. □

Proof of Theorem 1. Letting $\omega \in \mathcal{L}^{p^*(x)}(\Omega)$ and $\delta \in [0,1]$ where we choose $p^*(x)$ such that $E_1 \hookrightarrow \mathcal{L}^{p^*(x)}(\Omega)$ is compact, we consider the auxiliary problem

$$\mathrm{div}(|\nabla\triangle v|^{p(x)-2}\nabla\triangle v) - |\triangle v|^{q(x)-2}\triangle v = \delta g(x,\omega) \text{ in } \Omega,$$
$$v = \triangle v = 0 \text{ on } \partial\Omega.$$

Lemma 4 ensures its existence and so we can define the fixed point operator

$$T : [0,1]\times\mathcal{L}^{p^*(x)}(\Omega) \longrightarrow \mathcal{L}^{p^*(x)}(\Omega),$$
$$(\delta,\omega) \longmapsto v$$

and $T(\omega,0) = 0$.

If $\omega \in \mathcal{L}^{p^*(x)}(\Omega)$ satisfies $T(\omega,\delta) = \omega$, we can check that $\|\omega\|_{E_1} \leq C$ where $C > 0$ is independence of ω and δ from the idea of Lemma 5. The compact embedding $E_1 \hookrightarrow \mathcal{L}^{p^*(x)}(\Omega)$ can ensure that T is a continuous and compact operator. Leray-Schauder's fixed point theorem yields the existence of solutions of (6) and (7). □

3. Evolution Equation

In this section, we study the existence solutions of (1)–(3). For this purpose, we establish a semi-discrete problem at first:

$$\frac{1}{h}(v_k - v_{k-1}) + \mathrm{div}(|\nabla\triangle v_k|^{p(x)-2}\nabla\triangle v_k) - |\triangle v_k|^{q(x)-2}\triangle v_k$$
$$= g(x, v_{k-1}), x\in\Omega, \tag{19}$$
$$v_k(x,t) = \triangle v_k(x,t) = 0, x\in\partial\Omega, \tag{20}$$

where $v_k = v(x,kh), h = \frac{T}{n}, k = 1,2,\cdots,n$ and $n\in\mathbb{N}$.

Lemma 6. *Assume $v_0 \in E_2$.* (19) and (20) *admits a unique weak solution $v_k \in E_2$ satisfying*

$$\sum_{k=1}^{i}\int_\Omega |\nabla v_k|^2\mathrm{d}x + h\sum_{k=1}^{i}\int_\Omega |\nabla\triangle v_k|^{p(x)}\mathrm{d}x + h\sum_{k=1}^{i}\int_\Omega |\triangle v_k|^{q(x)}\mathrm{d}x$$
$$\leq \sum_{k=1}^{i}\int_\Omega |\nabla v_{k-1}|^2\mathrm{d}x + \frac{h}{2}\sum_{k=1}^{i}\int_\Omega |\nabla\triangle v_{k-1}|^{p(x)}\mathrm{d}x + CT, \tag{21}$$

and

$$\int_\Omega |\nabla v_i|^2 \mathrm{d}x + \frac{h}{2}\sum_{k=1}^{i}\int_\Omega |\nabla \triangle v_i|^{p(x)}\mathrm{d}x + \frac{h}{2}\int_\Omega |\nabla \triangle v_i|^{p(x)}\mathrm{d}x$$
$$+ h\sum_{k=1}^{i}\int_\Omega |\triangle v_k|^{q(x)}\mathrm{d}x$$
$$\leq \int_\Omega |\nabla v_0|^2 \mathrm{d}x + \frac{h}{2}\int_\Omega |\nabla \triangle v_0|^{p(x)}\mathrm{d}x + CT. \tag{22}$$

Proof. According to the argument of the Section 2, we conclude that the problem (19) and (20) has a unique weak solution $v_k \in E_2$ satisfying

$$\frac{1}{h}\int_\Omega \nabla(v_k - v_{k-1})\cdot\nabla\phi \mathrm{d}x + \int_\Omega |\nabla\triangle v_k|^{p(x)-2}|\nabla\triangle v_k|\cdot\nabla\triangle\phi \mathrm{d}x$$
$$+ \int_\Omega |\triangle v_k|^{q(x)-2}\triangle v_k \triangle\phi \mathrm{d}x = -\int_\Omega g(x, v_{k-1})\triangle\phi \mathrm{d}x \tag{23}$$

for any$\phi \in C_0^\infty(\Omega)$. Letting $\phi = v_k$ in (23), we have

$$\frac{1}{h}\int_\Omega |\nabla v_k|^2\mathrm{d}x + \int_\Omega |\nabla\triangle v_k|^{p(x)}\mathrm{d}x + \int_\Omega |\triangle v_k|^{q(x)}\mathrm{d}x$$
$$= \frac{1}{h}\int_\Omega \nabla v_k\cdot\nabla v_{k-1}\mathrm{d}x - \int_\Omega g(x, v_{k-1})\triangle v_k \mathrm{d}x$$
$$\leq \frac{1}{2h}\int_\Omega |\nabla v_{k-1}|^2\mathrm{d}x + \frac{1}{2h}\int_\Omega |\nabla v_k|^2\mathrm{d}x - \int_\Omega g(x, v_{k-1})\triangle v_k \mathrm{d}x. \tag{24}$$

Similar to the proof of (16) and (17), we get

$$-\int_\Omega g(x, v_{k-1})\triangle v_k\mathrm{d}x$$
$$\leq \frac{1}{2}\int_\Omega |\nabla\triangle v_k|^{p(x)}\mathrm{d}x + C\int_\Omega |g(x, v_{k-1})|^{p'(x)}\mathrm{d}x$$
$$\leq \frac{1}{2}\int_\Omega |\nabla\triangle v_k|^{p(x)}\mathrm{d}x + \frac{1}{4}\int_\Omega |\nabla\triangle v_{k-1}|^{p(x)}\mathrm{d}x + C. \tag{25}$$

By (24) and (25), one has

$$\frac{1}{2h}\int_\Omega |\nabla v_k|^2\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\triangle v_k|^{p(x)}\mathrm{d}x + \int_\Omega |\triangle v_k|^{q(x)}\mathrm{d}x$$
$$\leq \frac{1}{2h}\int_\Omega |\nabla v_{k-1}|^2\mathrm{d}x + \frac{1}{4}\int_\Omega |\nabla\triangle v_{k-1}|^{p(x)}\mathrm{d}x + C. \tag{26}$$

Hence, for any $1 \leq i \leq n$, we obtain

$$\sum_{k=1}^{i}\int_\Omega |\nabla v_k|^2\mathrm{d}x + h\sum_{k=1}^{i}\int_\Omega |\nabla\triangle v_k|^{p(x)}\mathrm{d}x + h\sum_{k=1}^{i}\int_\Omega |\triangle v_k|^{q(x)}\mathrm{d}x$$
$$\leq \sum_{k=1}^{i}\int_\Omega |\nabla v_{k-1}|^2\mathrm{d}x + \frac{h}{2}\sum_{k=1}^{i}\int_\Omega |\nabla\triangle v_{k-1}|^{p(x)}\mathrm{d}x + CT. \tag{27}$$

It completes the proof of (21) and (22) obtained from (21). □

Now, we are in the position to define the first approximate solution of (1)–(3)

$$\varpi^{(n)}(x,t) = \sum_{k=1}^{n}\chi_k(t)v_k(t) \tag{28}$$

where $\chi_k(t)$ is the characteristic function over the interval $((k-1)h, kh]$ for $k = 1, 2, ..., n$. For this approximate solution, we have the following uniform estimates.

Lemma 7. *One has*

$$\begin{aligned}
&\|\varpi^{(n)}\|_{\mathcal{L}^\infty(0,T;\mathcal{H}_0^1(\Omega))} + \|\triangle\varpi^{(n)}\|_{\mathcal{L}^{p^-}(0,T;\mathcal{W}_0^{1,p(x)}(\Omega))} \\
&+ \||\nabla\triangle\varpi^{(n)}|^{p(x)-2}\nabla\triangle\varpi^{(n)}\|_{\mathcal{L}^{p'^-}(\Omega_T)} + \||\triangle\varpi^{(n)}|^{q(x)-2}\triangle\varpi^{(n)}\|_{\mathcal{L}^{q'}(\Omega_T)} \\
&+ \|g(\cdot, \varpi^{(n)})\|_{\mathcal{L}^{p'^-}(\Omega_T)} \le C. \qquad (29)
\end{aligned}$$

Proof. By Lemma 6 and

$$\|\nabla\varpi^{(n)}\|^2_{\mathcal{L}^2(\Omega)} = \|\nabla v_k(x)\|^2_{\mathcal{L}^2(\Omega)} \le C, \qquad (30)$$

we have the estimate

$$\|\nabla\varpi^{(n)}\|_{\mathcal{L}^\infty(0,T;\mathcal{L}^2(\Omega))} \le C. \qquad (31)$$

On the other hand, we have

$$\begin{aligned}
&\|\triangle\varpi^{(n)}\|_{\mathcal{L}^{p^-}(0,T;\mathcal{W}_0^{1,p(x)})} = \left(\int_0^T \|\triangle\varpi^{(n)}\|^{p^-}_{1,p(x)} \mathrm{d}x\right)^{\frac{1}{p^-}} \\
&= \left(h\sum_{k=1}^n \|\triangle v_k\|^{p^-}_{1,p(x)} \mathrm{d}x\right)^{\frac{1}{p^-}} \\
&\le \left(Ch\sum_{k=1}^n \|\nabla\triangle v_k\|^{p^-}_{p(x)} \mathrm{d}x\right)^{\frac{1}{p^-}} \le C\left(h\sum_{k=1}^n (\int_\Omega |\nabla\triangle v_k|^{p(x)} \mathrm{d}x + 1)\right)^{\frac{1}{p^-}} \\
&\le C(C+T)^{\frac{1}{p^-}}. \qquad (32)
\end{aligned}$$

Letting $i = n$ in (22), we get

$$\begin{aligned}
&\int_0^T \int_\Omega |\nabla\triangle\varpi^{(n)}|^{p(x)} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\triangle\varpi^{(n)}|^{q(x)} \mathrm{d}x\mathrm{d}t \\
=&h\sum_{k=1}^n \int_\Omega |\nabla\triangle v_k|^{p(x)} \mathrm{d}x + h\sum_{k=1}^n \int_\Omega |\triangle v_k|^{q(x)} \mathrm{d}x \le C.
\end{aligned}$$

□

Another approximate solution is defined as follows:

$$v^{(n)}(x,t) = \sum_{k=1}^n \chi_k(t)[\vartheta_k(t)v_k(x) + (1-\vartheta_k(t))v_{k-1}(x)], \qquad (33)$$

where

$$\vartheta_k(t) = \begin{cases} \frac{t}{h} - (k-1), & \text{if } t \in ((k-1)h, kh], \\ 0, & \text{otherwise.} \end{cases}$$

We also obtain some uniform estimates for this approximate solution.

Lemma 8. *One has*

$$\left\|\frac{\partial v^{(n)}}{\partial t}\right\|_{\mathcal{L}^{p'^-}(0,T;\mathcal{W}^{-1,p'(x)}(\Omega))} + \|v^{(n)}\|_{\mathcal{L}^\infty(0,T;\mathcal{H}_0^1(\Omega))} \le C. \qquad (34)$$

Proof. By (33), we get

$$\frac{\partial v^{(n)}}{\partial t} = \frac{1}{h}\sum_{k=1}^{n}\chi_k(v_k - v_{k-1}) \tag{35}$$

and then

$$\begin{aligned}
\int_0^T \left\|\frac{\partial v^{(n)}}{\partial t}\right\|^{p'^-}_{\mathcal{W}^{-1,p'(x)}(\Omega)}\mathrm{d}t \leq & \int_0^T \left|\left\langle \frac{\partial v^{(n)}}{\partial t}, \psi\right\rangle\right|^{p'^-}\mathrm{d}t \\
\leq & C\int_0^T \left|\sum_{k=1}^{n}\chi_k(t)\int_\Omega |\nabla\triangle v_k|^{p(x)-2}\nabla\triangle v_k\nabla\psi\mathrm{d}x\right|^{p'^-}\mathrm{d}t \\
& + C\int_0^T \left|\sum_{k=1}^{n}\chi_k(t)\int_\Omega g(x,v_{k-1})\psi\mathrm{d}x\right|^{p'^-}\mathrm{d}t \\
& + C\int_0^T \left|\sum_{k=1}^{n}\chi_k(t)\int_\Omega |\triangle v_k|^{q(x)-2}\triangle v_k\psi\mathrm{d}x\right|^{p'^-}\mathrm{d}t \\
\leq & C\|\nabla\triangle\varpi^{(n)}\|^{p'^-}_{\mathcal{L}^{p'^-}(\Omega_T)} + C\|g(x,v_{k-1})\|^{p'^-}_{\mathcal{L}^{p'^-}(\Omega_T)} \\
\leq & C,
\end{aligned}$$

for any $\psi \in \mathcal{W}_0^{1,p(x)}(\Omega)$ with $\|\psi\|_{\mathcal{W}_0^{1,p(x)}(\Omega)} \leq 1$. It follows from (22) and (33) that

$$\begin{aligned}
&\|v^{(n)}\|^m_{\mathcal{L}^m(0,T;\mathcal{H}_0^1(\Omega))} \\
\leq & C^m\int_0^T\left(\int_\Omega |\nabla v^{(n)}|^2\mathrm{d}x\right)^{\frac{m}{2}}\mathrm{d}t \\
= & C^m\int_0^T\left(\int_\Omega \left|\sum_{k=1}^{n}\chi_k(t)[\vartheta_k(t)\nabla v_k(x) + (1-\vartheta_k(t))\nabla v_{k-1}(x)]\right|^2\mathrm{d}x\right)^{\frac{m}{2}}\mathrm{d}t \\
= & C^m\sum_{k=1}^{n}\int_{(k-1)h}^{(kh)}\left(\int_\Omega \left|[\vartheta_k(t)\nabla v_k(x) + (1-\vartheta_k(t))\nabla v_{k-1}(x)]\right|^2\mathrm{d}x\right)^{\frac{m}{2}}\mathrm{d}t \\
\leq & C^m\sum_{k=1}^{n}h\left(\int_\Omega (|\nabla v_k(x)|^2 + |\nabla v_{k-1}(x)|^2)\mathrm{d}x\right)^{\frac{m}{2}} \\
\leq & C^{m+\frac{m}{2}}T
\end{aligned}$$

with $C > 0$ independent of m. Perform the limit $m \to \infty$ to get

$$\|v^{(n)}\|_{\mathcal{L}^\infty(0,T;\mathcal{H}_0^1(\Omega))} = \lim_{m\to\infty}\|v^{(n)}\|_{\mathcal{L}^m(0,T;\mathcal{H}_0^1(\Omega))} \leq C.$$

□

Proof of Theorem 2. By (29), we can seek a subsequence of $\varpi^{(n)}$ (still denoted by itself) and two functions $\nu \in \mathcal{L}^{p'(x)}(\Omega_T), \nu' \in \mathcal{L}^{q'(x)}(\Omega_T)$ such that

$$\begin{aligned}
&\varpi^{(n)} \rightharpoonup v \text{ weakly * in } \mathcal{L}^\infty(0,T;\mathcal{H}_0^1(\Omega)), \\
&\triangle\varpi^{(n)} \rightharpoonup \triangle v \text{ weakly in } \mathcal{L}^{p^-}(0,T;\mathcal{W}_0^{1,p}(\Omega)), \\
&|\nabla\triangle\varpi^{(n)}|^{p(x)-2}\nabla\triangle\varpi^{(n)} \rightharpoonup \nu \text{ weakly in } \mathcal{L}^{p'(x)}(\Omega_T), \\
&|\triangle\varpi^{(n)}|^{q(x)-2}\triangle\varpi^{(n)} \rightharpoonup \nu' \text{ weakly in } \mathcal{L}^{q'(x)}(\Omega_T),
\end{aligned}$$

as $n \to \infty$.

It is easy to check that there exists a positive integer r such that $\mathcal{W}^{-1,p'}(\Omega) \hookrightarrow \mathcal{H}^{-r}(\Omega)$ and thus the embedding $\mathcal{H}_0^1(\Omega) \overset{\text{compact}}{\hookrightarrow} \mathcal{L}^2(\Omega) \hookrightarrow \mathcal{H}^{-r}(\Omega)$, the uniform estimate (34) and the Aubin lemma [17] yield the existence of a subsequence of $v^{(n)}$ and a function ϱ such that, as $n \to \infty$,

$$\begin{aligned}
&\frac{\partial v^{(n)}}{\partial t} \rightharpoonup \frac{\partial \varrho}{\partial t} \text{ weakly in } \mathcal{L}^{p'^-}(0,T;\mathcal{W}^{-1,p(x)}(\Omega)),\\
&v^{(n)} \rightharpoonup \varrho \text{ weakly } * \text{ in } \mathcal{L}^{\infty}(0,T;\mathcal{H}_0^1(\Omega)),\\
&v^{(n)} \to \varrho \text{ strongly in } C([0,T];\mathcal{L}^2(\Omega)),\\
&v^{(n)} \to \varrho \text{ a.e. in } \Omega_T.
\end{aligned}$$

Moreover, (23) gives, for any $\phi \in C_0^{\infty}(\Omega_T)$,

$$\begin{aligned}
&\left|\int_0^T\int_\Omega (\varpi^{(n)} - v^{(n)})\phi \mathrm{d}x\mathrm{d}t\right|\\
=&\left|\int_0^T\int_\Omega \sum_{k=1}^n \chi_k(t)(1-\vartheta_k(t))(v_k - v_{k-1})\phi \mathrm{d}x\mathrm{d}t\right|\\
=&\left|\int_0^T h\sum_{k=1}^n \chi_k(1-\vartheta_k)\int_\Omega (|\nabla\triangle v_k|^{p(x)-2}\nabla\triangle v_k\nabla\phi\right.\\
&\left. + |\triangle v_k|^{q(x)-2}\triangle v_k\phi + g(x,v_{k-1})\phi)\mathrm{d}x\mathrm{d}t\right|\\
\le& h\int_0^T\left(\left|\int_\Omega |\nabla\triangle\varpi^{(n)}|^{p(x)-2}\nabla\triangle\varpi^{(n)}\nabla\phi \mathrm{d}x\right| + \left|\int_\Omega |\triangle\varpi^{(n)}|^{q(x)-2}\triangle\varpi^{(n)}\phi \mathrm{d}x\right|\right.\\
&\left. + \left|\int_\Omega g(x,\varpi^{(n)})\phi \mathrm{d}x\right| + \left|\int_\Omega g(x,v_0)\phi \mathrm{d}x\right|\right)\mathrm{d}t\\
\le& Ch \to 0,
\end{aligned}$$

as $n \to \infty$.

By the continuity of g and $\varrho = v$ a.e. in Ω_T, we have $g(x,\varpi^{(n)}) \to g(x,v)$ a.e. in Ω_T. Furthermore, the estimate (see Lemma 7) $\|g(x,\varpi^{(n)})\|_{\mathcal{L}^{p'(x)}(\Omega_T)} < \infty$ gives

$$g(x,\varpi^{(n)}) \rightharpoonup g(x,v) \text{ weakly in } \mathcal{L}^{p'(x)}(\Omega_T).$$

Applying (23) and (35), we obtain, for any test function ϕ,

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial v^{(n)}}{\partial t}, \phi\right\rangle \mathrm{d}t\\
=&\int_0^T\int_\Omega |\nabla\triangle\varpi^{(n)}|^{p(x)-2}\nabla\triangle\varpi^{(n)}\nabla\phi \mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_\Omega |\triangle\varpi^{(n)}|^{q(x)-2}\triangle\varpi^{(n)}\phi \mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_\Omega g(x,\varpi^{(n)})\phi \mathrm{d}x\mathrm{d}t + h\int_\Omega g(x,v_0)\phi(x,0)\mathrm{d}x\\
&-\int_{(n-1)h}^T\int_\Omega g(x,v_n)\phi \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By taking $n \to \infty$, we have

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, \phi \right\rangle \mathrm{d}t = \int_0^T \int_\Omega \nu \nabla \phi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \nu' \phi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega g(x,v)\phi \mathrm{d}x\mathrm{d}t.$$

It remains to prove $\nu = |\nabla \triangle v|^{p(x)-2} \nabla \triangle v$ and $\nu' = |\triangle v|^{q(x)-2} \triangle v$. By using v as a test function in (23) and integrating by part, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla v(x,T)|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla v_0|^2 \mathrm{d}x + \int_0^T \int_\Omega \nu \nabla \triangle v \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \nu' \triangle v \mathrm{d}x\mathrm{d}t \\ = & - \int_0^T \int_\Omega g(x,v) \triangle v \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{36}$$

On the other hand, (23) implies

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla \varpi^{(n)}(x,T)|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla v_0|^2 \mathrm{d}x \\ &+ \int_0^T \int_\Omega |\nabla \triangle \varpi^{(n)}|^{p(x)} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\triangle \varpi^{(n)}|^{q(x)} \mathrm{d}x\mathrm{d}t \\ \leq & - \int_0^T \int_\Omega g(x, \varpi^{(n)}) \triangle \varpi^{(n)} \mathrm{d}x\mathrm{d}t - h\int_\Omega g(x,v_0)v_0 \mathrm{d}x \\ &+ \int_{(n-1)h}^T \int_\Omega g(x,v_n)v_n \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{37}$$

For any test functions ϕ, ϕ_1 and constant $\varepsilon > 0$, we have

$$\int_0^T \int_\Omega (|\rho^{(n)}|^{p(x)-2}\rho^{(n)} - |\eta_\varepsilon|^{p(x)-2}\eta_\varepsilon)(\rho^{(n)} - \eta_\varepsilon) \mathrm{d}x\mathrm{d}t \geq 0, \tag{38}$$

and

$$\int_0^T \int_\Omega (|\rho_1^{(n)}|^{q(x)-2}\rho_1^{(n)} - |\eta_\varepsilon|^{q(x)-2}\eta_{1\varepsilon})(\rho_1^{(n)} - \eta_{1\varepsilon}) \mathrm{d}x\mathrm{d}t \geq 0, \tag{39}$$

where $\rho^{(n)} = \nabla \triangle \varpi^{(n)}$, $\eta_\varepsilon = \nabla \triangle (v - \varepsilon\phi))$ and $\rho_1^{(n)} = \triangle \varpi^{(n)}$, $\eta_{1\varepsilon} = \triangle (v - \varepsilon\phi_1)$.

By (37)–(39), we arrive at

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla v(x,T)|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla v_0|^2 \mathrm{d}x - \int_0^T \int_\Omega |\nabla \triangle (v - \varepsilon\phi)|^{p(x)} \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Omega |\nabla \triangle (v - \varepsilon\phi)|^{p(x)-2} \nabla \triangle (v - \varepsilon\phi) \nabla \triangle v \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Omega \nu \nabla \triangle (v - \varepsilon\phi) \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega |\triangle (v - \varepsilon\phi_1)|^{q(x)} \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Omega |\triangle (v - \varepsilon\phi_1)|^{q(x)-2} \triangle (v - \varepsilon\phi_1) \triangle v \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Omega \nu' \triangle (v - \varepsilon\phi_1) \mathrm{d}x\mathrm{d}t \leq - \int_0^T \int_\Omega g(x,v) \triangle v \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Moreover, (36) gives

$$\begin{aligned} &\int_0^T \int_\Omega [|\nabla \triangle (v - \varepsilon\phi)|^{p(x)-2}] \nabla \triangle (v - \varepsilon\phi) - \nu] \nabla \triangle \phi \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Omega [|\triangle (v - \varepsilon\phi_1)|^{q(x)-2}] \triangle (v - \varepsilon\phi) - \nu'] \triangle \phi \mathrm{d}x\mathrm{d}t \leq 0. \end{aligned}$$

By letting $\varepsilon \to 0$, we obtain

$$\int_0^T \int_\Omega [|\nabla \triangle v|^{p(x)-2}] \nabla \triangle v - \nu] \nabla \triangle \phi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega [|\triangle v|^{q(x)-2} \triangle v - \nu'] \triangle \phi_1 \mathrm{d}x\mathrm{d}t \leq 0.$$

By letting $\phi_1 = 0$, we have

$$\int_0^T \int_\Omega [|\nabla \triangle v|^{p(x)-2}] \nabla \triangle v - \nu] \nabla \triangle \phi \mathrm{d}x\mathrm{d}t \leq 0.$$

The arbitrariness of ϕ yields $|\nabla \triangle v|^{p(x)-2} \nabla \triangle v = \nu$, a.e. in Ω_T. Similarly, we can obtain $|\triangle v|^{q(x)-2} \triangle v = \nu'$, a.e. in Ω_T. □

Proof of Uniqueness. Let $v_1(x)$ and $v_2(x)$ be two weak solutions to (1)–(3) and $\phi = v_1 - v_2$. By taking $\triangle \phi$ as the test function, we get

$$\begin{aligned} &\frac{1}{2} \int_0^t \int_\Omega \frac{d}{dt} |\nabla \phi|^2 \mathrm{d}x\mathrm{d}t \\ &+ \int_0^t \int_\Omega [|\nabla \triangle v_1|^{p(x)-2}] \nabla \triangle v_1 - |\nabla \triangle v_2|^{p(x)-2}] \nabla \triangle v_2] \nabla \triangle \phi \mathrm{d}x\mathrm{d}t \\ &+ \int_0^t \int_\Omega [|\triangle v_1|^{q(x)-2} \triangle v_1 - |\triangle v_2|^{q(x)-2} \triangle v_2] \triangle \phi \mathrm{d}x\mathrm{d}t = \mu \int_0^t \int_\Omega |\nabla \phi|^2 \mathrm{d}x\mathrm{d}t. \end{aligned}$$

It implies

$$\int_\Omega |\nabla \phi|^2 \mathrm{d}x \leq 2\mu \int_0^t \int_\Omega |\nabla \phi|^2 \mathrm{d}x\mathrm{d}t,$$

where we have used the fact $(|x|^{m-2}x - |y|^{m-2}y)(x-y) \geq 0$ for $m > 1$ and $x, y \in R$ (or R^N). By Gronwall's inequality, we obtain $v_1(x,t) = v_2(x,t)$ a.e. in Ω_T. □

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References

1. Bernis, F.; Friedman, A. Higher order nonlinear degenerate parabolic equations. *J. Differ. Equ.* **1990**, *83*, 179–206.
2. Myers, T.G. Thin films with high surface tension. *SIAM Rev.* **1998**, *40*, 441–462.
3. Elliott, C.M.; Zheng, S. On the Cahn Hilliard equation. *Arch Ration. Mech. Anal.* **1986**, *96*, 339–357.
4. Cahn, J.M.; Hilliard, J.E. Free energy of a non-uniform system I. Interfacial free energy. *J. Chem. Phys.* **1958**, *28*, 258–367.
5. Liang, B.; Zheng, S. Existence and asymptoticcs behavior of solutions to nonlinear parabolic equations of fourth order. *J. Math. Anal. Appl.* **2008**, *348*, 234–243.
6. Ayoujil, A.; Amrouss, A.R. On the spectrum of a fourth order elliptic equation with variable exponent. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 4916–4926.
7. Guo, B.; Gao, W. Study of weak solutions for a fourth-order parabolic equation with variable exponent of nonlinearity. *Z. Angew. Math. Phys.* **2011**, *62*, 909–926.
8. Guo, B.; Li, Y.; Gao, W. Singular phenomena of solutions for nonlinear diffusion equations involving p(x)-Laplace operator and nonlinear sources. *Z. Angew. Math. Phys.* **2015**, *66*, 989–1005.

9. Qu, C.; Zheng, S. Fujita type conditions for fast diffusion equation with variable source. *Appl. Anal.* **2009**, *88*, 1651–1664.
10. Carlen, E.A.; Ulusoy, S. An entropy dissipation-entropy estimate for a thin film type equation. *Commun. Math. Sci.* **2005**, *3*, 171–178.
11. Carrillo, J.A.; Toscani, G. Long-time asymptotics for strong solutions of the thin film equation. *Commun. Math. Phys.* **2002**, *225*, 551–571.
12. Bertozzi, A.L.; Pugh, M. The lubrication approximation for thin viscous films: Regularity and long time behavior of weak solutions. *Commun. Pure Appl. Math.* **1996**, *49*, 85–123.
13. Fan, X.L.; Zhang, Q.H. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal. Theory Methods Appl.* **2003**, *52*, 1843–1852.
14. Mihâilescu, M.; Râdulescu, V. On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. *Proc. Am. Math. Soc.* **2007**, *135*, 2929–2937.
15. Adams, R.A. *Sobolev Space*; Academic Press: Cambridge, MA, USA, 1975.
16. Gilbarg, D.; Trudinger, N.S. *Elliptic Partial Different Equations of Second Order*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 1983.
17. Simon, J. Compact sets in the space $\mathcal{L}^p(0,T;B)$. *Ann. Math. Pura Appl.* **1987**, *146*, 65–96.