

Article

The Resistance of an Arbitrary Body in Confined Unsteady Stokes Flow

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Abstract: In this article, we address resistance forces and torques acting onto a body with arbitrary shape moving in an unsteady Stokes flow. We start analyzing the functional form of the expressions for forces and torques, which depend on the frequency parameter and on the position of the body in the domain of the fluid, and determining the asymptotic limits for high and low frequencies. In this regard, we show that, for high frequencies (hence short times), forces and torques are obtained by the associated hydrodynamic problems considering ideal potential flows, independently of the geometry of the problem. Afterwards, with the aim of obtaining expressions for forces and torques valid in the entire range of frequencies, we extend to the unsteady case the *reflection method*, largely employed in the theory of the steady Stokes flows. In this way, general expressions are provided in terms of the Faxén operators of the body and the Green function associated with the geometry of the confinement, that are valid, to the leading order, at any frequency, independently of the geometry of the problem. Finally, as the application of the general expressions, explicit relations for the resistance forces acting onto a spherical body with no-slip boundary conditions near a plane wall with full-slip boundary conditions are obtained, valid over the entire frequency range, provided that the distance between the plane and the sphere is larger than one sphere radius.

Keywords: time-dependent Stokes regime; confined geometries; particle hydromechanics; inertial effects



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1. Introduction

Recent studies have shown that considering unsteadiness of fluid flows at microscale is crucial for understanding and controlling phenomena such as Brownian motion [1,2] and microswimming [3–5] or for improving investigation techniques, such as surfaces probing [6], and processes such as mixing and separation [7]. Although the very low Reynolds numbers allow us to neglect non-linear inertial effects at the microscale, whenever the characteristic time t_c , with which the conditions on the system change, is comparable with the momentum diffusion time $t_v = L_c^2/\nu$ (L_c being a characteristic length of the system and ν the kinematic viscosity of the fluid), the steady Stokesian approximation becomes inadequate. In these cases, the linear inertial term entering the Navier–Stokes equations needs to be considered, obtaining, hence, the so-called unsteady Stokes regime. More specifically, linear inertial effects become significant whenever the Womersley number of the system, defined as $\alpha = \sqrt{t_v/t_c}$, is not negligible.

Since the pioneering studies by Stokes on the effect of internal friction of fluids on the motion of pendulums [8], several contributions to the hydromechanics of bodies immersed in an unsteady Stokes flow have been presented. This problem is often formulated in the Fourier or Laplace domain, with the aim of eliminating the time variable in the differential equations governing the motion of the fluid, thus obtaining linear relations between hydrodynamic forces (and torques) and the velocity of immersed bodies. Classical results in this hydrodynamic regime include the force acting onto a sphere with no-slip boundary conditions translating with an arbitrary velocity provided by Boussinesq [9]

and Basset [10] and the torque acting onto a no-slip sphere rotating with an arbitrary angular velocity provided in the book by Lamb [11] (p. 642) (in the frequency formulation) and by Feuillebois and Lasek [12]. For the case of a sphere immersed in a generic ambient flow, Faxén laws for the force and torque acting onto a sphere with no-slip ones are available in the literature [13–16]. More recent works considered the case of a sphere with boundary conditions different from no-slip boundary conditions moving in the unbounded flows [17–20]. Expressions for forces and torques on bodies different from a sphere are provided in [21]. Lawrence and Weinbaum [22,23] discussed the asymptotic behavior (for low and high frequencies) of forces acting onto bodies different from spheres, proposing an approximate functional form valid in the entire range of frequencies. By enforcing the approximation of the Basset–Boussinesq integral with an instantaneous term, the dynamics of spheres have been analyzed in [24–26]. The Green function of unsteady Stokes flow in the unbounded domain (referred to as *unsteady Stokeslet* hereafter) was obtained in the literature using different approaches [16,27–32]. Moreover, higher-order internal and external singularities are provided by Pozrikidis in [16] and by Hsiao and Young in [33]. Pozrikidis [16] also discusses and extends (when-ever possible) to the unsteady Stokes flow useful theorems and results known in the singularity method for steady Stokes flows (see [32,34]), such as the slender body theory and the lower order Hinch–Kim dualism (see [35]) for solid particles with no-slip boundary conditions and drops.

In the case of confined fluid-particle interactions, hydrodynamic problems become mathematically more complicated, and exact expressions are rarely available in the literature. In [36], the authors provide the full Green function for the unsteady Stokes flow in the semi-space above a no-slip plane, completing and improving previous results available in the literature [16,37]. In [38], Simha et al. show that the Green function in the semi-space above a full-slip plane can be obtained by placing a mirror point force at the image point as in the case of steady Stokes flows. Although exact solutions provide forces and torques on immersed bodies in steady Stokes flows are known for several geometries, no solutions are available for bodies immersed in a confined unsteady Stokes flow, to the best of our knowledge. For example, while many results regarding a sphere moving in a steady Stokes flow confined by a plane wall are obtained by employing bispherical coordinates [39–43], separation of variables fails for the Helmholtz equation in this coordinate system, and consequently this approach cannot be extended to the unsteady Stokes regime [44].

An effective strategy for obtaining forces and torques on bodies immersed in confined flows consists in evaluating the leading order terms of the expansion in ℓ_p/ℓ_d , (ℓ_p being the characteristic length of the particle body and ℓ_d the characteristic distance between the body and the boundaries of the confinement), by using methods such as the image method [32,45,46], the reflection method [47,48] (see [46]) and asymptotic expansions [49]. There is a broad range of works in the literature that employ these approaches for obtaining forces and torques acting onto bodies immersed in confined steady Stokes flows; for example, [50–54]. As shown in [55], starting from the knowledge of the Faxén operators of a body in the unbounded Stokes flow and the Green function of the confinement without the body, it is possible to obtain the full expression of forces and higher-order moments acting on the body. As regards the unsteady Stokes flow, apart from a single contribution [56], most of the works have focused on the simpler problem of a sphere near a plane wall. However, even for the problem of a sphere oscillating near a plane wall, the results available in the literature are confusing and contradictory, and often their limits of applicability are not clearly defined. For example, the results provided in [57] do not preserve the symmetry of the grand-resistance matrix (it is straightforward to show that this property should hold also in the unsteady regime starting from the reciprocity theorem [32] (p. 154) and carrying out classical derivations applied in the steady case [32,46]). Furthermore, the force acting on a translating sphere provided by the author does not match the correction to the added mass due to the proximity of the no-slip plane

wall in the limit of high frequencies (the Womersley number α going to infinity). Similarly, the result provided in [58] does not match the expected force due to a sphere translating perpendicular to a no-slip plane wall in the limit for high frequencies. Additionally, as pointed out by Felderhof [37], by applying the fluctuation-dissipation theorem, the results in [58] provide an erroneous power law for the long-time decaying of the velocity autocorrelation function of a Brownian particle near a plane wall. In [37], Felderhof evaluated the diagonal entries of the Green function tensor in the semi-space and, by using the point particle approximation, obtained a closed-form expression for the force onto a sphere of radius r_p translating at distance h from a plane wall correct to the order $O(r_p/h)$. However, as pointed out in [38], hydrodynamic forces provided in [37] depend inconsistently on the particle mass, although the hydrodynamic problem can be formulated without considering this parameter.

In [38], the authors evaluated the leading order in the (ℓ_p/ℓ_d) -expansion of the forces acting onto a sphere translating near a plane wall with full-slip boundary conditions taking advantage of the simple expression of the Green function associated with the confinement. The results obtained by the authors match both the expected limits in the low—(where force in the stationary Stokes flow is recovered) and high-frequency range (where the force is provided by the added mass associated with the geometry of the problem). However, as we show in this article, the expressions for the hydrodynamic forces obtained by the authors [38] do not correspond to the exact forces over the entire frequency range (or Womersley numbers α) acting on an oscillating sphere near a full-slip plane even for $\ell_d \gg \ell_p$. More specifically, we show that a higher-order contribution to the reflected fields should be considered in order to obtain the exact leading-order solution over the entire range of frequencies.

Therefore, to the best of our knowledge, there are no exact solutions, valid for $\alpha \in [0, \infty)$, available in the literature concerning the motion of bodies in confined unsteady Stokes flows.

The aim of this article is to provide expressions for the hydrodynamic resistance forces and torques acting on an arbitrary body in unsteady Stokes flows valid over the entire range of frequencies. To this aim, we extend the reflection method to the unsteady Stokes flow and evaluate the forces and torques to the leading order in the series expansion in ℓ_d , by applying the Hinch–Kim dualism. These general expressions, valid over the entire range of the frequencies are applied to evaluate the hydrodynamic resistance forces in the case of a sphere with no-slip boundary conditions translating near a plane wall with full-slip boundary conditions.

The article is organized as follows. In Section 2, we state the problem of a body moving in a confined unsteady Stokes flow both in the time and frequency (Laplace) domain. In Section 3, we show that in the high-frequency limit the grand-resistance matrix of a body in unsteady confined Stokes flows can be obtained by corresponding solutions of a potential flow. In Section 4, we extend the reflection method to the unsteady Stokes flow. In Section 5, expressions for resistance forces and torques acting onto a body in a confined unsteady flow are provided correct to the order $O(\ell_d^{-3})$. In Section 6, the hydrodynamic forces acting onto a spherical particle with no-slip boundary conditions translating near a full-slip plane wall are derived by using analytical approximate expressions. The comparison of the analytical results with numerical Finite Element Method (FEM) simulations shows the validity of the former for any value of the Womersley number provided that the distance between the particle and the plane is greater than the particle radius.

2. Overview

Consider a particle moving in an incompressible Newtonian confined fluid (see Figure 1), and let μ and ρ be its viscosity and density, respectively. The Reynolds number is vanishingly small, and, whenever not otherwise specified, total adhesion is considered (no-slip boundary conditions) between the fluid and the surface of the particle

S_p and of the wall S_w . Under these conditions, fluid motion is governed by the unsteady Stokes equations

$$\rho \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} = \mu \Delta \mathbf{v}(\mathbf{x}, t) - \nabla p(\mathbf{x}, t); \quad \nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0 \quad \mathbf{x} \in V_f \quad (1)$$

equipped with the boundary conditions

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{U}(t) + \boldsymbol{\omega}(t) \times \mathbf{r} \quad \mathbf{x} \in S_p; \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{0} \quad \mathbf{x} \in S_w \quad (2)$$

where $\mathbf{v}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ are the velocity and the pressure fields of the fluid at point $\mathbf{x}$ and time t , V_f the fluid domain, $\mathbf{U}(t)$ and $\boldsymbol{\omega}(t)$ the time-dependent velocity and angular velocity of the particle, respectively, and $\mathbf{r}$ the position vector centered at a barycentric point of the particle.

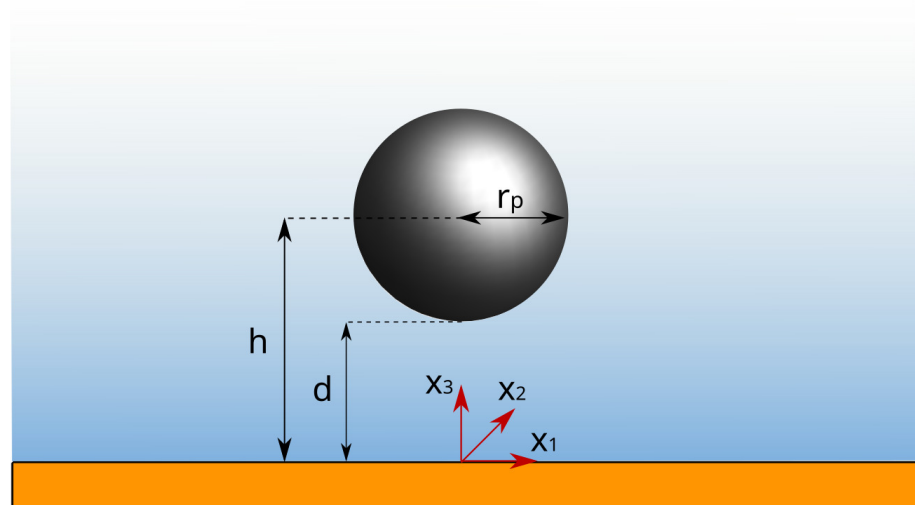


Figure 1. Schematic representation of a spherical particle near a plane wall.

The hydrodynamic force $\mathbf{F}(t)$ and torque $\mathbf{T}(t)$ acting onto the particle are respectively

$$\mathbf{F}(t) = - \int_{S_p} \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) dS; \quad \mathbf{T}(s) = - \int_{S_p} \mathbf{r} \times \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) dS \quad (3)$$

where $\mathbf{n}$ is the unit normal vector directed inward into the fluid, $\boldsymbol{\sigma}(\mathbf{x}, t) = p(\mathbf{x}, t) - 2\mu \mathbf{e}(\mathbf{x}, t)$ represents the stress tensor field at the point $\mathbf{x}$, with $\mathbf{e}(\mathbf{x}, t) = (\nabla \mathbf{v}(\mathbf{x}, t) + \nabla \mathbf{v}(\mathbf{x}, t)^T)/2$, the superscript T indicating the transposition operation.

Let us denote the Laplace transform of a generic time-dependent function $f(t)$ with the hatted notation $\hat{f}(s)$, hence

$$\hat{f}(s) = \int_0^\infty f(t) e^{-st} dt \quad (4)$$

The unsteady Stokes equations (1) in the Laplace domain becomes

$$\mu \left(\Delta - \frac{s}{\nu} \right) \hat{\mathbf{v}}(\mathbf{x}, s) - \nabla \hat{p}(\mathbf{x}, s) = \mathbf{0}; \quad \nabla \cdot \hat{\mathbf{v}}(\mathbf{x}, s) = 0 \quad \mathbf{x} \in V_f \quad (5)$$

equipped with the boundary conditions

$$\hat{\mathbf{v}}(\mathbf{x}, s) = \hat{\mathbf{U}}(s) + \hat{\boldsymbol{\omega}}(s) \times \mathbf{r} \quad \mathbf{x} \in S_p; \quad \hat{\mathbf{v}}(\mathbf{x}, s) = \mathbf{0} \quad \mathbf{x} \in S_w \quad (6)$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. The associated stress tensor is $\hat{\sigma}(\mathbf{x}, s) = \hat{p}(\mathbf{x}, s)\mathbf{I} - 2\mu\hat{\mathbf{e}}(\mathbf{x}, s)$, with $\mathbf{I}$ being the 3×3 identity matrix and $\hat{\mathbf{e}}(\mathbf{x}, s) = (\nabla\hat{\mathbf{v}}(\mathbf{x}, s) + \nabla\hat{\mathbf{v}}(\mathbf{x}, s)^T)/2$. The force and torque in the Laplace domain read as

$$\hat{\mathbf{F}}(s) = - \int_{S_p} \hat{\sigma}(\mathbf{x}, s) \cdot \mathbf{n}(\mathbf{x}) dS; \quad \hat{\mathbf{T}}(s) = - \int_{S_p} \mathbf{r} \times \hat{\sigma}(\mathbf{x}, s) \cdot \mathbf{n}(\mathbf{x}) dS \quad (7)$$

If we consider a spherical particle of radius r_p in an unbounded fluid domain (free space), the Laplace transform of the hydrodynamic force acting onto the particle is given by [32,59]

$$\hat{\mathbf{F}}(s) = -\hat{\mathbf{U}}(s) \left(6\pi\mu r_p + 6\pi r_p^2 \sqrt{\mu\rho s} + \frac{2}{3}\pi\rho r_p^3 s \right) \quad (8)$$

that, considering a vanishing initial velocity $\mathbf{U}(0) = \mathbf{0}$, corresponds to the time-domain expression [9,10,32]

$$\mathbf{F}(t) = -6\pi\mu r_p \mathbf{U}(t) - 6\sqrt{\pi\mu\rho} r_p^2 \int_0^t \frac{1}{\sqrt{t-t'}} \frac{d\mathbf{U}(t')}{dt'} dt' - \frac{2}{3}\pi\rho r_p^3 \frac{d\mathbf{U}(t)}{dt} \quad (9)$$

With reference to Equation (8), the term within parentheses consists of three contributions: (i) the first contribution is the Stokes resistance coefficient, obtained from the stationary Stokes equations setting $s = 0$ in Equation (5); (ii) the second one is the Basset–Boussinesq coefficient, providing the second term at the r.h.s of Equation (9), which depends on the history of particle acceleration up to time t ; (iii) the third contribution is proportional to the added mass of the sphere immersed in the fluid, corresponding in Equation (9) to a force increasing instantaneously the inertia of the particle. The Laplace transform of the hydrodynamic torque acting onto the spherical particle reads [11,32]

$$\hat{\mathbf{T}}(s) = -\hat{\boldsymbol{\omega}}(s) \left(8\pi\mu r_p^3 + \frac{8}{3}\pi\rho r_p^5 \frac{s}{1 + \sqrt{r_p^2 s/\nu}} \right) \quad (10)$$

corresponding to torque in the time domain for which the added moment of inertia is vanishing [12,32] (the expression reported in [32] contains some typos)

$$\mathbf{T}(t) = -8\pi\mu r_p^3 \boldsymbol{\omega}(t) - \frac{8}{3}\pi\rho r_p^5 \int_0^t \left[\frac{r_p}{\sqrt{\pi\nu(t-t')}} - e^{(t-t')\nu/r_p^2} \operatorname{erfc} \left(\sqrt{(t-t')\nu/r_p^2} \right) \right] \frac{d\boldsymbol{\omega}(t')}{dt'} dt' \quad (11)$$

When a body is immersed in a confined fluid, the hydrodynamic forces and torques should depend on the additional geometrical parameters representing the position $\boldsymbol{\zeta}$ of the particle in the confined domain a set of orientation angles $\boldsymbol{\theta}$. Furthermore, the isotropy break-up induced by confinement determines the occurrence of the coupling between forces and torques. Therefore, due to the linearity of Equations (5) and (6) with respect to the velocity $\hat{\mathbf{U}}(s)$ and the angular velocity $\hat{\boldsymbol{\omega}}(s)$, the forces and torques acting on the body is a linear function of the latter,

$$\begin{pmatrix} \hat{\mathbf{F}}(s) \\ \hat{\mathbf{T}}(s) \end{pmatrix} = -\mathbf{H}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s) \begin{pmatrix} \mathbf{U}(s) \\ \boldsymbol{\omega}(s) \end{pmatrix} \quad (12)$$

where $\mathbf{H}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s)$ is the hydrodynamic grand-resistance matrix of the body at the position $\boldsymbol{\zeta}$ and orientation $\boldsymbol{\theta}$ in the confined fluid. The grand-resistance matrix can be expressed in terms of the translational, rotational and roto-translational coupling friction matrices $\mathbf{R}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s)$, $\boldsymbol{\Omega}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s)$, $\mathbf{C}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s)$, as

$$\mathbf{H}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s) = \begin{pmatrix} \mathbf{R}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s) & \mathbf{C}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s) \\ \mathbf{C}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s)^T & \boldsymbol{\Omega}(\boldsymbol{\zeta}, \boldsymbol{\theta}, s) \end{pmatrix} \quad (13)$$

Since, for $s = 0$, steady Stokes equations should be recovered from Equation (5), we can express the grand resistance matrix entering Equation (12) in the form

$$H(\zeta, \theta, s) = \begin{pmatrix} \eta^{[t]}(\zeta, \theta) & \eta^{[c]}(\zeta, \theta) \\ \eta^{[c]}(\zeta, \theta)^T & \eta^{[r]}(\zeta, \theta) \end{pmatrix} + H^0(\zeta, \theta, s) \quad (14)$$

where

$$\begin{pmatrix} \eta^{[t]}(\zeta, \theta) & \eta^{[c]}(\zeta, \theta) \\ \eta^{[c]}(\zeta, \theta)^T & \eta^{[r]}(\zeta, \theta) \end{pmatrix} = H(\zeta, \theta, 0) \quad (15)$$

is the hydrodynamic grand-resistance matrix of the body at the position and orientation (ζ, θ) in the steady Stokes regime [46,60], and $\eta^{[t]}(\zeta, \theta)$, $\eta^{[r]}(\zeta, \theta)$, $\eta^{[c]}(\zeta, \theta)$ its translational, rotational, and roto-translational coupling friction matrices, respectively.

Equation (14) defines the 6×6 matrix $J(\zeta, \theta, s)$, which is a symmetric matrix (this symmetry is obtained straightforwardly following the same proof reported in [60] for the steady Stokes flow) possessing the property

$$H^0(\zeta, \theta, s) = o(1) \quad \text{for } s \rightarrow 0 \quad (16)$$

From Equations (8) and (10), it is possible to dissect the expression for the matrix $J(\zeta, \theta, s) = J(s)$ of a sphere in an unbounded flow as the sum of two separate contributions

$$H^0(s) = B(s) + \begin{pmatrix} m^{[t]} & m^{[c]} \\ m^{[c]T} & m^{[r]} \end{pmatrix} s \quad (17)$$

where $m^{[t]}$, $m^{[c]}$ and $m^{[r]}$ are the matrices collecting the Kirchhoff coefficients [61] (see [11] p. 164). More specifically, $m^{[t]}$ is the well-known added mass, firstly obtained by Green [62] in an unbounded potential flow, and $m^{[r]}$ and $m^{[c]}$ correspond to the added moment of inertia and the "added coupling of inertia", vanishing for a sphere (see [11,63,64]). In [65], Pozrikidis has shown that, in the limit $s \rightarrow \infty$ (hence, at short times $t \rightarrow 0$), the hydrodynamic force acting on a body in unsteady Stokes flows is due to the added mass independently of the shape of the body. In the next section, we show that, in this limit, the entire grand-resistance matrix is represented by the grand-resistance matrix in the potential flow regime, independently of the shape of the body and of the confinement.

3. Hydrodynamic Forces and Torques Acting onto a Particle at Short Time (In the Limit $s \rightarrow \infty$)

In this section, we show that, independently of the geometry of the configuration (body + confinement), it is always possible to express the matrix $H^0(\zeta, \theta, s)$ as

$$H^0(\zeta, \theta, s) = B(\zeta, \theta, s) + \begin{pmatrix} m^{[t]}(\zeta, \theta) & m^{[c]}(\zeta, \theta) \\ m^{[c]}(\zeta, \theta)^T & m^{[r]}(\zeta, \theta) \end{pmatrix} s \quad (18)$$

with $B(\zeta, s)$ possessing the properties

$$\begin{aligned} B(\zeta, s) &= o(1) & \text{for } s \rightarrow 0 \\ B(\zeta, s) &= o(s) & \text{for } s \rightarrow \infty \end{aligned} \quad (19)$$

and where the matrices of the Kirchhoff coefficients $m^{[t]}(\zeta, \theta)$, $m^{[c]}(\zeta, \theta)$ and $m^{[r]}(\zeta, \theta)$ can be obtained from the solution of a potential-flow bounded problem [66,67].

If the fluid is confined, the matrices entering the second term at the r.h.s of Equation (17) are, generically, position and orientation-dependent. In order to define $m^{[t]}(\zeta, \theta)$, $m^{[c]}(\zeta, \theta)$ and $m^{[r]}(\zeta, \theta)$, consider the following harmonic problems [11,61,66,67]

$$\begin{aligned}\Delta \phi_j^{[1]}(\mathbf{x}) &= 0 \quad \text{for } \mathbf{x} \in V_f \\ \frac{\partial \phi_j^{[1]}(\mathbf{x})}{\partial x_i} &= \delta_{ij} \quad \text{for } \mathbf{x} \in S_p \\ \frac{\partial \phi_j^{[1]}(\mathbf{x})}{\partial x_i} &= 0 \quad \text{for } \mathbf{x} \in S_w\end{aligned}\tag{20}$$

and

$$\begin{aligned}\Delta \phi_j^{[2]}(\mathbf{x}) &= 0 \quad \text{for } \mathbf{x} \in V_f \\ \frac{\partial \phi_j^{[2]}(\mathbf{x})}{\partial x_i} &= \varepsilon_{ijk} r_k \quad \text{for } \mathbf{x} \in S_p \\ \frac{\partial \phi_j^{[2]}(\mathbf{x})}{\partial x_i} &= 0 \quad \text{for } \mathbf{x} \in S_w\end{aligned}\tag{21}$$

so that the entries of $\mathbf{m}^{[t]}(\boldsymbol{\zeta}, \boldsymbol{\theta})$, $\mathbf{m}^{[c]}(\boldsymbol{\zeta}, \boldsymbol{\theta})$ and $\mathbf{m}^{[r]}(\boldsymbol{\zeta}, \boldsymbol{\theta})$ are [11,64]

$$m_{ij}^{[t]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) = -\rho \int_{S_p} \phi_j^{[1]}(\mathbf{x}) n_i dS\tag{22}$$

$$\begin{aligned}m_{ij}^{[c]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) &= -\rho \varepsilon_{ikl} \int_{S_p} \phi_j^{[1]}(\mathbf{x}) r_k n_l dS \\ &= -\rho \int_{S_p} \phi_i^{[2]}(\mathbf{x}) n_j dS\end{aligned}\tag{23}$$

$$m_{ij}^{[r]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) = -\rho \varepsilon_{ikl} \int_{S_p} \phi_j^{[2]}(\mathbf{x}) r_k n_l dS\tag{24}$$

where, in Equations (20)–(24) and throughout this article, the Einstein summation convention for repeated indexes is adopted, and ε_{ijk} represents the Ricci–Levi Civita symbol (fundamental antisymmetric tensor).

As well known, Equation (1) (and consequently Equation (5)) representing the unsteady Stokes regime of a Newtonian incompressible fluid, are valid in the case that the Reynolds number $Re = \rho U_c \ell_p / \mu$ (U_c being a characteristic velocity of the particle) is much smaller than one, $Re \ll 1$, and the Strouhal number $St = s \ell_p / U_c$ is of the same order of magnitude or greater than the reciprocal of the Reynolds number, $St \gtrsim 1/Re$. To investigate forces and torques acting on the particle in the limit $s \rightarrow \infty$ (corresponding to a Womersley number $\alpha^2 = St Re \gg 1$), it is convenient to normalize the equations governing the motion of the fluid Equations (5) and (6).

For the sake of simplicity, let us consider preliminary a translating body in the absence of rotation. Introducing the dimensionless variables $\tilde{\mathbf{v}}(\mathbf{x}, s) = \mathbf{v}(\mathbf{x}, s) / U_c$, $\tilde{p}(\mathbf{x}, s) = \hat{p}(\mathbf{x}, s) / P_c$, $\tilde{\mathbf{x}} = \mathbf{x} / \ell_p$, P_c being the characteristic pressure of the system, Equation (5) become

$$\alpha^2 \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) = \tilde{\Delta} \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) - \frac{P_c \alpha^2}{\rho s U r_p} \tilde{\nabla} \tilde{p}(\tilde{\mathbf{x}}, s); \quad \tilde{\nabla} \cdot \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) = 0 \quad \text{for } \tilde{\mathbf{x}} \in V_f\tag{25}$$

where $\tilde{\nabla} = (\partial / \partial \tilde{x}_1, \partial / \partial \tilde{x}_2, \partial / \partial \tilde{x}_3)$ and $\tilde{\Delta} = \partial^2 / \partial \tilde{x}_i \partial \tilde{x}_i$. In order to preserve incompressibility for $\alpha \rightarrow \infty$, we can choose $P_c = \rho s U_c r_p$, corresponding to the characteristic pressure of an accelerating fluid with characteristic acceleration $s U_c$. Therefore, Equation (25) reads

$$\alpha^2 \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) = \tilde{\Delta} \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) - \alpha^2 \tilde{\nabla} \tilde{p}(\tilde{\mathbf{x}}, s); \quad \tilde{\nabla} \cdot \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) = 0 \quad \text{for } \tilde{\mathbf{x}} \in V_f\tag{26}$$

defining $\tilde{\mathbf{U}}(s) = \hat{\mathbf{U}}(s)/U_c$, the boundary conditions Equation (6) attain the form

$$\tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) = \tilde{\mathbf{U}}(s) \quad \tilde{\mathbf{x}} \in S_p; \quad \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) = \mathbf{0} \quad \tilde{\mathbf{x}} \in S_w \quad (27)$$

As well known, the pressure of a Stokes flow is a harmonic field. In the limit for $s \rightarrow \infty$ (hence $\alpha^2 \rightarrow \infty$), the dimensionless Stokes Equation (26) reduces to the Laplacian equation

$$\tilde{\mathbf{v}}(\tilde{\mathbf{x}}, s) = -\tilde{\nabla} \tilde{p}(\tilde{\mathbf{x}}, s) + O\left(\frac{1}{\alpha^2}\right); \quad \tilde{\Delta} \tilde{p}(\tilde{\mathbf{x}}, s) = 0 \quad \text{for } \tilde{\mathbf{x}} \in V_f, \quad \alpha \rightarrow \infty \quad (28)$$

Due to the linearity of Equations (27) and (28) the solutions of these equations are homogeneous linear functions of the entries $\tilde{U}_j(s)$. Consequently, the tensorial/vectorial quantities $\tilde{v}_{ij}(\tilde{\mathbf{x}}, s)$ and $\tilde{p}_j(\tilde{\mathbf{x}}, s)$ can be introduced so that

$$\tilde{v}_i(\tilde{\mathbf{x}}, s) = \tilde{U}_j(s) \tilde{v}_{ij}(\tilde{\mathbf{x}}, s); \quad \tilde{p}(\tilde{\mathbf{x}}, s) = \tilde{U}_j(s) \tilde{p}_j(\tilde{\mathbf{x}}, s) \quad (29)$$

Substituting Equation (29) into Equations (27) and (28), the fields $\tilde{v}_{ij}(\tilde{\mathbf{x}}, s)$ and $\tilde{p}_j(\tilde{\mathbf{x}}, s)$, in the limit for $s \rightarrow \infty$, approach to $\tilde{v}_{ij}(\tilde{\mathbf{x}})$ and $\tilde{p}_j(\tilde{\mathbf{x}})$, solution of the problem

$$\tilde{v}_{ij}(\tilde{\mathbf{x}}) = -\frac{\partial \tilde{p}_j(\tilde{\mathbf{x}})}{\partial \tilde{x}_i}; \quad \tilde{\Delta} \tilde{p}(\tilde{\mathbf{x}}) = 0 \quad \text{for } \tilde{\mathbf{x}} \in V_f \quad (30)$$

equipped with the boundary conditions

$$\frac{\partial \tilde{p}_j(\tilde{\mathbf{x}})}{\partial \tilde{x}_i} = -\delta_{ij} \quad \tilde{\mathbf{x}} \in S_p; \quad \frac{\partial \tilde{p}_j(\tilde{\mathbf{x}})}{\partial \tilde{x}_i} = 0 \quad \tilde{\mathbf{x}} \in S_w \quad (31)$$

Consequently, the hydrodynamic force acting on the particle can be expressed in terms of $\tilde{p}_j(\tilde{\mathbf{x}}, s)$ and $\tilde{v}_{ij}(\tilde{\mathbf{x}}, s)$ as

$$\hat{F}_j(s) = -\rho s \hat{U}_i(s) \ell_p \int_{S_p} \left(\tilde{p}_i(\mathbf{x}, s) \delta_{jk} - \frac{\nu}{sr_p^2} \left(\frac{\partial \tilde{v}_{ji}(\mathbf{x}, s)}{\partial \tilde{x}_k} + \frac{\partial \tilde{v}_{ki}(\mathbf{x}, s)}{\partial \tilde{x}_j} \right) \right) n_k dS \quad (32)$$

By using the Gauss theorem and the incompressibility constraint, the viscous part of the integral on the r.h.s of Equation (32) becomes

$$-\frac{\nu}{s\ell_p^2} \int_{S_p} \left(\frac{\partial \tilde{v}_{ji}(\mathbf{x}, s)}{\partial \tilde{x}_k} + \frac{\partial \tilde{v}_{ki}(\mathbf{x}, s)}{\partial \tilde{x}_j} \right) n_k dS = \frac{\nu}{s\ell_p^2} \int_{V_f} \tilde{\Delta} \tilde{v}_{ji}(\mathbf{x}, s) dV \quad (33)$$

and in the limit for $\alpha \rightarrow \infty$ (i.e., $s \rightarrow \infty$), it vanishes due to Equation (30) (the velocity field is harmonic in this limit). Therefore, for $s \rightarrow \infty$, we have

$$\hat{F}_j(s) = -\rho s \hat{U}_i(s) \ell_p \int_{S_p} \tilde{p}_i(\mathbf{x}) n_j dS \quad (34)$$

Setting $\tilde{p}_i(\mathbf{x}) = -\phi_i^{[1]}(\mathbf{x})/\ell_p$, and comparing Equations (30), (31) and (34) with Equations (20) and (22), Equation (34) becomes

$$\hat{F}_j(s) = -s \hat{U}_i(s) m_{ji}^{[t]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) \quad (35)$$

which proves the above statement as regards the translation contribution to the force.

The torque on the particle defined by Equation (7) can be expressed as

$$\hat{T}_j(s) = -\rho s \hat{U}_i(s) \ell_p \epsilon_{jkl} \int_{S_p} \left(\tilde{p}_i(\mathbf{x}) r_k n_l - \frac{\nu}{sr_p^2} \left(\frac{\partial \tilde{v}_{li}(\mathbf{x}, s)}{\partial \tilde{x}_m} + \frac{\partial \tilde{v}_{mi}(\mathbf{x}, s)}{\partial \tilde{x}_l} \right) r_k n_m \right) dS \quad (36)$$

Using once again the Gauss theorem and the incompressibility constraint, the integral associated with the viscous term in Equation (36) vanishes for $s \rightarrow \infty$. Since by definition $\tilde{p}_i(\mathbf{x}) = -\phi_i^{[1]}(\mathbf{x})/\ell_p$ the comparison of Equation (36) with Equation (23) provides

$$\hat{T}_j(s) = -s \hat{U}_i(s) m_{ji}^{[c]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) \quad (37)$$

which proves the claim stated at the beginning of this section as regards the translational contribution to the torque.

Next, consider the body rotating with angular velocity $\boldsymbol{\omega}$ in the absence of translational motion. Reiterating the same procedure Equations (26)–(35), and defining in this case $\tilde{p}_i(\mathbf{x}) = -\phi_i^{[2]}(\mathbf{x})/\ell_p$, one finally obtains

$$\hat{F}_j(s) = -s \hat{\omega}_i(s) m_{ij}^{[c]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) \quad (38)$$

and

$$\hat{T}_j(s) = -s \hat{\omega}_i(s) m_{ji}^{[r]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) \quad (39)$$

that completes the proof of the initial claim.

In the specific case of a sphere near a plane wall depicted in Figure 1, for which the only configurational parameter is the distance h from the plane, the grand-resistance matrix can be represented as

$$\mathbf{H}(h, s) = \begin{pmatrix} \boldsymbol{\eta}^{[t]}(h) & \boldsymbol{\eta}^{[c]}(h) \\ \boldsymbol{\eta}^{[c]}(h)^T & \boldsymbol{\eta}^{[r]}(h) \end{pmatrix} + \mathbf{B}(h, s) + \begin{pmatrix} \mathbf{m}^{[t]}(h) & \mathbf{m}^{[c]}(h) \\ \mathbf{m}^{[c]}(h)^T & \mathbf{m}^{[r]}(h) \end{pmatrix} s \quad (40)$$

The expressions for $\boldsymbol{\eta}^{[t]}(h)$, $\boldsymbol{\eta}^{[r]}(h)$ and $\boldsymbol{\eta}^{[c]}(h)$ for a sphere near a plane wall are available in the literature [39,41,42,48]. Furthermore, since $\varepsilon_{ikl} r_k n_l$ is vanishing on the surface of a sphere, the added moment of inertia and the coupling terms are vanishing $\mathbf{m}^{[c]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) = \mathbf{0}$ and $\mathbf{m}^{[r]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) = \mathbf{0}$, while the added mass tensor assumes the structure $m_{ij}^{[t]}(\boldsymbol{\zeta}, \boldsymbol{\theta}) = m_{\parallel}(h)(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) + m_{\perp}(h)\delta_{i3}\delta_{j3}$ in the Cartesian coordinate system represented in Figure 1. The limit values for $h \gg r_p$ of the coefficients $m_{\parallel}(h)$ and $m_{\perp}(h)$ are well known and they have been obtained by G.G. Stokes [68]

$$m_{\parallel}(h) = \frac{2}{3}\pi r_p^3 \rho \left(1 + \frac{3}{16} \left(\frac{r_p}{h}\right)^3\right) \quad \text{for } h \gg r_p \quad (41)$$

and

$$m_{\perp}(h) = \frac{2}{3}\pi r_p^3 \rho \left(1 + \frac{3}{8} \left(\frac{r_p}{h}\right)^3\right) \quad \text{for } h \gg r_p \quad (42)$$

While several expressions for $m_{\perp}(h)$ have been presented in the literature, valid over the entire range of h -values [69–71], the counterpart for $m_{\parallel}(h)$ was lacking. Only recently, Van Der Geld [72] obtained compact explicit expressions for $m_{\perp}(h)$ and $m_{\parallel}(h)$ valid over the entire domain of h -values. Specifically,

$$m_{\parallel}(h) = -\frac{2}{3}\pi r_p^3 \rho \left(1 + \frac{3}{2} (\mathcal{P}^{-1})_{11}\right) \quad (43)$$

and

$$m_{\perp}(h) = -\frac{2}{3}\pi r_p^3 \rho \left(1 + \frac{3}{2} (\mathcal{Q}^{-1})_{22}\right) \quad (44)$$

$\mathcal{P}$ and $\mathcal{Q}$ being infinite matrices depending on r_p and h , the entries of which are

$$\mathcal{P}_{ij} = -\delta_{ij} + \binom{i+j}{i+1} \frac{i}{i+1} \left(\frac{r_p}{2h}\right)^{i+j+1} \quad (45)$$

$$Q_{ij} = -\delta_{ij} + \binom{i+j-2}{i-1} \frac{i-1}{i} \frac{r_p^{2i-1}}{(2h)^{i+j-1}} \quad (46)$$

where $i, j = 1, 2, 3, \dots$ and where the binomial coefficients are indicated by the parenthesis symbol according to the classical notation.

4. Reflection Method for Time Dependent Stokes Flow

Contrary to the steady Stokes-flow case, the linear inertial term in the equation governing the fluid motion makes so that the hydrodynamic forces and torques acting onto an oscillating particle are different from the forces and torques acting on a still particle immersed into an equivalently oscillating ambient flow.

More generally, if $\mathbf{F}(t)$ and $\mathbf{T}(t)$ are the force and the torque acting on a rigid body of external surface S_p , associated with the hydrodynamic problem

$$\rho \frac{\partial \mathbf{w}(\mathbf{x}, t)}{\partial t} = \mu \Delta \mathbf{w}(\mathbf{x}, t) - \nabla q(\mathbf{x}, t); \quad \nabla \cdot \mathbf{w}(\mathbf{x}, t) = 0 \quad \mathbf{x} \in V_f \quad (47)$$

equipped with the boundary conditions

$$\mathbf{w}(\mathbf{x}, t) = -\mathbf{u}(\mathbf{x}, t) \quad \mathbf{x} \in S_p; \quad \mathbf{w}(\mathbf{x}, t) = \mathbf{0} \quad \text{for } \mathbf{x} \rightarrow \infty \quad (48)$$

the force and the torque $\mathbf{F}'(t)$ and $\mathbf{T}'(t)$ acting on the same body associated with the hydrodynamic problem

$$\rho \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} = \mu \Delta \mathbf{v}(\mathbf{x}, t) - \nabla p(\mathbf{x}, t); \quad \nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0 \quad \mathbf{x} \in V_f \quad (49)$$

and equipped with the boundary conditions

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{0} \quad \mathbf{x} \in S_p; \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) \quad \text{for } \mathbf{x} \rightarrow \infty \quad (50)$$

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$$\mathbf{F}'(t) = \mathbf{F}(t) + \rho \int_{V_p} \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} dV \quad (51)$$

$$\mathbf{T}'(t) = \mathbf{T}(t) + \rho \int_{V_p} \mathbf{r} \times \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} dV \quad (52)$$

where V_p is the domain of the particle and dV the volume element. Equations (51) and (52) are obtained by considering that $\mathbf{v}(\mathbf{x}, t) = \mathbf{w}(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t)$, where $\mathbf{u}(\mathbf{x}, t)$ is a generic ambient unsteady Stokes flow defined within the domain $V_p \cup V_f$.

In extending the reflection method [46] to unsteady Stokes flows, it is necessary to take into account carefully the transformations Equations (51) and (52).

Let us consider the hydrodynamic problem Equations (1) and (2) of a particle moving in a confined unsteady Stokes flow with no-slip boundary conditions at the surface of the body and of the confinement. According to the reflection method [46], the velocity and pressure fields are expressed as the superposition of a countable system of fields

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= \mathbf{v}^{[1]}(\mathbf{x}, t) + \mathbf{v}^{[2]}(\mathbf{x}, t) + \mathbf{v}^{[3]}(\mathbf{x}, t) + \dots \\ p(\mathbf{x}, t) &= p^{[1]}(\mathbf{x}, t) + p^{[2]}(\mathbf{x}, t) + p^{[3]}(\mathbf{x}, t) + \dots \end{aligned} \quad (53)$$

with associated stress tensor $\boldsymbol{\sigma}^{[k]}(\mathbf{x}, t) = p^{[k]}(\mathbf{x}, t) - \mu(\nabla \mathbf{v}^{[k]}(\mathbf{x}, t) + \nabla \mathbf{v}^{[k]}(\mathbf{x}, t)^T)$, $k = 1, 2, 3, \dots$

Each field $(\mathbf{v}^{[k]}(\mathbf{x}, t), p^{[k]}(\mathbf{x}, t))$ is a solution of the unsteady Stokes equations with boundary conditions

$$\begin{aligned}\mathbf{v}^{[1]}(\mathbf{x}, t) &= \mathbf{U}(t) + \boldsymbol{\omega}(t) \times \mathbf{r} & \mathbf{x} \in S_p \\ \mathbf{v}^{[2k]}(\mathbf{x}, t) &= -\mathbf{v}^{[2k-1]}(\mathbf{x}, t) & \mathbf{x} \in S_w \\ \mathbf{v}^{[2k+1]}(\mathbf{x}, t) &= -\mathbf{v}^{[2k]}(\mathbf{x}, t) & \mathbf{x} \in S_p\end{aligned}\quad (54)$$

Owing to the linearity of the unsteady Stokes equations, each k -th field in Equation (53) contributes to the total force torque with an associated force $\mathbf{F}^{[k]}(t)$ and torque $\mathbf{T}^{[k]}(t)$,

$$\begin{aligned}\mathbf{F}(t) &= \mathbf{F}^{[1]}(t) + \mathbf{F}^{[2]}(t) + \mathbf{F}^{[3]}(t) + \dots \\ \mathbf{T}(t) &= \mathbf{T}^{[1]}(t) + \mathbf{T}^{[2]}(t) + \mathbf{T}^{[3]}(t) + \dots\end{aligned}\quad (55)$$

By definition of the field $\mathbf{v}^{[1]}(\mathbf{x}, t)$, the force $\mathbf{F}^{[1]}(t)$ and the torque $\mathbf{T}^{[1]}(t)$ are the force and torque experienced by the body moving in the unbounded fluid.

The force and the torque due to the field $\mathbf{v}^{[2]}(\mathbf{x}, t)$ are expressed by the integrals

$$\mathbf{F}^{[2]}(t) = - \int_{S_p} \boldsymbol{\sigma}^{[2]}(\mathbf{x}, t) \cdot \mathbf{n} dS \quad (56)$$

and

$$\mathbf{T}^{[2]}(t) = - \int_{S_p} \mathbf{r} \times \boldsymbol{\sigma}^{[2]}(\mathbf{x}, t) \cdot \mathbf{n} dS \quad (57)$$

The application of the Gauss divergence theorem provides

$$\mathbf{F}^{[2]}(t) = \rho \int_{V_p} \frac{\partial \mathbf{v}^{[2]}(\mathbf{x}, t)}{\partial t} dV \quad (58)$$

and

$$\mathbf{T}^{[2]}(t) = \rho \int_{V_p} \mathbf{r} \times \frac{\partial \mathbf{v}^{[2]}(\mathbf{x}, t)}{\partial t} dV \quad (59)$$

Let $\mathbf{F}'^{[3]}(t)$ and $\mathbf{T}'^{[3]}(t)$ be the force and the torque due to the field $\mathbf{v}^{[2]}(\mathbf{x}, t) + \mathbf{v}^{[3]}(\mathbf{x}, t)$. Comparing them with Equations (51) and (52), it is possible to recognize that

$$\mathbf{F}'^{[3]}(t) = \mathbf{F}^{[3]}(t) + \rho \int_{V_p} \frac{\partial \mathbf{v}^{[2]}(\mathbf{x}, t)}{\partial t} dV \quad (60)$$

and

$$\mathbf{T}'^{[3]}(t) = \mathbf{T}^{[3]}(t) + \rho \int_{V_p} \mathbf{r} \times \frac{\partial \mathbf{v}^{[2]}(\mathbf{x}, t)}{\partial t} dV \quad (61)$$

being these the force and the torque acting on the still body immersed in the ambient flow $\mathbf{v}^{[2]}(\mathbf{x}, t)$. More generally, the force and the torque $\mathbf{F}'^{[2k+1]}(t)$ and $\mathbf{T}'^{[2k+1]}(t)$, associated with the flow $\mathbf{v}^{[2k]}(\mathbf{x}, t) + \mathbf{v}^{[2k+1]}(\mathbf{x}, t)$,

$$\mathbf{F}'^{[2k+1]}(t) = \mathbf{F}^{[2k+1]}(t) + \rho \int_{V_p} \frac{\partial \mathbf{v}^{[2k]}(\mathbf{x}, t)}{\partial t} dV \quad (62)$$

and

$$\mathbf{T}'^{[2k+1]}(t) = \mathbf{T}^{[2k+1]}(t) + \rho \int_{V_p} \mathbf{r} \times \frac{\partial \mathbf{v}^{[2k]}(\mathbf{x}, t)}{\partial t} dV \quad (63)$$

are the force and the torque acting on the still body immersed in the ambient flow $\mathbf{v}^{[2k]}(\mathbf{x}, t)$.

Indicating with $\mathbf{F}^{[\infty]}(t)$ and $\mathbf{T}^{[\infty]}(t)$ the force and the torque acting on the particle moving with velocity $\mathbf{U}(t)$ and angular velocity $\boldsymbol{\omega}(t)$ in the unbounded fluid, the total force and torque in the confined case can be expressed as

$$\mathbf{F}(t) = \mathbf{F}^{[\infty]}(t) + \sum_{k=1}^{\infty} \mathbf{F}'^{[2k+1]}(t) \quad (64)$$

and

$$\mathbf{T}(t) = \mathbf{T}^{[\infty]}(t) + \sum_{k=1}^{\infty} \mathbf{T}'^{[2k+1]}(t) \quad (65)$$

The same expressions, Equations (64) and (65), are obtained in the case full-slip boundary conditions at the confinement walls are assumed, namely, in the case the boundary conditions, substituting Equation (2), read

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= \mathbf{U}(t) + \boldsymbol{\omega}(t) \times \mathbf{r} \quad \mathbf{x} \in S_p \\ \boldsymbol{\tau}^{(w)} \cdot \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}^{(w)} &= \mathbf{0}, \quad \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}^{(w)} = 0 \quad \mathbf{x} \in S_w \end{aligned} \quad (66)$$

In Equation (66), $\mathbf{n}^{(w)}$ is the unit vector normal to the surface of the confinement, $\boldsymbol{\tau}^{(w)} = \mathbf{I} - \mathbf{n}^{(w)} \otimes \mathbf{n}^{(w)}$ a set of unit vectors tangential to the surface of the confinement and $\otimes$ the tensor product operator. In fact, by defining the reflected fields $(\mathbf{v}^{[k]}(\mathbf{x}, t), p^{[k]}(\mathbf{x}, t))$ by the boundary conditions

$$\begin{aligned} \mathbf{v}^{[1]}(\mathbf{x}, t) &= \mathbf{U}(t) + \boldsymbol{\omega}(t) \times \mathbf{r} \quad \mathbf{x} \in S_p \\ \boldsymbol{\tau}^{(w)} \cdot \boldsymbol{\sigma}^{[2k]}(\mathbf{x}, t) \cdot \mathbf{n}^{(w)} &= -\boldsymbol{\tau}^{(w)} \cdot \boldsymbol{\sigma}^{[2k-1]}(\mathbf{x}, t) \cdot \mathbf{n}^{(w)} \\ \mathbf{v}^{[2k]}(\mathbf{x}, t) \cdot \mathbf{n}^{(w)} &= -\mathbf{v}^{[2k-1]}(\mathbf{x}, t) \cdot \mathbf{n}^{(w)} \quad \mathbf{x} \in S_w \\ \mathbf{v}^{[2k+1]}(\mathbf{x}, t) &= -\mathbf{v}^{[2k]}(\mathbf{x}, t) \quad \mathbf{x} \in S_p \end{aligned} \quad (67)$$

the same analytical procedure applied for Equations (55)–(63) can be used identically.

At the steady state, where $\mathbf{F}'^{[2k+1]}(t) = \mathbf{F}^{[2k+1]}(t)$ and $\mathbf{T}'^{[2k+1]}(t) = \mathbf{T}^{[2k+1]}(t)$, the classical expressions for the force and the torque [46,48] are recovered from Equations (64) and (65). Procopio and Giona [55], have analyzed the convergence properties of the series entering Equations (64) and (65) for steady Stokes flows developing a constructive method for evaluating each force and torque for arbitrary bodies. In this work, we assume the particle being far enough from the boundaries of the confinement, so that only the leading order terms need to be evaluated, and the series obtained from the "unsteady" reflection method converging, remanding to further works a more careful analysis of these aspects.

5. Approximate Expressions for Forces and Torques Acting onto a Particle Moving in a Confined Unsteady Stokes Flow

In this section, we provide approximate expressions for the hydrodynamic force and torque acting on bodies moving in unsteady Stokes flow in terms of the Green function of the confinement and of the Faxén operators of the body.

It is convenient to consider the hydrodynamic problem of a body moving in a confined unsteady Stokes flow in the frequency formulation (Laplace domain). The Laplace transform of the force and the torque on the particle due to the reflection fields obtained in Equations (64) and (65) provides

$$\hat{\mathbf{F}}(s) = \hat{\mathbf{F}}^{[\infty]}(s) + \sum_{k=1}^{\infty} \hat{\mathbf{F}}'^{[2k+1]}(s) \quad (68)$$

and

$$\hat{\mathbf{T}}(s) = \hat{\mathbf{T}}^{[\infty]}(s) + \sum_{k=1}^{\infty} \hat{\mathbf{T}}^{[2k+1]}(s) \quad (69)$$

Starting from Equations (68) and (69), the approximation provided in [55] Section 6.2 can be extended to unsteady Stokes flows by taking into account the differences due to the unsteadiness. This implies that an additional dimensionless distance $\ell_d / \sqrt{\nu/s}$ is present in the problem. Moreover, as discussed in the previous section, in unsteady conditions the problem of a translating body in a still fluid is not equivalent to the problem of a still body in a flow moving at the same constant velocity.

To begin with, let $\hat{\mathbf{G}}(\mathbf{x}, \boldsymbol{\zeta}, s) = \hat{\mathbf{S}}(\mathbf{x}, \boldsymbol{\zeta}, s) + \hat{\mathbf{W}}(\mathbf{x}, \boldsymbol{\zeta}, s)$ be the Laplace transform of the confined Green function of the unsteady Stokes equations (referred to as the *confined Green function* hereafter for brevity), centered at the pole $\boldsymbol{\zeta}$ and defined in the domain of the confinement $V_p \cup V_f$, where $\hat{\mathbf{S}}(\mathbf{x}, \boldsymbol{\zeta}, s)$ is the Laplace transform of the unsteady Stokeslet (referred to as *unsteady Stokeslet* hereafter) and $\hat{\mathbf{W}}(\mathbf{x}, \boldsymbol{\zeta}, s)$ the Laplace transform of the regular part of the Green function (referred to as *the regular part of the Green function* hereafter). More specifically, the unsteady Stokeslet, obtained by different approaches in the literature [16,27–32] and reported in Appendix A, is the solution of the problem

$$\begin{aligned} \left(\Delta - \frac{s}{\nu}\right) \hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) - \frac{\partial \hat{P}_{\beta}(\mathbf{x}, \boldsymbol{\zeta}, s)}{\partial x_b} &= -8\pi\delta_{b\beta} \delta(\mathbf{x} - \boldsymbol{\zeta}); \\ \frac{\partial \hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s)}{\partial x_b} &= 0 \quad \mathbf{x} \in \mathbb{R}^3 \end{aligned} \quad (70)$$

vanishing at infinity, hence

$$\hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) = 0 \quad \mathbf{x} \rightarrow \infty \quad (71)$$

where $\delta(\mathbf{x} - \boldsymbol{\zeta})$ is the Dirac delta function centered at the position of the pole $\boldsymbol{\zeta}$.

Whereas, the regular part $\hat{\mathbf{W}}(\mathbf{x}, \boldsymbol{\zeta}, s)$ is a bitensorial field solution of the unsteady Stokes equations

$$\left(\Delta - \frac{s}{\nu}\right) \hat{W}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) - \frac{\partial \hat{Q}_{\beta}(\mathbf{x}, \boldsymbol{\zeta}, s)}{\partial x_b} = 0; \quad \frac{\partial \hat{W}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s)}{\partial x_b} = 0 \quad \mathbf{x} \in V_p \cup V_f \quad (72)$$

with boundary conditions

$$\hat{W}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) = -\hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) \quad \mathbf{x} \in S_w \quad (73)$$

when no-slip boundary conditions are considered, and with boundary conditions

$$\begin{aligned} \hat{W}_{ab\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) n_b^{(w)} &= -\hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) n_b^{(w)} \\ \tau_{ac}^{(w)} \hat{T}_{ab\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) n_b^{(w)} &= -\tau_{ac}^{(w)} \hat{\Sigma}_{ab\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) n_b^{(w)} \quad \mathbf{x} \in S_w \end{aligned} \quad (74)$$

when full-slip boundary conditions are assumed at the surface of the confinement. Here, $\hat{T}_{ab\beta}(\mathbf{x}, \boldsymbol{\zeta}, s)$ represents the stress tensor associated with the regular part of the Green function $\hat{W}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s)$. In Equations (72)–(74) and in the remainder, the subscripts are distinguished according the bitensorial notation [73], using Latin letters $a, b, c, \dots = 1, 2, 3$ for the entries of bitensors at the field point $\mathbf{x}$ and Greek letters $\alpha, \beta, \gamma, \dots = 1, 2, 3$ for the entries at the pole point $\boldsymbol{\zeta}$.

Furthermore, assume that the velocity field around the body translating with velocity $\hat{\mathbf{U}}(s)$ can be expressed by a linear integro-differential operator $\mathcal{F}_{\beta\gamma}$ acting at the pole of the unsteady Stokeslet via the expression

$$\hat{v}_b(\mathbf{x}, s) = -\hat{U}_{\gamma}(s) \mathcal{F}_{\beta\gamma} \hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) \quad (75)$$

while the velocity field due to the rotation of the body with velocity $\hat{\omega}(s)$ by a linear integro-differential operator $\mathcal{T}_{\beta\gamma}$ acting at the pole of the unsteady Stokeslet via the expression

$$\hat{v}_b(\mathbf{x}, s) = \hat{\omega}_\gamma(s) \mathcal{T}_{\beta\gamma} \hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) \quad (76)$$

For example, in the case of a sphere with no-slip boundary conditions, we have

$$\mathcal{F}_{\beta\gamma} = -\delta_{\gamma\beta} \left(\left(3 + 3\alpha + \alpha^2 \right) \frac{r_p}{4} - \left(1 + \frac{3}{\alpha} + \frac{3e^\alpha}{\alpha^2} \right) \frac{r_p^3 \Delta}{4} \right) \quad (77)$$

and

$$\mathcal{T}_{\beta\gamma} = \frac{\varepsilon_{\gamma\delta\beta}}{2} \left(\frac{e^\alpha}{1 + \alpha} \right) \frac{\partial}{\partial \zeta_\delta} \quad (78)$$

where $\alpha^2 = r_p^2 s / \nu$.

Consider, for the sake of simplicity, the case of a particle translating without rotating (hence, $\hat{\omega}(s) = 0$). Applying the operator $\mathcal{F}_{\beta\gamma}$ to the system of Equations (72) and (73), and considering that the Laplace transform of the first reflection field (defined by the boundary conditions Equation (54) and consisting in the flow due to the particle translating into the unbounded fluid) is provided by the Equation (75), we have that the velocity field $\hat{v}^{[2]}(\mathbf{x}, s)$ is given by

$$\hat{v}_b^{[2]}(\mathbf{x}, s) = -\hat{U}_\gamma(s) \mathcal{F}_{\beta\gamma} \hat{W}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) \quad (79)$$

Equation (79) is valid also in the case that full-slip boundary conditions are assumed at the surface of the confinement, as can be shown by applying the operator $\mathcal{F}_{\beta\gamma}$ to Equation (74), considering the reflection fields provided by Equation (67).

The analysis developed below does not require any specification of the boundary conditions at the surface of the confinement. Therefore, either the cases of no-slip or full-slip boundary conditions at the surface of the confinement are included.

As shown in [16], for immersed bodies having no-slip boundary conditions, the same operator $\mathcal{F}_{\beta\gamma}$, providing the velocity field around a translating particle if applied to the pole of the unsteady Stokeslet Equation (A1), provides the force acting onto the still particle immersed in a generic ambient flow. This property, referred to as the Hinch–Kim dualism [35], has been shown also for the operator $\mathcal{T}_{\beta\gamma}$, providing both the velocity field around the rotating particle and the torque acting onto the still particle in a generic ambient flow. Therefore, since $\hat{\mathbf{F}}'^{[3]}(s)$ provides the force acting on the still particle immersed, at the position $\boldsymbol{\zeta}$, in the flow $\hat{v}^{[2]}(\mathbf{x}, s)$, we have [35]

$$\hat{F}_\gamma'^{[3]}(s) = -8\pi\mu \mathcal{F}_{\beta\gamma} \hat{v}_\beta^{[2]}(\boldsymbol{\zeta}, s) \quad (80)$$

and for the torque

$$\hat{T}_\gamma'^{[3]}(s) = 8\pi\mu \mathcal{T}_{\beta\gamma} \hat{v}_\beta^{[2]}(\boldsymbol{\zeta}, s) \quad (81)$$

Therefore, defining the symmetric matrices

$$\Phi_{\beta\gamma}(s) = -8\pi\mu \mathcal{F}_{\delta'\gamma} \mathcal{F}_{\delta\beta} \hat{W}_{\delta'\delta}(\boldsymbol{\zeta}', \boldsymbol{\zeta}, s) \Big|_{\boldsymbol{\zeta}'=\boldsymbol{\zeta}} \quad (82)$$

and

$$\Psi_{\beta\gamma}(s) = 8\pi\mu \mathcal{T}_{\delta'\gamma} \mathcal{F}_{\delta\beta} \hat{W}_{\delta'\delta}(\boldsymbol{\zeta}', \boldsymbol{\zeta}, s) \Big|_{\boldsymbol{\zeta}'=\boldsymbol{\zeta}} \quad (83)$$

it is possible to express $\hat{\mathbf{F}}'^{[3]}(s)$ and $\hat{\mathbf{T}}'^{[3]}(s)$ as

$$\hat{\mathbf{F}}'^{[3]}(s) = -\boldsymbol{\Phi}(s) \cdot \mathbf{U}(s) \quad (84)$$

and

$$\hat{\mathbf{T}}'^{[3]}(s) = -\mathbf{\Psi}(s)^T \cdot \mathbf{U}(s) \quad (85)$$

Higher-order reflected fields, and their associated forces, cannot be evaluated exactly solely from the knowledge of the operators $\mathcal{F}_{\beta\gamma}$ and $\mathcal{T}_{\beta\gamma}$, providing the flows due to the rigid motion of the particle. Therefore, if we restrict the analysis exclusively to the use of the Faxén operators $\mathcal{F}_{\beta\gamma}$ and $\mathcal{T}_{\beta\gamma}$, we are forced to approximate higher-order reflection fields by taking into account solely the leading-order terms. Of course a more general analysis of the unsteady Stokes problem is possible, by considering higher-order Faxén operators, and this will be developed in forthcoming works. Nevertheless, as shown below, the analysis restricted to the leading-order term provides fairly accurate results if the body is not too close to the confinement walls (for a spherical particle near a planar wall if the distance is in order of one particle radius). The velocity field $\hat{\mathbf{v}}^{[2]}(\mathbf{x}, s)$ can be approximated by a constant ambient flow acting on the particle with the velocity $\hat{\mathbf{v}}^{[2]}(\boldsymbol{\zeta}, s)$ evaluated at the position point $\boldsymbol{\zeta}$ of the particle, thus

$$\hat{\mathbf{v}}^{[2]}(\mathbf{x}, s) = \hat{\mathbf{v}}^{[2]}(\boldsymbol{\zeta}, s) + O(\ell_d^{-2}) \quad (86)$$

where the order of magnitude of the error has been evaluated using Equation (79), assuming a long-distance scaling for the regular part of the Green function analogous to that of the unsteady Stokeslet Equation (A1), hence $\hat{W}_{\beta\gamma}(\boldsymbol{\zeta}, \boldsymbol{\zeta}, s) = O(\ell_d^{-1})$. Using the same arguments adopted to obtain Equation (77), and substituting $\hat{\mathbf{U}}(s) \leftrightarrow -\hat{\mathbf{v}}^{[2]}(\boldsymbol{\zeta}, s) + O(\ell_d^{-2})$, we have

$$\hat{\mathbf{v}}_b^{[4]}(\mathbf{x}, s) = \hat{\mathbf{v}}_\alpha^{[2]}(\boldsymbol{\zeta}, s) \mathcal{F}_{\beta\alpha} \hat{W}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) + O(\ell_d^{-3}) \quad (87)$$

The force and torque due to $\hat{\mathbf{v}}_b^{[4]}(\mathbf{x}, s)$ can be thus expressed as

$$\hat{\mathbf{F}}_\gamma'^{[5]}(s) = -8\pi\mu \mathcal{F}_{\beta\gamma} \hat{\mathbf{v}}_\beta^{[4]}(\boldsymbol{\zeta}, s) \quad (88)$$

and

$$\hat{\mathbf{T}}_\gamma'^{[5]}(s) = 8\pi\mu \mathcal{T}_{\beta\gamma} \hat{\mathbf{v}}_\beta^{[4]}(\boldsymbol{\zeta}, s) \quad (89)$$

Approximating the force $\mathbf{F}'^{[3]}(s)$ in Equation (80) by using the translational resistance matrix in the unbounded fluid $\mathbf{R}(s)$ according

$$\mathbf{F}'^{[3]}(s) = -\mathbf{R}(s) \cdot \hat{\mathbf{v}}^{[2]}(\boldsymbol{\zeta}, s) + O(\ell_d^{-2}) \quad (90)$$

we obtain

$$\hat{\mathbf{F}}'^{[5]}(s) = \boldsymbol{\Phi}(s) \cdot \mathbf{R}(s)^{-1} \cdot \hat{\mathbf{F}}'^{[3]}(s) + O(\ell_d^{-3}) \quad (91)$$

and

$$\hat{\mathbf{T}}'^{[5]}(s) = \mathbf{\Psi}(s)^T \cdot \mathbf{R}(s)^{-1} \cdot \hat{\mathbf{F}}'^{[3]}(s) + O(\ell_d^{-3}) \quad (92)$$

The equivalent procedure applied to the evaluation of higher-order reflection forces and torques provides

$$\hat{\mathbf{F}}'^{[2k+3]}(s) = \boldsymbol{\Phi}(s) \cdot \mathbf{R}(s)^{-1} \cdot \hat{\mathbf{F}}'^{[2k+1]}(s) + O(\ell_d^{-3}) \quad (93)$$

and

$$\hat{\mathbf{T}}'^{[2k+3]}(s) = \mathbf{\Psi}(s)^T \cdot \mathbf{R}(s)^{-1} \cdot \hat{\mathbf{F}}'^{[2k+1]}(s) + O(\ell_d^{-3}) \quad (94)$$

Considering that $\mathbf{F}^{[\infty]}(s) = -\mathbf{R}(s) \cdot \mathbf{U}(s)$ and $\mathbf{T}^{[\infty]}(s) = -\mathbf{C}(s)^T \cdot \mathbf{U}(s)$, and substituting Equations (93) and (94) into Equations (68) and (69), we obtain the approximate expressions

$$\hat{\mathbf{F}}(s) = -\left(\mathbf{R}(s) + \boldsymbol{\Phi}(s) \cdot \left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1}\right) \cdot \hat{\mathbf{U}}(s) + O(\ell_d^{-3}) \quad (95)$$

and

$$\hat{\mathbf{T}}(s) = -\left(\mathbf{C}(s)^T + \boldsymbol{\Psi}(s)^T \cdot \left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1}\right) \cdot \hat{\mathbf{U}}(s) + O(\ell_d^{-3}) \quad (96)$$

By carrying out the same procedure, assuming the particle rotating with an angular velocity $\hat{\boldsymbol{\omega}}(s)$ without translating, and considering that, in the unbounded domain of the fluid, $\hat{\mathbf{F}}^{[\infty]}(s) = -\mathbf{C}(s) \cdot \hat{\boldsymbol{\omega}}(s)$ and $\hat{\mathbf{T}}^{[\infty]}(s) = -\boldsymbol{\Omega}(s) \cdot \hat{\boldsymbol{\omega}}(s)$, we obtain

$$\hat{\mathbf{F}}(s) = -\left(\mathbf{C}(s) + \left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1} \cdot \boldsymbol{\Psi}(s)\right) \cdot \hat{\boldsymbol{\omega}}(s) + O(\ell_d^{-3}) \quad (97)$$

and

$$\hat{\mathbf{T}}(s) = -\left(\boldsymbol{\Omega}(s) + \boldsymbol{\Theta}(s) + \boldsymbol{\Psi}(s)^T \cdot \left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1} \cdot \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Psi}(s)\right) \cdot \hat{\boldsymbol{\omega}}(s) + O(\ell_d^{-3}) \quad (98)$$

where the symmetric matrix $\boldsymbol{\Theta}(s)$ is defined as

$$\Theta_{\beta\gamma}(s) = -8\pi\mu\mathcal{T}_{\delta'\gamma}\mathcal{T}_{\delta\beta}\hat{W}_{\delta'\delta}(\boldsymbol{\zeta}', \boldsymbol{\zeta}, s) \Big|_{\boldsymbol{\zeta}'=\boldsymbol{\zeta}} \quad (99)$$

The expression for the force can be further simplified by simple matrix manipulations that finally lead to

$$\hat{\mathbf{F}}(s) = -\left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1} \cdot \hat{\mathbf{U}}(s) + O(\ell_d^{-3}) \quad (100)$$

Equations (95)–(98) are the equivalent expressions, for unsteady Stokes flows, of the extended Swan and Brady approximation derived in [55] for steady Stokes flows. As in the steady case, the order of magnitude of the error committed depends on the distance from the plane. However, in this case, there is not a single independent dimensionless group ℓ_p/ℓ_d which can be defined. As it is shown in the next section, where an explicit example is reported, the results obtained from Equations (95)–(98) depend on ℓ_d via different dimensionless groups, such as ℓ_p^3/ℓ_d^3 , $\ell_p^2\delta/\ell_d^3$, $\ell_p\delta^2/\ell_d^3$, with $\delta = \sqrt{\nu/s}$. In the same way, the order of magnitude reported in the approximations Equations (95)–(98) should be interpreted considering all possible dimensionless groups obtained by the products of the characteristic lengths $(\ell_p)^q$, δ^p and ℓ_d^{-3} , with $p, q \in \mathbb{Z}$ and with the constraint that $p \leq 2$ due to the result obtained in Section 3.

Since for the sphere the operator $\mathcal{F}_{\beta\gamma}$ (77) does not contain the first-order derivatives of the regular part of the Green function, the order of magnitude committed in the approximation Equation (86), becomes $O(\ell_d^{-3})$ instead of $O(\ell_d^{-2})$. As a consequence, all the estimates reported for generic objects as a function of ℓ_d , should be rescaled by an order of magnitude in the case of the sphere.

Furthermore, for a sphere the coupling matrix in the unbounded fluid vanishes ($\mathbf{C} = \mathbf{0}$), so that the 0-th order term in the expansion of the operator $\mathcal{T}_{\beta\gamma}$ in Equation (78) is also vanishing. As a consequence, the error committed in approximating the total torque by Equations (96) and (98) reduces by a further order of magnitude.

Therefore, for a sphere, Equations (95)–(98) read

$$\hat{\mathbf{F}}(s) = -\left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1} \cdot \hat{\mathbf{U}}(s) + O(\ell_d^{-4}) \quad (101)$$

$$\hat{\mathbf{T}}(s) = -\boldsymbol{\Psi}(s)^T \cdot \left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1} \cdot \hat{\mathbf{U}}(s) + O(\ell_d^{-5}) \quad (102)$$

$$\hat{\mathbf{F}}(s) = -\left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1} \cdot \boldsymbol{\Psi}(s) \cdot \hat{\boldsymbol{\omega}}(s) + O(\ell_d^{-4}) \quad (103)$$

$$\hat{\mathbf{T}}(s) = -\left(\boldsymbol{\Omega}(s) + \boldsymbol{\Theta}(s) + \boldsymbol{\Psi}(s)^T \cdot \left(\mathbf{I} - \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Phi}(s)\right)^{-1} \cdot \mathbf{R}(s)^{-1} \cdot \boldsymbol{\Psi}(s)\right) \cdot \hat{\boldsymbol{\omega}}(s) + O(\ell_d^{-5}) \quad (104)$$

6. Forces Acting onto a Sphere Translating near a Plane Wall with Full-Slip Boundary Conditions

In [38], Simha et al. evaluate the force acting onto a sphere with no-slip boundary conditions translating near a plane wall with full-slip boundary conditions by taking advantage of the simplicity of the Green function associated with this confinement. In fact, the regular part of the Green function, defined by Equations (72) and (74), assuming that S_w is a plane wall, can be expressed in terms of the unsteady Stokeslet Equation (A1), solution of the unbounded problem Equations (70) and (71), as

$$\hat{W}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) = J_{\beta\gamma} \hat{S}_{b\gamma}(\mathbf{x}, \mathbf{J} \cdot \boldsymbol{\zeta}, s) \quad (105)$$

where $\mathbf{J} = \mathbf{I} - 2\mathbf{n}^{(w)} \otimes \mathbf{n}^{(w)}$ is the mirror operator. Making use of the latter expression, considering that the resistance matrix for the sphere in the unbounded fluid is $R_{\beta\gamma} = 6\pi\mu r_p(1 + \alpha + \alpha^2/9)\delta_{\beta\gamma}$ and substituting the Faxén operator for the sphere Equation (77) into Equations (82) and (101), we finally obtain the expression for the force acting on a sphere (expressed in the Cartesian coordinate system depicted in Figure 1,

$$\hat{\mathbf{F}}(s) = -6\pi\mu r_p \begin{pmatrix} f_{\parallel}(\alpha, \epsilon) & 0 & 0 \\ 0 & f_{\parallel}(\alpha, \epsilon) & 0 \\ 0 & 0 & f_{\perp}(\alpha, \epsilon) \end{pmatrix} \cdot \mathbf{U}(s) + O(h^{-4}) \quad (106)$$

where

$$f_{\perp}(\alpha, \epsilon) = \frac{1 + \alpha + \alpha^2/9}{1 - \frac{A_{\perp}(\alpha, \epsilon)}{1 + \alpha + \alpha^2/9}}; \quad f_{\parallel}(\alpha, \epsilon) = \frac{1 + \alpha + \alpha^2/9}{1 - \frac{A_{\parallel}(\alpha, \epsilon)}{1 + \alpha + \alpha^2/9}} \quad (107)$$

and

$$\begin{aligned} A_{\perp}(\alpha, \epsilon) &= -\frac{3}{4} \frac{\epsilon^2}{\alpha} e^{2\alpha(1-\epsilon^{-1})} - \frac{3}{8} \frac{\epsilon^3}{\alpha^2} \left(e^{2\alpha(1-\epsilon^{-1})} - \left(1 + \alpha + \alpha^2/9 \right)^2 \right) \\ A_{\parallel}(\alpha, \epsilon) &= -\frac{3}{4} \epsilon e^{2\alpha(1-\epsilon^{-1})} - \frac{3}{8} \frac{\epsilon^2}{\alpha} e^{2\alpha(1-\epsilon^{-1})} - \frac{3}{16} \frac{\epsilon^3}{\alpha^2} \left(e^{2\alpha(1-\epsilon^{-1})} - \left(1 + \alpha + \alpha^2/9 \right)^2 \right) \end{aligned} \quad (108)$$

The dimensionless lengths entering Equations (107) and (108) are defined as $\epsilon = r_p/h$ and $\alpha = r_p/\delta$, where $\delta = \sqrt{\nu/s}$ is the viscous penetration depth.

In the limit of the steady Stokes flow, Equation (107) provide

$$\lim_{\alpha \rightarrow 0} f_{\perp}(\alpha, \epsilon) = \frac{1}{1 - \frac{3}{4}\epsilon + \frac{1}{8}\epsilon^3}; \quad \lim_{\alpha \rightarrow 0} f_{\parallel}(\alpha, \epsilon) = \frac{1}{1 + \frac{3}{8}\epsilon + \frac{1}{16}\epsilon^3} \quad (109)$$

consistently with the results available in the literature [51,74]. In the high-frequency limit, as expected by the results obtained in Section 3, a force proportional to the confined added mass is recovered from Equation (107). In fact

$$\lim_{s \rightarrow \infty} \frac{f_{\perp}(\alpha, \epsilon)}{\alpha^2/9} = \frac{1}{1 - \frac{3}{8}\epsilon^3}; \quad \lim_{s \rightarrow \infty} \frac{f_{\parallel}(\alpha, \epsilon)}{\alpha^2/9} = \frac{1}{1 - \frac{3}{16}\epsilon^3} \quad (110)$$

It is interesting to compare these results with the literature. The expressions obtained in [38] are based on two different approximation methods: (i) a point particle method developed by Felderhof (for the unsteady Stokes equations) [37], consisting in evaluating $\hat{\mathbf{F}}^{[3]}(s)$ via Equation (80) approximating $\hat{\mathbf{v}}^{(2)}(\mathbf{x}, s) \approx \hat{\mathbf{W}}(\mathbf{x}, \boldsymbol{\zeta}, s)/(8\pi\mu)$, and (ii) a reflection method, consisting in considering only $\hat{\mathbf{F}}^{[1]}(s)$ and $\hat{\mathbf{F}}'^{[3]}(s)$ in the series Equation (68), with $\hat{\mathbf{F}}'^{[3]}(s)$

evaluated via Equation (84). The expressions obtained by the reflection method in [38] (which is the most conservative regarding the error committed in the approximation) read in the present notation as

$$\begin{aligned} f_{\perp}(\alpha, \epsilon) &\approx \left(1 + \alpha + \alpha^2/9\right) \left(1 + \frac{A_{\perp}(\alpha, \epsilon)}{1 + \alpha + \alpha^2/9}\right) \\ f_{\parallel}(\alpha, \epsilon) &\approx \left(1 + \alpha + \alpha^2/9\right) \left(1 + \frac{A_{\parallel}(\alpha, \epsilon)}{1 + \alpha + \alpha^2/9}\right) \end{aligned} \quad (111)$$

From the analysis developed in Section 5, it follows that it is necessary to include higher-order reflection forces in order to obtain correct expressions to the order $O(h^{-3})$. For example, from Equation (91), the first term in the expansion of $\mathbf{F}^{[5]}(s)$ is $O(h^{-3})$ for a spherical particle (see the discussion at the end of Section 5). Therefore, neglecting this force does not allow us to evaluate correctly the effect of the confinement onto the inertial terms, which have an intensity order of magnitude of $O(h^{-3})$ (as emerges from the expression of the added mass Equations (41) and (42)). The expressions obtained by both the point particle approximation and the reflection method truncated to $\mathbf{F}^{[3]}(s)$ match both the expected limits for $s \rightarrow 0$ and for $s \rightarrow \infty$ (in the Fourier transform approach used in [38] $s = -i\omega$, where ω is the frequency of oscillation and i the imaginary unit). However, they differ, albeit slightly, from each other for values of $s \in (0, \infty)$.

Figure 2 depicts the comparison of the results obtained by FEM simulations with the analytical expressions derived using the higher order reflection method Equation (107) and with the expressions Equation (111) obtained in [38]. More specifically, Figure 2 refers to the dimensionless forces $f_{\perp}(\alpha, \epsilon) - f_{\perp}^0(\epsilon)$ and $f_{\parallel}(\alpha, \epsilon) - f_{\parallel}^0(\epsilon)$ after the subtraction of their stationary limits $f_{\perp}^0(\epsilon)$ and $f_{\parallel}^0(\epsilon)$,

$$f_{\perp}^0(\epsilon) = \lim_{s \rightarrow 0} f_{\perp}(\alpha, \epsilon); \quad f_{\parallel}^0(\epsilon) = \lim_{s \rightarrow 0} f_{\parallel}(\alpha, \epsilon) \quad (112)$$

It can be observed that the expressions Equation (111) do not match the results obtained via FEM numerical simulation at low frequencies especially as regards the perpendicular motion of the particle, whereas expressions Equation (107) provide valid results over the entire range of s -values.

The discrepancy of the expressions obtained by the reflection method truncated to $\mathbf{F}^{[3]}(s)$ Equation (111) with the FEM simulations is more evident in Figure 3, where the Basset–Boussinesq correction factors $b_{\perp}(\alpha, \epsilon)$ and $b_{\parallel}(\alpha, \epsilon)$, defined as

$$\begin{aligned} b_{\perp}(\alpha, \epsilon) &= \frac{f_{\perp}(\alpha, \epsilon) - f_{\perp}^0(\epsilon) - f_{\perp}^{\infty}(\epsilon)\alpha^2/9}{\alpha} \\ b_{\parallel}(\alpha, \epsilon) &= \frac{f_{\parallel}(\alpha, \epsilon) - f_{\parallel}^0(\epsilon) - f_{\parallel}^{\infty}(\epsilon)\alpha^2/9}{\alpha} \end{aligned} \quad (113)$$

are depicted. In Equation (113) we have set

$$f_{\perp}^{\infty}(\epsilon) = \lim_{s \rightarrow \infty} \frac{f_{\perp}(\alpha, \epsilon)}{\alpha^2/9}; \quad f_{\parallel}^{\infty}(\epsilon) = \lim_{s \rightarrow \infty} \frac{f_{\parallel}(\alpha, \epsilon)}{\alpha^2/9} \quad (114)$$

Figure 3 depicts the comparison of these quantities obtained by Equation (107), by Equation (111) and by FEM simulations.

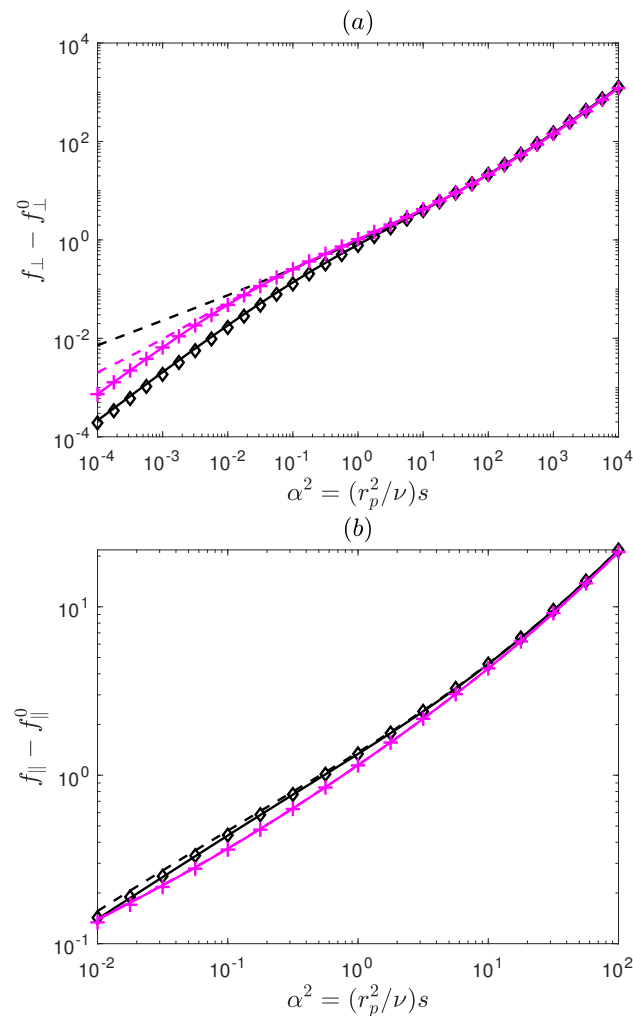


Figure 2. Dimensionless Laplace transform of forces, with the stationary limit subtracted, acting on a sphere with no-slip boundary conditions, translating into an unsteady Stokes flow bounded by a plane with full-slip boundary conditions. Black diamonds ($\diamond$) represent the results obtained via FEM simulations at $h = 2r_p$, solid black lines the results obtained using Equation (107) at $h = 2r_p$, and dashed black lines the results from [38] using eqs. (111) at $h = 2r_p$. Magenta plus signs (+) represent the results obtained via FEM simulations at $h = 10r_p$, solid magenta lines the results derived using Equation (107) for $h = 10r_p$, and dashed magenta lines the results from [38] using Equation (111) for $h = 10r_p$. (a) depicts the forces perpendicular to the plane due to a translation perpendicular to the plane, while (b) the forces parallel to the plane due to a translation parallel to the plane.

The Basset–Boussinesq correction factor $b_{\perp}(\alpha, \epsilon)$ reported in Figure 3a, which by FEM simulations and by Equation (107) is found to be vanishing for $s \rightarrow 0$, is instead different from zeros if evaluated by using Equation (111).

Results obtained by FEM simulations in the case of a sphere moving parallel to the plane wall are less accurate because it is not possible to take advantage by the axial symmetry possessed by the hydrodynamic problem of the sphere moving perpendicular to the plane wall. Since the numerical error is enhanced for extreme values of s ($s \rightarrow 0$ and $s \rightarrow \infty$), Figures 2b and 3b depict the results in the range $\alpha \in [10^{-2}, 10^2]$, in which numerical simulations are reliable and accurate. From the analysis of the data reported in Figure 3, it is possible to conclude that the expressions Equation (107) obtained using the approximation developed in Section 5 are accurate over the entire range of the frequencies (s values), given that the distance between the particle surface and the the plane wall is greater than one particle radius ($d \gtrsim r_p$, where $d = h - 1$ as shown in Figure 1). The expressions for the force onto a spherical particle near a full-slip plane wall Equation (107)

are, hence, an improvement of the results obtained by Simha et al. [38], which fail at low frequency even at large distances between sphere and plane ($d \gg r_p$).

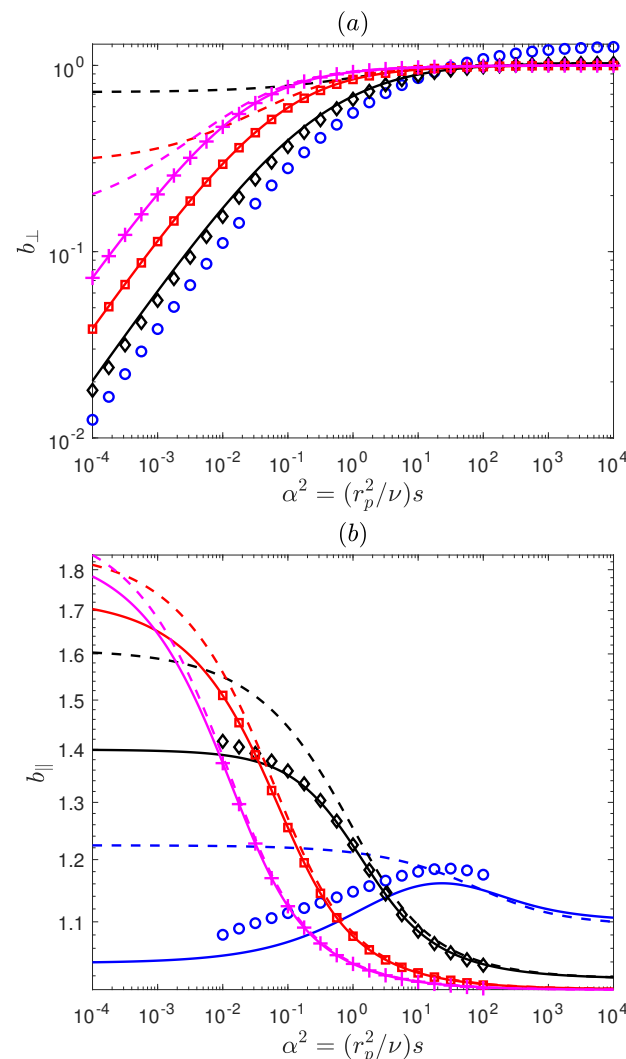


Figure 3. Basset–Boussinesq correction factors defined by Equation (113) for a sphere with no-slip boundary conditions translating near a plane wall with full-slip boundary conditions. Blue circles ($\circ$) represent the results obtained via FEM simulations at $h = 1.1r_p$, solid blue lines and dashed blue lines indicate the results derived using Equations (107) and (111), respectively, for the same distance. Black diamonds ($\diamond$) represent the results obtained from FEM simulations at $h = 2r_p$, solid black lines and dashed black lines indicate the results derived using Equations (107) and (111), respectively, for the same distance. Red squares ($\square$) represent the results obtained via FEM simulations at $h = 5r_p$, solid red lines and dashed red lines indicate the results derived using Equations (107) and (111), respectively, for the same distance. Magenta plus signs ($+$) represent the results obtained via FEM simulations at $h = 10r_p$, solid magenta lines, and dashed magenta lines indicate the results derived using Equations (107) and (111), respectively, for the same distance. (a) displays the correction factors due to a translation perpendicular to the plane, while (b) the correction factors due to a translation parallel to the plane.

7. Conclusions

In this article, we have proposed general expressions for evaluating the resistance forces and torques acting on bodies in unsteady Stokes flows to the order $O(\ell_d^{-3})$. Moreover, we have obtained for the first time (to the best of our knowledge) an expression, valid for any frequency and for distances $h \gtrsim 2r_p$, for the hydrodynamic forces acting on a body in a confined fluid. Specifically, the case refers to a sphere with no-slip bound-

ary conditions translating near a plane wall with full-slip boundary conditions. These results, useful in themselves in the study of particle transport and Brownian motion [1], represent the starting point for further developments in the theory of particle motion in confined unsteady Stokes flows, and admit both scientific and technological interest. Further developments may consider bodies with different boundary conditions and shapes once the associated Faxén operators are obtained either analytically or numerically. As well known from the case of steady Stokes flows regime, hydrodynamic forces and torques acting onto bodies with shapes different from the sphere (such as ellipsoidal and toroidal) will have non-vanishing coupling terms involving interesting rotational behavior if immersed in a shear flow [75–78] or close to a plane [79–81]. By obtaining forces and torques for bodies with these geometries it would be possible to evaluate the effect of the inertia of the fluid on their dynamics.

Jointly with the asymptotic analysis developed in Section 3 and with the extension of reflection method developed in Sections 4 and 5, the explicit expressions derived provide us a clear view of the elements necessary to describe the resistance forces and torques acting on a body in a confined unsteady Stokes flow, their asymptotic behaviors, and the errors committed in adopting certain assumptions and simplifications.

The extension of the reflection method to the unsteady Stokes flows, shows us that it is necessary to consider carefully the peculiarities of the unsteadiness as the direct extension of the expressions derived in the steady case (as, for example, performed in [58]) may lead to incorrect results.

Finally, the proof that, in the limit of high frequencies ($s \rightarrow \infty$), the resistance forces and torques acting on a body are given by the confined Kirchhoff coefficients, ensure us that we can obtain, in this limit (which is the most problematic due to very small viscous penetration depths δ), the forces and torques by solving much simpler potential problems.

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Abbreviations

ρ	Fluid density
μ	Dynamic viscosity
ν	Kinematic viscosity
t_c	Characteristic time of the problem
t_v	Momentum diffusion time
δ	Viscous penetration depth
L_c	Characteristic length of the problem
ℓ_p	Characteristic length of the particle body
ℓ_d	Characteristic distance between the body and the confinement boundaries
U_c	Characteristic velocity of the body
P_c	Characteristic pressure of the fluid
Re	Reynolds number
St	Strouhal number
α	Womersley number
$\mathbf{U}$	Translational velocity of the body
$\boldsymbol{\omega}$	Angular velocity of the body
r_p	Radius of a spherical body

h	Distance of a planar wall from the center of the spherical body
d	Distance of a planar wall from the surface of the spherical body
$\epsilon = h/r_p$	Dimensionless distance of a planar wall from the center of the spherical body
V_f	Domain of the fluid
V_p	Domain of the body
S_w	Surface of the confinement
S_p	Surface of the body
$\mathbf{n}$	Unitary normal vector to S_p
$\mathbf{n}^{(w)}$	Unitary normal vector to S_w
$\boldsymbol{\tau}^{(w)}$	Set of unitary tangential vectors to S_w
t	Time
s	Laplace variable
$\mathbf{x} = (x_1, x_2, x_3)$	Position of the fluid element
$\tilde{\mathbf{x}} = \mathbf{x}/\ell_p$	Dimensionless position of the fluid element
$\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$	Position of the body/pole of singularities
$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$	Set of orientation angles
$\mathbf{r} = (\mathbf{x} - \boldsymbol{\zeta})$	Position vector centered at the position of the body
$r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$	Norm of $\mathbf{r}$
$\hat{\bullet}$	Laplace transform of the function $\bullet$ (see Equation 4)
$\otimes$	Tensor product
$\bullet^T$	Transpose of the matrix $\bullet$
$\mathbf{I}$	Identity matrix
$\mathbf{J} = \mathbf{I} - 2\mathbf{n}^{(w)} \otimes \mathbf{n}^{(w)}$	Mirror operator
$\mathbf{v}$	Velocity field of the fluid
$\tilde{\mathbf{v}} = \hat{\mathbf{v}}/U_c$	Dimensionless Laplace transform of the velocity field of the fluid
p	Pressure field of the fluid
$\tilde{p} = \hat{p}/P_c$	Dimensionless Laplace transform of the pressure field of the fluid
$\boldsymbol{\sigma}$	Stress tensor of the fluid
$\mathbf{u}$	Ambient flow
$\mathbf{w} = \mathbf{v} - \mathbf{u}$	Disturbance flow
q	Pressure associated with the disturbance flow
$\hat{\mathbf{S}}$	Laplace transform of the unsteady Green function in the unbounded domain (<i>unsteady Stokeslet</i> for brevity, see the definition Equations (70) and (71))
$\hat{p}$	Pressure field associated with $\hat{\mathbf{S}}$
$\hat{\boldsymbol{\sigma}}$	Stress tensor associated with $\hat{\mathbf{S}}$
$\hat{\mathbf{W}}$	Laplace transform of the regular part of the confined unsteady Green function (<i>regular part of the Green function</i> for brevity, see the definition Equations (72) and (74))
$\hat{Q}$	Pressure field associated with $\hat{\mathbf{W}}$
$\hat{\mathbf{T}}$	Stress tensor associated with $\hat{\mathbf{W}}$
$\hat{\mathbf{G}} = \hat{\mathbf{S}} + \hat{\mathbf{W}}$	Laplace transform of the confined unsteady Green function
$\mathcal{F}_{\beta\gamma}$	Faxén operator for translations (see Equation (75))
$\mathcal{T}_{\beta\gamma}$	Faxén operator for rotations (see Equation (76))
$\mathbf{F}$	Hydrodynamic force acting on the body
$\mathbf{T}$	Hydrodynamic torque acting on the body
$\mathbf{H}$	Hydrodynamic grand-resistance matrix (see Equation (12))
$\mathbf{R}, \boldsymbol{\Omega}, \mathbf{C}$	Translational, rotational and roto-translational coupling friction matrices (see Equation (13))
$\boldsymbol{\eta}^{[t]}, \boldsymbol{\eta}^{[r]}, \boldsymbol{\eta}^{[c]}$	Translational, rotational and roto-translational coupling friction matrices at for the steady Stokes flow (see Equation (15))
$\mathbf{H}^0$	See Equation (14)
$\mathbf{B}$	Basset–Boussinesq hydrodynamic grand-resistance matrix (see Equation (18))

$\phi_j^{[1]}, \phi_j^{[2]}$	Potential flows (see Equations (20) and (21))
$\mathbf{m}^{[t]}, \mathbf{m}^{[r]}, \mathbf{m}^{[c]}$	Added mass, added moment of inertia and added coupling of inertia (see Equations (22)–(24))
$m_{\parallel}, m_{\perp}$	Entries of the added mass of a sphere close to a plane
$\mathcal{P}, \mathcal{Q}$	See Equations (45) and (46)
Φ, Ψ, Θ	See Equations (82), (83), (99)
$f_{\parallel}, f_{\perp}$	Dimensionless entries of the translational friction matrix of a sphere near a plane (see Equation (106))
$f_{\parallel}^0, f_{\perp}^0$	See Equation (112)
$f_{\parallel}^{\infty}, f_{\perp}^{\infty}$	See Equation (114)
$b_{\parallel}, b_{\perp}$	Basset–Boussinesq correction factors (see Equation (113))
$A_{\parallel}, A_{\perp}$	See Equation (108)

Appendix A. The Green Function of the Unsteady Stokes Flow in the Unbounded Domain

In this Appendix, the Laplace transform of the Green function of the unsteady Stokes flow in the unbounded domain obtained by different approaches in the literature (see for example [16,27,31,32]) is reported.

The solution of the problem Equations (70) and (71) is

$$\hat{S}_{b\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) = C_1(\alpha_r) \frac{\delta_{b\beta}}{r} + C_2(\alpha_r) \frac{(\mathbf{x} - \boldsymbol{\zeta})_b (\mathbf{x} - \boldsymbol{\zeta})_{\beta}}{r^3} \quad (\text{A1})$$

where

$$C_1(\alpha_r) = 2e^{-\alpha_r} \left(1 + \frac{1}{\alpha_r} + \frac{1}{\alpha_r^2} \right) - \frac{2}{\alpha_r^2} \quad (\text{A2})$$

and

$$C_2(\alpha_r) = -2e^{-\alpha_r} \left(1 + \frac{3}{\alpha_r} + \frac{3}{\alpha_r^2} \right) + \frac{6}{\alpha_r^2} \quad (\text{A3})$$

where $r^2 = (\mathbf{x} - \boldsymbol{\zeta}) \cdot (\mathbf{x} - \boldsymbol{\zeta})$ and $\alpha_r = r/\delta$ and $\delta = \sqrt{\nu/s}$. The betensorial pressure is

$$\hat{P}_{\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) = 2 \frac{(\mathbf{x} - \boldsymbol{\zeta})_{\beta}}{r^3} \quad (\text{A4})$$

and the associated stress tensor

$$\begin{aligned} \hat{\Sigma}_{ab\beta}(\mathbf{x}, \boldsymbol{\zeta}, s) = & -2C_3(\alpha_r) \frac{(\mathbf{x} - \boldsymbol{\zeta})_a (\mathbf{x} - \boldsymbol{\zeta})_b (\mathbf{x} - \boldsymbol{\zeta})_{\beta}}{r^5} + \\ & -\frac{2}{r^3} \left(\delta_{ab} (\mathbf{x} - \boldsymbol{\zeta})_{\beta} C_4(\alpha_r) + \left(\delta_{a\beta} (\mathbf{x} - \boldsymbol{\zeta})_b + \delta_{b\beta} (\mathbf{x} - \boldsymbol{\zeta})_a \right) C_5(\alpha_r) \right) \end{aligned} \quad (\text{A5})$$

where

$$C_3(\alpha_r) = 5C_2(\alpha_r) - 2e^{-\alpha_r}(\alpha_r + 1) \quad (\text{A6})$$

$$C_4(\alpha_r) = 1 - C_2(\alpha_r) \quad (\text{A7})$$

and

$$C_5(\alpha_r) = e^{-\alpha_r}(\alpha_r + 1) - C_2(\alpha_r) \quad (\text{A8})$$

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