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# On the Reliability Function of Variable-Rate Slepian-Wolf Coding <sup>†</sup>

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**Abstract:** The reliability function of variable-rate Slepian-Wolf coding is linked to the reliability function of channel coding with constant composition codes, through which computable lower and upper bounds are derived. The bounds coincide at rates close to the Slepian-Wolf limit, yielding a complete characterization of the reliability function in that rate region. It is shown that variable-rate Slepian-Wolf codes can significantly outperform fixed-rate Slepian-Wolf codes in terms of rate-error tradeoff. Variable-rate Slepian-Wolf coding with rate below the Slepian-Wolf limit is also analyzed. In sharp contrast with fixed-rate Slepian-Wolf codes for which the correct decoding probability decays to zero exponentially fast if the rate is below the Slepian-Wolf limit, the correct decoding probability of variable-rate Slepian-Wolf codes can be bounded away from zero.

**Keywords:** channel coding; duality; reliability function; Slepian-Wolf coding

## 1. Introduction

Consider the problem (see Figure 1) of compressing  $X^n = (X_1, X_2, \dots, X_n)$  with side information  $Y^n = (Y_1, Y_2, \dots, Y_n)$  available only at the decoder. Here  $\{(X_i, Y_i)\}_{i=1}^{\infty}$  is a joint memoryless source with zero-order joint probability distribution  $P_{XY}$  on finite alphabet  $\mathcal{X} \times \mathcal{Y}$ . Let  $P_X$  and  $P_Y$  be the marginal probability distributions of  $X$  and  $Y$  induced by the joint probability distribution  $P_{XY}$ . Without loss of generality, we shall assume  $P_X(x) > 0, P_Y(y) > 0$  for all  $x \in \mathcal{X}, y \in \mathcal{Y}$ . This problem was first studied by Slepian and Wolf in their landmark paper [1]. They proved a surprising result that the minimum rate for reconstructing  $X^n$  at the decoder with asymptotically zero error probability (as block length  $n$  goes to infinity) is  $H(X|Y)$ , which is the same as the case where the side information  $Y^n$  is also available at the encoder. The fundamental limit  $H(X|Y)$  is often referred to as the Slepian-Wolf limit. We shall assume  $H(X|Y) > 0$  throughout this paper.

Different from conventional lossless source coding, where most effort has been devoted to variable-rate coding schemes, research on Slepian-Wolf coding has almost exclusively focused on fixed-rate codes (see, e.g., [2–5] and the references therein). This phenomenon can be partly explained by the influence of channel coding. It is well known that there is an intimate connection between channel coding and Slepian-Wolf coding. Intuitively, one may view  $Y^n$  as the channel output generated by channel input  $X^n$  through discrete memoryless channel  $P_{Y|X}$ , where  $P_{Y|X}$  is the probability transition matrix from  $\mathcal{X}$  to  $\mathcal{Y}$  induced by the joint probability distribution  $P_{XY}$ . Since  $Y^n$  is not available at the encoder, Slepian-Wolf coding is, in a certain sense, similar to channel coding

without feedback. In a channel coding system, there is little incentive to use variable-rate coding schemes if no feedback link exists from the receiver to the transmitter. Therefore, it seems justifiable to focus on fixed-rate codes in Slepian-Wolf coding.

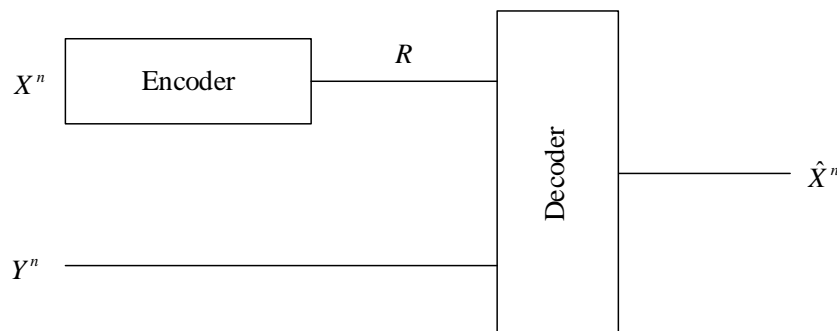


Figure 1. Slepian-Wolf coding.

This viewpoint turns out to be misleading. We shall show that variable-rate Slepian-Wolf codes can significantly outperform fixed-rate codes in terms of rate-error tradeoff. Specifically, it is revealed that variable-rate Slepian-Wolf codes can beat the sphere-packing bound for fixed-rate Slepian-Wolf codes at rates close to the Slepian-Wolf limit. It is known [6] that the correct decoding probability of fixed-rate Slepian-Wolf codes decays to zero exponentially fast if the rate is below the Slepian-Wolf limit. Somewhat surprisingly, the decoding error probability of variable-rate Slepian-Wolf codes can be bounded away from one even when they are operated below the Slepian-Wolf limit, and the performance degrades gracefully as the rate goes to zero. Therefore, variable-rate Slepian-Wolf coding is considerably more robust.

The rest of this paper is organized as follows. In Section 2, we review the existing bounds on the reliability function of fixed-rate Slepian-Wolf coding, and point out the intimate connections with their counterparts in channel coding. In Section 3, we characterize the reliability function of variable-rate Slepian-Wolf coding by leveraging the reliability function of channel coding with constant composition codes. Computable lower and upper bounds are derived. The bounds coincide at rates close to the Slepian-Wolf limit. The correct decoding probability of variable-rate Slepian-Wolf coding with rate below the Slepian-Wolf limit is studied in Section 4. An illustrative example is given in Section 5. We conclude the paper in Section 6. Throughout this paper, we assume the logarithm function is to base  $e$  unless specified otherwise.

## 2. Fixed-Rate Slepian-Wolf Coding and Channel Coding

To facilitate the comparisons between the performances of fixed-rate Slepian-Wolf coding and variable-rate coding, we shall briefly review the existing bounds on the reliability function of fixed-rate Slepian-Wolf coding. It turns out that a most instructive way is to first consider their counterparts in channel coding. The reason is two-fold. First, it provides the setup to introduce several important definitions. Second and more important, it will be clear that the reliability function of fixed-rate Slepian-Wolf coding is closely related to that of channel coding; indeed, such a connection will be further explored in the context of variable-rate Slepian-Wolf coding.

For any probability distributions  $P, Q$  on  $\mathcal{X}$  and probability transition matrices  $V, W : \mathcal{X} \rightarrow \mathcal{Y}$ , we use  $H(P)$ ,  $I(P, V)$ ,  $D(Q||P)$ , and  $D(W||V|P)$  to denote the standard entropy, mutual information, divergence, and conditional divergence functions; specifically, we have

$$H(P) = - \sum_x P(x) \log P(x),$$

$$I(P, V) = \sum_{x,y} P(x)V(y|x) \log \frac{V(y|x)}{\sum_{x'} P(x')V(y|x')},$$

$$D(Q||P) = \sum_x Q(x) \log \frac{Q(x)}{P(x)},$$

$$D(W||V|P) = \sum_{x,y} P(x)W(y|x) \log \frac{W(y|x)}{V(y|x)}.$$

The main technical tool we need is the method of types. First, we shall quote a few basic definitions from [7]. Let  $\mathcal{P}(\mathcal{X})$  denote the set of all probability distributions on  $\mathcal{X}$ . The type of a sequence  $x^n \in \mathcal{X}^n$ , denoted as  $P_{x^n}$ , is the empirical probability distribution of  $x^n$ . Let  $\mathcal{P}_n(\mathcal{X})$  denote the set consisting of the possible types of sequences  $x^n \in \mathcal{X}^n$ . For any  $P \in \mathcal{P}_n(\mathcal{X})$ , the type class  $\mathcal{T}_n(P)$  is the set of sequences in  $\mathcal{X}^n$  of type  $P$ . We will make frequent use of the following elementary results:

$$|\mathcal{P}_n(\mathcal{X})| \leq (n + 1)^{|\mathcal{X}|}, \tag{1}$$

$$\frac{1}{(n + 1)^{|\mathcal{X}|}} e^{nH(P)} \leq |\mathcal{T}_n(P)| \leq e^{nH(P)}, \quad P \in \mathcal{P}_n(\mathcal{X}), \tag{2}$$

$$\prod_{i=1}^n P(x_i) = e^{-n[D(Q||P)+H(Q)]}, \quad x^n \in \mathcal{T}_n(Q), Q \in \mathcal{P}_n(\mathcal{X}), P \in \mathcal{P}(\mathcal{X}). \tag{3}$$

A block code  $\mathcal{C}_n$  is an ordered collection of sequences in  $\mathcal{X}^n$ . We allow  $\mathcal{C}_n$  to contain identical sequences. Moreover, for any set  $\mathcal{A} \subseteq \mathcal{X}^n$ , we say that  $\mathcal{C}_n \subseteq \mathcal{A}$  if  $x^n \in \mathcal{A}$  for all  $x^n \in \mathcal{C}_n$ . Note that  $\mathcal{C}_n \subseteq \mathcal{A}$  does not imply  $|\mathcal{C}_n| \leq |\mathcal{A}|$ . The rate of  $\mathcal{C}_n$  is defined as

$$R(\mathcal{C}_n) = \frac{1}{n} \log |\mathcal{C}_n|.$$

Given a channel  $W_{Y|X} : \mathcal{X} \rightarrow \mathcal{Y}$ , a block code  $\mathcal{C}_n \subseteq \mathcal{X}^n$ , and channel output  $Y^n \in \mathcal{Y}^n$ , the output of the optimal *maximum likelihood* (ML) decoder is

$$\hat{X}^n = \arg \min_{x^n \in \mathcal{C}_n} - \sum_{i=1}^n \log W_{Y|X}(Y_i|x_i),$$

where the ties are broken in an arbitrary manner. The average decoding error probability of block code  $\mathcal{C}_n$  over channel  $W_{Y|X}$  is defined as

$$P_e(\mathcal{C}_n, W_{Y|X}) = \frac{1}{|\mathcal{C}_n|} \sum_{x^n \in \mathcal{C}_n} \Pr\{\hat{X}^n \neq x^n | x^n \text{ is transmitted}\}.$$

The maximum decoding error probability of block code  $\mathcal{C}_n$  over channel  $W_{Y|X}$  is defined as

$$P_{e,\max}(\mathcal{C}_n, W_{Y|X}) = \max_{x^n \in \mathcal{C}_n} \Pr\{\hat{X}^n \neq x^n | x^n \text{ is transmitted}\}.$$

The average correct decoding probability of block code  $\mathcal{C}_n$  over channel  $W_{Y|X}$  is defined as

$$P_c(\mathcal{C}_n, W_{Y|X}) = 1 - P_e(\mathcal{C}_n, W_{Y|X}).$$

**Definition 1.** Given a channel  $W_{Y|X} : \mathcal{X} \rightarrow \mathcal{Y}$ , we say that an error exponent  $E \geq 0$  is achievable with block codes at rate  $R$  if for any  $\delta > 0$ , there exists a sequence of block codes  $\{\mathcal{C}_n\}$  such that

$$\liminf_{n \rightarrow \infty} R(\mathcal{C}_n) \geq R - \delta,$$

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_e(\mathcal{C}_n, W_{Y|X}) \geq E - \delta. \tag{4}$$

The largest achievable error exponent at rate  $R$  is denoted by  $E(W_{Y|X}, R)$ . The function  $E(W_{Y|X}, \cdot)$  is referred to as the reliability function of channel  $W_{Y|X}$ .

Similarly, we say that a correct decoding exponent  $E^c \geq 0$  is achievable with block channel codes at rate  $R$  if for any  $\delta > 0$ , there exists a sequence of block codes  $\{C_n\}$  such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} R(C_n) &\geq R - \delta, \\ \liminf_{n \rightarrow \infty} -\frac{1}{n} \log P_c(C_n, W_{Y|X}) &\leq E^c + \delta. \end{aligned}$$

The smallest achievable correct decoding exponent at rate  $R$  is denoted by  $E^c(W_{Y|X}, R)$ . It will be seen that  $E^c(W_{Y|X}, R)$  is positive if and only if  $R > C(W_{Y|X})$ , where  $C(W_{Y|X}) \triangleq \max_{Q_X} I(Q_X, W_{Y|X})$  is the capacity of channel  $W_{Y|X}$ . Therefore, we shall refer to the function  $E^c(W_{Y|X}, \cdot)$  as the reliability function of channel  $W_{Y|X}$  above the capacity.

**Remark 1.** Given any block code  $C_n$  of average decoding error probability  $P_e(C_n, W_{Y|X})$ , we can expurgate the worst half of the codewords so that the maximum decoding error probability of the resulting code is bounded above by  $2P_e(C_n, W_{Y|X})$ . Therefore, the reliability function  $E(W_{Y|X}, \cdot)$  is unaffected if we replace  $P_e(C_n, W_{Y|X})$  by  $P_{e,\max}(C_n, W_{Y|X})$  in (4).

**Definition 2.** Given a probability distribution  $Q_X \in \mathcal{P}(\mathcal{X})$  and a channel  $W_{Y|X} : \mathcal{X} \rightarrow \mathcal{Y}$ , we say that an error exponent  $E \geq 0$  is achievable at rate  $R$  with constant composition codes of type approximately  $Q_X$  if for any  $\delta > 0$ , there exists a sequence of block codes  $\{C_n\}$  with  $C_n \subseteq \mathcal{T}_n(P_n)$  for some  $P_n \in \mathcal{P}_n(\mathcal{X})$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_n - Q_X\| &= 0, \\ \liminf_{n \rightarrow \infty} R(C_n) &\geq R - \delta, \\ \limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_e(C_n, W_{Y|X}) &\geq E - \delta, \end{aligned}$$

where  $\|\cdot\|$  is the  $l_1$  norm.

The largest achievable error exponent at rate  $R$  for constant composition codes of type approximately  $Q_X$  is denoted by  $E(Q_X, W_{Y|X}, R)$ . The function  $E(Q_X, W_{Y|X}, \cdot)$  is referred to as the reliability function of channel  $W_{Y|X}$  for constant composition codes of type approximately  $Q_X$ .

Similarly, we say that a correct decoding exponent  $E^c \geq 0$  is achievable at rate  $R$  with constant composition codes of type approximately  $Q_X$  if for any  $\delta > 0$ , there exists a sequence of block codes  $\{C_n\}$  with  $C_n \subseteq \mathcal{T}_n(P_n)$  for some  $P_n \in \mathcal{P}_n(\mathcal{X})$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_n - Q_X\| &= 0, \\ \liminf_{n \rightarrow \infty} R(C_n) &\geq R - \delta, \\ \liminf_{n \rightarrow \infty} -\frac{1}{n} \log P_c(C_n, W_{Y|X}) &\leq E^c + \delta. \end{aligned} \tag{5}$$

The smallest achievable correct decoding exponent at rate  $R$  for constant composition codes of type approximately  $Q_X$  is denoted by  $E^c(Q_X, W_{Y|X}, R)$ .

**Remark 2.** The reliability function  $E(Q_X, W_{Y|X}, \cdot)$  is unaffected if we replace  $P_e(C_n, W_{Y|X})$  by  $P_{e,\max}(C_n, W_{Y|X})$  in (5).

Let  $|t|^+ = \max\{0, t\}$  and  $d_{W_{Y|X}}(x, \tilde{x}) = -\log \sum_y \sqrt{W_{Y|X}(y|x)W_{Y|X}(y|\tilde{x})}$ . Define

$$E_{ex}(Q_X, W_{Y|X}, R) = \min_{Q_{\tilde{X}|X}: Q_X=Q_{\tilde{X}}, I(Q_X, Q_{\tilde{X}|X}) \leq R} \left[ \mathbb{E}_{Q_{X\tilde{X}}} d_{W_{Y|X}}(X, \tilde{X}) + I(Q_X, Q_{\tilde{X}|X}) - R \right], \tag{6}$$

$$E_{rc}(Q_X, W_{Y|X}, R) = \min_{V_{Y|X}} \left[ D(V_{Y|X} \| W_{Y|X} | Q_X) + |I(Q_X, V_{Y|X}) - R|^+ \right], \tag{7}$$

$$E_{sp}(Q_X, W_{Y|X}, R) = \min_{V_{Y|X}: I(Q_X, V_{Y|X}) \leq R} D(V_{Y|X} \| W_{Y|X} | Q_X), \tag{8}$$

where in (6),  $Q_{\tilde{X}}$  and  $Q_{X\tilde{X}}$  are respectively the marginal probability distribution of  $\tilde{X}$  and the joint probability distribution of  $X$  and  $\tilde{X}$  induced by  $Q_X$  and  $Q_{\tilde{X}|X}$ .

Let  $R_{ex}^\infty(Q_X, W_{Y|X})$  be the smallest  $R \geq 0$  with  $E_{ex}(Q_X, W_{Y|X}, R) < \infty$ . We have

$$R_{ex}^\infty(Q_X, W_{Y|X}) = \min_{Q_{\tilde{X}|X}: Q_X=Q_{\tilde{X}}, \mathbb{E}_{Q_{X\tilde{X}}} d_{W_{Y|X}}(X, \tilde{X}) < \infty} I(Q_X, Q_{\tilde{X}|X}). \tag{9}$$

It is known ([7], Exercise 5.18) that  $E_{ex}(Q_X, W_{Y|X}, R)$  is a decreasing convex function of  $R$  for  $R \geq R_{ex}^\infty(Q_X, W_{Y|X})$ ; moreover, the minimum in (9) is achieved at  $Q_{X\tilde{X}}$  if and only if

$$Q_{X\tilde{X}}(x, \tilde{x}) = \begin{cases} cQ(x)Q(\tilde{x}) & \text{if } d_{W_{Y|X}}(x, \tilde{x}) < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

where the probability distribution  $Q$  and the constant  $c$  are uniquely determined by the condition  $Q_X = Q_{\tilde{X}}$ .

It is shown in ([8], Lemma 3) that, for some  $R^*(Q_X, W_{Y|X}) \in [0, I(Q_X, W_{Y|X})]$ , we have

$$\max \left\{ E_{ex}(Q_X, W_{Y|X}, R), E_{rc}(Q_X, W_{Y|X}, R) \right\} = \begin{cases} E_{ex}(Q_X, W_{Y|X}, R) & \text{if } R \leq R^*(Q_X, W_{Y|X}), \\ E_{rc}(Q_X, W_{Y|X}, R) & \text{if } R > R^*(Q_X, W_{Y|X}). \end{cases} \tag{10}$$

It is also known ([7], Corollary 5.4) that

$$E_{rc}(Q_X, W_{Y|X}, R) = \begin{cases} E_{sp}(Q_X, W_{Y|X}, R) & \text{if } R \geq R_{cr}(Q_X, W_{Y|X}), \\ E_{sp}(Q_X, W_{Y|X}, R_{cr}) + R_{cr} - R & \text{if } 0 \leq R \leq R_{cr}(Q_X, W_{Y|X}), \end{cases} \tag{11}$$

where  $R_{cr} \triangleq R_{cr}(Q_X, W_{Y|X})$  is the smallest  $R$  at which the convex curve  $E_{sp}(Q_X, W_{Y|X}, R)$  meets its supporting line of slope  $-1$ . It is obvious that  $R_{cr}(Q_X, W_{Y|X}) \leq I(Q_X, W_{Y|X})$ .

**Proposition 1.**  $R_{cr}(Q_X, W_{Y|X}) = I(Q_X, W_{Y|X})$  if and only if the value of

$$\frac{W_{Y|X}(y|x)}{\sum_{x'} Q_X(x')W_{Y|X}(y|x')}$$

does not depend on  $y$  for all  $x, y$  such that  $Q_X(x)W_{Y|X}(y|x) > 0$ .

**Proof.** See Appendix A  $\square$

Define  $R_{sp}^\infty(Q_X, W_{Y|X}) = \inf\{R > 0 : E_{sp}(Q_X, W_{Y|X}, R) < \infty\}$ . It is known ([7], Exercise 5.3) that

$$R_{sp}^\infty(Q_X, W_{Y|X}) = \min I(Q_X, V_{Y|X}), \tag{12}$$

where the minimum is taken over those  $V_{Y|X}$ 's for which  $V_{Y|X}(y|x) = 0$  whenever  $W_{Y|X}(y|x) = 0$ ; in particular,  $R_{sp}^\infty(Q_X, W_{Y|X}) > 0$  if and only if for every  $y \in \mathcal{Y}$  there exists an  $x \in \mathcal{X}$  with  $Q_X(x) > 0$  and  $W_{Y|X}(y|x) = 0$ .

**Proposition 2.** *The minimum in (12) is achieved at  $V_{Y|X} = W_{Y|X}$  if and only if the value of*

$$\frac{W_{Y|X}(y|x)}{\sum_{x'} Q_X(x') W_{Y|X}(y|x')}$$

does not depend on  $y$  for all  $x, y$  such that  $Q_X(x) W_{Y|X}(y|x) > 0$ .

**Proof.** The proof is similar to that of Proposition 1. The details are omitted.  $\square$

One can readily prove the following result by combining Propositions 1 and 2.

**Proposition 3.** *The following statements are equivalent:*

1.  $R_{cr}(Q_X, P_{Y|X}) = I(Q_X, W_{Y|X})$ ;
2.  $R_{sp}^\infty(Q_X, P_{Y|X}) = I(Q_X, W_{Y|X})$ ;
3. *the value of*

$$\frac{W_{Y|X}(y|x)}{\sum_{x'} Q_X(x') W_{Y|X}(y|x')}$$

does not depend on  $y$  for all  $x, y$  such that  $Q_X(x) W_{Y|X}(y|x) > 0$ .

**Proposition 4.**

1.  $E(Q_X, W_{Y|X}, R) \geq \max\{E_{ex}(Q_X, W_{Y|X}, R), E_{rc}(Q_X, W_{Y|X}, R)\}$ ;
2.  $E(Q_X, P_{Y|X}, R) \leq E_{sp}(Q_X, W_{Y|X}, R)$  with the possible exception of  $R = R_{sp}^\infty(Q_X, W_{Y|X})$  at which point the inequality not necessarily holds;
3.  $E^c(Q_X, W_{Y|X}, R) = \min_{V_{Y|X}} \left[ D(V_{Y|X} \| W_{Y|X} | Q_X) + |R - I(Q_X; V_{Y|X})|^+ \right]$ .

**Remark 3.**  $E_{ex}(Q_X, W_{Y|X}, R)$ ,  $E_{rc}(Q_X, W_{Y|X}, R)$ , and  $E_{sp}(Q_X, W_{Y|X}, R)$  are respectively the expurgated exponent, the random coding exponent, and the sphere packing exponent of channel  $W_{Y|X}$  for constant composition codes of type approximately  $Q_X$ . The results in Proposition 4 are well known [7,9]. However, bounding the decoding error probability of constant composition codes often serves as an intermediate step in characterizing the reliability function for general block codes; as a consequence, the reliability function for constant composition codes is rarely explicitly defined. Moreover,  $E_{ex}(Q_X, W_{Y|X}, R)$ ,  $E_{rc}(Q_X, W_{Y|X}, R)$ , and  $E_{sp}(Q_X, W_{Y|X}, R)$  are commonly used to bound the decoding error probability of constant composition codes for a fixed block length  $n$ ; therefore, it is implicitly assumed that  $Q_X$  is taken from  $\mathcal{P}_n(\mathcal{X})$  (see, e.g., [7]). In contrast, we consider a sequence of constant composition codes with block length increasing to infinity and type converging to  $Q_X$  for some  $Q_X \in \mathcal{P}(\mathcal{X})$  (see Definition 2). A continuity argument is required for passing  $Q_X$  from  $\mathcal{P}_n(\mathcal{X})$  to  $\mathcal{P}(\mathcal{X})$ . For completeness, we supply the proof in Appendix B. Note that different from  $E(Q_X, W_{Y|X}, \cdot)$ , the function  $E^c(Q_X, W_{Y|X}, \cdot)$  has been completely characterized.

**Proposition 5.**

1.  $E(W_{Y|X}, R) = \sup_{Q_X} E(Q_X, W_{Y|X}, R)$ ,
2.  $E^c(W_{Y|X}, R) = \inf_{Q_X} E^c(Q_X, W_{Y|X}, R)$ .

**Remark 4.** In view of the fact that  $E^c(Q_X, W_{Y|X}, R)$  is a continuous function of  $Q_X$  defined on a compact set, we can replace “inf” with “min” in the above equation, i.e.,

$$E^c(W_{Y|X}, R) = \min_{Q_X} E^c(Q_X, W_{Y|X}, R). \tag{13}$$

**Proof.** It is obvious that  $E(W_{Y|X}, R) \geq \sup_{Q_X} E(Q_X, W_{Y|X}, R)$ ; the other direction follows from the fact that every block code  $C_n$  contains a constant composition code  $C'_n$  with  $P_{e,\max}(C'_n, W_{Y|X}) \leq P_{e,\max}(C_n, W_{Y|X})$  and  $R(C'_n) \geq R(C_n) - |\mathcal{X}| \frac{\log(n+1)}{n}$ . Similarly, it is clear that  $E^c(W_{Y|X}, R) \leq \inf_{Q_X} E^c(Q_X, W_{Y|X}, R)$ ; the other direction follows from the fact that given any block code  $C_n$ , one can construct a constant composition code  $C'_n$  with  $P_c(C'_n, W_{Y|X}) \leq (n + 1)^{|\mathcal{X}|} P_c(C_n, W_{Y|X})$  and  $R(C'_n) = R(C_n)$  [9]. □

The expurgated exponent, random coding exponent, and sphere packing exponent of channel  $W_{Y|X}$  for general block codes are defined as follows:

1. expurgated exponent

$$E_{ex}(W_{Y|X}, R) = \max_{Q_X} E_{ex}(Q_X, W_{Y|X}, R), \tag{14}$$

2. random coding exponent

$$E_{rc}(W_{Y|X}, R) = \max_{Q_X} E_{rc}(Q_X, W_{Y|X}, R), \tag{15}$$

3. sphere packing exponent

$$E_{sp}(W_{Y|X}, R) = \max_{Q_X} E_{sp}(Q_X, W_{Y|X}, R). \tag{16}$$

Let  $R_{sp}^\infty(W_{Y|X})$  be the smallest  $R$  to the right of which  $E_{sp}(W_{Y|X}, R)$  is finite. It is known ([7], Exercise 5.3) and [10] that

$$\begin{aligned} R_{sp}^\infty(W_{Y|X}) &= \max_{Q_X} R_{sp}^\infty(Q_X, W_{Y|X}) \\ &= -\log \left[ \min_{Q_X} \max_y \sum_{x \in \mathcal{X}: W_{Y|X}(y|x) > 0} Q_X(x) \right]. \end{aligned}$$

By Propositions 4 and 5, we recover the following well-known result [7,10]:

$$\max\{E_{ex}(W_{Y|X}, R), E_{rc}(W_{Y|X}, R)\} \leq E(W_{Y|X}, R) \leq E_{sp}(W_{Y|X}, R) \tag{17}$$

with the possible exception of  $R = R_{sp}^\infty(W_{Y|X})$  at which point the second inequality in (17) not necessarily holds.

Now we proceed to review the results on the reliability function of fixed-rate Slepian-Wolf coding. A fixed-rate Slepian-Wolf code  $\phi_n(\cdot)$  is a mapping from  $\mathcal{X}^n$  to a set  $\mathcal{A}_n$ . The rate of  $\phi_n(\cdot)$  is defined as

$$R(\phi_n) = \frac{1}{n} \log |\mathcal{A}_n|.$$

Given  $\phi_n(X^n)$  and  $Y^n$ , the output of the optimal *maximum a posteriori* (MAP) decoder is

$$\hat{X}^n = \arg \min_{x^n: \phi_n(x^n) = \phi_n(X^n)} - \sum_{i=1}^n \log P_{X|Y}(x_i|Y_i)$$

$$= \arg \min_{x^n: \phi_n(x^n) = \phi_n(X^n)} - \sum_{i=1}^n \log P_{XY}(x_i, Y_i),$$

where the ties are broken in an arbitrary manner. The decoding error probability of Slepian-Wolf code  $\phi_n(\cdot)$  is defined as

$$P_e(\phi_n, P_{XY}) = \Pr\{\widehat{X}^n \neq X^n\}.$$

The correct decoding probability of Slepian-Wolf code  $\phi_n(\cdot)$  is defined as

$$P_c(\phi_n, P_{XY}) = 1 - P_e(\phi_n, P_{XY}).$$

**Definition 3.** Given a joint probability distribution  $P_{XY}$ , we say that an error exponent  $E \geq 0$  is achievable with fixed-rate Slepian-Wolf codes at rate  $R$  if for any  $\delta > 0$ , there exists a sequence of fixed-rate Slepian-Wolf codes  $\{\phi_n\}$  such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\phi_n) &\leq R + \delta, \\ \limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_e(\phi_n, P_{XY}) &\geq E - \delta. \end{aligned}$$

The largest achievable error exponent at rate  $R$  is denoted by  $E_f(P_{XY}, R)$ . The function  $E_f(P_{XY}, \cdot)$  is referred to as the reliability function of fixed-rate Slepian-Wolf coding.

Similarly, we say that a correct decoding exponent  $E^c \geq 0$  is achievable with fixed-rate Slepian-Wolf codes at rate  $R$  if for any  $\delta > 0$ , there exists a sequence of fixed-rate Slepian-Wolf codes  $\{\phi_n\}$  such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\phi_n) &\leq R + \delta, \\ \liminf_{n \rightarrow \infty} -\frac{1}{n} \log P_c(\phi_n, P_{XY}) &\leq E^c + \delta. \end{aligned}$$

The smallest achievable correct decoding exponent at rate  $R$  is denoted by  $E_f^c(P_{XY}, R)$ . It will be seen that  $E_f^c(P_{XY}, R)$  is positive if and only if  $R < H(X|Y)$ . Therefore, we shall refer to the function  $E_f^c(P_{XY}, \cdot)$  as the reliability function of fixed-rate Slepian-Wolf coding below the Slepian-Wolf limit.

The expurgated exponent, random coding scheme, and sphere packing exponent of fixed-rate Slepian-Wolf coding are defined as follows:

1. expurgated exponent

$$E_{f,ex}(P_{XY}, R) = \min_{Q_X} \left[ D(Q_X \| P_X) + E_{ex}(Q_X, P_{Y|X}, H(Q_X) - R) \right], \tag{18}$$

2. random coding exponent

$$E_{f,rc}(P_{XY}, R) = \min_{Q_X} \left[ D(Q_X \| P_X) + E_{rc}(Q_X, P_{Y|X}, H(Q_X) - R) \right], \tag{19}$$

3. sphere packing exponent

$$E_{f,sp}(P_{XY}, R) = \min_{Q_X} \left[ D(Q_X \| P_X) + E_{sp}(Q_X, P_{Y|X}, H(Q_X) - R) \right]. \tag{20}$$



Equivalently, the random coding exponent and sphere packing exponent of fixed-rate Slepian-Wolf coding can be written as [11]:

$$E_{f,rc}(P_{XY}, R) = \max_{0 \leq \rho \leq 1} \left\{ -\log \sum_y \left[ \sum_x P_{XY}(x, y)^{\frac{1}{1+\rho}} \right]^{1+\rho} + \rho R \right\},$$

$$E_{f,sp}(P_{XY}, R) = \sup_{\rho > 0} \left\{ -\log \sum_y \left[ \sum_x P_{XY}(x, y)^{\frac{1}{1+\rho}} \right]^{1+\rho} + \rho R \right\}.$$

To see the connection between the random coding exponent and the sphere packing exponent, we shall write them in the following parametric forms [11]:

$$R = H(X^{(\rho)}|Y^{(\rho)}),$$

$$E_{f,sp}(P_{XY}, R) = D(P_{X^{(\rho)}Y^{(\rho)}} \| P_{XY}),$$

and

$$E_{f,rc}(P_{XY}, R) = \begin{cases} D(P_{X^{(\rho)}Y^{(\rho)}} \| P_{XY}) & \text{if } H(X|Y) \leq R \leq H(X_\rho|Y_\rho)|_{\rho=1}, \\ -\log \sum_y \left[ \sum_x \sqrt{P_{XY}(x, y)} \right]^2 + R & \text{if } R > H(X^{(\rho)}|Y^{(\rho)})|_{\rho=1}, \end{cases}$$

where the joint distribution of  $(X^{(\rho)}, Y^{(\rho)})$  is  $P_{X^{(\rho)}Y^{(\rho)}}$ , which is specified by

$$P_{Y^{(\rho)}}(y) = \frac{P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{\frac{1}{1+\rho}} \right]^{1+\rho}}{\sum_{y'} P_Y(y') \left[ \sum_x P_{X|Y}(x|y')^{\frac{1}{1+\rho}} \right]^{1+\rho}}, \quad y \in \mathcal{Y}, \tag{21}$$

$$P_{X^{(\rho)}|Y^{(\rho)}}(x|y) = \frac{P_{X|Y}(x|y)^{\frac{1}{1+\rho}}}{\sum_{x'} P_{X|Y}(x'|y)^{\frac{1}{1+\rho}}}, \quad x \in \mathcal{X}, y \in \mathcal{Y}. \tag{22}$$

Define the critical rate

$$R_{f,cr}(P_{XY}) = H(X^{(\rho)}|Y^{(\rho)})|_{\rho=1}.$$

Note that  $E_{rc}(P_{XY}, R)$  and  $E_{sp}(P_{XY}, R)$  coincide when  $R \in [H(X|Y), R_{f,cr}(P_{XY})]$ . Let  $R_{f,sp}^\infty(P_{XY}) = \sup\{R : E_{f,sp}(P_{XY}, R) < \infty\}$ . It is shown in [12] that

$$R_{f,sp}^\infty(P_{XY}) = \max_y \log |\{x \in \mathcal{X} : P_{X|Y}(x|y) > 0\}|.$$

It is well known [8,11,13] that the reliability function  $E_f(P_{XY}, \cdot)$  is upper-bounded by  $E_{f,sp}(P_{XY}, \cdot)$  and lower-bounded by  $E_{f,rc}(P_{XY}, \cdot)$  and  $E_{f,ex}(P_{XY}, \cdot)$ , i.e.,

$$\max\{E_{f,rc}(P_{XY}, R), E_{f,ex}(P_{XY}, R)\} \leq E_f(P_{XY}, R) \leq E_{f,sp}(P_{XY}, R) \tag{23}$$

with the possible exception of  $R = R_{f,sp}^\infty(P_{XY})$  at which point the second inequality in (23) not necessarily holds. Note that  $E_f(P_{XY}, R)$  is completely characterized for  $R \in [H(X|Y), R_{f,cr}(P_{XY})]$ .

Unlike  $E_f(P_{XY}, \cdot)$ , the function  $E_f^c(P_{XY}, \cdot)$  has been characterized for all  $R$ . Specifically, it is shown in [6,14] that

$$E_f^c(P_{XY}, R) = \min_{Q_X} \left[ D(Q_X \| P_X) + E^c(Q_X, P_{Y|X}, H(Q_X) - R) \right]. \tag{24}$$

Comparing (14) with (18), (15) with (19), (16) with (20), and (13) with (24), one can easily see that there exists an intimate connection between fixed-rate Slepian-Wolf coding for source distribution  $P_{XY}$  and channel coding for channel  $P_{Y|X}$ . This connection can be roughly interpreted as the manifestation of the following facts [15].

1. Given, for each type  $Q_X \in \mathcal{P}_n(\mathcal{X})$ , a constant composition code  $\mathcal{C}_n(Q_X) \subseteq \mathcal{T}_n(Q_X)$  with  $R(\mathcal{C}_n(Q_X)) \approx H(Q_X) - R$  and  $P_{e,\max}(\mathcal{C}_n(Q_X), P_{Y|X}) \approx e^{-nE(Q_X)}$ , one can use  $\mathcal{C}_n(Q_X)$  to partition type class  $\mathcal{T}_n(Q_X)$  into approximately  $e^{nR}$  disjoint subsets such that each subset is a constant composition code of type  $Q_X$  with the maximum decoding error probability over channel  $P_{Y|X}$  approximately equal to or less than that of  $\mathcal{C}_n(Q_X)$ . Note that these partitions, one for each type class, yield a fixed-rate Slepian-Wolf code of rate approximately  $R$  with  $\Pr\{\hat{X}^n \neq X^n | X^n \in \mathcal{T}_n(Q_X)\} \lesssim e^{-nE(Q_X)}$ . Since  $\Pr\{X^n \in \mathcal{T}_n(Q_X)\} \approx e^{-nD(Q_X \| P_X)}$  (cf. (2) and (3)), it follows that  $\Pr\{\hat{X}^n \neq X^n, X^n \in \mathcal{T}_n(Q_X)\} \lesssim e^{-n[D(Q_X \| P_X) + E(Q_X)]}$ . The overall decoding error probability  $\Pr\{\hat{X}^n \neq X^n\}$  of the resulting Slepian-Wolf code can be upper-bounded, on the exponential scale, by  $e^{-n[D(Q_X^* \| P_X) + E(Q_X^*)]}$ , where  $Q_X^* = \arg \min_{Q_X} D(Q_X \| P_X) + E(Q_X)$ . In contrast, one has the freedom to choose  $Q_X$  in channel coding, which explains why maximization (instead of minimization) is used in (14)–(16).
2. Given a fixed-rate Slepian-Wolf code  $\phi_n(\cdot)$  with  $R(\phi_n) \approx R$  and  $P_e(\phi_n, P_{XY}) \approx e^{-nE}$ , one can, for each type  $Q_X \in \mathcal{P}_n(\mathcal{X})$ , lift out a constant composition code  $\mathcal{C}_n(Q_X) \subseteq \mathcal{T}_n(Q_X)$  with  $R(\mathcal{C}_n(Q_X)) \gtrsim H(Q_X) - R$  and  $P_e(\mathcal{C}_n(Q_X), P_{Y|X}) \lesssim e^{-n[E - D(Q_X \| P_X)]}$ .
3. The correct decoding exponents for channel coding and fixed-rate Slepian-Wolf coding can be interpreted in a similar way. Note that in channel coding, to maximize the correct decoding probability one has to minimize the correct decoding exponent; this is why in (13) minimization (instead of maximization) is used.

Therefore, it should be clear that to characterize the reliability functions for channel coding and fixed-rate Slepian-Wolf coding, it suffices to focus on constant composition codes. It will be shown in the next section that a similar reduction holds for variable-rate Slepian-Wolf coding. Indeed, the reliability function for constant component codes plays a predominant role in determining the fundamental rate-error tradeoff in variable-rate Slepian-Wolf coding.

### 3. Variable-Rate Slepian-Wolf Coding: Above the Slepian-Wolf Limit

A variable-rate Slepian-Wolf code  $\phi_n(\cdot)$  is a mapping from  $\mathcal{X}^n$  to a binary prefix code  $\mathcal{B}_n$ . Let  $l(\phi_n(x^n))$  denote the length of binary string  $\phi_n(x^n)$ . The rate of variable-rate Slepian-Wolf code  $\phi_n(\cdot)$  is defined as

$$R(\phi_n, P_{XY}) = \frac{1}{n \log_2 e} \mathbb{E}[l(\phi_n(X^n))].$$

It is worth noting that  $R(\phi_n, P_{XY})$  depends on  $P_{XY}$  only through  $P_X$ .

Given  $\phi_n(X^n)$  and  $Y^n$ , the output of the optimal *maximum a posteriori* (MAP) decoder is

$$\begin{aligned} \hat{X}^n &= \arg \min_{x^n: \phi_n(x^n) = \phi_n(X^n)} - \sum_{i=1}^n \log P_{X|Y}(x_i | Y_i) \\ &= \arg \min_{x^n: \phi_n(x^n) = \phi_n(X^n)} - \sum_{i=1}^n \log P_{XY}(x_i, Y_i), \end{aligned}$$

where the ties are broken in an arbitrary manner. The decoding error probability of variable-rate Slepian-Wolf code  $\varphi_n(\cdot)$  is defined as

$$P_e(\varphi_n, P_{XY}) = \Pr\{\widehat{X}^n \neq X^n\}.$$

The correct decoding probability of Slepian-Wolf code  $\varphi_n(\cdot)$  is defined as

$$P_c(\varphi_n, P_{XY}) = 1 - P_e(\varphi_n, P_{XY}).$$

**Definition 4.** Given a joint probability distribution  $P_{XY}$ , we say that an error exponent  $E \geq 0$  is achievable with variable-rate Slepian-Wolf codes at rate  $R$  if for any  $\delta > 0$ , there exists a sequence of variable-rate Slepian-Wolf codes  $\{\varphi_n\}$  such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\varphi_n, P_{XY}) &\leq R + \delta, \\ \limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_e(\varphi_n, P_{XY}) &\geq E - \delta. \end{aligned}$$

The largest achievable error exponent at rate  $R$  is denoted by  $E_v(P_{XY}, R)$ . The function  $E_v(P_{XY}, \cdot)$  is referred to as the reliability function of variable-rate Slepian-Wolf coding.

The power of variable-rate Slepian-Wolf coding results from its flexibility in rate allocation. Since there are only polynomial number of types for any given  $n$  (cf. (1)), the encoder can convey the type information to the decoder using negligible amount of rate when  $n$  is large enough. Therefore, without loss of much generality, we can assume that the type of  $X^n$  is known to the decoder. Under this assumption, an optimal fixed-rate Slepian-Wolf encoder of rate  $R$  should partition  $\mathcal{T}_n(P)$  into  $\min\{|\mathcal{T}_n(P)|, e^{nR}\}$  disjoint subsets for each  $P \in \mathcal{P}_n$ . It can be seen that the rate allocated to  $\mathcal{T}_n(P)$  is always  $R$  if  $|\mathcal{T}_n(P)| \geq e^{nR}$ . In general, the type  $Q_X^*$  that dominates the error probability of fixed-rate Slepian-Wolf coding is different from  $P_X$ . In contrast, for variable-rate Slepian-Wolf coding, we can losslessly compress the sequences of types that are bounded away  $P_X$  by allocating enough rate to those type classes (but its contribution to the overall rate is still negligible since the probability of those type classes are extremely small), and therefore, effectively eliminate the dominant error event in fixed-rate Slepian-Wolf coding. As a consequence, the types that can cause decoding error in variable-rate Slepian-Wolf coding must be very close to  $P_X$ . This is the main intuition underlying the proof of the following theorem. A similar argument has been used in the context of variable-rate Slepian-Wolf coding under mismatched decoding [16].

**Theorem 1.**  $E_v(P_{XY}, R) = E(P_X, P_{Y|X}, H(P_X) - R)$ .

**Proof.** The proof is divided into two parts. Firstly, we shall show that  $E_v(P_{XY}, R) \geq E(P_X, P_{Y|X}, H(P_X) - R)$ . The main idea is that one can use a constant composition code  $\mathcal{C}_n$  of type approximately  $P_X$  and rate approximately  $H(P_X) - R$  to construct a variable-rate Slepian-Wolf code  $\varphi_{n'}(\cdot)$  with  $n' \approx n$ ,  $R(\varphi_{n'}, P_{XY}) \approx R$ , and  $P_e(\varphi_{n'}, P_{XY}) \leq P_{e,\max}(\mathcal{C}_n, P_{Y|X})$ .

By Definition 2, for any  $\delta > 0$ , there exists a sequence of constant composition codes  $\{\mathcal{C}_n\}$  with  $\mathcal{C}_n \subseteq \mathcal{T}_n(P_n)$  for some  $P_n \in \mathcal{P}_n(\mathcal{X})$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_n - P_X\| &= 0, \\ \liminf_{n \rightarrow \infty} R(\mathcal{C}_n) &\geq H(P_X) - R - \delta, \\ \limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_{e,\max}(\mathcal{C}_n, P_{Y|X}) &\geq E(P_X, P_{Y|X}, H(P_X) - R) - \delta. \end{aligned}$$

Since  $P_X(x) > 0$  for all  $x \in \mathcal{X}$ , we have

$$\max_{P \in \mathcal{P}_n(\mathcal{X}) \cap \mathcal{E}(\delta)} \max_x \frac{P_n(x)}{P(x)} \leq (1 + \delta)^2$$

for all sufficiently  $n$ , where

$$\mathcal{E}(\delta) = \left\{ P \in \mathcal{P}(\mathcal{X}) : \max_x \frac{P_X(x)}{P(x)} \leq 1 + \delta, H(P) \leq H(P_X) + \delta, D(P||P_X) \leq \delta \right\}.$$

Let  $k_n = \lceil (1 + \delta)^2 n \rceil$ . When  $n$  is large enough, we can, for each  $P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)$ , construct a constant composition code  $\mathcal{C}'_{k_n}(P)$  of length  $k_n$  and type  $P$  by appending a fixed sequence in  $\mathcal{X}^{k_n-n}$  to each codeword in  $\mathcal{C}_n$ . It is easy to see that

$$|\mathcal{C}'_{k_n}(P)| = |\mathcal{C}_n|, \tag{25}$$

$$P_{e,\max}(\mathcal{C}'_{k_n}(P), P_{Y|X}) = P_{e,\max}(\mathcal{C}_n, P_{Y|X}) \tag{26}$$

for all  $P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)$ . One can readily show by invoking the covering lemma in [17] that for each  $P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)$ , there exist  $L(k_n)$  permutations  $\pi_1, \dots, \pi_{L(k_n)}$  of the integers  $1, \dots, k_n$  such that

$$\bigcup_{i=1}^{L(k_n)} \pi_i(\mathcal{C}'_{k_n}(P)) = \mathcal{T}_{k_n}(P),$$

where

$$L(k_n) \triangleq \max_{P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)} \left\lfloor |\mathcal{C}'_{k_n}(P)|^{-1} |\mathcal{T}_{k_n}(P)| \log |\mathcal{T}_{k_n}(P)| + 1 \right\rfloor.$$

In view of (25), we can rewrite  $L(k_n)$  as

$$L(k_n) = \max_{P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)} \left\lfloor |\mathcal{C}_n|^{-1} |\mathcal{T}_{k_n}(P)| \log |\mathcal{T}_{k_n}(P)| + 1 \right\rfloor.$$

Note that

$$P_{e,\max}(\pi_i(\mathcal{C}'_{k_n}(P)), P_{Y|X}) = P_{e,\max}(\mathcal{C}'_{k_n}(P), P_{Y|X}), \quad i = 1, 2, \dots, L(k_n). \tag{27}$$

Given  $\pi_1(\mathcal{C}'_{k_n}(P)), \dots, \pi_{L(k_n)}(\mathcal{C}'_{k_n}(P))$ , we can partition  $\mathcal{T}_{k_n}(P)$  into  $L(k_n)$  disjoint subsets:

$$\begin{aligned} \mathcal{T}_{k_n}(P, 1) &= \pi_1(\mathcal{C}'_{k_n}(P)), \\ \mathcal{T}_{k_n}(P, i) &= \pi_i(\mathcal{C}'_{k_n}(P)) \setminus \bigcup_{j=1}^{i-1} \pi_j(\mathcal{C}'_{k_n}(P)), \quad i = 2, \dots, L(k_n). \end{aligned}$$

It is clear that

$$P_{e,\max}(\mathcal{T}_{k_n}(P, i), P_{Y|X}) \leq P_{e,\max}(\pi_i(\mathcal{C}'_{k_n}(P)), P_{Y|X}), \quad i = 1, 2, \dots, L(k_n). \tag{28}$$

Now construct a sequence of variable-rate Slepian-Wolf codes  $\{\phi_{k_n}(\cdot)\}$  as follows.

1. The encoder sends the type of  $x^{k_n}$  to the decoder, where each type is uniquely represented by a binary sequence of length  $m_1(k_n)$ .
2. If  $x^{k_n} \in \mathcal{T}_{k_n}(P)$  for some  $P \notin \mathcal{E}(\delta)$ , the encoder sends  $x^{k_n}$  losslessly to the decoder, where each  $x^{k_n} \in \mathcal{T}_{k_n}(P)$  is uniquely represented by a binary sequence of length  $m_2(k_n)$ .

3. If  $x^{k_n} \in \mathcal{T}_{k_n}(P)$  for some  $P \in \mathcal{E}(\delta)$ , the encoder finds the set  $\pi_{i^*}(\mathcal{C}'_{k_n}(P))$  that contains  $x^{k_n}$  and sends the index  $i^*$  to the decoder, where each index in  $\{1, 2, \dots, L(k_n)\}$  is uniquely represented by a binary sequence of length  $m_3(k_n)$ .

Specifically, we choose

$$\begin{aligned} m_1(k_n) &= \lceil \log_2 |\mathcal{P}_{k_n}(\mathcal{X})| \rceil, \\ m_2(k_n) &= \max_{P \in \mathcal{P}_{k_n}(\mathcal{X})} \lceil \log_2 |\mathcal{T}_{k_n}(P)| \rceil, \\ m_3(k_n) &= \lceil \log_2 L(k_n) \rceil. \end{aligned}$$

Note that

$$R(\phi_{k_n}, P_{XY}) = \frac{m_1(k_n) + (1 - \theta)m_2(k_n) + \theta m_3(k_n)}{k_n \log_2 e},$$

where

$$\theta = \sum_{P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)} \Pr\{X^{k_n} \in \mathcal{T}_{k_n}(P)\}.$$

It is easy to verify (cf. (1)–(3)) that

$$\begin{aligned} m_1(k_n) &\leq |\mathcal{X}| \log_2(k_n + 1) + 1, \\ m_2(k_n) &\leq k_n \log_2 |\mathcal{X}| + 1, \\ 1 - \theta &\leq (k_n + 1)^{|\mathcal{X}|} e^{-k_n \delta}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\phi_{k_n}, P_{XY}) &= \limsup_{n \rightarrow \infty} \frac{m_3(k_n)}{k_n \log_2 e} \\ &\leq \max_{P \in \mathcal{E}(\delta)} H(P) - \frac{1}{(1 + \delta)^2} \liminf_{n \rightarrow \infty} R(\mathcal{C}_n) \\ &\leq H(P_X) + \delta - \frac{H(P_X) - R - \delta}{(1 + \delta)^2}. \end{aligned} \tag{29}$$

By (26)–(28) and the construction of  $\phi_{k_n}(\cdot)$ , it is clear that

$$\begin{aligned} P_e(\phi_{k_n}, P_{XY}) &= \sum_{P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)} \sum_{i=1}^{L(k_n)} \Pr\{X^{k_n} \in \mathcal{T}_{k_n}(P, i)\} \Pr\{\hat{X}^n \neq X^n | X^{k_n} \in \mathcal{T}_{k_n}(P, i)\} \\ &\leq \sum_{P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)} \sum_{i=1}^{L(k_n)} \Pr\{X^{k_n} \in \mathcal{T}_{k_n}(P, i)\} P_{e, \max}(\mathcal{T}_{k_n}(P, i), P_{Y|X}) \\ &\leq \sum_{P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)} \sum_{i=1}^{L(k_n)} \Pr\{X^{k_n} \in \mathcal{T}_{k_n}(P, i)\} P_{e, \max}(\pi_i(\mathcal{C}'_{k_n}(P)), P_{Y|X}) \\ &= \sum_{P \in \mathcal{P}_{k_n}(\mathcal{X}) \cap \mathcal{E}(\delta)} \sum_{i=1}^{L(k_n)} \Pr\{X^{k_n} \in \mathcal{T}_{k_n}(P, i)\} P_{e, \max}(\mathcal{C}_n, P_{Y|X}) \\ &\leq P_{e, \max}(\mathcal{C}_n, P_{Y|X}), \end{aligned}$$

which implies

$$\begin{aligned} \limsup_{n \rightarrow \infty} -\frac{1}{k_n} \log P_e(\varphi_{k_n}, P_{XY}) &\geq \limsup_{n \rightarrow \infty} -\frac{1}{k_n} \log P_{e, \max}(\mathcal{C}_n, P_{Y|X}) \\ &\geq \frac{E(P_X, P_{Y|X}, H(P_X) - R) - \delta}{(1 + \delta)^2}. \end{aligned} \tag{30}$$

In view of (29), (30), and the fact that  $\delta > 0$  is arbitrary, we must have  $E_v(P_{XY}, R) \geq E(P_X, P_{Y|X}, H(P_X) - R)$  (cf. Definition 4).

Now we proceed to show that  $E_v(P_{XY}, R) \leq E(P_X, P_{Y|X}, H(P_X) - R)$ . The main idea is that one can extract a constant composition code of type approximately  $P_X$  and rate approximately  $H(X) - R$  or greater from a given variable-rate Slepian-Wolf code  $\varphi_n(\cdot)$  of rate approximately  $R$  such that the average decoding error probability of this constant composition code over channel  $P_{Y|X}$  is bounded from above by  $\gamma P_e(\varphi_n, P_{XY})$ , where  $\gamma$  is a constant that does not depend on  $n$ .

By Definition 4, for any  $\delta > 0$ , there exists a sequence of variable-rate Slepian-Wolf codes  $\{\varphi_n\}$  such that

$$\limsup_{n \rightarrow \infty} R(\varphi_n, P_{XY}) \leq R + \delta, \tag{31}$$

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_e(\varphi_n, P_{XY}) \geq E_v(P_{XY}, R) - \delta. \tag{32}$$

Suppose  $\varphi_n(\cdot)$  induces a partition of  $\mathcal{T}_n(P)$ ,  $P \in \mathcal{P}_n(\mathcal{X})$ , into  $N_n(P)$  disjoint subsets  $\mathcal{T}_n(P, 1), \dots, \mathcal{T}_n(P, N_n(P))$ . Here the partition is defined as follows:  $\varphi_n(x^n) = \varphi_n(\tilde{x}^n)$  if  $x^n, \tilde{x}^n \in \mathcal{T}_n(P, i)$  for some  $i$ , and  $\varphi_n(x^n) \neq \varphi_n(\tilde{x}^n)$  if  $x^n \in \mathcal{T}_n(P, i), \tilde{x}^n \in \mathcal{T}_n(P, j)$  for  $i \neq j$ . Let

$$r(\mathcal{T}_n(P)) = \frac{1}{n \log_2 e} \mathbb{E}[l(\varphi_n(X^n)) | X^n \in \mathcal{T}_n(P)], \quad P \in \mathcal{P}_n(\mathcal{X}).$$

It follows from the source coding theorem that

$$r(\mathcal{T}_n(P)) \geq \frac{1}{n} \sum_{i=1}^{N_n(P)} \frac{|\mathcal{T}_n(P, i)|}{|\mathcal{T}_n(P)|} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P, i)|}. \tag{33}$$

Define

$$\begin{aligned} \mathcal{F}_n(\delta) &= \left\{ (P, i) : \frac{1}{n} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P, i)|} \leq R + 2\delta, P \in \mathcal{P}_n(\mathcal{X}), i = 1, 2, \dots, N_n(P) \right\}, \\ \mathcal{F}_n^c(\delta) &= \{ (P, i) \notin \mathcal{F}_n(\delta) : P \in \mathcal{P}_n(\mathcal{X}), i = 1, 2, \dots, N_n(P) \}, \\ \mathcal{G}_n(\gamma) &= \left\{ (P, i) : \Pr\{\hat{X}^n \neq X^n | X^n \in \mathcal{T}_n(P, i)\} \leq \gamma P_e(\varphi_n, P_{XY}), P \in \mathcal{P}_n(\mathcal{X}), i = 1, 2, \dots, N_n(P) \right\}, \\ \mathcal{G}_n^c(\gamma) &= \{ (P, i) \notin \mathcal{G}_n(\gamma) : P \in \mathcal{P}_n(\mathcal{X}), i = 1, 2, \dots, N_n(P) \}, \end{aligned}$$

where

$$\gamma > \frac{R + 2\delta}{\delta}. \tag{34}$$

Note that

$$\begin{aligned} R(\varphi_n, P_{XY}) &= \sum_{P \in \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} r(\mathcal{T}_n(P)) \\ &\geq \frac{1}{n} \sum_{P \in \mathcal{P}_n(\mathcal{X})} \sum_{i=1}^{N_n(P)} \Pr\{X^n \in \mathcal{T}_n(P, i)\} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P, i)|} \end{aligned} \tag{35}$$

$$\begin{aligned} &\geq \frac{1}{n} \sum_{(P,i) \in \mathcal{F}_n^c(\delta)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P,i)|} \\ &\geq (R + 2\delta) \sum_{(P,i) \in \mathcal{F}_n^c(\delta)} \Pr\{X^n \in \mathcal{T}_n(P,i)\}, \end{aligned} \tag{36}$$

where (35) is due to (33). Combing (31) and (36) yields

$$\limsup_{n \rightarrow \infty} \sum_{(P,i) \in \mathcal{F}_n^c(\delta)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \leq \frac{R + \delta}{R + 2\delta}. \tag{37}$$

Moreover, we have

$$\sum_{(P,i) \in \mathcal{G}_n^c(\gamma)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \leq \frac{1}{\gamma} \tag{38}$$

since otherwise

$$\begin{aligned} P_e(\varphi_n, P_{XY}) &= \sum_{P \in \mathcal{P}_n(\mathcal{X})} \sum_{i=1}^{N_n(P)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \Pr\{\hat{X}^n \neq X^n | X^n \in \mathcal{T}_n(P,i)\} \\ &\geq \sum_{(P,i) \in \mathcal{G}_n^c(\gamma)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \Pr\{\hat{X}^n \neq X^n | X^n \in \mathcal{T}_n(P,i)\} \\ &> \gamma P_e(\varphi_n, P_{XY}) \sum_{(P,i) \in \mathcal{G}_n^c(\gamma)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \\ &\geq P_e(\varphi_n, P_{XY}), \end{aligned}$$

which is absurd.

Define

$$\begin{aligned} \mathcal{S}_n(\delta) &= \left\{ P \in \mathcal{P}_n(\mathcal{X}) : H(P) \geq H(P_X) - \delta, \max_x \frac{P(x)}{P_X(x)} \leq 1 + \delta \right\}, \\ \mathcal{S}_n^c(\delta) &= \mathcal{P}_n(\mathcal{X}) \setminus \mathcal{S}_n(\delta), \\ \mathcal{D}_n(\delta, \gamma) &= \{(P, i) : (P, i) \in \mathcal{F}_n(\delta) \cap \mathcal{G}_n(\gamma), P \in \mathcal{S}_n(\delta)\}, \\ \mathcal{D}_n^c(\delta, \gamma) &= \{(P, i) \notin \mathcal{D}_n(\delta, \gamma) : P \in \mathcal{P}_n(\mathcal{X}), i = 1, 2, \dots, N_n(P)\}. \end{aligned}$$

It follows from the weak law of large numbers that

$$\lim_{n \rightarrow \infty} \sum_{P \in \mathcal{S}_n^c(\delta)} \Pr\{X^n \in \mathcal{T}_n(P)\} = 0. \tag{39}$$

We have

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \sum_{(P,i) \in \mathcal{D}_n(\delta, \gamma)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \\ &= \liminf_{n \rightarrow \infty} \left\{ 1 - \sum_{(P,i) \in \mathcal{D}_n^c(\delta, \gamma)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \right\} \\ &\geq \liminf_{n \rightarrow \infty} \left\{ 1 - \sum_{(P,i) \in \mathcal{F}_n^c(\delta)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} - \sum_{(P,i) \in \mathcal{G}_n^c(\gamma)} \Pr\{X^n \in \mathcal{T}_n(P,i)\} \right. \\ &\quad \left. - \sum_{P \in \mathcal{S}_n^c(\delta)} \Pr\{X^n \in \mathcal{T}_n(P)\} \right\} \end{aligned}$$

$$\geq 1 - \frac{R + \delta}{R + 2\delta} - \frac{1}{\gamma} \tag{40}$$

$$> 0, \tag{41}$$

where (40) is due to (37)–(39), and (41) is due to (34). Therefore,  $\mathcal{D}_n(\delta, \gamma)$  is non-empty for all sufficiently large  $n$ . Pick an arbitrary  $(P_n^*, i^*)$  from  $\mathcal{D}_n(\delta, \gamma)$  for each sufficiently large  $n$ . We can construct a constant composition code  $\mathcal{C}_{m_n}$  of length  $m_n = \lceil (1 + \delta)n \rceil$  and type  $P_{m_n}$  for some  $P_{m_n} \in \mathcal{P}_{m_n}(\mathcal{X})$  by concatenating a fixed sequence in  $\mathcal{X}^{m_n-n}$  to each sequence in  $\mathcal{T}_n(P_n^*, i^*)$  such that

$$\lim_{n \rightarrow \infty} \|P_{m_n} - P_X\| = 0. \tag{42}$$

Note that

$$\begin{aligned} \liminf_{n \rightarrow \infty} R(\mathcal{C}_{m_n}) &= \liminf_{n \rightarrow \infty} \frac{1}{m_n} \log |\mathcal{T}_n(P_n^*, i^*)| \\ &\geq \liminf_{n \rightarrow \infty} \frac{n}{m_n} \left[ \frac{1}{n} \log |\mathcal{T}_n(P_n^*)| - R - 2\delta \right] \\ &\geq \frac{H(P_X) - R - 3\delta}{1 + \delta}. \end{aligned} \tag{43}$$

Moreover, since

$$P_e(\mathcal{C}_{m_n}, P_{Y|X}) = \Pr\{\widehat{X}^n \neq X^n | X^n \in \mathcal{T}_n(P_n^*, i^*)\} \leq \gamma P_e(\varphi_n, P_{XY}),$$

it follows from (32) that

$$\limsup_{n \rightarrow \infty} -\frac{1}{m_n} \log P_e(\mathcal{C}_{m_n}, P_{Y|X}) \geq \frac{E_v(P_{XY}, R) - \delta}{1 + \delta}. \tag{44}$$

In view of (42)–(44) and the fact that  $\delta > 0$  is arbitrary, we must have  $E_v(P_{XY}, R) \leq E(P_X, P_{Y|X}, H(P_X) - R)$  (cf. Definition 2). The proof is complete.  $\square$

The following result is an immediate consequence of Theorem 1 and Proposition 4.

**Corollary 1.** Define

$$\begin{aligned} E_{v,ex}(P_{XY}, R) &= E_{ex}(P_X, P_{Y|X}, H(P_X) - R), \\ E_{v,rc}(P_{XY}, R) &= E_{rc}(P_X, P_{Y|X}, H(P_X) - R), \\ E_{v,sp}(P_{XY}, R) &= E_{sp}(P_X, P_{Y|X}, H(P_X) - R). \end{aligned}$$

We have

1.  $E_v(P_{XY}, R) \geq \max\{E_{v,ex}(P_{XY}, R), E_{v,rc}(P_{XY}, R)\};$
2.  $E_v(P_{XY}, R) \leq E_{v,sp}(P_{XY}, R)$  with the possible exception of  $R = H(P_X) - R_{sp}^\infty(P_X, P_{Y|X})$  at which point the inequality not necessarily holds.

**Remark 5.**

1. We have  $E_v(P_{XY}, R) = \infty$  for  $R > H(P_X) - R_{ex}^\infty(P_X, P_{Y|X})$ , and  $E_v(P_{XY}, R) < \infty$  for  $R < H(P_X) - R_{sp}^\infty(P_X, P_{Y|X})$ . Therefore,  $H(P_X) - R_{ex}^\infty(P_X, P_{Y|X})$  and  $H(P_X) - R_{sp}^\infty(P_X, P_{Y|X})$  are respectively the upper bound and the lower bound on the zero-error rate of variable-rate Slepian-Wolf coding.
2. In view of (11), we have

$$E_v(P_{XY}, R) = E_{v,sp}(P_{XY}, R) = E_{sp}(P_X, P_{Y|X}, H(P_X) - R)$$



for  $R \in [H(X|Y), H(P_X) - R_{cr}(P_X, P_{Y|X})]$ . Note that

$$E_{v,sp}(P_{XY}, R) \geq E_{f,sp}(P_{XY}, R) \geq E_f(P_{XY}, R),$$

where the first inequality is strict unless the minimum in (20) is achieved at  $Q_X = P_X$ , (i.e.,  $P_{X^{(\rho)}} = P_X$ , where  $P_{X^{(\rho)}}$  is the marginal distribution of  $X^{(\rho)}$  induced by  $P_{Y^{(\rho)}}$  and  $P_{X^{(\rho)}|Y^{(\rho)}}$  in (21) and (22)). Therefore, variable-rate Slepian-Wolf coding can outperform fixed-rate Slepian-Wolf coding in terms of rate-error tradeoff.

For  $R > H(P_X) - R_{cr}(P_X, P_{Y|X})$ , it is possible to obtain upper bounds on  $E_v(P_{XY}, R)$  that are tighter than  $E_{v,sp}(P_{XY}, R)$ . Let  $E_{ex}(P_{Y|X}, R)$  and  $E_{sp}(P_{Y|X}, R)$  be respectively the expurgated exponent and the sphere packing exponent of channel  $P_{Y|X}$ . The straight-line exponent  $E_{sl}(P_{Y|X}, R)$  of channel  $P_{Y|X}$  [10] is the smallest linear function of  $R$  which touches the curve  $E_{sp}(P_{Y|X}, R)$  and also satisfies

$$E_{sl}(P_{Y|X}, 0) = E_{ex}(P_{Y|X}, 0),$$

where  $E_{ex}(P_{Y|X}, 0)$  is assumed to be finite. Let  $R_{sl}(P_{Y|X})$  be the point at which  $E_{sl}(P_{Y|X}, R)$  and  $E_{sp}(P_{Y|X}, R)$  coincide. It is well known [10] that  $E(P_{Y|X}, R) \leq E_{sl}(P_{Y|X}, R)$  for  $R \in (0, R_{sl}(P_{Y|X})]$ . Since  $E(P_X, P_{Y|X}, R) \leq E(P_{Y|X}, R)$ , it follows from Theorem 1 that

$$E_v(P_{XY}, R) \leq E_{sl}(P_{Y|X}, H(P_X) - R)$$

for  $R \in [\max\{H(P_X) - R_{sl}(P_{Y|X}), 0\}, H(P_X))$ .

Note that the straight-line exponent holds for arbitrary block codes; one can obtain further improvement at high rates by leveraging bounds tailored to constant composition codes. Let  $E_{ex}^*(Q_X, P_{Y|X}, 0)$  be the concave upper envelope of  $E_{ex}(Q_X, P_{Y|X}, 0)$  considered as a function of  $Q_X$ . In view of ([7], Exercise 5.21), we have

$$E(Q_X, P_{Y|X}, R) \leq E_{ex}^*(Q_X, P_{Y|X}, 0)$$

for any  $Q_X \in \mathcal{P}(\mathcal{X})$  and  $R > 0$ . Now it follows from Theorem 1 that

$$E_v(P_{XY}, R) \leq E_{ex}^*(P_X, P_{Y|X}, 0)$$

for  $R < H(P_X)$ .

The following theorem provides the second order expansion of  $E_v(P_{XY}, R)$  at the Slepian-Wolf limit.

**Theorem 2.** Assuming  $R_{cr}(P_X, P_{X|Y}) < I(P_X, P_{Y|X})$  (see Proposition 1 for the necessary and sufficient condition), we have

$$\lim_{r \downarrow 0} \frac{E_v(P_{XY}, H(X|Y) + r)}{r^2} = \frac{1}{2} \left[ \sum_{x,y} P_{XY}(x,y) \tau^2(x,y) - \sum_x P_X(x) \left( \sum_y \tau(x,y) P_{Y|X}(y|x) \right)^2 \right]^{-1},$$

where  $\tau(x,y) = \log P_Y(y) - \log P_{Y|X}(y|x)$ .

**Remark 6.** If  $R_{cr}(P_X, P_{Y|X}) = I(P_X, P_{Y|X})$ , then we have  $E_{v,rc}(P_{XY}, R) = R - H(X|Y)$  for  $R \geq H(X|Y)$ , which implies

$$\lim_{r \downarrow 0} \frac{E_v(P_{XY}, H(X|Y) + r)}{r^2} = \infty.$$

It is also worth noting that the second order expansion of  $E_v(P_{XY}, R)$  at the Slepian-Wolf limit yields the redundancy-error tradeoff constant of variable-rate Slepian-Wolf coding derived in [18].

**Proof.** Since  $R_{cr}(P_X, P_{X|Y}) < I(P_X; P_{Y|X})$ , it follows that  $H(X|Y) + r \in (H(X|Y), H(P_X) - R_{cr}(P_X, P_{Y|X}))$  when  $r$  ( $r > 0$ ) is sufficiently close to zero. In this case, we have

$$\begin{aligned} \frac{E_v(P_{XY}, H(X|Y) + r)}{r^2} &= \frac{E_{sp}(P_X, P_{Y|X}, I(P_X, P_{Y|X}) - r)}{r^2} \\ &= \min_{Q_{Y|X}: I(P_X, Q_{Y|X}) \leq I(P_X, P_{Y|X}) - r} \frac{D(Q_{Y|X} \| P_{Y|X} | P_X)}{r^2} \\ &= \min_{Q_{Y|X}: I(P_X, Q_{Y|X}) = I(P_X, P_{Y|X}) - r} \frac{D(Q_{Y|X} \| P_{Y|X} | P_X)}{r^2}, \end{aligned}$$

where the last equality follows from the fact that  $E_{sp}(P_X, P_{Y|X}, R)$  is a strictly decreasing convex function of  $R$  for  $R \in (R_{sp}^\infty(P_X, P_{Y|X}), I(P_X, P_{Y|X})]$ .

Let  $\Delta(x, y) = Q_{Y|X}(y|x) - P_{Y|X}(y|x)$  for  $x \in \mathcal{X}, y \in \mathcal{Y}$ . Let  $\Delta(y) = \sum_x P_X(x)\Delta(x, y)$  for  $y \in \mathcal{Y}$ . By the Taylor expansion,

$$\begin{aligned} I(P_X, Q_{Y|X}) &= \sum_{x,y} P_X(x)(P_{Y|X}(y|x) + \Delta(x, y)) \log(P_{Y|X}(y|x) + \Delta(x, y)) \\ &\quad - \sum_y (P_Y(y) + \Delta(y)) \log(P_Y(y) + \Delta(y)) \\ &= \sum_{x,y} P_X(x)(P_{Y|X}(y|x) + \Delta(x, y)) \left( \log P_{Y|X}(y|x) + \frac{\Delta(x, y)}{P_{Y|X}(y|x)} + o(\Delta(x, y)) \right) \\ &\quad - \sum_y (P_Y(y) + \Delta(y)) \left( \log P_Y(y) + \frac{\Delta(y)}{P_Y(y)} + o(\Delta(y)) \right) \\ &= I(P_X, P_{Y|X}) - \sum_y (\Delta(y) + \Delta(y) \log P_Y(y) + o(\Delta_y)) \\ &\quad + \sum_{x,y} P_X(x)(\Delta(x, y) + \Delta(x, y) \log P_{Y|X}(y|x) + o(\Delta(x, y))) \end{aligned}$$

and

$$\begin{aligned} D(Q_{Y|X} \| P_{Y|X} | P_X) &= \sum_{x,y} P_X(x)Q_{Y|X}(y|x) \log \frac{Q_{Y|X}(y|x)}{P_{Y|X}(y|x)} \\ &= \sum_{x,y} P_X(x)(P_{Y|X}(y|x) + \Delta(x, y)) \log \left( 1 + \frac{\Delta(x, y)}{P_{Y|X}(y|x)} \right) \\ &= \sum_{x,y} P_X(x)(P_{Y|X}(y|x) + \Delta(x, y)) \left( \frac{\Delta(x, y)}{P_{Y|X}(y|x)} - \frac{\Delta^2(x, y)}{2P_{Y|X}^2(y|x)} + o(\Delta^2(x, y)) \right) \\ &= \sum_{x,y} P_X(x) \left( \frac{\Delta^2(x, y)}{2P_{Y|X}(y|x)} + o(\Delta^2(x, y)) \right). \end{aligned}$$

Here  $f(z) = o(z)$  means  $\lim_{z \rightarrow 0} \frac{f(z)}{z} = 0$ .

As  $r \downarrow 0$ , we have  $\Delta(y) \rightarrow 0, \Delta(x, y) \rightarrow 0$  for all  $x \in \mathcal{X}, y \in \mathcal{Y}$ . Therefore, by ignoring the high order terms which do not affect the limit, we get

$$\lim_{r \downarrow 0} \frac{E_v(P_{XY}, H(X|Y) + r)}{r^2} = \lim_{r \downarrow 0} \min \sum_{x,y} \frac{P_X(x)\Delta^2(x, y)}{2P_{Y|X}(y|x)r^2}, \tag{45}$$

where the minimization is over  $\Delta(x, y)$  ( $x \in \mathcal{X}, y \in \mathcal{Y}$ ) subject to the constraints

1.  $\sum_y \Delta(x, y) = 0$  for all  $x \in \mathcal{X}$ ,
2.  $\sum_{x,y} P_X(x) \tau(x, y) \Delta(x, y) = r$ .

Introduce the Lagrange multipliers  $\alpha(x)$  ( $x \in \mathcal{X}$ ) and  $\beta$  for these constraints, and define

$$G = \sum_{x,y} \frac{P_X(x) \Delta^2(x, y)}{2P_{Y|X}(y|x)} - \sum_{x,y} \alpha(x) \Delta(x, y) - \beta \sum_{x,y} P_X(x) \tau(x, y) \Delta(x, y).$$

The Karush-Kuhn-Tucker conditions yield

$$\frac{\partial G}{\partial \Delta(x, y)} = -\alpha(x) - \beta P_X(x) \tau(x, y) + \frac{P_X(x) \Delta(x, y)}{P_{Y|X}(y|x)} = 0, \quad x \in \mathcal{X}, y \in \mathcal{Y}.$$

Therefore, we have

$$\Delta(x, y) = \beta \tau(x, y) P_{Y|X}(y|x) + \frac{P_{Y|X}(y|x)}{P_X(x)} \alpha(x). \tag{46}$$

Substituting (46) into constraint 1, we obtain

$$\alpha(x) = -\beta P_X(x) \sum_y \tau(x, y) P_{Y|X}(y|x),$$

which, together with (46), yields

$$\Delta(x, y) = \beta \tau(x, y) P_{Y|X}(y|x) - \beta P_{Y|X}(y|x) \sum_{y'} \tau(x, y') P_{Y|X}(y'|x). \tag{47}$$

Therefore, we have

$$\begin{aligned} & \sum_{x,y} \frac{P_X(x) \Delta^2(x, y)}{2P_{Y|X}(y|x)} \\ &= \frac{\beta^2}{2} \sum_{x,y} P_{XY}(x, y) \left[ \tau(x, y) - \sum_{y'} \tau(x, y') P_{Y|X}(y'|x) \right]^2 \\ &= \frac{\beta^2}{2} \sum_{x,y} P_{XY}(x, y) \left[ \tau^2(x, y) - 2\tau(x, y) \sum_{y'} \tau(x, y') P_{Y|X}(y'|x) + \left( \sum_{y'} \tau(x, y') P_{Y|X}(y'|x) \right)^2 \right] \\ &= \frac{\beta^2}{2} \left[ \sum_{x,y} P_{XY}(x, y) \tau^2(x, y) - \sum_x P_X(x) \left( \sum_y \tau(x, y) P_{Y|X}(y|x) \right)^2 \right]. \end{aligned} \tag{48}$$

Constraint 2 and (47) together yield

$$\begin{aligned} \frac{r^2}{\beta^2} &= \frac{1}{\beta^2} \left( \sum_{x,y} P_X(x) \tau(x, y) \Delta(x, y) \right)^2 \\ &= \left[ \sum_{x,y} P_X(x) \tau(x, y) \left( \tau(x, y) P_{Y|X}(y|x) - P_{Y|X}(y|x) \sum_{y'} \tau(x, y') P_{Y|X}(y'|x) \right) \right]^2 \\ &= \left[ \sum_{x,y} P_{XY}(x, y) \tau^2(x, y) - \sum_x P_X(x) \left( \sum_y \tau(x, y) P_{Y|X}(y|x) \right)^2 \right]^2. \end{aligned} \tag{49}$$

The proof is complete by substituting (48) and (49) back into (45).  $\square$

#### 4. Variable-Rate Slepian-Wolf Coding: Below the Slepian-Wolf Limit

**Definition 5.** Given a joint probability distribution  $P_{XY}$ , we say that a correct decoding exponent  $E^c \geq 0$  is achievable with variable-rate Slepian-Wolf codes at rate  $R$  if for any  $\delta > 0$ , there exists a sequence of variable-rate Slepian-Wolf codes  $\{\varphi_n\}$  such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\varphi_n, P_{XY}) &\leq R + \delta, \\ \liminf_{n \rightarrow \infty} -\frac{1}{n} \log P_c(\varphi_n, P_{XY}) &\leq E^c + \delta. \end{aligned}$$

The smallest achievable correct decoding exponent at rate  $R$  is denoted by  $E_v^c(P_{XY}, R)$ .

In view of Theorem 1, it is tempting to conjecture that  $E_v^c(P_{XY}, R) = E^c(P_X, P_{Y|X}, H(P_X) - R)$ . It turns out this is not true. We shall show that  $E_v^c(P_{XY}, R) = 0$  for all  $R$ . Actually we have a stronger result—the correct decoding probability of variable-rate Slepian-Wolf coding can be bounded away from zero even when  $R < H(X|Y)$ . This is in sharp contrast with fixed-rate Slepian-Wolf coding for which the correct decoding probability decays to zero exponentially fast if the rate is below the Slepian-Wolf limit. To make the statement more precise, we need the following definition.

**Definition 6.** Given a joint probability distribution  $P_{XY}$ , we say that a correct decoding probability  $P_{c,v}(P_{XY}, R)$  is achievable with variable-rate Slepian-Wolf codes at rate  $R$  if for any  $\delta > 0$ , there exists a sequence of variable-rate Slepian-Wolf codes  $\{\varphi_n\}$  such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\varphi_n, P_{XY}) &\leq R + \delta, \\ \limsup_{n \rightarrow \infty} P_c(\varphi_n, P_{XY}) &\geq P_{c,v}(P_{XY}, R) - \delta. \end{aligned}$$

The largest achievable correct decoding probability at rate  $R$  is denoted by  $P_{c,v}^{\max}(P_{XY}, R)$ .

**Theorem 3.**  $P_{c,v}^{\max}(P_{XY}, R) = \frac{R}{H(X|Y)}$  for  $R \in (0, H(X|Y)]$ .

**Remark 7.** It is obvious that  $P_{c,v}^{\max}(P_{XY}, R) = 1$  for  $R > H(X|Y)$ . Moreover, since  $P_{c,v}^{\max}(P_{XY}, R)$  is a monotonically increasing function of  $R$ , it follows that  $P_{c,v}^{\max}(P_{XY}, 0) = 0$ .

**Proof.** The intuition underlying the proof is as follows. Assume the rate is below the Slepian-Wolf limit, i.e.,  $R < H(X|Y)$ . For each type  $P$  in the neighborhood of  $P_X$ , the rate allocated to the type class  $\mathcal{T}_n(P)$  should be no less than  $H(X|Y)$  in order to correctly decode the sequences in  $\mathcal{T}_n(P)$ . However, since almost all the probability are captured by the type classes whose types are in the neighborhood of  $P_X$ , there is not enough rate to protect all of them. Note that if the rate is evenly allocated among these type classes, none of them can get enough rate; consequently, the correct decoding probability goes to zero. A good way is to protect only a portion of them to accumulate enough rate. Specifically, we can protect  $\frac{R}{H(X|Y)}$  fraction of these type classes so that the rate allocated to each of them is about  $H(X|Y)$  and leave the remaining type classes unprotected. It turns out this strategy achieves the maximum correct decoding probability as the block length  $n$  goes to infinity. Somewhat interestingly, although  $E_v^c(P_{XY}, R) \neq E^c(P_X, P_{Y|X}, H(P_X) - R)$ , the function  $E^c(P_X, P_{Y|X}, \cdot)$  does play a fundamental role in establishing the correct result.

The proof is divided into two parts. Firstly, we shall show that  $P_{c,v}^{\max}(P_{XY}, R) \geq \frac{R}{H(X|Y)}$ . For any  $\epsilon > 0$ , define

$$\mathcal{U}(\epsilon) = \{P \in \mathcal{P}(\mathcal{X}) : \|P - P_X\| \leq \epsilon\}.$$

Since  $P_X(x) > 0$  for all  $x \in \mathcal{X}$ , we can choose  $\epsilon$  small enough so that

$$q_{\min}(\epsilon) \triangleq \min_{P \in \mathcal{U}(\epsilon), x \in \mathcal{X}} P(x) > 0.$$

Using Stirling’s approximation

$$\sqrt{2\pi m} \left(\frac{m}{e}\right)^m e^{\frac{1}{12m+1}} < m! < \sqrt{2\pi m} \left(\frac{m}{e}\right)^m e^{\frac{1}{12m}},$$

we have, for any  $P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})$ ,

$$\begin{aligned} \Pr(X^n \in \mathcal{T}_n(P)) &= \frac{n!}{\prod_x (nP(x))!} \prod_x [P_X(x)]^{nP(x)} \\ &\leq \frac{\sqrt{2\pi n} e^{\frac{1}{12n}}}{\prod_x \sqrt{2\pi n P(x)}} e^{-nD(P||P_X)} \\ &\leq \frac{\sqrt{2\pi} e^{\frac{1}{12n}}}{\prod_x \sqrt{2\pi P(x)}} n^{-\frac{|\mathcal{X}|-1}{2}} \\ &\leq \frac{\sqrt{2\pi} e^{\frac{1}{12n}}}{\prod_x \sqrt{2\pi q_{\min}(\epsilon)}} n^{-\frac{|\mathcal{X}|-1}{2}}, \end{aligned}$$

which implies that  $\Pr(X^n \in \mathcal{T}_n(P))$  converges uniformly to zero as  $n \rightarrow \infty$  for all  $P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})$ . Moreover, it follows from the weak law of large numbers that

$$\lim_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr(X^n \in \mathcal{T}_n(P)) = 1.$$

Therefore, for any  $\delta > 0$ ,  $R \in (0, H(X|Y)]$ , and sufficiently large  $n$ , we can find a set  $\mathcal{S}_n \subseteq \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})$  such that

$$\frac{R}{H(X|Y)} - \delta \leq \sum_{P \in \mathcal{S}_n} \Pr(X^n \in \mathcal{T}_n(P)) \leq \frac{R}{H(X|Y)}.$$

Now consider a sequence of variable-rate Slepian-Wolf codes  $\{\varphi_n(\cdot)\}$  specified as follows.

1. The encoder sends of type of  $X^n$  to the decoder, where each type is uniquely represented by a binary sequence of length  $\lceil \log_2 |\mathcal{P}_n(\mathcal{X})| \rceil$ .
2. For each  $P \in \mathcal{S}_n$ , the encoder partitions the type class  $\mathcal{T}_n(P)$  into  $L_n$  subsets  $\mathcal{T}_n(P, 1), \mathcal{T}_n(P, 2), \dots, \mathcal{T}_n(P, L_n)$ . If  $X^n \in \mathcal{T}_n(P)$  for some  $P \in \mathcal{S}_n$ , the encoder finds the subset  $\mathcal{T}_n(P, i^*)$  that contains  $X^n$  and sends the index  $i^*$  to the decoder, where each index in  $\{1, 2, \dots, L_n\}$  is uniquely represented by a binary sequence of length  $\lceil \log_2 L_n \rceil$ .
3. The remaining type classes are left uncoded.

Specifically, we let

$$L_n = \left\lceil \left( 2(n+1)^{|\mathcal{X}|^2} e^{n(H(X|Y)+\delta)} \right) \right\rceil.$$

It follows from ([8], Theorem 2) that for each  $P \in \mathcal{S}_n$ , it is possible to partition the type class  $\mathcal{T}_n(P)$  into  $L_n$  disjoint subsets  $\mathcal{T}_n(P, 1), \mathcal{T}_n(P, 2), \dots, \mathcal{T}_n(P, L_n)$  so that

$$-\frac{1}{n} \log \Pr(\hat{X}^n \neq X^n | X^n \in \mathcal{T}_n(P)) \geq \min_{Q_X \in \mathcal{U}(\epsilon)} \left[ E_{rc}(Q_X, P_{Y|X}, H(Q_X) - H(X|Y) - \delta) - \epsilon \right]$$

uniformly for all  $P \in \mathcal{S}_n$  when  $n$  is sufficiently large. In view of the fact that  $E_{rc}(P_X, P_{Y|X}, I(P_X, P_{Y|X}) - \delta) > 0$  and that  $E_{rc}(Q_X, P_{Y|X}, R)$  as a function of the pair  $(Q_X, R)$  is uniformly equicontinuous, we have

$$\min_{Q_X \in \mathcal{U}(\epsilon)} [E_{rc}(Q_X, P_{Y|X}, H(Q_X) - H(X|Y) - \delta) - \epsilon] \triangleq \kappa_1 > 0$$

for sufficiently small  $\epsilon$ .

For this sequence of constructed variable-rate Slepian-wolf codes  $\{\varphi_n(\cdot)\}$ , it can be readily verified that

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\varphi_n, P_{XY}) &= \limsup_{n \rightarrow \infty} \frac{1}{n \log_2 e} \left[ \lceil \log_2 |\mathcal{P}_n(\mathcal{X})| \rceil + \sum_{P \in \mathcal{S}_n} \Pr\{X^n \in \mathcal{T}_n(P)\} \lceil \log_2 L_n \rceil \right] \\ &\leq \frac{R}{H(X|Y)} (H(X|Y) + \delta) \end{aligned}$$

and

$$\begin{aligned} \limsup_{n \rightarrow \infty} P_c(\varphi_n, R) &\geq \limsup_{n \rightarrow \infty} \sum_{P \in \mathcal{S}_n} \Pr\{X^n \in \mathcal{T}_n(P)\} \left[ 1 - \Pr\{\widehat{X}^n \neq X^n | X^n \in \mathcal{T}_n(P)\} \right] \\ &\geq \limsup_{n \rightarrow \infty} \sum_{P \in \mathcal{S}_n} \Pr\{X^n \in \mathcal{T}_n(P)\} (1 - e^{-n\kappa_1}) \\ &\geq \frac{R}{H(X|Y)} - \delta. \end{aligned}$$

Since  $\delta > 0$  is arbitrary, it follows from Definition 6 that  $P_{c,v}^{\max}(P_{XY}, R) \geq \frac{R}{H(X|Y)}$ .

Now we proceed to prove the other direction. It follows from Definition 6 that for any  $\delta > 0$ , there exists a sequence of variable-rate Slepian-Wolf codes  $\{\varphi_n(\cdot)\}$  with

$$\begin{aligned} \limsup_{n \rightarrow \infty} R(\varphi_n, P_{XY}) &\leq R + \delta, \\ \limsup_{n \rightarrow \infty} P_c(\varphi_n, P_{XY}) &\geq P_{c,v}^{\max}(R) - \delta. \end{aligned}$$

Recall the definition of  $\mathcal{T}_n(P, 1), \dots, \mathcal{T}_n(P, N_n(P))$  as well as  $r(\mathcal{T}_n(P))$  in the proof of Theorem 1. For  $P \in \mathcal{P}_n(\mathcal{X})$ , define

$$\begin{aligned} \mathcal{I}_n(P, \delta) &= \left\{ i : \frac{1}{n} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P, i)|} \leq H(X|Y) - \delta, i = 1, 2, \dots, N_n(P) \right\}, \\ \mathcal{I}_n^c(P, \delta) &= \left\{ i : \frac{1}{n} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P, i)|} > H(X|Y) - \delta, i = 1, 2, \dots, N_n(P) \right\}. \end{aligned}$$

Note that

$$\sum_{i \in \mathcal{I}_n(P, \delta)} \frac{|\mathcal{T}_n(P, i)|}{|\mathcal{T}_n(P)|} \geq 1 - \frac{r(\mathcal{T}_n(P))}{H(X|Y) - \delta}$$

since

$$\begin{aligned} r(\mathcal{T}_n(P)) &\geq \frac{1}{n} \sum_{i=1}^{N_n(P)} \frac{|\mathcal{T}_n(P, i)|}{|\mathcal{T}_n(P)|} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P, i)|} \\ &\geq \frac{1}{n} \sum_{i \in \mathcal{I}_n^c(P, \delta)} \frac{|\mathcal{T}_n(P, i)|}{|\mathcal{T}_n(P)|} \log \frac{|\mathcal{T}_n(P)|}{|\mathcal{T}_n(P, i)|} \\ &\geq (H(X|Y) - \delta) \sum_{i \in \mathcal{I}_n^c(P, \delta)} \frac{|\mathcal{T}_n(P, i)|}{|\mathcal{T}_n(P)|}, \end{aligned} \tag{50}$$

where (50) is due to (33).

Each  $\mathcal{T}_n(P, i)$  can be viewed as a constant composition code of type  $P$  and we have

$$\Pr\{\widehat{X}^n = X^n | X^n \in \mathcal{T}_n(P, i)\} = P_c(\mathcal{T}_n(P, i), P_{Y|X}).$$

Note that for  $P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})$  and  $i \in \mathcal{I}_n(P, \delta)$ ,

$$\begin{aligned} \frac{1}{n} \log |\mathcal{T}_n(P, i)| &\geq \frac{1}{n} \log |\mathcal{T}_n(P)| - H(X|Y) + \delta \\ &\geq H(P) - H(X|Y) + \delta - |\mathcal{X}| \frac{\log(n+1)}{n}. \end{aligned}$$

Therefore, it follows from ([9], Lemma 5) that

$$-\frac{1}{n} \log P_c(\mathcal{T}_n(P, i), P_{Y|X}) \geq \min_{Q_X \in \mathcal{U}(\epsilon)} E^c(Q_X, P_{Y|X}, H(Q_X) - H(X|Y) + \delta - \epsilon) - \epsilon$$

uniformly for all  $P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})$  and  $i \in \mathcal{I}_n(P, \delta)$  when  $n$  is sufficiently large. In view of the fact that  $E^c(P_X, P_{Y|X}, I(P_X, P_{Y|X}) + \delta) > 0$  and that  $E^c(Q_X, P_{Y|X}, R)$  as a function of the pair  $(Q_X, R)$  is uniformly equicontinuous, we have

$$\min_{Q_X \in \mathcal{U}(\epsilon)} \left[ E^c(Q_X, P_{Y|X}, H(Q_X) - H(X|Y) + \delta - \epsilon) - \epsilon \right] \triangleq \kappa_2 > 0$$

for sufficiently small  $\epsilon$ .

Now it is easy to see that

$$\begin{aligned} &\liminf_{n \rightarrow \infty} P_e(\varphi_n, P_{XY}) \\ &\geq \liminf_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} \sum_{i \in \mathcal{I}_n(P, \delta)} \frac{|\mathcal{T}_n(P, i)|}{|\mathcal{T}_n(P)|} \left( 1 - \Pr\{\widehat{X}^n = X^n | X^n \in \mathcal{T}_n(P, i)\} \right) \\ &\geq \liminf_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} \sum_{i \in \mathcal{I}_n(P, \delta)} \frac{|\mathcal{T}_n(P, i)|}{|\mathcal{T}_n(P)|} (1 - e^{-n\kappa_2}) \\ &\geq \liminf_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} \left( 1 - \frac{r(\mathcal{T}_n(P))}{H(X|Y) - \delta} \right) (1 - e^{-n\kappa_2}) \\ &\geq \liminf_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} (1 - e^{-n\kappa_2}) \\ &\quad - \limsup_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} \frac{r(\mathcal{T}_n(P))}{H(X|Y) - \delta} (1 - e^{-n\kappa_2}) \\ &\geq \liminf_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} (1 - e^{-n\kappa_2}) \\ &\quad - \limsup_{n \rightarrow \infty} \sum_{P \in \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} \frac{r(\mathcal{T}_n(P))}{H(X|Y) - \delta} (1 - e^{-n\kappa_2}) \\ &= \liminf_{n \rightarrow \infty} \sum_{P \in \mathcal{U}(\epsilon) \cap \mathcal{P}_n(\mathcal{X})} \Pr\{X^n \in \mathcal{T}_n(P)\} (1 - e^{-n\kappa_2}) \\ &\quad - \limsup_{n \rightarrow \infty} \frac{R(\varphi_n, P_{XY})}{H(X|Y) - \delta} (1 - e^{-n\kappa_2}) \\ &= 1 - \frac{R + \delta}{H(X|Y) - \delta} \end{aligned}$$

which implies

$$\limsup_{n \rightarrow \infty} P_c(\varphi_n, P_{XY}) \leq \frac{R + \delta}{H(X|Y) - \delta}.$$

Therefore, we have

$$P_{c,v}^{\max}(P_{XY}, R) - \delta \leq \frac{R + \delta}{H(X|Y) - \delta}.$$

Since  $\delta > 0$  is arbitrary, this completes the proof.  $\square$

### 5. Example

Consider the joint distribution  $P_{XY}$  over  $\mathbb{Z}_2 \times \mathbb{Z}_2$  with  $P_{X|Y}(1|0) = P_{X|Y}(0|1) = p$  and  $P_Y(0) = \tau$ . We assume  $p \in (0, \frac{1}{2})$ ,  $\tau \in (0, \frac{1}{2}]$ . It is easy to compute that

$$\begin{aligned} P_X(0) &= 1 - P_X(1) = \tau(1 - p) + (1 - \tau)p, \\ P_{Y|X}(1|0) &= 1 - P_{Y|X}(0|0) = \frac{(1 - \tau)p}{\tau(1 - p) + (1 - \tau)p}, \\ P_{Y|X}(0|1) &= 1 - P_{Y|X}(1|1) = \frac{\tau p}{\tau p + (1 - \tau)(1 - p)}. \end{aligned}$$

For this joint distribution, we have  $H(X|Y) = H_b(p)$ , where  $H_b(\cdot)$  is the binary entropy function (i.e.,  $H_b(p) = -p \log p - (1 - p) \log(1 - p)$ ). Given  $R \in [0, \log 2]$ , let  $q$  be the unique number satisfying  $H_b(q) = R$  and  $q \leq \frac{1}{2}$ . It can be verified that

$$\begin{aligned} E_{f,sp}(P_{XY}, R) &= D(q||p), \quad R \in [H_b(p), \log 2], \\ E_f^c(P_{XY}, R) &= D(q||p), \quad R \in [0, H_b(p)]. \end{aligned}$$

Note that

$$\begin{aligned} E_{ex}(Q_X, P_{Y|X}, 0) &= - \sum_{x,x'} Q_X(x) Q_X(x') \log \left[ \sum_y \sqrt{P_{Y|X}(y|x) P_{Y|X}(y|x')} \right] \\ &= -2Q_X(0)Q_X(1) \log \left[ \sum_y \sqrt{P_{Y|X}(y|0) P_{Y|X}(y|1)} \right], \end{aligned}$$

which is a concave function of  $Q_X$ . Therefore,

$$E_{ex}^*(P_X, P_{Y|X}, 0) = E_{ex}(P_X, P_{Y|X}, 0).$$

Moreover, we have

$$\begin{aligned} E_{ex}(P_{Y|X}, 0) &= \max_{Q_X} E_{ex}(Q_X, P_{Y|X}, 0) \\ &= -\frac{1}{2} \log \left[ \sum_y \sqrt{P_{Y|X}(y|0) P_{Y|X}(y|1)} \right]. \end{aligned}$$

It is easy to show that

$$\begin{aligned} E_{v,sp}(P_{XY}, H(P_X)) &= E_{sp}(P_X, P_{X|Y}, 0) \\ &= \min_{Q_Y} \sum_x P_X(x) \sum_y Q_Y(y) \log \frac{Q_Y(y)}{P_{Y|X}(y|x)}, \end{aligned}$$



where the minimizer  $Q_Y^*$  is given by

$$Q_Y^*(y) = \frac{\prod_x P_{Y|X}(y|x)P_X(x)}{\sum_{y'} \prod_x P_{Y|X}(y'|x)P_X(x)}, \quad y \in \mathcal{Y}.$$

Define

$$E_{f,er}(P_{XY}, R) = \max\{E_{f,ex}(P_{XY}, R), E_{f,rc}(P_{XY}, R)\},$$

$$E_{v,er}(P_{XY}, R) = \max\{E_{v,ex}(P_{XY}, R), E_{v,rc}(P_{XY}, R)\}.$$

We have

$$E_f(P_{XY}, R) \geq E_{f,er}(P_{XY}, R),$$

$$E_v(P_{XY}, R) \geq E_{v,er}(P_{XY}, R).$$

It can be seen from Figure 2 that the achievable error exponent  $E_{v,er}(P_{XY}, R)$  of variable-rate Slepian-Wolf coding can completely dominate the sphere packing exponent  $E_{f,sp}(P_{XY}, R)$  of fixed-rate Slepian-Wolf coding. The gain of variable-rate coding gradually diminishes as  $\tau \rightarrow \frac{1}{2}$  (see Figures 3 and 4).

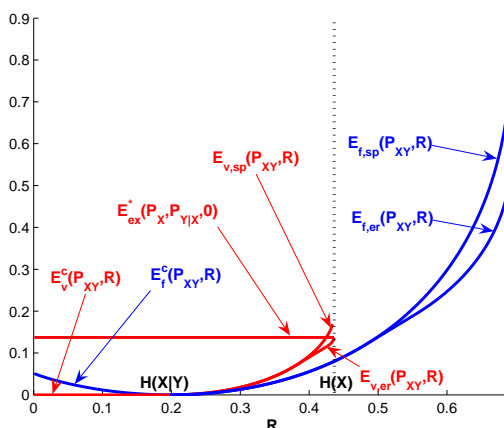


Figure 2.  $p = 0.05, \tau = 0.12$ .

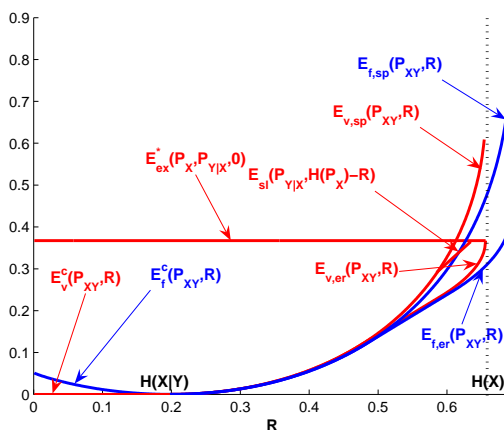


Figure 3.  $p = 0.05, \tau = 0.35$ .

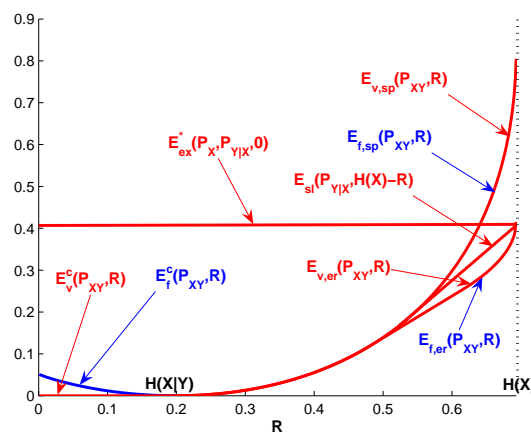


Figure 4.  $p = 0.05, \tau = 0.50$ .

### 6. Concluding Remarks

We have studied the reliability function of variable-rate Slepian-Wolf coding. An intimate connection between variable-rate Slepian-Wolf codes and constant composition codes has been revealed. It is shown that variable-rate Slepian-Wolf coding can outperform fixed-rate Slepian-Wolf coding in terms of rate-error tradeoff. Finally, we would like to mention that Theorem 1 has been generalized by Weinberger and Merhav in their recent paper on the optimal tradeoff between the error exponent and the excess-rate exponent of variable-rate Slepian-Wolf coding [19].

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### Appendix A. Proof of Proposition 1

In view of (7) and (11), we have  $R_{cr}(Q_X, W_{Y|X}) = I(Q_X, W_{Y|X})$  if and only if the minimum of the convex optimization problem

$$\min_{V_{Y|X}} D(V_{Y|X} \| W_{Y|X} | Q_X) + I(Q_X, V_{Y|X}) \tag{A1}$$

is achieved at  $V_{Y|X} = W_{Y|X}$ . Let  $V_{Y|X}^*$  be a minimizer to the above optimization problem. Note that for  $x, y$  such that  $Q_X(x)W_{Y|X}(y|x) = 0$ , there is no loss of generality in setting  $V_{Y|X}^*(y|x) = W_{Y|X}(y|x)$ . Let  $\mathcal{A} = \{x \in \mathcal{X} : Q_X(x) > 0\}$  and  $\mathcal{B}_x = \{y \in \mathcal{Y} : W_{Y|X}(y|x) > 0\}$  for  $x \in \mathcal{A}$ . We can rewrite (A1) in the following equivalent form:

$$\min_{V_{Y|X}(y|x): x \in \mathcal{A}, y \in \mathcal{B}_x} \sum_{x \in \mathcal{A}, y \in \mathcal{B}_x} Q_X(x)V_{Y|X}(y|x) \log \frac{V_{Y|X}^2(y|x)}{W_{Y|X}(y|x) \sum_{x' \in \mathcal{A}} Q_X(x')V_{Y|X}(y|x')}$$

subject to

$$\begin{aligned} V_{Y|X}(y|x) &\geq 0 \quad \text{for all } x \in \mathcal{A}, y \in \mathcal{B}_x, \\ \sum_{y \in \mathcal{B}_x} V_{Y|X}(y|x) &= 1 \quad \text{for all } x \in \mathcal{A}. \end{aligned}$$

Define

$$G' = \sum_{x \in \mathcal{A}, y \in \mathcal{B}_x} Q_X(x) V_{Y|X}(y|x) \log \frac{V_{Y|X}^2(y|x)}{W_{Y|X}(y|x) \sum_{x' \in \mathcal{A}} Q_X(x') V_{Y|X}(y|x')} - \sum_{x \in \mathcal{A}, y \in \mathcal{B}_x} \alpha(x, y) V_{Y|X}(y|x) - \sum_{x \in \mathcal{A}, y \in \mathcal{B}_x} \beta(x) V_{Y|X}(y|x),$$

where  $\alpha(x, y) \in \mathbb{R}_+$  ( $x \in \mathcal{A}, y \in \mathcal{B}_x$ ) and  $\beta(x) \in \mathbb{R}$  ( $x \in \mathcal{A}$ ). The Karush-Kuhn-Tucker conditions yield

$$\begin{aligned} \left. \frac{\partial G'}{\partial V_{Y|X}(y^*|x^*)} \right|_{V_{Y|X}(y^*|x^*)=V_{Y|X}^*(y^*|x^*)} &= 2Q_X(x^*) \log V_{Y|X}^*(y^*|x^*) + Q_X(x^*) - Q_X(x^*) \log W_{Y|X}(y^*|x^*) \\ &\quad - Q_X(x^*) \log \sum_{x' \in \mathcal{A}} Q_X(x') V_{Y|X}^*(y^*|x') - \alpha(x^*, y^*) - \beta(x^*) \\ &= 0 \quad \text{for all } x^* \in \mathcal{A}, y^* \in \mathcal{B}_{x^*}, \end{aligned}$$

$$V_{Y|X}^*(y^*|x^*) \geq 0 \quad \text{for all } x^* \in \mathcal{A}, y^* \in \mathcal{B}_{x^*},$$

$$\sum_{y^* \in \mathcal{B}_{x^*}} V_{Y|X}^*(y^*|x^*) = 1 \quad \text{for all } x^* \in \mathcal{A},$$

$$\alpha(x^*, y^*) V_{Y|X}^*(y^*|x^*) = 0 \quad \text{for all } x^* \in \mathcal{A}, y^* \in \mathcal{B}_{x^*}.$$

By the complementary slackness conditions (i.e.,  $V_{Y|X}^*(y^*|x^*) > 0 \Rightarrow \alpha(x^*, y^*) = 0$ ), we have  $V_{Y|X}^* = W_{Y|X}$  if and only if for all  $x^* \in \mathcal{A}, y^* \in \mathcal{B}_{x^*}$ ,

$$Q_X(x^*) \log W_{Y|X}(y^*|x^*) + Q_X(x^*) - Q_X(x^*) \log \sum_{x' \in \mathcal{A}} Q_X(x') W_{Y|X}(y^*|x') - \beta(x^*) = 0,$$

i.e., the value of

$$\frac{W_{Y|X}(y|x)}{\sum_{x'} Q_X(x') W_{Y|X}(y|x')}$$

does not depend on  $y$  for all  $x, y$  such that  $Q_X(x) W_{Y|X}(y|x) > 0$ .

### Appendix B. Proof of Proposition 4

1. It is known ([7], Exercise 5.17) that for every  $R > 0, \delta > 0$ , and  $P \in \mathcal{P}_n(\mathcal{X})$  there exists a constant composition code  $\mathcal{C}_n \subseteq \mathcal{T}_n(P)$  such that

$$\begin{aligned} R(\mathcal{C}_n) &\geq R - \delta, \\ -\frac{1}{n} \log P_{e, \max}(\mathcal{C}_n, W_{Y|X}) &\geq E_{ex}(P, W_{Y|X}, R) - \delta \end{aligned}$$

whenever  $n \geq n_0(|\mathcal{X}|, |\mathcal{Y}|, \delta)$ . Let  $P_n$  be a sequence of types with  $P_n \in \mathcal{P}_n(\mathcal{X})$  and

$$\lim_{n \rightarrow \infty} \|P_n - Q_X\| = 0.$$

Define

$$V_n^* = \arg \min \left[ \sum_{x, \tilde{x}} P_n(x) V_n(\tilde{x}|x) d_{W_{Y|X}}(x, \tilde{x}) + I(P_n, V_n) - R \right],$$

where the minimization is over  $V_n : \mathcal{X} \rightarrow \mathcal{X}$  subject to the constraints

$$\sum_x P_n(x) V_n(\tilde{x}|x) = P_n(\tilde{x}) \quad \text{for all } \tilde{x} \in \mathcal{X},$$

$$I(P_n, V_n) \leq R.$$

Note that  $\{V_n^*\}$  must contain a converging subsequence  $\{V_{n_k}^*\}$ . Define

$$V^* = \lim_{k \rightarrow \infty} V_{n_k}^*.$$

It is easy to verify that

$$\begin{aligned} \sum_{x \in \mathcal{X}} Q_X(x) V^*(\tilde{x}|x) &= \lim_{k \rightarrow \infty} \sum_{x \in \mathcal{X}} P_{n_k}(x) V_{n_k}^*(\tilde{x}|x) \\ &= \lim_{k \rightarrow \infty} P_{n_k}(\tilde{x}) \\ &= Q_X(\tilde{x}) \quad \text{for all } \tilde{x} \in \mathcal{X}, \\ I(Q_X, V^*) &= \lim_{k \rightarrow \infty} I(P_{n_k}, V_{n_k}^*) \\ &\leq R. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\limsup_{n \rightarrow \infty} Ex(P_n, W_{Y|X}, R) \\ &\geq \limsup_{k \rightarrow \infty} Ex(P_{n_k}, W_{Y|X}, R) \\ &= \limsup_{k \rightarrow \infty} \sum_{x, \tilde{x} \in \mathcal{X}} P_{n_k}(x) V_{n_k}^*(\tilde{x}|x) d_{W_{Y|X}}(x, \tilde{x}) + I(P_{n_k}, V_{n_k}^*) - R \\ &\geq \sum_{x, \tilde{x} \in \mathcal{X}} Q_X(x) V^*(\tilde{x}|x) d_{P_{Y|X}}(x, \tilde{x}) + I(Q_X, V^*) - R \\ &\geq Ex(Q_X, W_{Y|X}, R). \end{aligned}$$

It is also known ([7], Theorem 5.2) that for every  $R > 0$ ,  $\delta > 0$ , and  $P \in \mathcal{P}_n(\mathcal{X})$  there exists a constant composition code  $\mathcal{C}_n \subseteq \mathcal{T}_n(P)$  such that

$$\begin{aligned} R(\mathcal{C}_n) &\geq R - \delta, \\ -\frac{1}{n} \log P_{e, \max}(\mathcal{C}_n, W_{Y|X}) &\geq E_{rc}(P, W_{Y|X}, R) - \delta \end{aligned}$$

whenever  $n \geq n_0(|\mathcal{X}|, |\mathcal{Y}|, \delta)$ . So it can be readily shown that

$$E(Q_X, W_{Y|X}, R) \geq E_{rc}(Q_X, W_{Y|X}, R)$$

by invoking the fact that  $E_{rc}(P, W_{Y|X}, R)$  as a function of the pair  $(P, R)$  is uniformly equicontinuous ([7], Lemma 5.5). The proof is complete.

- By Definition 2, for every  $R > 0$ ,  $\delta > 0$  there exists a sequence of block channel codes  $\{\mathcal{C}_n\}$  with  $\mathcal{C}_n \subseteq \mathcal{T}_n(P_n)$  for some  $P_n \in \mathcal{P}_n(\mathcal{X})$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_n - Q_X\| &= 0, \\ \liminf_{n \rightarrow \infty} R(\mathcal{C}_n) &\geq R - \delta, \end{aligned}$$

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_{e,\max}(\mathcal{C}_n, W_{Y|X}) \geq E(Q_X, W_{Y|X}, R) - \delta. \tag{A2}$$

For simplicity, we assume  $R(\mathcal{C}_n) \geq R - \delta$  for all  $n$ . Now it follows from ([7], Theorem 5.3) that

$$-\frac{1}{n} \log \left[ 2P_{e,\max}(\mathcal{C}_n, W_{Y|X}) \right] \leq E_{sp}(P_n, W_{Y|X}, R - 2\delta)(1 + \delta) \tag{A3}$$

whenever  $n \geq n_0(|\mathcal{X}|, |\mathcal{Y}|, \delta)$ . Let

$$V_{Y|X}^* = \arg \min_{V_{Y|X}: I(Q_X, V_{Y|X}) \leq R - 3\delta} D(V_{Y|X} \| W_{Y|X} | Q_X).$$

Without loss of generality, we can set  $V_{Y|X}^*(\cdot|x) = W_{Y|X}(\cdot|x)$  for all  $x \in \{x' \in \mathcal{X} : Q_X(x') = 0\}$ . It is easy to see that there exists an  $\epsilon > 0$  such that

$$\begin{aligned} I(P, V_{Y|X}^*) &\leq R - 2\delta, \\ D(V_{Y|X}^* \| W_{Y|X} | P) &\leq D(V_{Y|X}^* \| W_{Y|X} | Q_X) + \delta \end{aligned}$$

for all  $P \in \mathcal{P}(\mathcal{X})$  with  $\|P - Q_X\| \leq \epsilon$ . Therefore, for all sufficiently large  $n$ ,

$$\begin{aligned} E_{sp}(P_n, W_{Y|X}, R - 2\delta) &= \min_{V_{Y|X}: I(P_n, V_{Y|X}) \leq R - 2\delta} D(V_{Y|X} \| W_{Y|X} | P_n) \\ &\leq D(V_{Y|X}^* \| W_{Y|X} | P_n) \\ &\leq D(V_{Y|X}^* \| W_{Y|X} | Q_X) + \delta \\ &= E_{sp}(Q_X, W_{Y|X}, R - 3\delta) + \delta. \end{aligned} \tag{A4}$$

Combining (A2)–(A4), we get

$$E(Q_X, W_{Y|X}, R) - \delta \leq [E_{sp}(Q_X, W_{Y|X}, R - 3\delta) + \delta](1 + \delta).$$

In view of the fact that  $\delta > 0$  is arbitrary and that for fixed  $P$  and  $W_{Y|X}$ ,  $E_{sp}(P, W_{Y|X}, R)$  is a decreasing continuous convex function of  $R$  in the interval where it is finite ([7], Lemma 5.4), the proof is complete.

- It is known ([9], Lemma 5) that every constant composition code  $\mathcal{C}_n$  of common type  $P$  for some  $P \in \mathcal{P}_n(\mathcal{X})$  and rate  $R(\mathcal{C}_n) \geq R + \delta$  (with  $R > 0$  and  $\delta > 0$ ) has

$$-\frac{1}{n} \log P_c(\mathcal{C}_n, W_{Y|X}) \geq \min_{V_{Y|X}} \left[ D(V_{Y|X} \| W_{Y|X} | P) + |R - I(P, V_{Y|X})|^+ \right] - \delta$$

whenever  $n \geq n_0(|\mathcal{X}|, |\mathcal{Y}|, \delta)$ . Moreover, it is also known ([9], Lemma 2) ([7], Exercise 5.16) that for every  $R > 0$ ,  $\delta > 0$ , and  $P \in \mathcal{P}_n(\mathcal{X})$  there exists a constant composition code  $\mathcal{C}_n \subseteq \mathcal{T}_n(P)$  such that

$$\begin{aligned} R(\mathcal{C}_n) &\geq R - \delta, \\ -\frac{1}{n} \log P_c(\mathcal{C}_n, W_{Y|X}) &\leq \min_{V_{Y|X}} \left[ D(V_{Y|X} \| W_{Y|X} | P) + |R - I(P, V_{Y|X})|^+ \right] + \delta \end{aligned}$$

whenever  $n \geq n_0(|\mathcal{X}|, |\mathcal{Y}|, \delta)$ . In view of the fact that  $\min_{V_{Y|X}} [D(V_{Y|X} \| W_{Y|X} | P) + |R - I(P, V_{Y|X})|^+]$  as a function of the pair  $(P, R)$  is uniformly equicontinuous, it can be readily shown that

$$E_c(Q_X, W_{Y|X}, R) = \min_{V_{Y|X}} \left[ D(V_{Y|X} \| W_{Y|X} | Q_X) + |R - I(P, V_{Y|X})|^+ \right].$$

The proof is complete.

## References

1. Slepian, D.; Wolf, J.K. Noiseless coding of correlated information sources. *IEEE Trans. Inf. Theory* **1973**, *19*, 471–480.
2. Csiszár, I. Linear codes for sources and source networks: Error exponents, universal coding. *IEEE Trans. Inf. Theory* **1982**, *28*, 585–592.
3. Pradhan, S.S.; Ramchandran, K. Distributed source coding using syndromes (DISCUS): Design and construction. *IEEE Trans. Inf. Theory* **2003**, *49*, 626–643.
4. Chen, J.; He, D.-K.; Jagmohan, A. The equivalence between Slepian-Wolf coding and channel coding under density evolution. *IEEE Trans. Commun.* **2009**, *57*, 2534–2540.
5. Sun, Z.; Tian, C.; Chen, J.; Wong, K.M. LDPC code design for asynchronous Slepian-Wolf coding. *IEEE Trans. Commun.* **2010**, *58*, 511–520.
6. Oohama, Y.; Han, T.S. Universal coding for the Slepian-Wolf data compression system and the strong converse theorem. *IEEE Trans. Inf. Theory* **1994**, *40*, 1908–1919.
7. Csiszár, I.; Körner, J. *Information Theory: Coding Theorems for Discrete Memoryless Systems*; Academic: New York, NY, USA, 1981.
8. Csiszár, I.; Körner, J. Graph decomposition: A new key to coding theorems. *IEEE Trans. Inf. Theory* **1981**, *27*, 5–12.
9. Dueck, G.; Körner, J. Reliability function of a discrete memoryless channel at rates above capacity. *IEEE Trans. Inf. Theory* **1979**, *25*, 82–85.
10. Gallager, R.G. *Information Theory and Reliable Communication*; Wiley: New York, NY, USA, 1968.
11. Gallager, R.G. *Source Coding with Side Information and Universal Coding*; MIT LIDS Technical Report (LIDS-P-937). Unpublished work, 1976.
12. Chen, J.; He, D.-K.; Jagmohan, A.; Lastras-Montaña, L. On the redundancy-error tradeoff in Slepian-Wolf coding and channel coding. In Proceedings of the 2007 IEEE International Symposium on Information Theory, Nice, France, 24–29 January 2007; pp. 1326–1330.
13. Csiszár, I.; Körner, J. Towards a general theory of source networks. *IEEE Trans. Inf. Theory* **1980**, *26*, 155–165.
14. Chen, J.; He, D.-K.; Jagmohan, A.; Lastras-Montaña, L.A.; Yang, E.-H. On the linear codebook-level duality between Slepian-Wolf coding and channel coding. *IEEE Trans. Inf. Theory* **2009**, *55*, 5575–5590.
15. Ahlswede, R.; Dueck, G. Good codes can be produced by a few permutations. *IEEE Trans. Inf. Theory* **1982**, *28*, 430–443.
16. Chen, J.; He, D.-K.; Jagmohan, A. On the duality between Slepian-Wolf coding and channel coding under mismatched decoding. *IEEE Trans. Inf. Theory* **2009**, *55*, 4006–4018.
17. Ahlswede, R. Coloring hypergraphs: A new approach to multi-user source coding—II. *J. Comb. Inf. Syst. Sci.* **1980**, *5*, 220–268.
18. He, D.-K.; Lastras-Montaña, L.; Yang, E.-H.; Jagmohan, A.; Chen, J. On the redundancy of Slepian-Wolf coding. *IEEE Trans. Inf. Theory* **2009**, *55*, 5607–5627.
19. Weinberger, N.; Merhav, N. Optimum tradeoffs between the error exponent and the excess-rate exponent of variable-rate Slepian-Wolf coding. *IEEE Trans. Inf. Theory* **2015**, *61*, 2165–2190.

