

Article

Mathematical Model and Numerical Analysis of Hydraulic Shock Attenuation by a Damper in Pipeline Transportation Systems

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Abstract

This study presents a computationally efficient quasi-one-dimensional mathematical model based on the traveling wave method to investigate hydraulic shock attenuation using a gas-hydraulic damper in pipeline systems. Unlike conventional models, this formulation accounts for fluid compressibility and incorporates a non-linear boundary condition strictly satisfying gas mass conservation within the damper. The model was successfully validated against a MATLAB 2024 Simulink benchmark, demonstrating a maximum pressure amplitude discrepancy of only 5–8%. A parametric analysis evaluated the effects of damper volume, initial gas pressure, and pipe diameter on surge suppression. Results show that insufficient damper volume causes extreme negative pressure drops, risking severe cavitation and fluid column separation. Conversely, excessive volume induces “over-damping,” undesirably increasing system inertia and delaying steady-state recovery. Crucially, scaling analysis reveals that a damper optimized for a specific pipe diameter loses efficacy in larger pipes, as the flow’s kinetic energy scales with the diameter’s square. This model provides a robust, precise computational tool for the optimal and safe design of pipeline networks.

Keywords: hydraulic shock; gas-hydraulic damper; air vessel



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1. Introduction

Ensuring the operational reliability and technical safety of main pipeline systems transporting water, oil, and other liquid media is one of the central tasks of modern applied fluid mechanics and power engineering [1,2]. During the operation of such systems, transient hydrodynamic regimes frequently occur, caused by the sudden shutdown of pump units, the rapid closure of control valves, or changes in consumption [3]. These phenomena initiate a water hammer—a process characterized by the rapid propagation of high-frequency waves of increased and decreased pressure [4]. The fundamental principles of the water hammer theory were laid down in the classic works of N.E. Zhukovsky, whose equations still serve as the foundation for the calculation of hydrodynamic systems.

The rarefaction phases (negative pressure half-waves) pose a particular danger during a water hammer. When the local pressure in the pipeline drops below the saturated vapor pressure of the pumped liquid, flow continuity is disrupted, resulting in the intensive

formation of cavitation bubbles [5,6]. The subsequent violent collapse of these cavities upon the return of the wave generates secondary micro-shocks of colossal destructive power, leading to fatigue failure of the pipe walls and serious accidents [7].

To effectively protect pipeline networks from such dynamic overloads, gas-hydraulic dampers, particularly air vessels [8,9], are widely used in global engineering practice. In recent years, researchers have paid considerable attention to optimizing the parameters of such devices. A number of authors have utilized the method of characteristics (MOC) to model transient processes in water conduits equipped with air vessels, analyzing the effect of air volume on the maximum pressure [10,11]. Other researchers have applied complex computational fluid dynamics (CFD) methods in a 3D modeling format to study local turbulent structures inside the damper [12,13].

Despite the extensive literature in this field, most existing quasi-one-dimensional mathematical models rely on a series of simplifications. Often, the density of the liquid medium is assumed to be strictly constant, and the thermodynamic state of the gas in the damper is described by an empirical polytropic process with a constant exponent [14]. Furthermore, many models neglect the strict condition of mass conservation for the trapped gas and fail to account for the complex influence of the pipeline diameter on the efficiency of inertial energy dissipation. On the other hand, the use of cumbersome 3D-CFD models requires enormous computational resources and is not always suitable for the multi-factor engineering optimization of extensive main networks.

In this regard, there is an urgent need to develop rigorous yet computationally efficient mathematical models capable of precisely describing the dynamic interaction between a compressible fluid and a gas damper. The purpose of this study is to develop an improved quasi-one-dimensional mathematical model and a numerical solution algorithm based on the traveling wave method to analyze the water hammer mitigation process using an air vessel. For the first time, this paper proposes a nonlinear boundary condition strictly based on the law of conservation of gas mass in the damper. Following verification against a reference model in the MATLAB Simulink environment, an in-depth analysis was conducted on the effect of the damper volume, initial gas pressure, and pipeline diameter on the risks of cavitation and the overall inertia of the hydraulic system.

2. Formulation and Mathematical Model of the Problem

We examined an elementary section of the relief pipeline with a length l and internal diameter D , functioning in the quadratic resistance mode (fully rough turbulent regime). In this developed turbulent flow regime, the friction resistance coefficient λ has a constant value and is independent of the Reynolds number. It can be determined using the Shifrinson formula: $\lambda = 0.11(k_{ekv}/D)^{0.25}$.

Where k_{ekv} is the equivalent absolute surface roughness of the internal pipeline wall (in meters). Because both k_{ekv} and D are expressed in the same units of length, their ratio is dimensionless, thereby ensuring the dimensional consistency of the friction factor λ . The applicability range for this adopted relation is valid for highly turbulent flows satisfying the condition $Re \geq 500(D/k)$.

When transporting high-density fluids, it is essential to consider the variation in the elevation of the pipeline axis, $y(x)$, as a function of the longitudinal distance, x . For this purpose, the pipeline slope, defined as $\sin \alpha = \frac{dy}{dx}$, is utilized along the length of the pipeline (Figure 1).

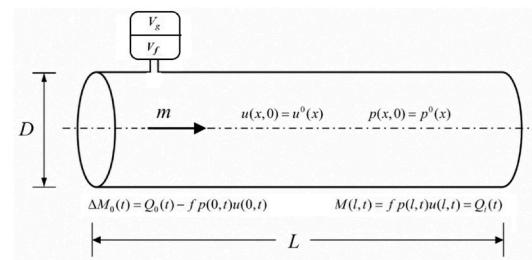


Figure 1. Geometric diagram of the problem and boundary conditions.

The unsteady state of a fluid in a field of inertial, pressure gradient, resistance, and gravity forces is described by quasi-one-dimensional equations of conservation and transfer of momentum, and the continuity equation:

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = -\frac{\lambda}{2D} \rho u |u| - \rho g \sin \alpha, \\ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \end{cases} \quad (1)$$

Equation (1) is derived as a quasi-one-dimensional reduction in the Navier–Stokes equations. Since the pipeline length is significantly greater than its radius $L \gg R$, radial velocity variations are neglected. The derivation incorporates internal friction and pipe inclination, building upon the foundational methodologies of Charny and Kurbatova [15]. Furthermore, Equation (2) adequately captures the real compressibility of the fluid for the transient pressure ranges investigated in this study, avoiding unnecessary computational complexity while maintaining high accuracy.

Often, in particular in the primary model of N.E. Zhukovsky, the density of the liquid medium is assumed to be constant. In our case, the state of the liquid is described using the following relation:

$$p - p_* = \frac{k}{\rho_*} (\rho - \rho_*) \quad (2)$$

where k is the volumetric compression ratio of the considered liquid; p_* , ρ_* are the pressure and density of the liquid in an undisturbed state.

Specifically, for water at $T = 293.15$ K, the reference values are taken as $p_* = 101,325$ Pa, $\rho_* = 998.15$ kg/m³, and $k = 2.12$ GPa. These parameters are also utilized to calculate the propagation velocity of small pressure perturbations (the speed of sound) in the fluid, given by the expression $c_f = \sqrt{k/\rho_*}$.

The initial conditions of the problem are formed for the flow velocity and pressure along the length of the section:

$$u(x, 0) = u^0(x), \quad (3)$$

$$p(x, 0) = p^0(x). \quad (4)$$

This is particularly true if a steady-state operation with a constant mass flow rate of the fluid is taken as the initial condition.

$$M(x, 0) = f \rho(x, 0) u(x, 0) = M^0(x) = \text{const} \quad (5)$$

Here, $f = \pi D^2/4$ is the cross-sectional area of the pipeline. The initial conditions of the problem are formulated by a numerical solution of the equation:

$$\frac{dp^0}{dx} + \rho^0 g \sin \alpha + \frac{\lambda}{2D} \rho^0 (u^0)^2 + \rho^0 u^0 \frac{du^0}{dx} = 0 \quad (6)$$

Using dependencies $u^0 = \frac{M^0}{f\rho^0}$, $\frac{d\rho^0}{dx} = c^{-2}\frac{dp^0}{dx}$, $\frac{du^0}{dx} = -\frac{M^0}{f(\rho^0)^2 c^2}\frac{dp}{dx}$, the equation is presented in the form

$$\frac{dp^0}{dx} + \rho^0 g \sin \alpha + \frac{\lambda(M^0)^2}{2Df^2} \frac{1}{\rho^0} - \frac{(M^0)^2}{f^2 c^2 (\rho^0)^2} \frac{dp^0}{dx} = 0. \quad (7)$$

Here and further,

$$\rho^0 = \rho_* \left(1 + \frac{p^0 - p_*}{k} \right). \quad (8)$$

From the last equation, we find

$$\frac{dp^0}{dx} = \left(-\rho^0 g \sin \alpha - \frac{\lambda(M^0)^2}{2Df^2 \rho^0} \right) / \left(1 - \frac{(M^0)^2}{(fc\rho^0)^2} \right). \quad (9)$$

This equation is integrated numerically from 0 to x ($\leq l$) subject to the condition $p^0(0) = p_0^0$. With the known grid values p_i^0 , the fluid density at the computational nodes is determined using the corresponding formula, and the fluid flow velocity is calculated as follows:

$$u_i^0 = \frac{M^0}{f\rho_* \left(1 + \frac{p_i^0 - p_*}{k} \right)}. \quad (10)$$

Let us establish the boundary condition at $x = 0$.

The mass flow rate of the fluid entering the cross-section $x = 0$ equipped with an air vessel is $Q_0(t)$. At the outlet of the pipe section, the mass flow rate of the fluid is given by:

$$M(l, t) = f\rho(l, t)u(l, t) = Q_l(t). \quad (11)$$

3. Method for Solving the Problem

For the numerical solution of the problem, a method was used that had previously been employed in solving problems for main gas pipelines [8]. The equations and conditions regarding the non-dimensionalized variables were transformed into nonlinear transfer equations for unilateral disturbances.

The following scale variables were adopted: pipeline section length l , propagation velocity of small pressure perturbations c , time scale l/c , and reference density ρ_* . For the non-dimensionalization process, a reference pressure of $p_0 = 100,000$ Pa was employed, which was subsequently used to determine the speed of sound in the fluid.

$$c = \sqrt{\frac{p - p_* + k}{\rho_*}}. \quad (12)$$

Introduction of an auxiliary function [15]

$$\varphi = \ln(\rho/\rho_*), \quad (13)$$

where ρ_* is the characteristic density of the liquid, and the equations are presented in a non-dimensional form:

$$\begin{cases} \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{\varphi}}{\partial \bar{x}} = -\frac{\lambda l}{2D} |\bar{u}| \bar{u} - \frac{l_0 g}{c^2} \sin \alpha, \\ \frac{\partial \bar{\varphi}}{\partial \bar{t}} + \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{u} \frac{\partial \bar{\varphi}}{\partial \bar{x}} = 0. \end{cases} \quad (14)$$

The initial conditions were presented as

$$\bar{u}(\bar{x}, 0) = u^0(\bar{x})/c, \bar{\rho}(\bar{x}, 0) = \rho^0(\bar{x})/\rho_*, \quad (15)$$

and boundary conditions as

$$\bar{\rho}(0, \bar{t})\bar{u}(0, \bar{t}) + s_0\tau \frac{\partial \bar{\rho}(0, \bar{t})}{\partial \bar{t}} = -\bar{M}_0(t) \quad (16)$$

$$\bar{\rho}(1, \bar{t})\bar{u}(1, \bar{t}) = \frac{Q_l(\bar{t})}{f\rho_*c} = \bar{Q}_l(\bar{t}) \quad (17)$$

Analogues of running waves were introduced through

$$f_1 = \bar{u} + \varphi, \quad (18)$$

$$f_2 = \bar{u} - \varphi, \quad (19)$$

the application of which to the equations leads to the following equations:

$$\frac{\partial f_1}{\partial \bar{t}} + (1 + \bar{u}) \frac{\partial f_1}{\partial \bar{x}} = -\frac{\lambda l}{2D} |\bar{u}| \bar{u} - \frac{l_0 g}{c^2} \sin \alpha, \quad (20)$$

$$\frac{\partial f_2}{\partial \bar{t}} + (\bar{u} - 1) \frac{\partial f_2}{\partial \bar{x}} = -\frac{\lambda l}{2D} |\bar{u}| \bar{u} - \frac{l_0 g}{c^2} \sin \alpha. \quad (21)$$

The initial and boundary conditions were transformed according to the expressions f_1 and f_2 .

A step at the non-dimensional distance was taken as $h = 1/N_x$ ($N_x = 1000$), and in non-dimensional time it was $\tau = 1/5000$.

The difference between these quantities is the intensity of mass transfer between the pipeline and the air cap:

$$\Delta M_0(t) = Q_0(t) - f\rho(0, t)u(0, t). \quad (22)$$

It represents the rate of change in the mass of the liquid $\rho_f^0(t)V_f^0(t)$ in the air cap over time:

$$\Delta M_0(t) = \frac{d}{dt} [\rho_f^0(t)V_f^0(t)] = \frac{d\rho_f^0(t)}{dt} V_f^0(t) + \rho_f^0(t) \frac{dV_f^0(t)}{dt}. \quad (23)$$

where $\rho_f^0(t)$ is the liquid density; $V_f^0(t)$ is the volume of liquid in the inlet air cap. The remaining part of the volume of the air cap, $V_{BK}^0 - V_g^0(t) = V_g^0(t)$, is occupied by air. We neglected the change in the volume of the air cap: $V_{BK}^0 = \text{const}$.

It is preferable to determine the value of $\Delta M_0(t)$ through the volume of air in the air cap $V_g^0(t)$, since the mass of air in the air cap is practically constant and amounts to

$$m_g^0 = \frac{p_g^0(t)V_g^0(t)}{zRT} = \text{const}. \quad (24)$$

We assume that the pressure $p(0, t)$ also applies to the volume of the air cap. Therefore, the volume of air in the air cap is as follows [15,16]:

$$V_g^0(t) = \frac{zRTm_g^0}{p(0, t)}. \quad (25)$$

Taking this fact into account, we transform

$$\begin{aligned}\Delta M_0(t) &= \frac{d\rho_f^0(t)}{dt} [V_{BK}^0 - V_g^0(t)] + \rho_f^0(t) \frac{d}{dt} [V_{BK}^0 - V_g^0(t)] = \left[V_{BK}^0 - \frac{zRTm_g^0}{p(0,t)} \right] \frac{\partial \rho(0,t)}{\partial t} - \\ &- \rho(0,t) \frac{\partial}{\partial t} \frac{zRTm_g^0}{p(0,t)} = \left[V_{BK}^0 - \frac{c_g^2 m_g^0}{p(0,t)} \right] \frac{\partial \rho(0,t)}{\partial t} + \rho(0,t) \frac{c_g^2 m_g^0 c_f^2}{p^2(0,t)} \frac{\partial \rho(0,t)}{\partial t} = \\ &= \left[V_{BK}^0 - \frac{c_g^2 m_g^0}{p(0,t)} + \frac{c_g^2 m_g^0 c_f^2 \rho(0,t)}{p^2(0,t)} \right] \frac{\partial \rho(0,t)}{\partial t} = \left[V_{BK}^0 + \frac{c_g^2 m_g^0 (k - p_*)}{p^2(0,t)} \right] \frac{\partial \rho(0,t)}{\partial t} = \\ &= \left[V_{BK}^0 + \frac{c_g^2 m_g^0 (k - p_*)}{(p_* + c_f^2 \rho(0,t) - k)^2} \right] \frac{\partial \rho(0,t)}{\partial t}.\end{aligned}\quad (26)$$

Here, the notation $c_g^2 = zRT = \text{const}$ was introduced, and it was taken into account:

$$p_g(t) = p_f(t) = p(0,t) = p_* + c_f^2 \rho(0,t) - k.$$

Then, the mass balance in the cross-section $x = 0$ is expressed as an equality:

$$f\rho(0,t)u(0,t) + \left[V_{BK}^0 + \frac{c_g^2 m_g^0 (k - p_*)}{(p_* + c_f^2 \rho(0,t) - k)^2} \right] \frac{\partial \rho(0,t)}{\partial t} = Q_0(t). \quad (27)$$

Also, both the size and the condition at the input of the plot are considered:

$$\bar{\rho}(0,\bar{t})\bar{u}(0,\bar{t}) + \left[V_{BK}^0 + \frac{c_g^2 m_g^0 (k - p_*)}{(p_0 + c_f^2 \rho_* \bar{\rho}(0,\bar{t}) - k)^2} \right] \frac{1}{f l} \frac{\partial \bar{\rho}(0,\bar{t})}{\partial \bar{t}} = \frac{Q_0(t)}{f \rho_* c} \quad (28)$$

Here, we introduce the notations:

$$\bar{M}_0(\bar{t}) = \frac{Q_0(\bar{t})}{f \rho_* c}, \quad (29)$$

$$s_0 = \left[V_{BK}^0 + \frac{c_g^2 m_g^0 (k - p_*)}{(p_0 + c_f^2 \rho_* \bar{\rho}(0,\bar{t}) - k)^2} \right] \frac{1}{\tau f l} (> 0). \quad (30)$$

Here, τ is a step through timelessness. Then the condition takes the following form [15]:

$$\bar{\rho}(0,\bar{t})\bar{u}(0,\bar{t}) + s_0 \tau \frac{\partial \bar{\rho}(0,\bar{t})}{\partial \bar{t}} = -\bar{M}_0(t). \quad (31)$$

From the derived condition, the value of $\bar{u}(0,\bar{t})$ can be determined. To achieve this, $\bar{\rho}(0,\bar{t})$ must be expressed in terms of $\bar{u}(0,\bar{t})$, taking into account that for the new time $\bar{t}$, the value $f_2(0,\bar{t}) = \bar{u}(0,\bar{t}) - \varphi(0,\bar{t})$ is known. Therefore, $\varphi(0,\bar{t}) = f_2(0,\bar{t}) - \bar{u}(0,\bar{t})$, and the dimensionless density is defined as follows:

$$\bar{\rho}(0,\bar{t}) = e^{f_2(0,\bar{t}) - \bar{u}(0,\bar{t})}. \quad (32)$$

Taking into account this substitution, the condition approximated by the explicit scheme takes the form [16]:

$$e^{f_{2,0}^{n+1} - \bar{u}_0^{n+1}} \bar{u}_0^{n+1} + s_0 \left(e^{f_{2,0}^{n+1} - \bar{u}_0^{n+1}} - \bar{\rho}_0^n \right) = -\bar{M}_0^{n+1}. \quad (33)$$

We will create an auxiliary function relative to $\bar{u} = \bar{u}_{N_x}^{n+1}$:

$$F(\bar{u}) = e^{\bar{u}}(\bar{u} + s_0) - e^{f_{2,0}^{n+1}}(s_0 \bar{\rho}_0^n - \bar{M}_0^{n+1}). \quad (34)$$

Let us evaluate the first derivative:

$$F'(\bar{u}) = e^{\bar{u}}(\bar{u} + s_0 + 1) > 0 \quad (35)$$

and the second derivative

$$F''(\bar{u}) = e^{\bar{u}}(\bar{u} + s_0 + 2) > 0. \quad (36)$$

According to the first estimate, the auxiliary function is monotonically increasing, and according to the second estimate, the graph of the function is concave in the physical domain of definition $[-1; 1]$. This information facilitates the application of the tangent (Newton's) method with a zero initial approximation. $\bar{u}^0 = +1$.

From the expression of the slope (k) of the tangent to the graph of the function at zero abscissa ($F(\bar{u}^{k+1}) = 0$),

$$F'(\bar{u}^k) = \frac{F(\bar{u}^{k+1}) - F(\bar{u}^k)}{\bar{u}^{k+1} - \bar{u}^k} \quad (37)$$

it follows that a recurrent formula is appropriate:

$$\bar{u}^{k+1} = \bar{u}^k - \frac{F(\bar{u}^k)}{F'(\bar{u}^k)} = \bar{u}^k - \frac{(s_0 \bar{\rho}_0^n - \bar{M}_0^{n+1})e^{\bar{u}^k + f_{2,0}^{n+1}} - \bar{u}^k - s_0}{\bar{u} + 1 + s_0}. \quad (38)$$

The iterative process according to this formula continues until the condition is met [17]. $|F(\bar{u}^{k+1})| < 10^{-12}$ or $|\bar{u}^{k+1} - \bar{u}^k| < 10^{-12}$.

Next, the values are determined based on the newly found value of $\bar{u}_0^{n+1} = \bar{u}^{k+1}$, $f_{1,0}^{n+1}$, φ_0 and

$$p(0, t) = p_* + c_f^2 \rho_* e^{f_{2,0}^{n+1} - \bar{u}^k} - k. \quad (39)$$

The equations are approximated using an implicit time scheme of first-order accuracy, and their convective terms are approximated using a flux opposition scheme. The coefficient and the right-hand side of the first equation are taken from the previous time step, which allowed the formulation of a recursive relation:

$$f_{1i}^{n+1} = \frac{f_{1i}^n + \sigma(1 + \bar{u}_i^n)f_{1i-1}^{n+1} + \tau\Phi(\bar{u}_i^n, \alpha_i)}{1 + \sigma(1 + \bar{u}_i^n)}, \quad (40)$$

where $\sigma = \tau/h$, $\Phi(\bar{u}_i^n, \alpha_i) = -\frac{\lambda l}{2D}\bar{u}_i^n|\bar{u}_i^n| - \frac{lg}{c^2}\sin\alpha_i$ $i = 1, 2, \dots, N_x$.

The second change to the task and algorithm relates to the condition at $x = l$. In dimensional form, it is given by [18].

$$M(l, t) = f\rho(l, t)u(l, t) = Q_l(t). \quad (41)$$

In a dimensionless form, the condition takes the form

$$\bar{\rho}(1, \bar{t})\bar{u}(1, \bar{t}) = \frac{Q_l(\bar{t})}{f\rho_*c} = \bar{Q}_l(\bar{t}). \quad (42)$$

For $Q_l(\bar{t}) = 0$, the condition dictates that $\bar{u}(1, \bar{t}) = 0$. Since the new value f_{1,N_x}^{n+1} is known, it follows that $\varphi_{N_x}^{n+1} = -f_{1,N_x}^{n+1}$ and $f_{2,N_x}^{n+1} = \varphi_{N_x}^{n+1}$.

With $Q_l(\bar{t}) \neq 0$, we have a finite difference equation of the form:

$$e^{f_{1,N_x}^{n+1} - f_{2,N_x}^{n+1}} \bar{u}_{N_x}^{n+1} = \bar{Q}_l^{n+1}. \quad (43)$$

Since it is $f_{2,N_x}^{n+1} = 2\bar{u}_{N_x}^{n+1} - f_{1,N_x}^{n+1}$, the condition takes the form:

$$e^{f_{1,N_x}^{n+1} - \bar{u}_{N_x}^{n+1}} \bar{u}_{N_x}^{n+1} = \bar{Q}_l^{n+1}. \quad (44)$$

Let us denote $\bar{u} = \bar{u}_{N_x}^{n+1}$ and construct an auxiliary function

$$\Phi(\bar{u}) = \bar{Q}_l^{n+1} e^{-f_{1,N_x}^{n+1}} - \bar{u} e^{-\bar{u}}. \quad (45)$$

The zero of this function was determined using the tangent method; we need to find an initial approximation $\bar{u}^{(0)}$.

We find the first as follows:

$$\Phi'(\bar{u}) = e^{-\bar{u}} - \bar{u} e^{-\bar{u}} = e^{-\bar{u}}(\bar{u} - 1) (< 0), \quad (46)$$

and the second derivatives:

$$\Phi''(\bar{u}) = -e^{-\bar{u}}(\bar{u} - 1) + e^{-\bar{u}} = e^{-\bar{u}}(2 - \bar{u}) (> 0). \quad (47)$$

The function $\Phi(\bar{u})$ decreases within the interval $[-1; 1]$, and its graph is concave. In this case, it is appropriate to set the initial approximation as $\bar{u}^{(0)} = -1$.

For $i = N_x - 1, N_x - 2, \dots, 0$, the values of the second auxiliary function are determined according to formula [19]:

$$f_{2i}^{n+1} = \frac{f_{2i}^n + \sigma(1 + \bar{u}_i^n) f_{2i-1}^{n+1} + \tau \phi(u_i^n, \alpha_i)}{1 + \sigma(1 - \bar{u}_i^n)}. \quad (48)$$

The relevant changes have been made to the program and a series of computational experiments have been conducted. Let us discuss the results.

4. Results and Discussion

To ensure the adequacy of the model, block diagrams of the benchmark visual-mathematical model of the system were developed in the MATLAB environment using the Simscape library, in accordance with the specified conditions (see Figure 2). The created block diagram serves as a computational environment representing a system of complex differential-algebraic equations with specific boundary conditions, simulating the state of the transported medium within the system. Furthermore, it is fundamentally based on the one-dimensional Navier–Stokes and continuity equations governing the fluid flow.

The wave reflection periods in both models coincided perfectly, confirming the validity of the selected system parameters and the calculated wave propagation velocity. The discrepancy in the maximum pressure amplitudes is merely 5–8%. This falls well within the acceptable engineering tolerance of 10–15% and can be attributed to the fact that MATLAB incorporates more complex empirical correlations to calculate fluid friction. Furthermore, the dissipation of shock energy and the subsequent stabilization of the transient process occurred over identical time intervals in both computational approaches (Figure 3).

In the next stage, we analyze the effect of various damper volumes on pressure surges. The studies demonstrated that increasing the damper volume (from $V_{BK}^0 = 0.5$ to $V_{BK}^0 = 4.0 \text{ m}^3$) clearly results in a reduction in the maximum wave amplitude and the smoothing of oscillations. In particular, large-volume dampers absorb the

abrupt impact energy of the flow, converting dangerous oscillations into safe, low-frequency pulsations. Concurrently, the process dynamics revealed that an excessive increase in the damper volume (e.g., the case with $V_{BK}^0 = 4.0 \text{ m}^3$) increases the system's inertia and significantly prolongs the time required for the hydraulic circuit to transition to a new steady-state operating regime with a mass flow rate of 152 kg/s .

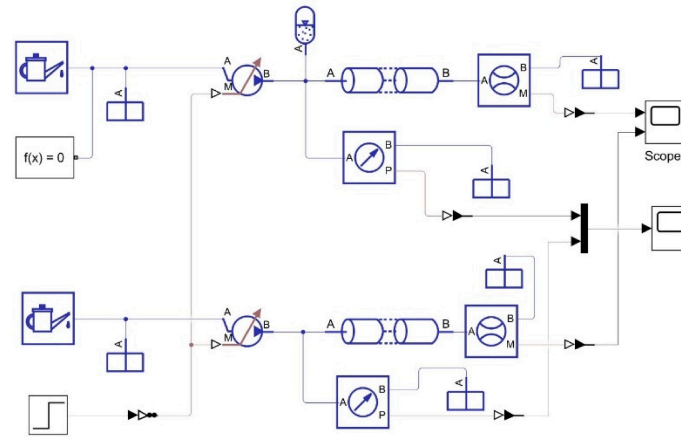


Figure 2. Hydraulic system and damper block diagram in the Simulink environment (Simscape).

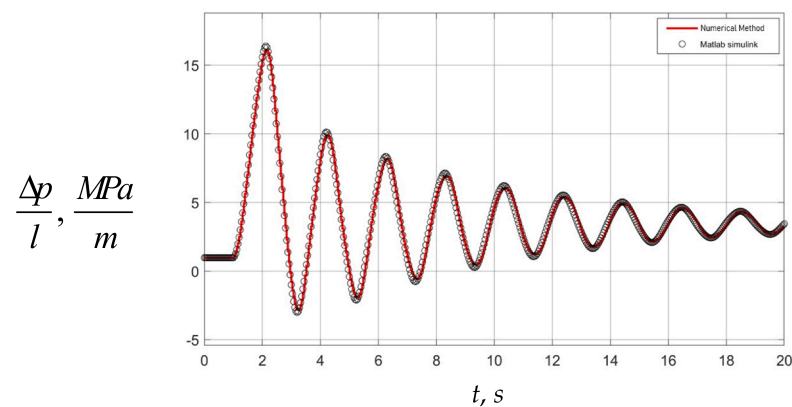


Figure 3. Graph comparing the results of the reference model (Simulink) and the proposed numerical model (Pascal ABC). $l = 500 \text{ m}$, $D = 0.2 \text{ m}$, $V_{BK}^0 = 0.2 \text{ m}^3$, $M = 76.2056 \text{ kg/s}$, $p_0 = 7.5 \text{ MPa}$, $p_d = 7 \text{ MPa}$.

The obtained results scientifically substantiate that selecting a damper by optimally calculating its volume is a critical task in engineering practice to ensure the technical safety of the system while preserving its operational responsiveness.

Figure 4 presents the attenuation dynamics of pressure oscillations in the system during a hydraulic shock under the influence of dampers with varying volumes. The graphical data clearly reflect the distribution of dynamic stresses exerted on the pipe walls.

In scenarios where the system operates without a damper or with an insufficiently small damper volume (ranging from $V_{BK}^0 = 0.001$ to $V_{BK}^0 = 0.1 \text{ m}^3$), the amplitude of relative pressure oscillations escalated critically up to 1.6 MPa/m . Furthermore, during the wave reflection phase, the relative pressure amplitude plunged into the negative region, reaching -0.4 MPa/m .

Conversely, increasing the damper volume from $V_{BK}^0 = 0.5$ to $V_{BK}^0 = 2.0 \text{ m}^3$ resulted in a drastic reduction in the maximum relative pressure surges. Specifically, the implementation of large-capacity dampers effectively smoothed the sharp wave peaks, curtailed the oscillation amplitude by more than half, and completely eliminated the risk of the pressure transitioning into the negative zone.

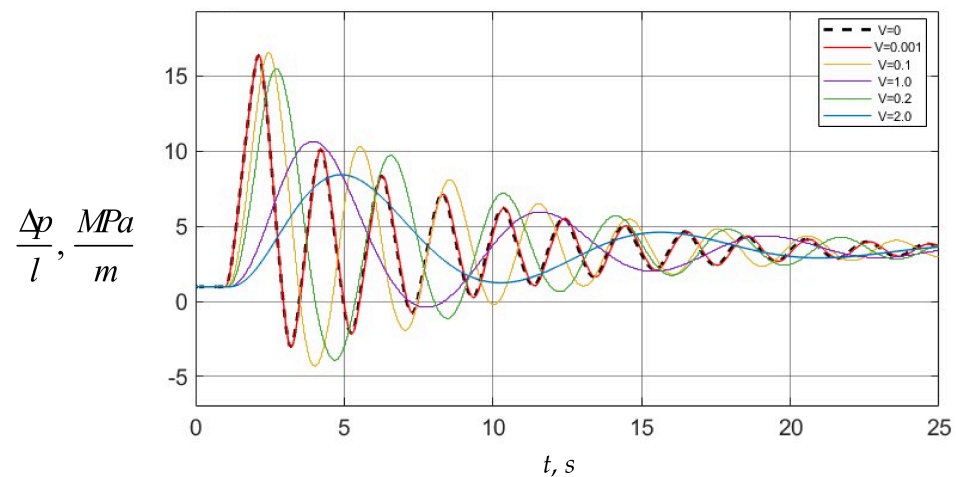


Figure 4. The influence of the damper volume on pressure fluctuations. $l = 500$ m, $D = 0.2$ m, $M = 76.2056$ kg/s, $p_0 = 7.5$ MPa, $p_d = 7$ MPa.

The graphical dynamics demonstrate that increasing the damper volume to $V_{BK}^0 = 2.0$ m³ subsequently lowers the natural oscillation frequency of the system. However, as previously noted, an excessive expansion of the absorbing volume amplifies the inertial properties of the hydraulic circuit. This fundamentally prolongs the time required for complete oscillation attenuation and significantly delays the system's return to its steady-state operating pressure.

Based on the results of the conducted computer simulations, a graph illustrating the dependence of the damper's efficiency in mitigating hydraulic shock on its volume is presented (Figure 5). It is well established that the maximum oscillation amplitude is observed during the initial phase of perturbation propagation and reflection. Therefore, the efficiency of the damper was defined as the ratio of the first-phase amplitudes of the fluid mass flow (momentum) waves generated in the pipeline with and without the damper. An analysis of this graph reveals that the efficiency index varies according to an asymptotic, rather than linear, relationship.

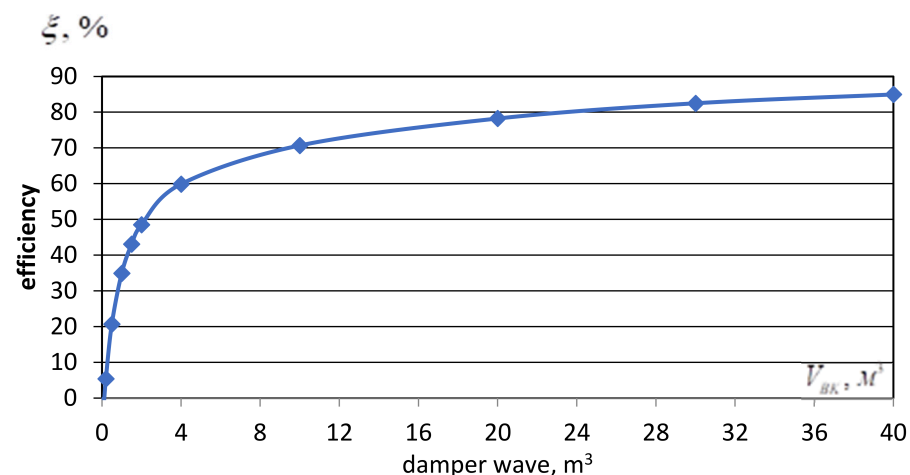


Figure 5. Evaluation and optimization of the damper's efficiency.

When the damper is installed but the initial gas pressure inside it is low (e.g., 1.0 MPa), the gas provides too soft a resistance. The impact energy compresses the gas excessively and rapidly. Following this compression, the gas violently expands, releasing the stored potential energy back into the fluid. This rebounding effect creates a strong secondary pressure spike in the system up to 1.3 MPa, as clearly illustrated in Figure 6.

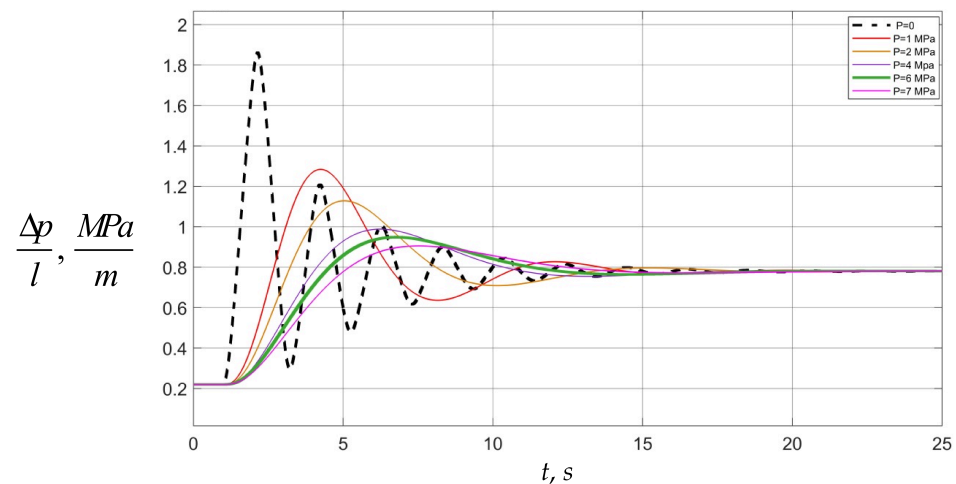


Figure 6. The influence of the initial gas pressure in the damper on the overall pressure gradient over time.

When the initial gas pressure in the damper was increased to the optimal level, the sharp pressure jump was completely eliminated. Most importantly, with correct calibration of the gas pressure, oscillations lost their dangerous wave-like nature and transitioned into a mode of monotonic damping. In this state, the system reached its steady stable operating pressure without excessive shocks, following a smooth and safe trajectory.

Figure 7 presents a graph of the disturbance damping efficiency depending on the initial gas pressure in the damper. The graph shows that the obtained results are nonlinear in nature, approaching an asymptotic regime. Analyzing it, the following patterns can be identified.

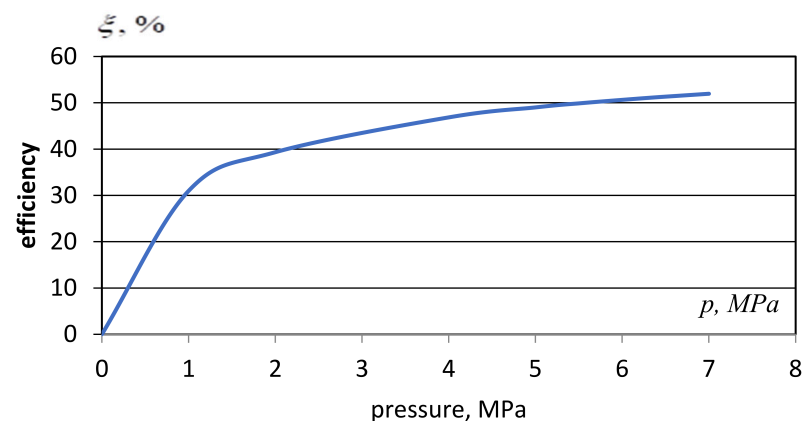


Figure 7. The influence of initial gas pressure on the damper's efficiency.

A steep rise in the curve is observed at initial argument values ranging from 0 to 1.5 – 2 MPa. As the initial pressure in the damper increased from 0.1 MPa to 2 MPa, its efficiency grew from zero to 40%. This indicates that even at low initial pressures, the damper is capable of ensuring system safety by 40% compared to a pipeline without a damper.

An analysis of the black dashed line indicates that following the flow interruption, the maximum pressure in the system surged abruptly to approximately $16 \cdot 10^5$ Pa/m. Furthermore, during the wave reflection phase, the pressure plummeted into the negative domain, reaching approximately $-5 \cdot 10^5$ Pa/m. This represents a highly critical state, indicating the onset of cavitation and the potential for severe structural damage to the pipeline walls.

A damper with a volume of $V_{BK}^0 = 1.0 \text{ m}^3$, which demonstrated 100% efficiency in a pipeline with a diameter of $D = 0.1 \text{ m}$, failed to adequately fulfill its protective function when the pipe diameter was increased to $D = 0.3 \text{ m}$. As our simulation results demonstrate, for this larger pipe diameter, the damper was only capable of reducing the initial maximum pressure peak to $12.5 \cdot 10^5 \text{ Pa}$.

The most hazardous aspect of this scenario is the emergence of negative pressure values during the reverse phase. This indicates that the $V_{BK}^0 = 1.0 \text{ m}^3$ damper lacked the capacity to dissipate the massive inertial energy generated within the larger pipe, thereby failing to eliminate the risk of destructive cavitation zones forming within the system.

As a logical culmination of the preceding multi-factor computer simulations and experimental studies, Figure 8 presents the dependence of the hydraulic shock absorption efficiency ξ , % by a damper with a volume of $V_{BK}^0 = 1.0 \text{ m}^3$ on the pipeline cross-sectional diameter, D . The abscissa of the graph represents the pipe diameters, while the ordinate indicates the pressure reduction efficiency coefficient.

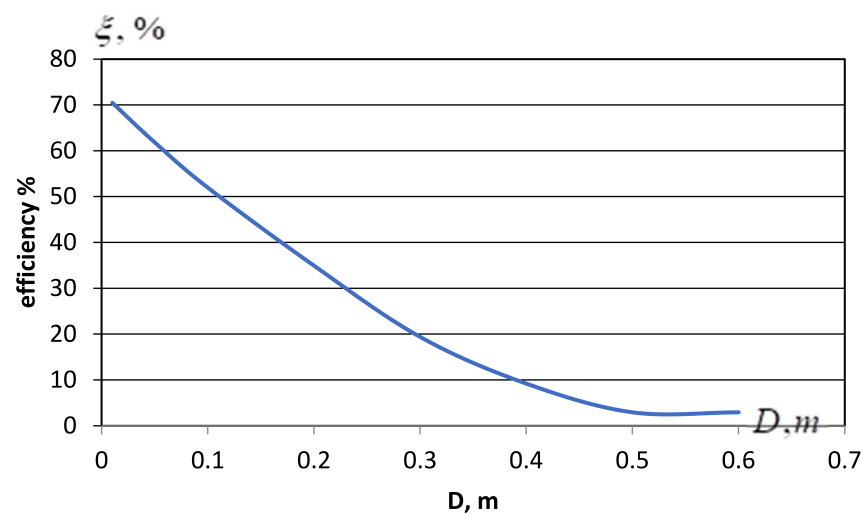


Figure 8. The influence of the changing pipe diameter and the corresponding mass flow rate on the efficiency of a constant-volume damper $V_{BK}^0 = 1.0 \text{ m}^3$.

5. Conclusions

Traditional methods for designing hydraulic shock protection systems often rely on simplified static or linearized dependencies, which fail to fully capture the complex, nonlinear dynamics of compressible fluid and gas interactions. To address this gap, this study developed an advanced numerical algorithm based on the traveling wave method. The primary mathematical novelty lies in the formulation of a nonlinear boundary condition that strictly enforces the conservation of air mass within the gas-hydraulic damper, enabling high-precision modeling of transient hydrodynamic processes in main pipelines. The model was successfully verified against a high-fidelity benchmark simulation, confirming its accuracy, reliability, and suitability for complex engineering calculations.

The conducted simulations underscore that the precise optimization of damper parameters is a critical factor for ensuring the technical safety of pipeline networks. The results demonstrate that an insufficient damper volume leads to extreme pressure fluctuations and severe rarefaction phases, thereby creating a high risk of fluid column separation and the onset of destructive cavitation. Conversely, an excessive expansion of the damper volume induces a pronounced “over-damping” effect. This significantly amplifies the overall inertia of the hydraulic system and drastically prolongs the time required for the transient process to stabilize into a new steady-state operating regime.

The efficiency of the damper is highly dependent on the pipeline diameter due to the scaling laws of fluid kinetic energy. The inertial energy involved in a hydraulic shock scales nonlinearly with both the pipeline diameter and the flow velocity. This necessitates a precise, individualized design for dampers, as an absorbing device optimized for one pipe diameter may lose its protective efficacy when applied to conduits of different dimensions.

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