

Article

A Correction Term for the Asymptotic Scaling of Drag in Flat-Plate Turbulent Boundary Layers

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Abstract

Dixit et al. proposed an asymptotic drag scaling method for zero-pressure-gradient flat-plate turbulent boundary layers based on the approximation $M \sim U_\tau^2 \delta$, where M is the kinematic momentum rate through the boundary layer, U_τ is the friction velocity, and δ is the boundary-layer thickness. In the present paper, an explicit Reynolds-number-dependent correction to this approximation is derived from the logarithmic mean-velocity profile. Integration of the log law across the layer yields $M \sim U_\tau^2 \delta f(Re_\tau)$, where $Re_\tau = \delta U_\tau / \nu$ is the friction Reynolds number and $f(Re_\tau)$ is given analytically. Application of the correction to the dataset compiled by Dixit et al. shows that the corrected scaling gives an exponent consistent with the asymptotic value $-1/2$ within bootstrap confidence intervals, whereas the uncorrected formulation does not. The correction should be viewed as a leading-order amendment, since the derivation uses the logarithmic law outside its strict range of validity.

Keywords: asymptotic drag scaling; zero-pressure-gradient turbulent boundary layers; logarithmic mean-velocity profile; Reynolds-number-dependent correction term

1. Introduction

Skin-friction scaling in canonical wall-bounded turbulence remains a central problem in fluid mechanics because drag directly determines pressure loss, pumping power, and aerodynamic performance. For zero-pressure-gradient (ZPG) turbulent boundary layers, Dixit et al. [1] recently proposed a compact asymptotic drag law based on the kinematic momentum rate through the boundary layer:

$$M = \int_0^\delta U^2 dz \sim U_\tau^2 \delta \quad (1)$$

where δ is the boundary-layer thickness (defined here, following Ref. [1], as the wall-normal distance at which the mean velocity reaches 99 % of the freestream value U_∞), U is the mean streamwise velocity, z is the wall-normal coordinate, and U_τ is the friction velocity. The attractiveness of Equation (1) lies in its simplicity. However, if the mean streamwise velocity obeys a logarithmic law over a substantial part of the boundary layer, then the integral in Equation (1) should retain a weak but systematic dependence on the friction Reynolds number $Re_\tau = \delta U_\tau / \nu$.

The purpose of the present paper is to derive this missing Reynolds-number dependence and to assess its effect on the scaling proposed in Ref. [1]. The resulting amendment is small enough that the original scaling remains a useful first approximation, but large enough that it materially affects the exponent inferred from finite-Reynolds-number data. This interpretation is consistent with earlier scaling work in wall turbulence, where weak



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Reynolds-number dependencies can influence the apparent asymptotic behavior [2,3]. The question of how the logarithmic law affects skin-friction correlations has a long history, starting from the classical analyses by Clauser [4] and Coles [5]; more recent assessments of the impact of log-law parameters on friction predictions include those by Nagib et al. [6].

The paper is organized as follows. In Section 2, the correction term is derived from the logarithmic law for the mean streamwise velocity. In Section 3, the correction is applied to the measurements compiled by Dixit et al. [1]. The implications and limitations are discussed in Section 4, and the conclusions are given in Section 5.

2. Derivation of the Correction Term

As a starting point, I introduce the logarithmic law in the form used by Marusic et al. [7]:

$$U^+ = \frac{1}{\kappa} \ln(z^+) + A, \quad (2)$$

where $U^+ = U/U_\tau$, $z^+ = zU_\tau/\nu$, ν is the kinematic viscosity, κ is the von Kármán constant, and A is an additive constant that depends on wall condition. Equation (2) does not hold arbitrarily close to the wall or throughout the outer wake region, but it is nevertheless informative to ask what correction follows if it is used as a leading-order approximation for the layer integral defining M .

Using $\eta = z/\delta$, so that $z^+ = Re_\tau \eta$, Equation (2) becomes

$$\frac{U}{U_\tau} = \frac{1}{\kappa} \ln(Re_\tau \eta) + A = \frac{1}{\kappa} \ln(Re_\tau) + \frac{1}{\kappa} \ln(\eta) + A. \quad (3)$$

Substitution into Equation (1) gives

$$\begin{aligned} M &= U_\tau^2 \delta \int_0^1 \left[\frac{1}{\kappa} \ln(Re_\tau \eta) + A \right]^2 d\eta \\ &= U_\tau^2 \delta \left[\frac{2}{\kappa^2} - \frac{2A}{\kappa} + A^2 + \ln(Re_\tau) \left(\frac{2A}{\kappa} - \frac{2}{\kappa^2} \right) + \frac{\ln^2(Re_\tau)}{\kappa^2} \right], \end{aligned} \quad (4)$$

where $\int_0^1 \ln(\eta) d\eta = -1$ and $\int_0^1 \ln^2(\eta) d\eta = 2$ have been used.

The factor in square brackets in Equation (4) was effectively treated as constant in Ref. [1]. Figure 1 shows that this is not supported by the data: $M/(U_\tau^2 \delta)$ varies systematically with Re_τ and increases by roughly a factor of three over approximately two decades in Re_τ . The two curves shown correspond to the following:

- Red solid line: Original log-law constants from Ref. [7], $\kappa = 0.39$ and $A = 4.3$.
- Black dashed line: Fitted constants, $\kappa_{\text{fit}} = 0.39$ and $A_{\text{fit}} = 5.7$.

This motivates the amended scaling

$$M \sim U_\tau^2 \delta f(Re_\tau), \quad (5)$$

where

$$f(Re_\tau) = \left[\frac{2}{\kappa^2} - \frac{2A}{\kappa} + A^2 + \ln(Re_\tau) \left(\frac{2A}{\kappa} - \frac{2}{\kappa^2} \right) + \frac{\ln^2(Re_\tau)}{\kappa^2} \right]. \quad (6)$$

The fitted constants provide a substantially larger coefficient of determination than the original constants; see Table 1. These fitted values should be interpreted as effective parameters within the simplified full-layer model, not as replacement universal constants.

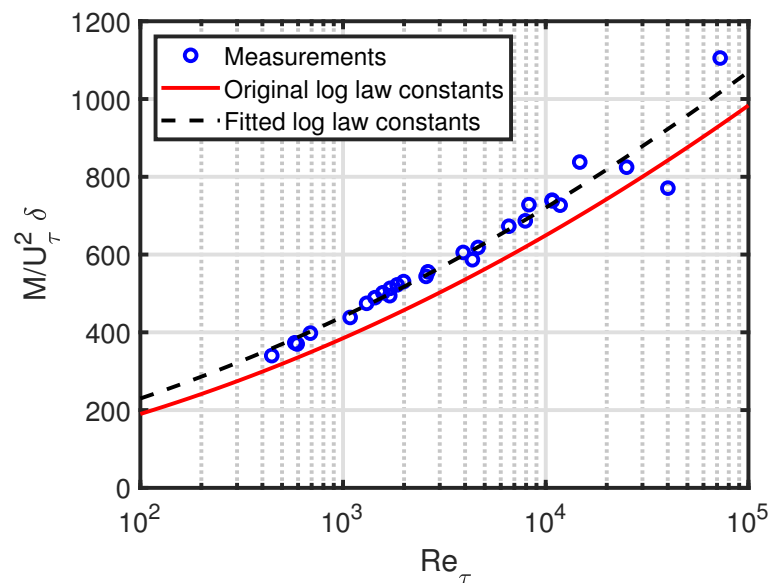


Figure 1. $M/(U_\tau^2 \delta)$ as a function of Re_τ . Blue circles are measurements from Table I in Ref. [1]. The red solid line uses the original log-law constants from Ref. [7], while the black dashed line uses fitted constants.

Table 1. Log-law parameters and coefficient of determination (R^2) for $f(Re_\tau)$.

Log-Law Constants	κ	A	R^2
Original	0.39	4.3	0.80464
Fitted	0.39	5.7	0.94960

The asymptotic scaling law derived in Ref. [1] is

$$\tilde{U}_\tau \sim \frac{1}{\sqrt{\tilde{\delta}}}, \quad (7)$$

where

$$\tilde{U}_\tau = \frac{U_\tau \nu}{M} \sim \frac{\nu}{U_\tau \delta} = \frac{1}{Re_\tau} \quad (8)$$

is the dimensionless drag and

$$\tilde{\delta} = \frac{\delta M}{\nu^2} \sim \frac{\delta^2 U_\tau^2}{\nu^2} = Re_\tau^2 \quad (9)$$

scales as the friction Reynolds number squared.

If Equation (5) is used instead of Equation (1), then Equation (7) becomes

$$\tilde{U}_\tau \sqrt{f(Re_\tau)} \sim \frac{1}{\sqrt{\tilde{\delta}}}, \quad (10)$$

where $\sqrt{f(Re_\tau)}$ is the correction term.

3. Application of the Correction Term

I first fit all measurements in Ref. [1] to

$$\tilde{U}_\tau = C \tilde{\delta}^D, \quad (11)$$

where C and D are fit parameters. The results are summarized in Table 2 and Figure 2. Equations (7) and (8) in Ref. [1] are discrete power-law models fitted over lower and higher $\tilde{\delta}$ ranges, whereas Equation (9) in Ref. [1] is a continuous model across the full range. The

continuous model reported in Ref. [1] has a slightly larger coefficient of determination than either discrete fit.

Table 2. Fit parameters and coefficient of determination (R^2) for the fits in Ref. [1] and in the present work.

Equation	C	D	R^2
Equation (7) in Ref. [1]	0.15144	−0.55745	0.99982
Equation (8) in Ref. [1]	0.10869	−0.54261	0.99992
Equation (9) in Ref. [1]	—	—	0.99998
Equation (11)	0.17291	−0.56439	0.99991
Equation (12)	1.06598	−0.50629	0.99992
Equation (13)	1.23257	−0.51017	0.99992

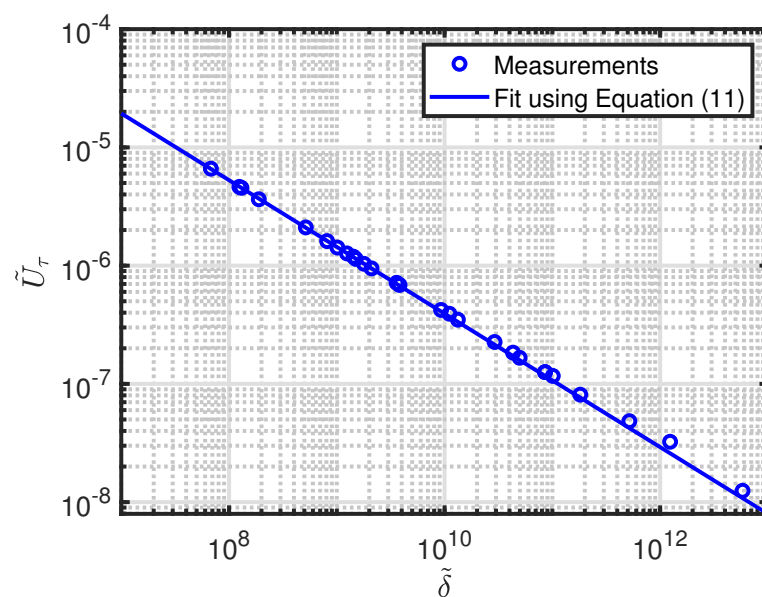


Figure 2. Measurements from Ref. [1] and fit to Equation (11).

The next two fits include the correction term $\sqrt{f(Re_\tau)}$, either with the original log-law constants,

$$\tilde{U}_\tau \sqrt{f(Re_\tau)_{\text{original constants}}} = C \tilde{\delta}^D, \quad (12)$$

or with the fitted log-law constants,

$$\tilde{U}_\tau \sqrt{f(Re_\tau)_{\text{fitted constants}}} = C \tilde{\delta}^D. \quad (13)$$

The results are shown in Table 2 and Figure 3. The fit quality remains essentially unchanged, but the fitted exponents are now much closer to the asymptotic value $-1/2$. It is important to distinguish the algebraic and empirical content of this result. The algebraic consequence of the correction is that dividing by $\sqrt{f(Re_\tau)}$ partially absorbs the systematic curvature in the $\tilde{U}_\tau$ – $\tilde{\delta}$ relationship, which follows from the definitions alone. The empirical finding is that the magnitude of this absorption is consistent with the data, so that the residual power law has an exponent indistinguishable from $-1/2$ within the measurement uncertainty. This suggests that the deviation from $-1/2$ in Equation (11) is not only a finite-Reynolds-number effect in the available data, but also partly a consequence of omitting the correction term.

To quantify the uncertainty in the fitted exponents, a bootstrap resampling analysis with 10,000 iterations was performed. The 95% confidence intervals for the exponent D are: $[-0.553, -0.542]$ for Equation (11), $[-0.502, -0.492]$ for Equation (12), and $[-0.505, -0.495]$

for Equation (13). The asymptotic value $D = -1/2$ lies outside the confidence interval for the uncorrected fit but inside the intervals for both corrected fits. A sensitivity test removing the five lowest- Re_τ points shifts all exponents by less than 0.02, confirming robustness.

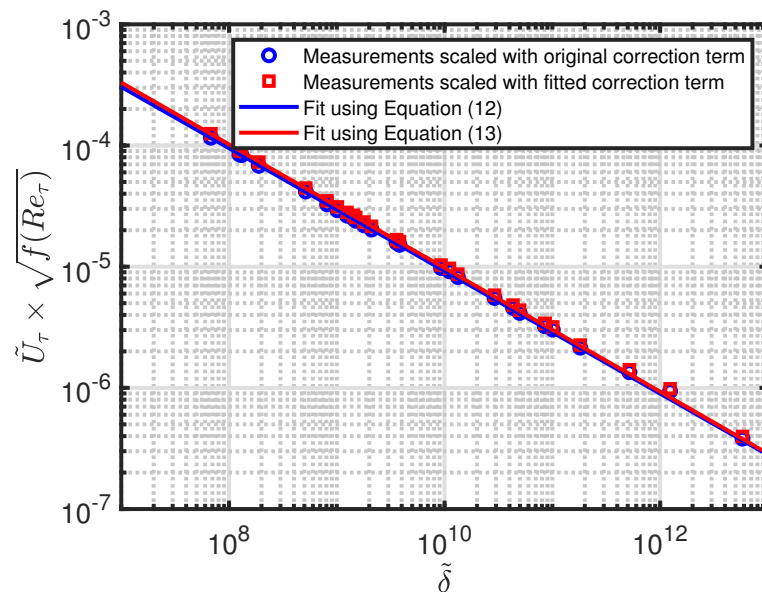


Figure 3. Measurements from Ref. [1] with correction terms applied and fits to Equations (12) and (13).

It should be noted that the correction term based on the *original* log-law constants ($\kappa = 0.39$, $A = 4.3$) already shifts the exponent to -0.506 (Table 2), which is within the 95% bootstrap confidence interval of $-1/2$. Since these constants are independently specified from the literature [7] and are not fitted to the present dataset, this provides an independent validation of the correction.

4. Discussion

It is useful to relate the present approach to the classical momentum integral formulation. The general two-dimensional momentum-integral equation is conventionally written using the boundary-layer edge velocity $U_e(x)$; for the present zero-pressure-gradient case, $U_e = U_\infty = \text{const}$ and $dU_e/dx = 0$. The von Kármán momentum integral equation then reads $\tau_w = \rho U_e^2 d\theta/dx$, where τ_w is the wall shear stress, ρ is the density, x is the streamwise coordinate, and $\theta = \int_0^\infty (U/U_e)(1 - U/U_e) dz$ is the momentum thickness. The upper limit ∞ is admissible here because the integrand vanishes outside the boundary layer. The displacement thickness $\delta^* = \int_0^\infty (1 - U/U_e) dz$ is convergent for the same reason. By contrast, the quantity $M = \int_0^\delta U^2 dz$ used here is intrinsically a finite-domain integral, since $U^2 \rightarrow U_e^2$ outside the boundary layer and $\int_0^\infty U^2 dz$ would diverge. The convergent deficit counterpart is $\int_0^\infty (U_e^2 - U^2) dz = U_e^2(\delta^* + \theta)$, so that M and the conventional momentum-deficit quantities are complementary rather than equivalent. The two formulations are therefore not independent, but M does not replace θ in the exact von Kármán balance; it instead supplies a finite-domain integrated momentum content whose Reynolds-number-dependent profile shape is captured by the correction term $f(Re_\tau)$ derived above. A systematic comparison of the two approaches is left for future work.

Comparison of the fit based on Equation (11) with those based on Equations (12) and (13) shows that the correction term scales only weakly with $\tilde{\delta}$ over the available data range:

$$\sqrt{f(Re_\tau)} \sim \tilde{\delta}^{0.05}. \quad (14)$$

This explains why the corrected fits remain close to power laws while nevertheless shifting the exponent toward $-1/2$.

For the correction term based on the original log-law constants, the fitted prefactor is close to unity ($C = 1.06598$; Table 2). This motivates the approximate closed-form relation

$$\tilde{U}_\tau \sqrt{f(Re_\tau)_{\text{original constants}}} \approx \frac{1}{\sqrt{\delta}}, \quad (15)$$

which reproduces the data rather well despite the simplicity of the model.

The correction term should not be interpreted as a universal asymptotic modification that can be extrapolated to arbitrarily high Reynolds numbers. Rather, it quantifies, within the confines of the log-law approximation, the leading-order Reynolds-number dependence that is absorbed into the prefactor when M is approximated as $U_\tau^2 \delta$.

The limitations of the approach should remain explicit. First, Equation (4) is based on using the logarithmic law across the full boundary-layer thickness, so it neglects both the viscous sublayer and the outer wake contribution. A more complete composite profile (e.g., the wake formulation of Coles [5]) would likely modify the numerical coefficients in $f(Re_\tau)$ and could provide a more physically grounded correction. Second, the fitted additive constant $A_{\text{fit}} = 5.7$ differs from the canonical smooth-wall value $A = 4.3$ reported by Marusic et al. [7]. This difference arises because A_{fit} acts as an effective parameter that compensates for the missing wake and sublayer contributions in the simplified full-layer integral; it should therefore not be regarded as a replacement for the universal log-law constant. Third, part of the high- Re_τ scatter in Figure 1 may reflect differences in effective roughness or in the definitions of δ used across the underlying experiments. In this sense, the present result is best interpreted as a leading-order asymptotic amendment to the drag scaling of Ref. [1], rather than as a complete theory for the mean-velocity profile.

5. Conclusions

A correction term to the asymptotic drag scaling in ZPG turbulent boundary layers has been derived by integrating the logarithmic mean-velocity law across the boundary layer. The resulting amendment, $M \sim U_\tau^2 \delta f(Re_\tau)$, shows that the ratio $M/(U_\tau^2 \delta)$ is not constant but increases systematically with friction Reynolds number. When applied to the dataset of Dixit et al. [1], the correction moves the fitted drag exponent closer to the asymptotic value $-1/2$: bootstrap confidence intervals show that $D = -1/2$ is consistent with the corrected fits but not with the uncorrected fit. Importantly, this improvement is already obtained with the independently specified original log-law constants, without fitting to the present dataset. The main contribution of the present analysis is therefore to identify an explicit Reynolds-number dependence that is hidden when M is approximated solely by $U_\tau^2 \delta$. This correction should be understood as a leading-order result within the log-law approximation, not as a universal asymptotic law.

A natural next step would be to repeat the derivation using a composite inner–outer profile (incorporating, for example, the wake function of Coles [5]), so that the separate contributions from the logarithmic region and the wake region can be quantified more systematically.

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Data Availability Statement: The experimental data analysed in this study were taken from Table 1 of Ref. [1]. The fitting scripts used to produce the figures and tables are available from the author upon reasonable request.

Conflicts of Interest: The author declares no conflicts of interest.

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