

Article

The Operational Choi–Jamiołkowski Isomorphism

Emily Adlam

Basic Research Community for Physics, 04315 Leipzig, Germany; eadlam90@gmail.com

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Abstract: In this article, I use an operational formulation of the Choi–Jamiołkowski isomorphism to explore an approach to quantum mechanics in which the state is not the fundamental object. I first situate this project in the context of generalized probabilistic theories and argue that this framework may be understood as a means of drawing conclusions about the intratheoretic causal structure of quantum mechanics which are independent of any specific ontological picture. I then give an operational formulation of the Choi–Jamiołkowski isomorphism and show that, in an operational theory which exhibits this isomorphism, several features of the theory which are usually regarded as properties of the quantum state can be derived from constraints on non-local correlations. This demonstrates that there is no need to postulate states to be the bearers of these properties, since they can be understood as consequences of a fundamental equivalence between multipartite and temporal correlations.

Keywords: quantum foundations; generalized probabilistic theories; ontology; quantum state

1. Introduction

In the standard mathematical formulation of quantum mechanics, the quantum state is the most fundamental object. However, there exists a longstanding debate about whether we should regard the quantum state as an element of reality [1,2]. A recent result known as the PBR theorem purported to settle this debate once and for all by showing that no interpretation of quantum mechanics where the quantum state is not an element of reality can reproduce all of the theory’s empirical predictions [3]. However, in order to prove the PBR theorem it is assumed that information about the past must be carried into the future by a mediating state, so in fact the setup for the theorem takes it for granted that the fundamental object in quantum mechanics is some sort of temporally evolving state, whether or not that state turns out to be the quantum state.

The assumption that temporal correlations must be mediated by states is seldom seriously questioned, but it is in fact in tension with certain elements of modern physics [4], and this suggests that we ought to take seriously the possibility of re-imagining quantum mechanics with an ontology that does not employ states at all. Quantum states have certain fundamental temporal properties (for example, they cannot be broadcasted, they are contextual, they exhibit interference effects, and they are subject to uncertainty relations) and a literal reading of the formalism of quantum mechanics would suggest that the characteristic features of the theory are derived from these properties of the quantum state. Thus as a first step towards a stateless ontology, we should determine whether it is necessary to postulate some sort of state to be the bearer of these properties, or whether they can be accounted for in another way.

In recent years there have been moves within the quantum foundations community toward a type of research where quantum mechanics is placed in the context of a space of generalized probabilistic theories (GPTs), with the aim of showing that certain features of quantum mechanics can be derived from some chosen axioms [5–13]. However, most of these projects use axioms which employ a concept of ‘state’, and hence they don’t offer an obvious route to a stateless ontology. In this paper, I employ

the methodology of the generalized probabilistic theories framework, but I develop an alternative approach which is explicitly geared towards eliminating the concept of state.

I begin in Section 2 by elaborating on the rationale for studying generalized probabilistic theories. Then in Section 3 I set out an operational form of the Choi–Jamiołkowski isomorphism, and demonstrate that this operational isomorphism can be used to derive temporal properties of states from constraints on the types of non-local correlations that are permitted within the theory. This demonstrates that in fact we don't require a state to be the bearer of the aforementioned temporal properties, and thus paves the way for a new way of thinking about the intratheoretic causal structure of quantum mechanics where states are not fundamental objects.

2. Intratheoretic Causal Structure

2.1. Ontology

The issue of ontology is a vexed one in physics. It is a well-established fact that ontology is always underdetermined by the empirical results, and therefore no amount of experimentation can give us certainty about the nature of the unobservable entities associated with our theory [14]. Furthermore, when one theory is replaced by another, it is often the case that the theoretical objects postulated by the original theory do not feature in the new theory, and thus unless we believe our current theory is the correct final theory of reality we should probably assume that the theoretical objects posited by our current theory will not feature in future theories [15]. Considerations such as these have led many physicists to adopt variants of anti-realism, positivism or empiricism, holding that physics should be concerned only with empirical predictions and should be silent on questions of ontology [16,17].

However, it is important to recognise that the ontology we associated with our theories affects the way in which science progresses. For example, recall that classical physics can be written either in the standard Newtonian form, where we start with initial positions and calculate trajectories from those positions, or in a Lagrangian form where we define an 'action' over an entire history and find the trajectory which optimizes the action [18,19]. Historically most physicists have considered the Newtonian formulation of classical physics to be more fundamental than the Lagrangian one, and in this context it was natural for the founders of quantum mechanics to formulate the new theory in terms of a state evolving forwards in time, even though we now know that an alternative Lagrangian formulation in terms of path integrals exists [18,20]. Quantum mechanics might look very different (and indeed might well be less difficult to 'interpret') had we taken the Lagrangian formulation seriously from the start.

Moreover, simply becoming an operationalist and refusing to talk about ontology doesn't confer immunity to these sorts of biases, because ontological assumptions are baked into the way in which we think about and do physics [4,18,21], and so those who decline to think seriously about ontology are effectively choosing to uphold the status quo ontological assumptions - which, in the case of quantum mechanics, entails upholding the assumption of an ontology based on states.

Thus reconsidering the ontology associated with our theories can be a useful way of flushing out unjustified assumptions and ultimately coming up with new physics. However, in adopting this strategy we risk getting caught up in the details of some specific ontology, which due to the problem of underdetermination will inevitably not be correct in every particular. Ideally, what we would like to do is extract the significant structural content of various different ontologies and discard the specifics, which don't really contribute to the project of coming up with new physics and are essentially just window-dressing. This is exactly what the framework of generalized probabilistic theories allows us to do [5–13]: It offers a middle way between realism and empiricism, where we impose hierarchies within our theories by regarding certain features of the theory as consequences of other features, but do not actually specify the details of the ontology underlying the hierarchy.

2.2. Intratheoretic Causation

Hierarchical structures of this kind occur in many places throughout the sciences. They are most straightforward in theories which come with an ontology attached, because in such cases a hierarchy of fundamentality can be read directly off the ontology. For example, the ontology of classical physics postulates objects which move in inertial reference frames and are subject to certain mechanical forces, but centrifugal force is not one of those mechanical forces—rather it is a ‘fictitious’ force which arises when we try to describe an object in a reference frame which is accelerating relative to an inertial reference frame [22]. This ontology encodes a hierarchy where the centrifugal forces are to be understood as consequences of the more fundamental laws governing motion relative to inertial frames. Indeed, the hierarchy licences counterfactual assertions about the relationship between these features: If the laws governing motion relative to inertial frames had been relevantly different, the centrifugal forces would also have been different. Thus we observe that these hierarchies of fundamentality are roughly analogous to causal structures, with ‘causation’ to be understood in the counterfactual sense: In a given theory, feature A causes feature B if feature B would not have obtained if A had not obtained (There are many subtleties that have to be dealt with in any counterfactual account of causation; I will not deal with any of them here, but refer to reference [23] for a general discussion of counterfactual approaches to causation. It is likely that further subtleties would arise in the application of this approach to intratheoretic causation; I will return to these in forthcoming work.). Therefore I will henceforth refer to these hierarchies as ‘intratheoretic causal structures’ though I emphasize that the word ‘causal’ is here being used in a specialised sense, since the relata of the putative causal relation are not immanent.

We are already quite familiar with the case where the intratheoretic causal structure can be read off the ontology. However, what if there is no general consensus regarding the ontology of a theory? Quantum mechanics is one such case—to this day it is not even clear whether quantum mechanics requires the existence of the wavefunction [2], let alone other proposed features such as spontaneous collapses [24], de Broglie-Bohm particles [25] or ‘many-worlds’ [26]. As a result the intratheoretic causal structure of quantum mechanics is much less transparent - witness the continuing controversy over the reality of the quantum state, which can be understood as a discussion about whether we should adopt an intratheoretic causal structure in which the characteristic features of quantum mechanics are derived from the properties of the quantum state, or whether we should take it that the quantum state itself is derived from some deeper properties.

Obviously it is open to us to resolve these questions by adopting an interpretation of quantum mechanics and drawing the conclusions implied by its ontology. For example, the Everett interpretation tells us that the quantum state is indeed the fundamental object of quantum mechanics [26], whereas Quantum Bayesianism suggests that the quantum state is simply a description of an agent’s degrees of belief about the outcomes of measurements [27]. However, the study of generalized probabilistic theories offers an alternative, allowing us to draw conclusions about intratheoretic causal structure which are independent of any specific ontological assumptions. Research of this kind works by placing quantum mechanics in the context of a wider space of GPTs so that we can meaningfully consider counterfactual questions about how changing one aspect of a theory might lead to other changes [5–13]. This often involves proving statements of the form: ‘any generalized probabilistic theory which has feature X must also have feature Y’. For example, reference [12] proves that any GPT which obeys no-signalling must also obey a form of monogamy of correlations, thus demonstrating the existence of a (putatively causal) relationship between these features which holds regardless of what the underlying ontology might be. The framework of generalized probabilistic theories therefore provides exactly the formalism needed for us to study the counterfactual reasoning associated with intratheoretic causal structure in a way that does not depend on any particular ontology.

Of course, showing that certain features of quantum mechanics can be derived from other features in this way doesn’t actually prove anything about the intratheoretic causal structure of quantum mechanics, because there exists no unique axiomatization of quantum mechanics and each different

axiomatization will suggest different intratheoretic causal relations. However, that is just to restate the old problem of underdetermination of theory by data, and a similar response can be made: The point is not necessarily to know for certain, but to understand the implications of different possible causal structures as a way of exploring alternative ways forward for physics. One need not believe that any particular causal structure is the correct one, or indeed that a unique correct causal structure even exists, to see the value of the exercise.

2.3. Structural Realism

I have suggested that one motivation for research in the space of generalized probabilistic theories is the idea that we don't have much hope of coming to know the true ontology of our theories, so instead of focusing on the details of specific ontologies we should instead think in terms of intratheoretic causal structures, which plausibly we could get right even if the true ontology remains epistemically inaccessible. This suggestion clearly has common ground with the philosophical view known as structural realism [15], which proposes that since the specific ontologies of our theories will likely be discarded when we move to a new theory, we should not be epistemically committed to specific ontologies but instead commit to the structural relations implied by the ontologies of the theories, since these relations are often maintained when we move to a new theory. However, the relations to which structural realism refers are usually instantiated by specific objects in the world, whereas the intratheoretic relations that I discuss here hold between parts of a theory, so the domain of application of this form of structuralism is importantly different.

3. The Choi–Jamiołkowski Isomorphism

In the remainder of this article, I will elaborate on an intratheoretic causal structure for quantum mechanics where the quantum state is not the fundamental object. My approach is based on the Choi–Jamiołkowski isomorphism, which describes the mathematical correspondence between quantum channels and entangled bipartite states [28]. The Choi–Jamiołkowski isomorphism can be derived from the standard mathematical formalism of quantum mechanics, and the fact that it is usually taught in this way encourages physicists to regard it as a consequence of the properties of the quantum state. In particular, it is often interpreted by appeal to an operational procedure known as 'noisy gate teleportation' in which agents A and B share entanglement and agent A is in possession of a system in an unknown state ρ and the aim is for B to end up with a system in the state $T\rho$ where T is some fixed transformation; this cannot usually be achieved with perfect certainty, but insofar as it is possible it works because of the Choi–Jamiołkowski isomorphism [29,30]. This presentation implies that the isomorphism is just one of the many surprising operational consequences which follows from the properties of the quantum state, so it is of no special foundational interest.

However, consider: What the isomorphism actually tells us (roughly speaking), is that in quantum mechanics, the set of possible multipartite correlations exhibited by entangled states is equivalent to the set of possible temporal correlations exhibited by sequences of measurements on a single system over time. This equivalence has the air of a fact about ontology. Indeed, I would argue that and indeed, any choice for the ontology of quantum mechanics which failed to reflect such a striking equivalence would surely be suspect on the grounds of exhibiting 'asymmetries which do not appear to be inherent in the phenomena'. [31] Thus in this paper I will suppose that the Choi–Jamiołkowski isomorphism is indeed a deep fact about the underlying ontology of quantum mechanics, and therefore causally prior to most other features of the theory.

In accordance with the discussion of Section 2, rather than proposing a specific ontology which has this feature, I will proceed directly to an investigation of the consequences of this conjecture for the intratheoretic causal relations in quantum mechanics. In Section 3, I give an operational formulation of the Choi–Jamiołkowski isomorphism, which relates measurements on different parts of a multipartite system to sequences of measurements on the same system and vice versa. In Sections 5–7, I use it to prove results of the form 'Any generalized probabilistic theory which obeys the operational

Choi–Jamiołkowski isomorphism, and which has property A, must also have property B', where A is a constraint on multipartite measurements and B is a constraint on sequences of measurements, as shown in the table below (Table 1). That is, I show that several features of quantum mechanics which are typically interpreted as temporal properties of the quantum state could in fact be derived from the properties of some more general entity which is the substratum for both multipartite states and time evolution.

Table 1. Summary of results.

Property	Implies
Strong monogamy of correlations and Bell nonlocality	No-broadcasting and quantum interference
No-signalling and Bell nonlocality	Preparation contextuality
Information causality	Fine-grained uncertainty relations

Each of the properties in this table will be given an operational definition in subsequent sections. Note that the concept of 'operational state' will feature in the definition of some of the properties on the right-hand side of the table, since they have been chosen specifically as properties which are normally attached to states. However, crucially, the concept of state is not used in the operational definitions of the concepts on the left-hand side, or in the operational Choi–Jamiołkowski isomorphism. Thus as promised, these results demonstrate features that appear to be properties of states can in fact be understood as consequences of features of the theory which do not depend in any way on a concept of state.

3.1. Related Work

While the mainstream literature on quantum mechanics is still very dominated by state-based approaches, in recent years a variety of interesting work has been done on non-standard temporal pictures. For example, there has been growing interest in retrocausal approaches to quantum theory [32,33], including a proof by Pusey and Leifer demonstrating that if quantum mechanics obeys a certain sort of time-symmetry then it must exhibit retrocausality [34] and a model due to Wharton which suggests a natural resolution to the quantum reality problem using the "all-at-once"-style analysis of action principles [35]. Elsewhere, Oreshkov and Cerf [36] have set out the process matrix formalism, which allows us to generalize the framework of operational theories in a way that does not depend on a predefined time or causal structure, thus giving us the mathematical resources to deal with theories that might contain indefinite causal order, causal loops or other structures that don't fit into our familiar notions of time and causality. Similarly, Shrapnel and Costa have used the process matrix formalism to set out a generalisation of the ontological models approach which does not assume that information must be carried through time by a mediating state, and have used this generalisation to demonstrate that even without mediating states quantum mechanics must still exhibit a generalized form of contextuality [37]. Thus the ideas set out here add to a growing body of work on the ways in which quantum mechanics could be embedded into global and temporally non-local structures.

It has been recognised that the Choi–Jamiołkowski isomorphism is likely to play an important role in such non-standard temporal pictures—in particular, the process matrix is defined using the Choi–Jamiołkowski representations of the relevant CP maps [36]. The process matrix formalism therefore assigns to the isomorphism an implicit ontological significance, and so the results presented here can be understood as complementary to that line of research, since I have made the ontological significance of the isomorphism explicit and used an operational formulation to explore its consequences for intratheoretic causal structure. A variety of other authors have also suggested that the isomorphism should be understood in ontological terms, so for example it is noted in [38] that the Choi–Jamiołkowski representation for quantum operations 'gives a nice way of unifying statics and dynamics in one framework: The future is entangled (or at least correlated) with the past', and likewise,

reference [39] puts forward a theoretical model in which ‘one particle at N times is ... equivalent to N particles at one time’.

Finally, a different version of an ‘operational Choi–Jamiołkowski isomorphism’ was put forward in reference [6], where it is shown that in any operational theory which (a) is causal, and (b) has the property that every mixed state has a purification, it is necessarily the case that there exists an isomorphism between transformations and bipartite states which has the same structural properties as the Choi–Jamiołkowski isomorphism. This paper is a good example of what I have described as the study of intratheoretic causal structure, but the structure proposed is quite different to the one I have suggested here. First, I have taken the ‘operational Choi–Jamiołkowski isomorphism’ to be fundamental and derived other features of quantum mechanics from it, whereas in reference [6] the operational Choi–Jamiołkowski is derived from other features of the theory, so the intratheoretic causal structure is in fact precisely reversed. Second, I have specifically avoided using the concept of state in my formulation of the operational Choi–Jamiołkowski isomorphism and instead derived properties of the state from the isomorphism together with constraints on possible correlations, whereas in reference [6] one of the axioms from which the rest of quantum mechanics is derived is ‘every mixed state has a purification’, so in this approach it seems natural to take the state as a fundamental object and not an emergent feature.

4. Preliminaries

4.1. Operational Theories

In attempting to understand some of the special features of quantum mechanics, it is often helpful to make comparisons between quantum theory and other possible theories. In this context it is common to employ the language of generalized probabilistic theories, where theories are defined entirely in terms of preparation procedures and measurements, eschewing abstract mathematical constructions. This has been a very active area of study in recent years, with many interesting results emerging to show how quantum mechanics relates to the broader field of possible operational theories. For example, it was shown that quantum mechanics is not the maximally non-local theory without signalling [10]: There exists a gap between the numerical bound on the set of non-local correlations which can be produced in a theory limited only by no-signalling, and the corresponding ‘Tsirelson’ bound on the set of non-local correlations which can be produced in quantum theory, and much effort has gone into trying to explain this difference (see for example refs [11,40–42]).

In the framework of operational theories, a given theory is specified as a quadruple $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ where \mathcal{P} is a set of preparations, \mathcal{M} is a set of measurements, \mathcal{T} is a set of transformations, and the function $p(M^x|P, T)$ specifies, for every possible combination of preparation P , transformation T , and measurement outcome M^x , the probability of obtaining outcome M^x to the measurement M if it is performed on a system prepared according to P and then subjected to transformation T . [43]

I will sometimes write $p(M^x|P)$ as shorthand for $p(M^x|P, \mathbb{I})$, so $p(M^x|P)$ denotes the probability of obtaining outcome M^x to the measurement M if it is performed directly on a system prepared according to P . For the operational theories considered in this paper, I stipulate that performing a transformation followed by a given measurement is equivalent to simply performing some other measurement, i.e., for any $P, T, \{M^x\}$, there exists some $\{N^x\}$ such that $p(M^x|P, T) = p(N^x|P, \mathbb{I})$; for any operational theory where this is not the case, we can trivially make it the case by expanding the set \mathcal{M} .

4.2. Ensemble Preparations

The operational formulation of the Choi–Jamiołkowski isomorphism will make use of the following concept:

Definition 1. An ensemble preparation, P , specified by a probability distribution $p(i) : I \in \{1, 2, \dots, N\}$ and a set of preparations $\{Q_i : I \in \{1, 2, \dots, N\}\}$, is a procedure in which an observer draws a number i from $\{1, 2, \dots, N\}$ with probability distribution $p(i)$, and then performs the corresponding preparation Q_i .

In particular, when the operational theory in question is quantum mechanics, every possible ensemble preparation can be described by a POVM $\{M_i\}$ and density operator ρ , where $p(i) = \text{Tr}(\rho M_i)$ and P_i is a preparation which produces the quantum state $\rho_i = \frac{\sqrt{\rho} M_i \sqrt{\rho}}{\text{Tr}(M_i \rho)}$ [44].

For brevity, I will also use the following definitions:

Definition 2. For any ensemble preparation P for a given system S , any set of channels T_2, T_3, \dots, T_n which may then be applied to S , and any set of measurements M_2, M_3, \dots, M_n which can be applied to the distinct outputs of these channels, I denote by $p_{P;T_2 \dots T_n;M_2 \dots M_n}$ the joint probability distribution over the outcome of the random choice in the ensemble preparation P and the outcome of the measurements M_2, M_3, \dots, M_n .

Definition 3. For any joint preparation $P_{12 \dots n}$ for a set of systems S_1, S_2, \dots, S_n , and any set of measurements M_1, M_2, \dots, M_n which can be applied separately to systems S_1, S_2, \dots, S_n , I denote by $p_{P_{12 \dots n};M_1 \dots M_n}$ the joint probability distribution over the outcomes of the measurements M_1, M_2, \dots, M_n .

4.3. The Choi–Jamiołkowski Isomorphism

In its original form, the Choi–Jamiołkowski isomorphism is a one-to-one map between unnormalized quantum states and completely positive (CP) operators. Specifically, for a given CP operator $\epsilon_{B|A}$ mapping a Hilbert space \mathcal{H}_A to a Hilbert space \mathcal{H}_B , the corresponding Choi–Jamiołkowski state ρ^ϵ is obtained by applying $\epsilon_{B|A}$ to half of the maximally entangled state $\rho_{AA'} = \frac{1}{d} \sum_i \sum_{i'} |i\rangle_A |i'\rangle_{A'} \langle i'|_A \langle i|_{A'}$, where A' is a quantum system with the same Hilbert space as A , and d is the dimension of this Hilbert space: [44,45] (The formula presented here is for the Choi version of the isomorphism; I will not need to employ the -Jamiołkowski version here.)

$$\rho_{AB}^\epsilon = (\epsilon_{B|A} \otimes \mathbb{I}) \rho_{AA'}$$

However, in reference [44] the isomorphism is reformulated as a map between conditional density operators and completely positive trace-preserving (CPTP) operators. This formulation, which I paraphrase below, is more transparent in its physical interpretation and hence I will take it as the starting point for the operational approach:

Lemma 1. For any bipartite state ρ_{AB}^ϵ , there exists a CPTP map ϵ and a reduced state $\rho_A = \text{Tr}_B(\rho_{AB}^\epsilon)$ such that given any two POVMs M and O , if M^T is obtained by taking the transpose of all the measurement operators in M with respect to some fixed basis, then when we perform the ensemble preparation described by the POVM M^T and the density operator ρ_A , then evolve the state according to ϵ , then perform the measurement O , the probability that state ρ_i is prepared and then the measurement M has outcome j is the same as the joint probability of obtaining outcomes M_i and O_j when the POVM M is performed on system A and the POVM O is performed on system B for a bipartite system AB in the state ρ_{AB}^ϵ .

Conversely, for any pair of a CPTP map and state ρ there exists a bipartite state ρ_{AB}^ϵ such that the same conditions hold, so we have defined an isomorphism between bipartite states and pairs (ρ_A, ϵ^r) , where ϵ^r denotes the restriction of the CPTP map ϵ to the support of ρ_A .

4.4. Reformulation

Using these concepts, we can define an operational version of the Choi–Jamiołkowski isomorphism.

Definition 4. Operational Choi–Jamiołkowski Isomorphism:

For any joint preparation $P_{123\dots n}$ on a set of systems S, S_2, \dots, S_n , there exists a set of channels T_2, T_3, \dots, T_n which may simultaneously be applied to the system S , such that for any set of measurements M, M_2, \dots, M_n which may be performed on S, S_2, \dots, S_n , there exists an ensemble preparation P for the system S such that the distribution $p_{P_{123\dots n}; M, M_2, \dots, M_n}$ is the same as the distribution $p_{P; T_2, \dots, T_n; M_2, \dots, M_n}$.

Conversely, for any set of channels T_2, T_3, \dots, T_n which may simultaneously be applied to the system S to produce a set of systems S_2, \dots, S_n , there exists a joint preparation $P_{123\dots n}$ for systems S, S_2, S_3, \dots, S_n such that for any ensemble preparation P which may be performed for system S and any set of measurements M_2, M_3, \dots, M_n which may be performed on the products S_2, \dots, S_n , there exists a measurement M on S such that the distribution $p_{P_{123\dots n}; M, M_2, \dots, M_n}$ is the same as the distribution $p_{P; T_2, \dots, T_n; M_2, \dots, M_n}$.

In graphical terms, this says that for any scenario as in Figure 1 there exists a scenario as in Figure 2 and vice versa; likewise, for any scenario as in Figure 3 there exists a scenario as in Figure 4 and vice versa.

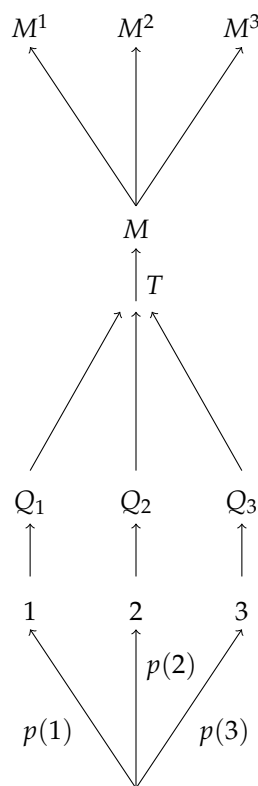


Figure 1. A schematic diagram of an ensemble preparation P with three possible results for the random number generation step, followed by a transformation T , followed by a measurement M with three possible outcomes M^1, M^2, M^3 . This scenario can be described by the probability distribution $p_{P; T; M}$.

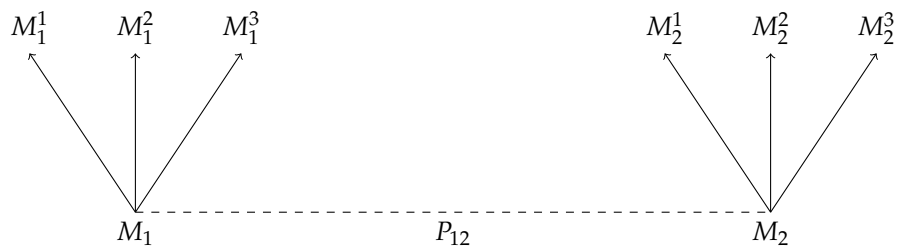


Figure 2. A schematic diagram of a joint preparation P_{12} on a bipartite state, followed by measurement M_1 with possible outcomes M_1^1, M_1^2, M_1^3 on one system, and measurement M_2 with possible outcomes M_2^1, M_2^2, M_2^3 on the other system. This scenario can be described by the probability distribution $p_{P_{12};M_1,M_2}$.

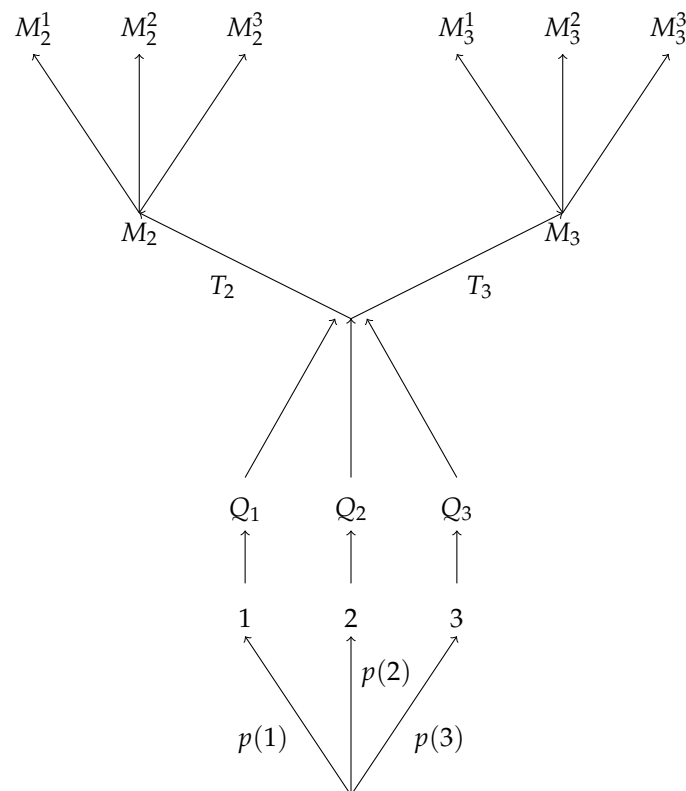


Figure 3. A schematic diagram of an ensemble P with three possible results for the random number generation step, followed by two channels T_2 and T_3 each producing a distinct system, followed by a measurement M_2 with three possible outcomes M_2^1, M_2^2, M_2^3 on one system, and a measurement M_3 with three possible outcomes M_3^1, M_3^2, M_3^3 on the other system. This scenario can be described by the probability distribution $p_{P;T_2,T_3;M_2,M_3}$.

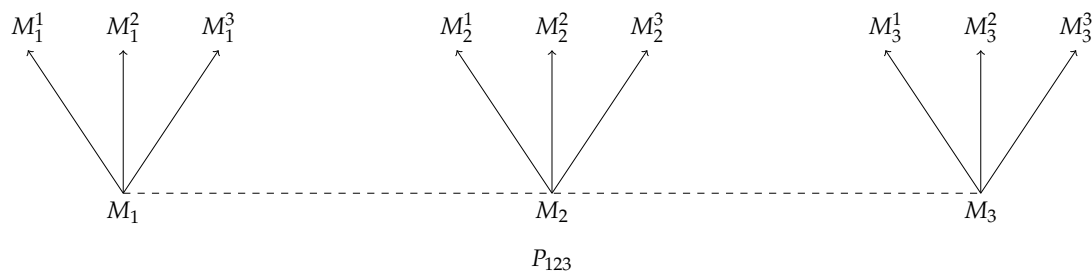


Figure 4. A schematic diagram of a joint preparation P_{12} on a bipartite state, followed by followed by a measurement M_1 with possible outcomes M_1^1, M_1^2, M_1^3 on one system, and measurement M_2 with possible outcomes M_2^1, M_2^2, M_2^3 on the other system. This scenario can be described by the probability distribution $p_{P_{123}; M_1, M_2, M_3}$.

Relation to Original Choi–Jamiołkowski Isomorphism

This operational definition of the isomorphism differs from the original in several respects. It is of course to be expected that an operational formulation of the isomorphism will not be exactly equivalent to the original quantum-mechanical statement of it, since the quantum-mechanical version is heavily theory-dependent and it is not possible to jettison these theory-dependent elements without losing some content. I believe nonetheless that this formulation captures much of what is important about the original Choi–Jamiołkowski isomorphism.

First, it should be noted that the operational formulation does not actually postulate the existence of an isomorphism: It makes a statement of the form ‘for each x there exists a y , and for each y there exists an x' ’, but this does not imply that there is an isomorphism between the sets $\{x\}$, $\{y\}$, since I do not insist that each distinct x should be mapped to a different y . One could of course make this further stipulation and it might turn out that further interesting results would arise from the correspondingly stronger constraint, but it was not necessary to do this for any of the results obtained here and I thought it best to employ the weakest possible version of the constraint that was still sufficient to derive these results. I have continued to use the term ‘isomorphism’ for consistency with the established terminology.

Second, the operational formulation applies to all multipartite states, whereas the quantum-mechanical Choi–Jamiołkowski isomorphism applies only to bipartite states. This change has been made in order to make it possible to derive the no-cloning result in a particularly simple and transparent way. This particular generalization has in fact been considered before: It was used to demonstrate the connection between no-cloning/no-broadcasting theorems and the monogamy of entanglement in reference [46]. Furthermore, recall that the motivation for putting forward this operational Choi–Jamiołkowski isomorphism is the intuition that the quantum isomorphism expresses a deep fact about the underlying ontology of the theory - channels behave like preparations because they are in fact the same sort of thing at the level of the underlying ontology. If this is the case, then we would naturally expect the constraint to apply to all multipartite states rather than just bipartite ones, since bipartite states are simply a special case of multipartite states.

5. No-Broadcasting Theorem and the Monogamy of Entanglement

5.1. Background

It is a particularly distinctive feature of quantum information that quantum states cannot be cloned—that is, there is no quantum operation which makes perfect copies of two non-orthogonal quantum states. In fact, the impossibility of cloning quantum states is a special case of the impossibility of broadcasting quantum states:

Definition 5. A map M from the Hilbert space \mathcal{H} to the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ broadcasts the set of states S iff for any state $\rho \in S$, $\text{Tr}_A(M(\rho)) = \rho$ and $\text{Tr}_B(M(\rho)) = \rho$.

Definition 6. A universal broadcasting map broadcasts any set of states.

Theorem 1. No-Broadcasting Theorem: A set of quantum states can be broadcasted if and only if they commute pairwise [47] (A pair of quantum states ρ_0, ρ_1 are said to commute iff $\rho_0\rho_1 - \rho_1\rho_0 = 0$ [47])

Corollary 1. There is no universal broadcasting map in quantum mechanics

5.2. Operational Formulations

In reference [46], it is shown that Corollary 1 can be derived directly from the monogamy of entanglement using the Choi–Jamiołkowski isomorphism:

Theorem 2. Supposing the existence of a universal broadcasting map is equivalent to supposing the existence of a tripartite state of a system A, B, C where both the bipartite reduced state of A, B and the bipartite reduced state of B, C are pure and maximally entangled.

Here I will derive a similar result, but my derivation will be presented in the context of general operational theories without presupposing the Hilbert space structure of quantum mechanics. Therefore before embarking on the proof, I need to provide definitions of broadcasting and monogamy which are suited to the operational context.

In order to define ‘broadcasting’ we will use the concept of as an ‘operational state’; note, again, that this concept will be used only to define broadcasting, and will not appear in the definitions of the features from which no-broadcasting will be derived.

Definition 7. Given two preparation procedures P_a, P_b which appear in an operational theory, I say that these procedures produce the same operational state iff when a system is prepared using one of these procedures, there is no subsequent measurement or sequence of measurements which can give us any information about whether the system was prepared using P_a or P_b .

Definition 8. A transformation T broadcasts the set of operational states S iff when T is applied to a system whose operational state is $\rho \in S$, together with one or more ancilla systems, the output includes two distinct systems both having the operational state ρ .

Definition 9. A universal broadcasting map broadcasts any set of operational states.

Defining the relevant monogamy property for an operational context is less straightforward. In quantum mechanics, the monogamy of entanglement refers to the fact that the amount of entanglement a quantum system has with a given system limits the amount of entanglement it can have with any other system - in the most extreme case, a quantum system which is maximally entangled with a given quantum system cannot be entangled at all with any other quantum system. However, entanglement is a property of quantum states, usually quantified using state-dependent measures like the ‘concurrence’ (or ‘tangle’) [48], and these measures do not have a straightforward physical interpretation, making it difficult to translate them to the framework of operational theories.

Therefore it is more helpful for us to focus on the ‘monogamy of correlations’, which is a property of certain theories in which there are limits on how strongly a system can be correlated with other systems. The monogamy of correlations is defined purely in terms of observable statistics, and is therefore much more easily cast in an operational framework.

In order to facilitate comparison to the quantum case, I will define monogamy for a general operational theory using the CHSH quantity:

Definition 10. CHSH quantity: For any joint preparation P_{AB} of two systems, and any pair of measurements M_A^0, M_A^1 which may be performed on system A , and any pair of measurements M_B^0, M_B^1 which may be performed

on system B, the CHSH quantity $\mathcal{B}_{AB}(P_{AB}, M_A^0, M_A^1, M_B^0, M_B^1)$ for this combination of preparation and measurements is defined by the following sum of expectation values:

$$\mathcal{B}_{AB}(P_{AB}, M_A^0, M_A^1, M_B^0, M_B^1) := \langle AB \rangle_{P_{AB}, M_A^0, M_B^0} + \langle AB \rangle_{P_{AB}, M_A^0, M_B^1} + \langle AB \rangle_{P_{AB}, M_A^1, M_B^0} - \langle AB \rangle_{P_{AB}, M_A^1, M_B^1} \quad (1)$$

Note that is also possible to define a related CHSH quantity $\mathcal{B}_{AB}(P^0, P^1, T, M_B^0, M_B^1)$ for the scenario where an experimenter applies one of two ensemble preparations P^0, P^1 to a system, then applies a fixed transformation T , then performs one of two measurements M_B^0, M_B^1 . This quantity is defined just as in Equation (1), where the result of the measurement M_A is replaced with the result of the probabilistic choice in the ensemble preparation. This definition ensures that if for some transformation T , some set of preparations P_{AB}, P^0, P^1 and some set of measurements $M_1^0, M_1^1, M_2^0, M_2^1$, for any $x, y \in \{0, 1\}$ the distributions $p_{P_{AB}; M_1^x M_2^y}$ and $p_{P^x; T; M_2^y}$ are the same, then $\mathcal{B}_{AB}(P_{AB}, M_A^0, M_A^1, M_B^0, M_B^1) = \mathcal{B}_{AB}(P^0, P^1, T, M_B^0, M_B^1)$.

With a definition for the CHSH quantity in hand, we may now define two types of monogamy which might be obeyed by an operational theory:

Definition 11. *Non-signalling monogamy of correlations: An operational theory obeys non-signalling monogamy of correlations iff for any joint preparation of three systems S_A, S_B, S_C , for any choice of measurements M_A^0, M_A^1 on S_A , any choice of measurements M_B^0, M_B^1 on S_B , and any choice of measurements M_C^0, M_C^1 on S_C , the associated CHSH quantities satisfy:*

$$\mathcal{B}_{AB}(P_{ABC}, M_A^0, M_A^1, M_B^0, M_B^1) + \mathcal{B}_{BC}(P_{ABC}, M_B^0, M_B^1, M_C^0, M_C^1) \leq 4$$

It can be shown that this bound is necessarily obeyed by any operational theory which does not allow signalling (see Definition 15) [12].

Definition 12. *Strong monogamy of correlations: An operational theory obeys strong monogamy of correlations iff for any joint preparation of three systems S_A, S_B, S_C , for any choice of measurements M_A^0, M_A^1 on S_A , any choice of measurements M_B^0, M_B^1 on S_B , and any choice of measurements M_C^0, M_C^1 on S_C , the associated CHSH quantities satisfy:*

$$\mathcal{B}_{AB}(P_{ABC}, M_A^0, M_A^1, M_B^0, M_B^1)^2 + \mathcal{B}_{BC}(P_{ABC}, M_B^0, M_B^1, M_C^0, M_C^1)^2 \leq 8$$

Strong monogamy of correlations need not be obeyed by all non-signalling operational theories, but it is obeyed by quantum mechanics [49].

5.3. Bell Nonlocality

We will also require the notion of Bell nonlocality, which is again defined by analogy with the quantum case:

Definition 13. *An operational theory exhibits Bell nonlocality iff there exists at least one joint preparation P_{AB} for two systems S_A and S_B , such that for some pair of measurements M_A^0, M_A^1 on S_A and some pair of measurements M_B^0, M_B^1 on S_B , the CHSH quantity satisfies:*

$$\mathcal{B}_{AB}(P_{AB}, M_A^0, M_A^1, M_B^0, M_B^1) > 2$$

5.4. Theorem

Using these concepts, we have the following theorem (see Appendix A.1 for the proof).

Theorem 3. *In an operational theory which obeys the operational Choi–Jamiołkowski isomorphism and exhibits Bell nonlocality, the existence of a universal broadcasting map implies that the theory violates strong monogamy of correlations*

Note that if we impose the ontological Choi–Jamiołkowski isomorphism as a fundamental constraint on an operational theory, it follows by modus tollens that the strong monogamy of correlations together with the existence of Bell nonlocality implies the absence of broadcasting, as advertised in the table above.

5.5. Quantum Interference

The results of Section 5.4 may also be used to derive the existence of quantum interference from the monogamy of correlations together with the existence of Bell non-locality.

To see this, let us limit ourselves to considering GPTs whose observables have the algebraic structure of the self-adjoint operators in a C^* algebra. Not all GPTs meet this criterion (See reference [50] for a discussion of the general conditions under which a theory can be represented as a C^* algebra), but nonetheless it is a reasonable heuristic framework since a large class of important GPTs, including classical mechanics and quantum mechanics, can indeed be thus represented. Then observe that for any theory represented as a C^* -algebra, it can be shown that if the C^* algebra is commutative then the theory has a universal broadcasting map [13,51]. Thus if a GPT does not have a universal broadcasting map, any C^* algebra that represents it must be non-commutative. Finally, note that theories represented by commutative C^* algebras are classical phase space theories, whereas theories represented by non-commutative C^* algebras have observables which cannot be simultaneously measured and hence exhibit superposition and interference [51].

Thus, if we limit ourselves to GPTs which have the algebraic structure of the self-adjoint operators in a C^* algebra and which obey the operational Choi–Jamiołkowski isomorphism, it follows from Theorem 3 that any GPT which exhibits strong monogamy of correlations and Bell non-locality must also exhibit superposition and interference. This means that superposition and interference need not be regarded as properties of the quantum state - with the help of the operational Choi–Jamiołkowski isomorphism they can be derived directly from properties of non-local correlations, with no need to appeal to a concept of state at all.

6. Preparation Contextuality and No-Signalling

6.1. Background

One of the most puzzling feature of quantum theory is the fact that it is contextual: That is, it is not possible to interpret the outcomes of quantum-mechanical measurements as if they simply reveal the pre-existing value of some property of the system being measured.

The original formulation of contextuality [52] tied the concept of contextuality to the concept of determinism, but the notion was subsequently generalized by Spekkens [43]. Spekkens' form of contextuality is most easily defined within the ontological models framework, where one supposes that every system has a single real 'ontic state', which determines the probabilities for the outcomes of any measurement on that system (The use of this framework does not necessarily imply the endorsement of a description of quantum reality in terms of underlying states; the ontological models approach could simply be regarded as a helpful language in which we may express mathematical facts about the structure of quantum theory, such as its contextual character.). An ontological model thus consists of a space Λ of ontic states λ , a set of probability distributions $\mu^P(\lambda)$ giving the probability that the system ends up in the state λ when we perform the preparation procedure P , a set of response

functions $\xi_{M,X}(\lambda)$ giving the probability that we obtain outcome M^x when we perform measurement M on a system whose ontic state is λ , and a set of column-stochastic matrices T^O representing the way in which the ontic state is transformed when some operation O is applied to the system. A valid ontological model must satisfy the following positivity and normalization conditions:

1. $\forall P, \lambda \quad \mu_P(\lambda) \in [0, 1]$
2. $\forall P \quad \sum_{\lambda} \mu_P(\lambda) = 1$
3. $\forall M, X \quad \xi_{M,X}(\lambda) \in [0, 1]$
4. $\forall M \quad \sum_X \xi_{M,X}(\lambda) = 1$
5. $\forall O \quad T_O$ is a column-stochastic matrix.

An ontological model is preparation contextual if it does not represent every quantum state by a unique probability distribution $\mu(\lambda)$; transformation contextual iff it does not represent every possible quantum operation O by a unique transformation matrix T_O ; and measurement contextual if it does not represent every possible quantum measurement element M^x by a unique set of response functions $\xi_{M,X}(\lambda)$. Spekkens proved [43] that any ontological model of quantum mechanics which reproduces all the correct measurement statistics must exhibit preparation and transformation contextuality, but examples such as the Beltrametti-Bugajski model [53] show that this is not true for measurement contextuality.

6.2. Operational Formulations

I now generalize the definition of preparation contextuality to the framework of operational theories. (Similar generalizations can be made for measurement and transformation contextuality, but I will not need these concepts here).

Definition 14. *We say that an operational theory is preparation contextual iff it is not possible to represent the theory by a valid ontological model in which every operational state is represented by a unique probability distribution over ontic states.*

I will also employ a definition of the no-signalling principle suited for an operational framework. This generalization is a straightforward one, since it is just the operational version of the no-signalling principle as it is used in the context of relativistic cryptography [54]:

Definition 15. *Operational no-signalling principle: In a process involving a set of non-communicating devices $\{D_i\} : i \in \{1 \dots N\}$ such that device D_i accepts an input variable N_i and produces an output variable O_i , let J be any subset of $\{1 \dots N\}$, let O_J be the set of variables $\{O_j : j \in J\}$, let N_J be the set of variables $\{N_j : j \in J\}$; then if the inputs $\{N_i\}$ are uncorrelated, the outcomes satisfy $p(O_J | N_1, \dots, N_n) = p(O_J | N_J)$.*

6.3. Theorem

Thus we have the following theorem (see Appendix A.1 for the proof).

Theorem 4. *Given an operational theory which obeys the operational Choi–Jamiołkowski isomorphism and the no-signalling principle, if the theory is preparation non-contextual, it does not exhibit Bell nonlocality.*

Note that if we impose the ontological Choi–Jamiołkowski isomorphism as a fundamental constraint on an operational theory, it follows by modus tollens that if the theory exhibits Bell nonlocality but obeys the no-signalling principle, it must be preparation contextual; that is, preparation contextuality is a necessary feature of any such operational theory describing a world which is non-local but also non-signalling.

7. Uncertainty Relations

7.1. Background

‘Heisenberg’s Uncertainty Principle’ is one of the most well-known features of quantum physics, not least because the fact that there exist incompatible measurements whose results cannot be simultaneously predicted with certainty seems strongly in conflict with our classical intuitions. This provides a clear imperative to seek an explanation for the existence of such apparently arbitrary limitations on the properties that quantum systems can possess.

To be more precise, there are in fact a number of different uncertainty relations which characterize tradeoffs between the precision with which the outcomes of various sets of measurements can be predicted. Most of these relations were originally formulated in terms of standard deviations and commutators, but this approach has been criticized on the grounds that the use of commutators makes the relations state-dependent, which should not be the case given that in quantum mechanics there always exists a dynamical evolution which transforms any initial pure state into any other pure state [55]. Thus more recently there has been a move towards expressing uncertainty relations as a constraint on the Shannon entropies corresponding to the probability distributions over outcomes associated with various possible measurements [55–57]. However these entropic uncertainty relations have their own limitations—in particular, they are not capable of distinguishing between different possible combinations of outcomes [58]—and therefore in certain cases it is more useful to consider fine-grained uncertainty relations, which directly constrain particular sets of outcomes of different measurements.

The derivation of this section is inspired by the work of reference [58], where it is shown that it is possible to derive the Tsirelson bound from the following fine-grained entropic uncertainty relation, which applies to any pair of orthogonal two-outcome measurements M and M' on a two-dimensional quantum system (such as the binary spin-half observables X and Z):

$$\forall m \forall n \forall \rho \in \{0, 1\} \quad p(m|M)_\rho + p(n|M')_\rho \leq 1 + \frac{1}{\sqrt{2}} \quad (2)$$

Here, $p(m|M)_\rho$ denotes the probability that I obtain outcome $m \in \{0, 1\}$ when I perform measurement M on a system in quantum state ρ .

In the operational framework, Equation (2) becomes a constraint applying to all preparations in the operational theory which prepare systems of dimension two, according to the operational definition of ‘dimension’ which I provide in Section 7.2. I also reverse the argument of reference [58], in order to derive the uncertainty relation from the Tsirelson bound. I then note that it was shown in reference [11] that the Tsirelson bound can be derived from the principle of information causality:

Definition 16. *Information causality: if Alice and Bob pre-share a set of devices which exhibit nonlocal correlations, and Alice receives a bit string $N_0N_1\dots N_r$ and sends Bob a classical message M of m bits, and Bob performs a measurement with some setting k and obtains outcome O , then $\sum_r I(MO : N_r | k = r) > m$ [11]*

Thus, if we take it that information causality is a fundamental constraint on quantum theory, as suggested in [11], the fact that quantum states obey this particular fine-grained uncertainty relation in quantum theory may be understood as a consequence of the fundamental constraint. Moreover, in quantum mechanics, the inequality in Equation (2) is tight (for example, the inequality is saturated if we choose measurements X , Z and prepare the system to be measured in an eigenstate of $\frac{1}{\sqrt{2}}(X + Z)$ or $\frac{1}{\sqrt{2}}(X - Z)$) and thus argue in this particular case information causality may be understood as setting the exact limits of quantum uncertainty.

7.2. Operational Formulations

Since the fine-grained uncertainty relation of Equation (2) applies only for systems of dimension two, we need an operational way to single out systems of dimension two. To do so, we must first add some structure to the usual framework of operational theories. Thus far, we have assumed that for a given operational theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ it is possible to perform any preparation followed by any transformation followed by any measurement, but in a theory with a meaningful concept of dimension this is not so, because we can only perform measurements which match the dimension of the system I have prepared. Thus in this section I consider an operational theory to be a collection of subtheories $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$, such that within any given subtheory it is possible to perform any preparation followed by any transformation followed by any measurement, but it is not possible to perform a preparation from one subtheory followed by a transformation or measurement from another subtheory. I then define the dimension of a sub-theory as follows.

Definition 17. A sub-theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ of an operational theory is d -dimensional iff d is the smallest number such that there exists a set of $d^2 - 1$ continuous parameters in $[0, 1]$ with the following properties:

1. Specifying the values of all $d^2 - 1$ parameters for any preparation $P \in \mathcal{P}$ fully determines the probabilities $p(M^x|P)$ for every outcome M^x of every measurement M in \mathcal{M} .
2. For every possible set of values of the $d^2 - 1$ parameters, there exists a preparation $P \in \mathcal{P}$ described by those parameters.

Obviously, the choice to associate d dimensions with $d^2 - 1$ parameters is inspired by quantum theory: A d -dimensional quantum state is described by a set of d complex parameters, or d^2 continuous real parameters, minus 1 to account for normalization. However, this particular mapping is a naming convention only - I could equally well have chosen to say that a subtheory described by a set of d parameters is d -dimensional, which might have seemed the natural choice if it were not for the example of quantum theory. Therefore this definition does not take for granted any particular structure for the state space of the subtheory, other than the continuity implied by the second property.

The fine-grained uncertainty relation also applies only for pairs of orthogonal measurements, and therefore we require a notion of ‘orthogonal measurement’ for the operational framework:

Definition 18. In some subtheory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ of an operational theory, two measurements $M_1, M_2 \in \mathcal{M}$ are orthogonal iff given an arbitrary unknown preparation P , the set of probabilities $\{p(M_1^x|P)\}$ and $\{p(M_2^x|P)\}$ are independent.

This definition implies that given a preparation P , and any set of q orthogonal measurements $\{M_i\}$ each having two outcomes M_i^0, M_i^1 , we need q independent parameters to specify the set of q probabilities $p(M_i^0|M_i, P)$. Therefore I can set q out of the $d^2 - 1$ parameters needed to specify the full set of outcome probabilities to be equal to the q probabilities $p(M_i^0|M_i, P)$. In particular, for a two-dimensional system, given any set of three orthogonal measurements $\{M_1, M_2, M_3\}$ each having two outcomes, specifying the three probabilities $p(M_1^0|M_1, P), p(M_2^0|M_2, P), p(M_3^0|M_3, P)$ is sufficient to fix the probabilities for any other measurement which may be performed after the preparation P .

7.3. Theorem

Using these concepts, we have the following theorem (see Appendix A.3 for the proof).

Theorem 5. If an operational theory obeys the operational Choi–Jamiołkowski isomorphism and information causality, then given any subtheory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ of dimension two, for any preparation $P \in \mathcal{P}$ and any

pair of orthogonal measurements $M_1, M_2 \in \mathcal{M}$, and any two outcomes M_1^m, M_2^n of the measurements M_1, M_2 , we must have:

$$p(M_1^m|M) + p(M_2^n|M') \leq 1 + \frac{1}{\sqrt{2}}$$

8. Discussion

I have suggested that the study of intratheoretic causal structures offers us a way of rethinking the structural assumptions built into our theories in a way that is independent of any particular ontology. In particular, I have focused here on a particularly stubborn ontological assumption which is deeply woven into both classical and quantum physics: The idea that information about the past must be carried into the future by a mediating state.

While it is still very common to assume that temporal correlations must be mediated, we do not usually suppose that the non-local multipartite correlations in quantum mechanics are mediated by an intervening process, not least because any such process would either have to travel faster than light or go backwards in time. Thus by considering an intratheoretic causal structure for quantum mechanics in which the equivalence between multipartite correlations and temporal correlations is treated as fundamental, we are moving towards a picture of quantum mechanics where temporal correlations are not mediated. Therefore the structure set up here offers a way of thinking about quantum mechanics where neither the quantum state nor any other sort of state is a fundamental object.

In particular, I have demonstrated that several of the characteristic properties of the quantum state can in fact be derived from features of non-local correlations without appeal to any concept of state. For example, Theorem 4 shows that using the operational Choi–Jamiołkowski isomorphism, the fact that quantum states exhibit preparation contextuality can be derived from the fact that we live in a world which is non-local but non-signalling. Intuitions will vary, of course, but to me non-locality and no-signalling seem like very fundamental features of reality, and therefore I find it plausible to regard preparation contextuality as a consequence of non-locality and no-signalling. Thus preparation contextuality need not be regarded as a fundamental property of some entity known as the quantum state—it can be thought of as a behaviour that is in fact caused by very general facts about space and time together with the equivalence between multipartite correlations and temporal correlations.

The same goes for the other derivations presented here. There is no need to postulate a quantum state to be the bearer of the properties of ‘no-broadcasting’, ‘interference’, and ‘obeying uncertainty relations’, since these features can be regarded as consequences of constraints on non-local correlations. Thus these derivations point towards a picture of quantum mechanics where the state is not a fundamental entity but instead emerges from deeper properties of the theory. There do exist several specific proposals for the ontology of quantum mechanics which eliminate states from the picture—particularly those approaches where the ontology consists entirely of events, such as the GRW flash ontology [59,60] or Kent’s solution to the Lorentzian quantum reality problem [61]—but by couching the discussion in terms of intratheoretic causal structure it is possible to understand some general consequences of the elimination of states without committing to specific ontological details.

I will finish by reinforcing that the possibility of a stateless ontology for quantum mechanics has significant consequences for physics outside quantum foundations, because the attitude that we take toward the quantum state has a strong influence on the way we are likely to proceed when working on extensions of the theory, such as quantum field theory and quantum gravity. If we think of the quantum state as the fundamental object of quantum theory, it is natural to deal with gravity by writing down quantum states of spacetime - that is, by ‘quantizing gravity’, as is done in most of the mainstream approaches to quantum gravity, including covariant quantum gravity, canonical quantum gravity and string theory. On the other hand, if we think that the quantum state is actually just a codification of relations between events or some other ontological substratum, it seems more natural to regard spacetime itself as emerging from the underlying substratum, as in causal set theory. Which direction we take will depend to a large extent on the beliefs we hold, implicitly or explicitly,

about the underlying intratheoretic causal structure of quantum mechanics, and so exploring new structures where states are not fundamental may shed new light on outstanding problems in quantum field theory and quantum gravity.

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Appendix A

Appendix A.1. Proof of Theorem 3

Proof. Consider any operational theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ which obeys the operational Choi–Jamiołkowski isomorphism. Suppose the theory exhibits Bell-nonlocality. Then for some joint preparation P_{AB} and some pair of measurements M_A^0, M_A^1 on S_A and some pair of measurements M_B^0, M_B^1 on S_B , the CHSH quantity satisfies:

$$\mathcal{B}_{AB}(P_{AB}, M_A^0, M_A^1, M_B^0, M_B^1) > 2$$

From the operational Choi–Jamiołkowski isomorphism, there exist ensemble preparations P^0, P^1 and a transformation T which give rise to the same probability distributions, and therefore

$$\mathcal{B}_{AB}(P^0, P^1, T, M_B^0, M_B^1) > 2$$

By stipulation (recall Section 4.1), I can always combine the transformation T with the measurements P^0, P^1 to produce new measurements N^0, N^1 , thus there exist measurements N^0, N^1 such that $\mathcal{B}_{AB}(P^0, P^1, \mathbb{I}, N^0, N^1) > 2$.

Now suppose the theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ has a universal broadcasting map, i.e., a channel which takes any given operational state to two perfect copies of the same operational state. This channel may be represented as two copies of the identity transformation \mathbb{I} acting on the input state.

Let us apply this universal broadcasting map to the result of an ensemble preparation P , producing two copies of the input state on two systems S_B, S_C . I may then apply measurements N_B, N_C , giving rise to a joint distribution $p_{P; \mathbb{I}; N_B}$ over the outcome of the ensemble preparation and the outcome of the measurement N_B , and another joint distribution $p_{P; \mathbb{I}; N_C}$ over the outcome of the ensemble preparation and the outcome of the measurement N_C , and hence a joint distribution $p_{P; N_B, N_C}$ over the outcome of the ensemble preparation and the two outcomes of the two measurements.

From the operational Choi–Jamiołkowski isomorphism, it follows that there exists a joint preparation P_{ABC} of three systems S_A, S_B, S_C and a measurement N_A on system S_A such that if I perform P_{ABC} and then the measurements N_A, N_B, N_C on the three systems respectively, the joint distribution $p_{P_{ABC}; N_A N_B N_C}$ over the measurement outcomes is identical to the distribution $p_{P; N_B, N_C}$.

Thus consider applying the universal broadcasting map to either of the two ensemble preparations P^0, P^1 and then choosing from the set of measurements N^0, N^1 for the systems S_B, S_C . I have that:

$$\mathcal{B}_{AB}(P^0, P^1, \mathbb{I}, N^0, N^1) > 2$$

$$\mathcal{B}_{AC}(P^0, P^1, \mathbb{I}, N^0, N^1) > 2$$

It follows that there exists a preparation P'_{ABC} such that if I perform P'_{ABC} and then choose from the measurements N^0, N^1 for the systems S_A, S_B, S_C , I have:

$$\mathcal{B}_{AB}(P'_{ABC}, N_A^0, N_A^1, N_B^0, N_B^1) > 2$$

$$\mathcal{B}_{AC}(P'_{ABC}, \mathbb{I}, N_A^0, N_A^1, N_C^0, N_C^1) > 2$$

Therefore:

$$\mathcal{B}_{AB}(P'_{ABC}, N_A^0, N_A^1, N^0, N^1)^2 + \mathcal{B}_{AC}(P'_{ABC}, \mathbb{I}, N_A^0, N_A^1, N^0, N^1)^2 > 8$$

Thus the monogamy of correlations is necessarily violated in the theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$. \square

Appendix A.2. Proof of Theorem 4

Proof. Consider an operational theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ which obeys the operational Choi–Jamiołkowski isomorphism and the no-signalling principle.

Within such a theory, suppose we perform a joint preparation P_{AB} for two distinct systems A, B , then perform one measurement from a set of n measurements $\{M_i\}$ on system A , and a fixed measurement F_b on system B .

It follows from the operational Choi–Jamiołkowski isomorphism that there exists a set of n ensemble preparations $\{P_i\}$ on a system S and a channel T such that if we perform the preparation P_i on S , then input S to the channel T , then perform measurement F_b on the product of the channel, each of the resulting distributions $p_{P_i;T;F_b}$ is identical to the corresponding distribution $p_{P_{AB};M_i;F_b}$. Thus each P_i is associated with a probability distribution $p^i(x)$ such that $p^i(x) = p^A(M_i^x | M_i, P_{AB})$.

Since the theory is no-signalling, there is no choice of measurement F_b on system B which gives any information about which measurement M_i was performed on system A in the former scenario, and therefore there is no choice of measurement F_b which gives any information about which ensemble preparation P_i was chosen in the latter scenario. This means that any one of the preparation procedures in $\{P_i\}$, followed by the transformation T_{AB} , gives rise to the same operational state, which I denote by χ . I will henceforth denote by P_i^i the composite preparation procedure obtained by performing the procedure P_i followed by the transformation T_{AB} .

Let us now write down an ontological model for this operational theory.

Each ensemble preparation P_i is associated with a set of preparations P_i^x each being chosen with probability $p^i(x)$. Each preparation P_i^x is associated with a probability distribution $\mu(\lambda | P_i^x)$ in the ontological model, and the result of performing a preparation P_i^x and then applying the transformation T is also associated with a probability distribution, $\mu(\lambda | T, P_i^x)$ (We need not assume that there exists a unique column-stochastic matrix in the ontological model which represents the transformation T —if there is more than one such transformation, we simply use any one which produces the correct statistics for this particular scenario—and therefore we need not assume that the theory is transformation contextual.). Thus each of the ensemble preparations $P_i \in \{P_i\}$ gives rise to a representation of χ as a probability distribution over ontic states:

$$\begin{aligned} p(\lambda | \chi) &= \sum_x p^i(x) \mu(\lambda | T, P_x^i) \\ &= \sum_x p^A(M_i^x | M_i, P_{AB}) \mu(\lambda | T, P_x^i) \end{aligned}$$

For any $M_i \in \{M_i\}$, the joint probability distribution $p_{P_{AB};M_i;F_b}$ over the outcomes of the measurements on the systems A, B after the preparation P_{AB} can be written as follows (Note that we need not assume that there exists a unique set of response functions $p(F_b^y | F_b, \lambda)$ for the measurement F_b —if there is more than one such response function, we can simply use any one which produces the correct statistics for this particular scenario—and therefore we need not assume that the theory is measurement non-contextual.):

$$\begin{aligned} p\left(M_i^x, F_b^y\right)_{P_{AB}, M_i, F_b} &= p^A\left(M_i^x | M_i, P_{AB}\right) p\left(F_b^y | M_i^x, M_i, F_b, P_{AB}\right) \\ &= p^A\left(M_i^x | M_i, P_{AB}\right) p\left(F_b^y | T, P_x^i\right) \\ &= p^A\left(M_i^x | M_i, P_{AB}\right) \sum_{\lambda} \xi\left(F_b^y | F_b, \lambda\right) p\left(\lambda | T, P_x^i\right) \end{aligned} \tag{A1}$$

Now suppose the operational theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ is not preparation contextual. This means that it is possible to choose the ontological model in such a way that all representations of χ are identical, so there exists a unique probability distribution $p(\lambda)$ such that for any $i \in \{1, 2, \dots, n\}$ I have:

$$\sum_x p^A(M_i^x | M_i, P_{AB}) p(\lambda | T, P_x^i) = p(\lambda)$$

Now for any $i \in \{1, 2, \dots, n\}$ let us define:

$$\xi(M_i^x | M_i, \lambda) := \frac{p^A(M_i^x | M_i, P_{AB}) p(\lambda | T, P_x^i)}{p(\lambda)}$$

Note that $\forall i \sum_x \xi(M_i^x | M_i, \lambda) = \sum_x \frac{p^A(M_i^x | M_i, P_{AB}) p(\lambda | T, P_x^i)}{p(\lambda)} = \frac{p(\lambda)}{p(\lambda)} = 1$, and since every $\xi(M_i^x | M_i, \lambda)$ is nonnegative, it follows that $\forall i, x \xi(M_i^x | M_i, \lambda) \in [0, 1]$. These are the sufficient conditions for a function to be a valid response function of an ontological model, so $\xi(M_c^x | M_a, \lambda)$ is a valid response function.

The joint probability distribution over the outcomes of the measurements on the systems A, B after the preparation P_{AB} can then be written as follows:

$$p(M_i^x, F_b^y)_{P_{AB}, M_i, F_b} = \sum_{\lambda} \xi(F_b^y | F_b, \lambda) \xi(M_i^x | M_i, \lambda) p(\lambda) \tag{A1}$$

Thus any joint probability distributions within the theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ are factorizable, and it is a well-known result [62] that measurements described by factorizable probability distributions cannot satisfy $\mathcal{B}_{AB} > 2$, so the theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ cannot exhibit Bell nonlocality. \square

Appendix A.3. Proof of Theorem 5

Proof. Let $M_1, M_2, M_3 \in \mathcal{P}$ be a set of three orthogonal measurements for the two-dimensional subtheory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$, each having two outcomes which I label by 0, 1.

Let $P_a \in \mathcal{P}$ be a preparation which maximizes $p(0|M_1, P_a) + p(0|M_2, P_a)$, with the maximization performed over all preparations in \mathcal{P} . Let us say that $p(0|M_1, P_a) = i$, $p(0|M_2, P_a) = j$ and $p(0|M_3, P_a) = k$. Since I can choose $p(0|M_1, P)$, $p(0|M_2, P)$, $p(0|M_3, P)$ to be the set of three parameters which suffice to fix the measurement outcomes for any other measurement performed after P , and by definition, for any valid set of values of these parameters, there exists a preparation $P \in \mathcal{P}$ described by those parameters, it follows that there exist preparations P_b, P_c, P_d with outcome probabilities as follows:

$$\begin{array}{c|c|c|c|c} & P_a & P_b & P_c & P_d \\ p(0|M_1) & i & 1-i & i & 1-i \\ p(0|M_2) & j & 1-j & 1-j & j \\ p(0|M_3) & k & k & k & k \end{array}$$

Suppose Alice performs either preparation P_a or P_b , chosen uniformly at random, and then sends the resulting system to Bob. If Bob then performs the measurement M_1 , he will obtain the outcome 0 with probability $\frac{1}{2}$ and if Bob then performs the measurement M_2 , he will obtain the outcome 0 with probability $\frac{1}{2}$. Likewise, if Alice performs P_c or P_d chosen uniformly at random and then sends the system to Bob, if Bob performs the measurement M_1 Bob will obtain the outcome 0 with probability $\frac{1}{2}$ and if Bob then performs the measurement M_2 , he will obtain the outcome 0 with probability $\frac{1}{2}$. Clearly the probabilities for the outcomes of measurement M_3 are also the same for both decompositions. Since the probabilities $p(0|M_1, P)$, $p(0|M_2, P)$, $p(0|M_3, P)$ suffice to fix the measurement outcomes for any other measurement performed after P , this implies that these two probabilistic preparations give rise to the same measurement statistics for any subsequent

measurements, and hence both decompositions produce the same operational state, which I denote by χ .

It follows from the operational Choi–Jamiołkowski isomorphism that there exists a preparation P_{AB} and a measurement S_0 such that when S_0 is performed on one subsystem A of a bipartite system AB jointly prepared using preparation P_{AB} with probability $\frac{1}{2}$ I obtain outcome 0 and the resulting operational state of subsystem B is the same as the operational state produced by preparation P_a , and with probability $\frac{1}{2}$ I obtain outcome 1 and the resulting operational state of subsystem B is the same as the operational state produced by preparation P_b . Likewise, there exists a measurement S_1 defined similarly for preparations P_c, P_d .

Now consider a CHSH game in which Alice and Bob are each given input bits c, d and are required to produce outcomes a, b such that $a \oplus b = cd$. Alice and Bob may employ the following strategy:

1. Before the start of the game, Alice and Bob perform the preparation P_{AB} , and then Alice takes subsystem A and Bob takes subsystem B .
2. When Alice is given input $c \in \{0, 1\}$, she performs measurement S_c and then returns her measurement outcome.
3. When Bob is given input $d \in \{0, 1\}$, he performs measurement M_{d+1} and then returns his measurement outcome.

If the inputs are chosen uniformly at random, the probability that Alice and Bob win the game using this strategy is:

$$\frac{1}{4} \sum_{c,d,a} p(a|c)p(x(acd)|acd)$$

where $p(a|s)$ the probability that Alice obtains outcome a when she performs measurement s , $x(acd)$ is defined such that for any a, c, d , $a \oplus x(a, c, d) = cd$, and $p(q|acd)$ is the probability that Bob obtains outcome q when Alice performs measurement S_c and obtains outcome a and Bob performs measurement M_{d+1} .

S_0 and S_1 are defined such that $\forall a \forall s p(a|s) = \frac{1}{2}$. Moreover, by the definition of S_0 , if Alice performs measurement S_0 and obtains outcome 0, the resulting probabilities $p(q|0, 0, d)$ for the outcomes of Bob's measurement M_{d+1} are the same as if Alice were to simply perform preparation P_a and hand the prepared system over to Bob. That is, $p(0|0, 0, 0) = i$ and $p(0|0, 0, 1) = j$. Similar relationships hold for other possible values of a and c , and using the table above, it is straightforward to obtain:

$$\forall a, c \quad \sum_{d=1}^2 p(x(a, c, d)|a, c, d) = i + j$$

Hence the probability that Alice and Bob win the game using this strategy is $\frac{1}{2}(i + j)$.

Ex hypothesi the operational theory satisfies information causality. It is known that information causality implies the Tsirelson bound [11], and the Tsirelson bound implies that the maximum probability of winning this CHSH game is $1 + \frac{1}{\sqrt{2}}$; hence $i + j \leq 1 + \frac{1}{\sqrt{2}}$.

Since we chose P_a to be the preparation which maximizes $p(0|M_1, P_a) + p(0|M_2, P_a)$, it follows that for any preparation P , $p(0|M_1, P) + p(0|M_2, P) \leq 1 + \frac{1}{\sqrt{2}}$.

Then note that for any $m \in \{0, 1\}, n \in \{0, 1\}$ I can make a similar argument by starting from the requirement that P_a is the state which maximizes $p(m|0) + p(n|1)$, and hence: \square

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