Article

A Framework for Analyzing Individual-Tree and Whole-Stand Growth by Fusing Multilevel Data: Stochastic Differential Equation and Copula Network

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Abstract: In forestry, growth functions form the basis of research and are widely used for the mathematical modeling of stand variables, e.g., tree or stand basal area, stand height, stand volume, site index, and many more. In this study, to estimate five-dimensional dependencies between tree diameter at breast height, potentially available area, height, crown area, and crown base height, we used a normal copula approach whereby the growths of individual variables are described using a stochastic differential equation with mixed-effect parameters. The normal copula combines the marginal distributions of tree diameter at breast height, potentially available area, height, crown area, and crown base height into a joint multivariate probability distribution. Copula models have the advantage of being able to use collected longitudinal, multivariate, and discrete data for which the number of measurements of individual variables does not match. This study introduced a normalized multivariate interaction information measure based on differential entropy to assess the causality between tree size variables. In order to accurately and quantitatively assess the stochastic processes of the tree size variables’ growth and to provide a scientific basis for the formalization of models, an analysis method of the synergetic theory of information entropy has been proposed. Theoretical findings are illustrated using an uneven-aged, mixed-species empirical dataset of permanent experimental plots in Lithuania.

Keywords: tree growth; diffusion process; diameter; potentially available area; height; crown base height; crown area

1. Introduction

It is obvious that when analyzing forests growing in different territories with different soil fertility, the most important thing is to understand the interdependencies of individual tree or stand size variables. Dependencies may be unequal for trees of different species and ages. The most commonly used tree size variables are diameter at breast height, height, crown base height, crown area, and potentially available area. The question naturally arises whether any one or a combination of these variables is more appropriate for estimating another tree size variable or a certain attribute (e.g., basal area, volume, etc.), and perhaps certain variables may be considered redundant in the simulation of other size variables or certain attributes thereof.

Multivariate statistical analysis in forestry uses statistical techniques that simultaneously examine two or more interrelated tree size variables in order to explain or define the interrelationships of one of them with other variables. These methods can be classified as dependency methods, which study the relationship between one tree size dependent variables and their independent predictors. Several researchers have reported allometric and linear equations between diameter at breast height and height for biomass estimation [1–3].
Others have reported crown width models using diameter at breast height, crown base height, and height as predictor variables [4–6]. There have also been studies involving height and diameter models [7,8], tree size variables (diameter at breast height, height, basal area, and stem volume) as a function of age [9], and forest carbon accumulation [10]. Bivariate analysis investigates the relationship between a paired dataset of an individual tree or stand variable measurements, for example, the height–diameter relationships [11,12], the height and basal area models [13,14], and others [15,16]. Linear multivariate analysis uses two or more tree or stand variables and analyzes which, if any, are correlated with a specific predictor variable [17,18]. The goal of multivariate analysis is to determine which tree or stand size variables significantly influence or cause a causal relationship with the predictor variable [19–22]. Despite the inherently stochastic nature of biological pathways, most existing mathematical models, until now, have relied on deterministic frameworks with a priori assumptions about the regression form, and disregarded the stochastic fluctuations in the environment. Real-world biological growth processes are always multidimensional and open to environmental perturbations, so their evolution in time is governed by the phenomenon of randomness. Anything happening in the flow of time is the result of the effects and outcomes of many different independent random events. Thus, the phenomenon of randomness and dynamism confirms that the regression analysis of forest modeling, which is based on the concept of a random variable, should be transformed into a time-driven growth in the form of a random process. The formulation of changes in a tree or stand variable over time in the form of a stochastic process is associated with the probability density function of the modeled variable. Knowledge of tree size distribution is very useful for forming modern forest management strategies and analyzing the mechanisms of stand evolution [23–27]. It should be emphasized that many distributions have been used for tree diameter distributions of different tree species [28–32], for tree height distributions of different tree species [33–36], and for their joint distributions [37–41].

In forestry, a stochastic process is conceptualized as a family of some random variables indexed by some set, usually time (age). In practical applications, it is usually not enough to study separate random processes because certain multidimensional combinations are required. For example, when forming height–diameter or tree volume relationships, we have to formalize a two-dimensional random process (tree height and diameter), and when forming stand volume relationships, we have to analyze a three-dimensional process (tree height, diameter, and potentially available area), etc. A stochastic process assigns an age function (trajectory) to each experimental outcome of a tree or stand size variable. The value of the trajectory at a given age is a random variable with a given density function for the distribution, but at another moment in the trajectory, it may be another random variable with a different density for the distribution. However, stochastic multivariate calculus methods require a large number of calculations and a lot of computer time, so they have not been applied to forestry problems to the extent they should have been [42]. Thanks to the significant increase in computing power and the availability of software to the user in recent decades, interest in stochastic calculus techniques and their use has increased. The pioneers of the application of stochastic calculus in forestry should be considered the Japanese researchers who started by analyzing the one-dimensional evolution of tree diameter described by a partial differential equation [43–46]. Further research on stochastic processes in forestry has included continuous and discrete time models described by diffusion processes and Markov chains [47–51].

Most studies of multivariate models assume that the interactions between tree size variables are described as a collection of pairwise interactions. This framework has been very fruitful, yielding important insights into the effects of the model structure on tree height growth, tree diameter growth, and many other dynamical phenomena. This paradigm of mathematical modeling used in forestry has allowed the creation of a very large number of alternative regression relationships to model a particular variable of a tree or stand size variable. Unfortunately, when comparing them with various statistical measures, no significant changes in accuracy were observed. While the assumption that interactions
between dynamical tree or size variables occur in pairs is often valid, there are many situations wherein interactions between more than two variables vary via time and occur simultaneously in a way that cannot be reduced to multiple pair interactions. One of the goals of our research is to identify the structure of dependencies between tree size variables in order to better formalize the mathematical relationship of one tree size variable with others. Three techniques are applied in this work: (1) the dynamics of the individual tree size variable are described in the frame of a diffusion process and the corresponding transition probability function is obtained; (2) dependencies of individual tree size variables are estimated by joining the transition probability density functions via the copula method; and (3) the influence of a single variable or a mix of variables on the response variable is determined using information measures. In the last decade, in order to simulate the applicability of tree size variable growth models defined by stochastic differential equations, diffusion processes were used exclusively, for which a transition probability density function exist in an exact form. Stochastic differential equation models can predict the growth and mortality of individual trees in a forest, including diameter at breast height, tree height, crown area, crown base height, potentially available area, stem taper, number of trees per hectare, biomass, volume and basal area, and much more [52,53]. The sigmoidal growth of tree size variables was analyzed by the Gompertz and Bertalanffy lognormal diffusion processes [54–56], and the exponential growth was analyzed by the Vasicek normal diffusion process [57,58]. The copula method can be used as a multiskilled tool to model dependent variables of tree size variables [59–61]. In addition, our copula-based method can be used for the augmentation of real data with a new set of observed data to build robust statistical and deep-learning models more rapidly and at lower cost. Differential entropy theory is a powerful information quantification tool for identifying information transmission between individual tree size variables and defining the structure of size dependencies [20].

In this context, the current work formalizes mathematical models for the interpretation and prediction of the tree size variables. (1) Stand simulation models will be defined by mathematically linking tree size variables with response stand variables. (2) Mathematical prediction models are abstractions of the real forest system, and they allow us to generalize knowledge about the dynamics of forest development and provide information for forest management and planning. The models of tree size variable growth as a function of age are based on a diffusion process defined by a mixed-effect parameters stochastic differential equation, the solution of which has an exact form transition probability density function. The two-dimensional, three-dimensional, four-dimensional, and five-dimensional probability density functions of tree size variables (diameter, height, potentially available area, crown base height, and crown area) are defined using the normal copula function. All conditional distributions of tree size variable are defined by Bayes’ Theorem [62]. The mean, variance, and quantile trajectories of tree size variable are obtained from the derived probability density function and an integration procedure.

The main goal is to study in a general way the methods of cross comparisons of all newly developed growth models. Following an information–theoretic line of thought, Shannon’s differential entropy [63] provides knowledge of what happens to the tree size variable interactions in state space. Concerning tree size variable dynamics, we can calculate the differential Shannon entropies from the one-dimensional, multi-dimensional, and conditional probability densities of tree size variables (diameter, height, potentially available area, crown base height, and crown area). It is the purpose of this paper to use the normalized information measure for comparing different tree size variable growth models. The paper is organized as follows. In Section 2, we describe the observed datasets and the stochastic differential equation growth models used in the study, and we also provide the basic normal copula functions to study the various dependencies. The analytical and numerical results are collected in Section 3, and a summary is given in Section 4. The results are illustrated using an observed dataset from uneven mixed-species permanent
2. Materials and Methods

In this paper we have presented a novel method for tree size variable analysis using multivariate normal copula models in a mixed-effect parameters stochastic differential equation framework to address a major limitation in the individual-tree and whole-stand growth literature: the inability to consider the age-dependent, continuous pathway of the tree diameter, potentially available area, height, crown area, and crown base height as correlated outcomes. Most of the individual-tree and whole-stand modeling literature either considers the static outcome or continuous tree diameter, potentially available area, height, crown area, and crown base height outcomes in separate models. Our stochastic differential equation and copula function model enabled the exposure effect of correlation to vary smoothly along the distributions of the tree diameter, potentially available area, height, crown area, and crown base height, offering much more flexibility than mean regression models. This multivariate framework makes it possible to explain statements about the joint effects of explanatory variables (tree diameter, potentially available area, height, crown area, and crown base height), and allows for probabilistic estimates regarding all five responses (e.g., the conditional probability an individual tree has diameter higher than 14 cm and height higher than 17 m). The basic structure of a stochastic differential equation and copula network for analyzing individual-tree and whole-stand growth is shown in Figure 1.

![Figure 1](image.png)

**Figure 1.** Scheme of the algorithm of the growth models (SDE—stochastic differential equation, pdf—probability density function, AMLP—approximated maximum likelihood procedure, and MLP—maximum likelihood procedure).

2.1. Study Area

The Kazlų Rūda forests (54°44′ N, 23°29′ E) are located in the southwest of Lithuania. Mean temperatures vary from −16.4 °C in winter to +22 °C in summer. Precipitation is distributed throughout the year, although predominantly occurs in the summer, and averages approximately 680 mm a year. The Kazlų Rūda forests are rich in forest and plant resources. The forests cover 47,000 ha, ranking third largest in Lithuania. This article focuses on the modeling of uneven and mixed-species (pine (*Pinus sylvestris*), spruce (*Picea abies*) and silver birch (*Betula pendula* Roth and *Betula pubescens* Ehrh.)) tree datasets. In terms of regeneration regime, all plots varied between naturally regenerated and artificially regenerated and were located in pure or mixed stands. Plots were measured between one and seven times every five or more years. At the establishment of 52 permanent experimental plots, the following data were recorded for sample trees: age, diameter at breast height, tree position, height, crown base height, and crown width. The age of the *i*th tree (ranging from all trees to the 10th) in the first measurement cycle was recorded by counting its growth rings in the growth core (for even aged stands, from the records in the documents), and the ages of the remaining trees were obtained from the arithmetic mean. The position accuracy of the plane coordinates was 1 dcm, and the diameter measurements were made to the nearest 1 mm. The height, crown base height, and crown width measurements were made with an accuracy of approximately 1 dcm. The potentially...
available area of each tree was calculated based on Voronoi polygons. The tree crown area was calculated using the formula for the area of an ellipse whose semi-axes are measured. The breakdown of the obtained measurements is presented in Figure 2. The first level (52 plots; 53,958 mixed-species trees) was used to estimate the fixed effect parameters for the tree diameter and the potentially available area stochastic differential equation, and to estimate the correlation between the tree diameter and the potentially available area; the second level (50 plots; 10,905 mixed-species trees) was used to estimate the fixed effect parameters for the tree height stochastic differential equation, and to estimate the correlations of diameter–height and potentially available area–height; lastly, the third level (44 plots; 1995 mixed-species trees) was used to estimate the fixed effect parameters for the tree crown base height and crown width stochastic differential equation, and to estimate the remaining elements of the correlation matrix of the five-dimensional normal copula by maximizing the pseudo maximum likelihood procedure. The summary of measurements is presented in Table 1.

Table 1. Descriptive statistics of observed datasets used for modeling.

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<th>Data Level</th>
<th>Tree Species</th>
<th>Data</th>
<th>Number of Trees</th>
<th>Min</th>
<th>Max</th>
<th>Mean</th>
<th>St. Dev.</th>
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<td>10.62</td>
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Tree age (t), diameter (d), potentially available area (p), height (h), crown base height (hc), and crown area (wc).
2.2. Stochastic Differential Equations of Tree Size Variables

It is common to rely on an empirical approach when deriving a mathematical model of an individual tree size variable, namely, to fit a particular regression model to the available “observations” of that variable [64]. The accuracy of these individual-tree models appears to be satisfactory, but the complexity of the model limits its direct use when aggregating into whole-stand models. Moreover, such models do not attempt to directly develop the pathway of the variables over time. It is more important for planning and forest management purposes to accurately forecast the growth of a tree size variable than to simulate the relationship between tree size variables. Governed by this idea, we change the simulated tree variable (random variable, \( X \)) to a random process, \( X(t) \), which changes in time \( t \) and described by a certain stochastic differential equation. The evolution of tree diameter, potentially available area, height, crown area, and crown base height, \( X_1(t), X_2(t), X_3(t), X_4(t), X_5(t) \), is defined using stochastic differential equations, the solutions of which are diffusion processes. In this study, the tree diameter diffusion process \( \{X_2^i(t) \mid t \in [t_0, T]\} \), the tree potentially available area diffusion process \( \{X_3^i(t) \mid t \in [t_0, T]\} \), the tree height diffusion process \( \{X_4^i(t) \mid t \in [t_0, T]\} \), and the crown area diffusion process \( \{X_5^i(t) \mid t \in [t_0, T]\} \) \((i = 1, \ldots, M\), where \( M \) is the number of individuals, \( t_0 \) is the initial time, and \( T \) is any finite number) are described by the 4-parameter Gompertz-type stochastic differential equation,

\[
dX_1^i(t) = \left( (a_j + \phi_j^i) - \beta_j \ln(X_1^i(t) - \gamma_j) \right) (X_1^i(t) - \gamma_j) dt + \sqrt{\sigma_j^2} (X_1^i(t) - \gamma_j) \cdot dW_j^i(t), j = 1, \ldots, 4,
\]

\[
X_1^i(t_0) = x_{1o}^i, \quad X_2^i(t_0) = x_{2o}^i = \delta, \quad X_3^i(t_0) = x_{3o}^i, \quad X_4^i(t_0) = x_{4o}^i,
\]

and the tree crown base height diffusion process \( \{X_5^i(t) \mid t \in [t_0, T]\} \) is described by the 3-parameter Vasicek-type stochastic differential equation,

\[
dX_5^i(t) = \left( a_5 + \phi_5^i - X_5^i(t) \right) dt + \sqrt{\sigma_5} \cdot dW_5^i(t), X_5^i(t_0) = x_{50}^i
\]

where \( \delta \) is an unknown fixed-effect parameter to be estimated, \( W_j^i(t), j = 1, \ldots, 5 \) are independent one-dimensional Brownian motions, the random effects \( \phi_j^i, j = 1, \ldots, 5 \) are independent and normally distributed random variables with zero mean and constant variances \((\phi_j^i \sim N(0; \sigma_j^2))\), and the unknown fixed effect parameters \( \theta_0 = \{a_j, \beta_j, \gamma_j, \sigma_j \mid j = 1, \ldots, 5, \gamma_j, j = 1, \ldots, 3, \delta \} \) must be estimated.

After performing the process transformation, we find that the conditional stochastic processes \( (X_1^i(t) \mid X_1^i(t_0) = x_0^i), j = 1, \ldots, 4 \) have lognormal distributions \( LN_1(\mu_j^i(t); \sigma_j^2(t)) \), and the conditional stochastic process \( X_5^i(t) \mid X_5^i(t_0) = x_{50}^i \) has normal distribution \( N_1(\mu_j^i(t); \sigma_j^2(t)) \), with the mean \( \mu_j^i(t) \), variance \( \sigma_j^2(t) \), and probability density function \( f_j^i(x_j, t; \theta, \phi_j^i) \) defined as follows:

\[
\mu_j^i(t) = \ln(x_0^i - \gamma_j) - \beta_j(t-t_0) + \frac{1}{\beta_j} \left( 1 - e^{-\beta_j(t-t_0)} \right) \left( a_j + \phi_j^i - \sigma_j^2 \right), j = 1, \ldots, 4.
\]
The methodology used in this work is expressed by defining five separate stochastic processes instead of a five-dimensional stochastic process, which allows us to reduce the number of estimated unknown parameters and provide the convergence of the approximated maximum likelihood procedure. Given that \( \{x_{j1}, x_{j2}, \ldots, x_{jm_j}\} \) are the directly observed values of the \( j \)th separate stochastic process at discrete times \( \{t_{i1}, t_{i2}, \ldots, t_{in_j}\} \) (\( n_{ij} \) is the number of observed trees of the \( i \)th stand, \( i = 1, \ldots, M, j = 1, \ldots, 5 \)), then the maximum log-likelihood function takes the following form:

\[
LL^2_j(\theta'_j, \phi'_j) = \sum_{i=1}^{M} \int_R \left( \sum_{k=1}^{n_{ij}} \ln \left( f_j^i(x_{jk}, t_{ik} \mid \theta_j, \phi_j) \right) + \ln \left( p(\phi_j \mid \sigma_{\theta_j}^2) \right) \right) d\phi_j \quad \theta'_j = (\theta_j, \phi_j), \quad \phi'_j = (\phi_j^1, \ldots, \phi_j^M),
\]

where the probability density function \( f_j^i(x_{jk}, t_{ik} \mid \theta_j, \phi_j) \) takes the form defined by Equations (6)–(10), and \( p(\phi_j \mid \sigma_{\theta_j}^2) \) is the normal probability density function with zero mean and constant variance, \( \sigma_{\theta_j}^2 \).

In this study, for the maximization of \( LL^2_j(\theta'_j, \phi'_j) \), a two-step approximate maximum likelihood procedure is used [62]. The inner optimization step estimates the random effects \( \hat{\phi}_j \) for each \( j = 1, \ldots, 5, i = 1, \ldots, M \) for the given values of the fixed effects \( \bar{\theta}_j \) with Equation (12):

\[
\hat{\phi}_j = \max_{\phi_j} \left( s(\phi_j \mid \bar{\theta}_j) \right)
\]

where

\[
s(\phi_j \mid \bar{\theta}_j) = \sum_{k=1}^{n_{ij}} \ln \left( f_j^i(x_{jk}, t_{ik} \mid \bar{\theta}_j, \phi_j) \right) + \ln \left( p(\phi_j \mid \sigma_{\theta_j}^2) \right)
\]

The outer optimization step maximizes \( LL^2_j(\theta'_j, \hat{\phi}_j) \) defined by Equation (14) after we insert random effects \( \hat{\phi}_j \) into Equation (14):

\[
LL^2_j(\theta'_j, \hat{\phi}_j) \approx \sum_{i=1}^{M} \left( s' \left( \hat{\phi}_j \mid \theta'_j \right) + \frac{1}{2} \ln(2\pi) \right) - \frac{1}{2} \ln \left( \det \left( -\frac{\partial^2 s' \left( \hat{\phi}_j \mid \theta_j, \sigma_{\theta_j}^2 \right)}{\partial \phi_j^2} \hat{\phi}_j = \bar{\theta}_j \right) \right).
\]
In the first iteration, the values of the random effects \( \tilde{\phi}_j \) are set to zero, as the mean values \( E(\phi^*_j) = 0, j = 1, \ldots, 5, i = 1, \ldots, M \). These two steps are iterated until convergence. Parameter estimates using the approximated maximum likelihood procedure are costly processes in terms of time and computational power, as many more iterations are required.

2.3. Dependence of Tree Size Variables by Copula Approach

In this work, the copula method is very useful because it facilitates the technique of parameter estimates of tree size variables processes and explains the structure of the dependencies of these processes. A multivariate distribution from the point of view of the copula method can be completely described by its marginal distributions, and the copula also defines the dependence of individual variables \([65]\). In this article, we will use the five-dimensional normal copula function to combine the five different transition probability densities defined by Equations (6)–(10). A normal copula is elliptical, symmetric, and completely determined by its correlation matrix \( P \). In the first iteration, the values of the random effects \( \tilde{\phi}_j \) are set to zero, as the mean values \( E(\phi^*_j) = 0, j = 1, \ldots, 5, i = 1, \ldots, M \). These two steps are iterated until convergence. Parameter estimates using the approximated maximum likelihood procedure are costly processes in terms of time and computational power, as many more iterations are required.

\[ C_{u_1, \ldots, u_m}(u_1, \ldots, u_m) = \Phi_m(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_m)) \]

where \( \Phi^{-1} \) is the inverse cumulative distribution function of the standard normal distribution, and \( \Phi_m \) is the \( m \)-dimensional normal cumulative distribution function with a mean vector 0 and correlation matrix \( P \). The density can be written as

\[ c_{u_1, \ldots, u_m}(u_1, \ldots, u_m) = \frac{1}{\sqrt{|P|}} \exp \left( -\frac{1}{2} \left( \Phi^{-1}(u_1) \right)^T \left( P^{-1} - I \right) \left( \Phi^{-1}(u_1) \right) \right) \]

where \( I \) is the identity matrix and \( P \) takes the following form:

\[ P = \begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1m} \\
\rho_{12} & 1 & \cdots & \rho_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{1m} & \rho_{2m} & \cdots & 1
\end{pmatrix} \]

We estimate the density parameters of the \( m \)-variate copula by maximizing the log-likelihood copula function defined by:

\[ l(P) = \sum_{i=1}^{M} \sum_{j=1}^{n_{ij}} \ln \left( c_{u_1, \ldots, u_m}(\Phi^{-1}(F_i(x_{ij}^t, t^i)), \ldots, \Phi^{-1}(F_m(x_{mj}^t, t^j))) | P \right) \]

2.4. Normalized Information Measures

From Shannon’s original definition \([66]\), it can be seen that the interest in information theory and entropy as a new concept was driven by the desire to understand the uncertainty of information sources. In the newly presented tree size variable growth systems (1)–(2), the interpretation of this uncertainty leads to more robust models and a better understanding of complex phenomena. The information–theoretic approach can be used for the development of new information measures for comparing derived theoretical models of tree size variable dynamic.

We now define the joint copula-type two-dimensional, three-dimensional, four-dimensional, and five-dimensional probability density functions in the form:

\[ f_{j,k}^i(x^i_j, x^i_k, t^i | \hat{P}, \hat{\phi}_j^i, \hat{\phi}_k^i) = c_{j,k}^i(G^i_j(x^i_j, t^i), G^i_k(x^i_k, t^i) | \hat{P}) \hat{f}_j^i(x^i_j, t^i) \hat{f}_k^i(x^i_k, t^i), j, k \in \{1, 2, 3, 4, 5\}, \]

(19)
\[
\begin{align*}
&f_{j,k,m}^i \left( x_{j,t}^i, x_{j,t}^k, x_{m,t}^i \left| \hat{P} \left( \tilde{\phi}_1^i, \tilde{\phi}_k^i, \tilde{\phi}_m^i \right) \right. \right) \\
&= c_{j,k,m}^i \left( G_{j}^i (x_{j,t}^i), G_{k}^i (x_{j,t}^k), G_{m}^i (x_{m,t}^i) \left| \hat{P} \left( \tilde{\phi}_1^i, \tilde{\phi}_k^i, \tilde{\phi}_m^i \right) \right. \right) \\
&= c_{j,k,m,n}^i \left( G_{j}^i (x_{j,t}^i), G_{k}^i (x_{j,t}^k), G_{m}^i (x_{m,t}^i), G_{n}^i (x_{n,t}^i) \left| \hat{P} \right. \right) \\
&\times f_{j}^i \left( x_{j,t}^i \right) f_{k}^i \left( x_{j,t}^k \right) f_{m}^i \left( x_{m,t}^i \right) f_{n}^i \left( x_{n,t}^i \right), \quad j, k, m, n \in \{1, 2, 3, 4, 5\}. \tag{20}
\end{align*}
\]

\[
\begin{align*}
&f_{j,k,m,n}^i \left( x_{j,t}^i, x_{j,t}^k, x_{j,t}^m, x_{n,t}^i \left| \hat{P} \left( \tilde{\phi}_1^i, \tilde{\phi}_k^i, \tilde{\phi}_m^i, \tilde{\phi}_n^i \right) \right. \right) \\
&= c_{j,k,m,n}^i \left( G_{j}^i (x_{j,t}^i), G_{k}^i (x_{j,t}^k), G_{m}^i (x_{m,t}^i), G_{n}^i (x_{n,t}^i) \left| \hat{P} \right. \right) \\
&\times f_{j}^i \left( x_{j,t}^i \right) f_{k}^i \left( x_{j,t}^k \right) f_{m}^i \left( x_{m,t}^i \right) f_{n}^i \left( x_{n,t}^i \right) \prod_{i=1}^{5} f_{j}^i \left( x_{j,t}^i \right), \tag{21}
\end{align*}
\]

where “cap” indicates that fixed-effect parameter estimates from Tables 2 and 3 are taken and random factor estimates are calculated using Equation (12), \( \hat{P} \left( \tilde{\phi}_1^i, \tilde{\phi}_k^i, \tilde{\phi}_m^i, \tilde{\phi}_n^i \right) = f_{j}^i \left( x_{j,t}^i, t \delta_j, \tilde{\phi}_j^i \right), \\
\]

\[
F_{j}^i \left( x_{j,t}^i \right) = \int_{\gamma_j} f_{j}^i \left( y, t \right) dy, \quad z = \gamma_j \text{ if } j = 1, \ldots, 4, \quad z = 0, \text{ if } j = 5, \quad z = -\infty, \quad \text{and} \quad G_{j}^i (x_{j,t}^i) = \Phi^{-1} \left( F_{j}^i \left( x_{j,t}^i \right) \right). 
\]

Table 2. Parameter estimates (standard errors) for the mixed-effect mode of the stochastic differential Equations (1)–(2).

<table>
<thead>
<tr>
<th>Equations</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \sigma )</th>
<th>( \delta )</th>
<th>( \sigma_{\phi} )</th>
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<td></td>
<td></td>
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<td>-22.9305</td>
<td>0.0007</td>
<td>-</td>
<td>0.0027</td>
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<td></td>
<td>(0.005)</td>
<td>(0.0001)</td>
<td>(0.2589)</td>
<td>(1.1 \times 10^{-5})</td>
<td>-</td>
<td>(0.0004)</td>
</tr>
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<td>Potentially area</td>
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<td>0.0078</td>
<td>1.6698</td>
<td>0.0082</td>
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<td>(0.0006)</td>
<td>(0.0002)</td>
<td>(0.0447)</td>
<td>(0.0001)</td>
<td>(0.0286)</td>
<td>(0.0012)</td>
</tr>
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<td>Height</td>
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<td>0.0337</td>
<td>-11.7954</td>
<td>0.0007</td>
<td>-</td>
<td>0.0047</td>
</tr>
<tr>
<td></td>
<td>(0.0007)</td>
<td>(0.0002)</td>
<td>(0.2856)</td>
<td>(2.1 \times 10^{-5})</td>
<td>-</td>
<td>(0.0007)</td>
</tr>
<tr>
<td>Crown base height</td>
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<td>-</td>
<td>0.3654</td>
<td>-</td>
<td>4.3306</td>
</tr>
<tr>
<td></td>
<td>(0.2373)</td>
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<td>(0.2233)</td>
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<td>-</td>
<td>(0.4721)</td>
</tr>
<tr>
<td>Crown width</td>
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<td>-1.5622</td>
<td>0.0800</td>
<td>-</td>
<td>0.0059</td>
</tr>
<tr>
<td></td>
<td>(0.0021)</td>
<td>(0.0009)</td>
<td>(0.2233)</td>
<td>(0.0007)</td>
<td>-</td>
<td>(0.0009)</td>
</tr>
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<td>-</td>
<td>0.0134</td>
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<tr>
<td></td>
<td>(0.011)</td>
<td>(0.0004)</td>
<td>(0.0566)</td>
<td>(0.0002)</td>
<td>-</td>
<td>(0.0022)</td>
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<tr>
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<td>(0.0550)</td>
<td>(0.0003)</td>
<td>(0.0448)</td>
<td>(0.0017)</td>
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<td>Height</td>
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<td>-3.3666</td>
<td>0.0047</td>
<td>-</td>
<td>0.0071</td>
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<tr>
<td></td>
<td>(0.0003)</td>
<td>(0.0008)</td>
<td>(0.2551)</td>
<td>(0.0003)</td>
<td>-</td>
<td>(0.0013)</td>
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<td>Crown base height</td>
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<td>-</td>
<td>0.1755</td>
<td>-</td>
<td>7.0771</td>
</tr>
<tr>
<td></td>
<td>(1.3368)</td>
<td>(0.0004)</td>
<td>(0.0114)</td>
<td>-</td>
<td>(0.0141)</td>
<td>(1.0541)</td>
</tr>
<tr>
<td>Crown width</td>
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</tr>
<tr>
<td></td>
<td>(0.0034)</td>
<td>(0.0016)</td>
<td>(0.2429)</td>
<td>(0.0011)</td>
<td>-</td>
<td>(0.0024)</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diameter</td>
<td>0.1316</td>
<td>0.0393</td>
<td>-5.8221</td>
<td>0.0056</td>
<td>-</td>
<td>0.0146</td>
</tr>
<tr>
<td></td>
<td>(0.023)</td>
<td>(0.0008)</td>
<td>(0.2373)</td>
<td>(0.0002)</td>
<td>-</td>
<td>(0.0022)</td>
</tr>
<tr>
<td>Potentially area</td>
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<td>0.0189</td>
<td>-2.0214</td>
<td>0.0092</td>
<td>1.9502</td>
<td>0.0010</td>
</tr>
<tr>
<td></td>
<td>(0.0028)</td>
<td>(0.0011)</td>
<td>(0.2128)</td>
<td>(0.0006)</td>
<td>(0.1361)</td>
<td>(0.0016)</td>
</tr>
<tr>
<td>Height</td>
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<td>0.0404</td>
<td>-39.3068</td>
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<td>-</td>
<td>0.0052</td>
</tr>
<tr>
<td></td>
<td>(0.0099)</td>
<td>(0.0022)</td>
<td>(2.1701)</td>
<td>(3.9 \times 10^{-5})</td>
<td>-</td>
<td>(0.0009)</td>
</tr>
<tr>
<td>Crown base height</td>
<td>13.4070</td>
<td>0.1782</td>
<td>-</td>
<td>10.4929</td>
<td>-</td>
<td>3.0937</td>
</tr>
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<td></td>
<td>(0.5751)</td>
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<td>(2.7817)</td>
<td>(2.717)</td>
<td>-</td>
<td>(0.5152)</td>
</tr>
<tr>
<td>Crown width</td>
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<td>-0.9590</td>
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<td>0.6791</td>
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<tr>
<td></td>
<td>(0.4316)</td>
<td>(0.1543)</td>
<td>(0.0421)</td>
<td>(0.2524)</td>
<td>-</td>
<td>(0.1702)</td>
</tr>
</tbody>
</table>
The conditional probability density functions with one covariate, two covariates, three covariates, and four covariates \( k, m, n, s \in \{1, 2, 3, 4, 5\} \) \( j \) are defined as:

\[
f^i_{j,k,m}(x^i_j, t | \bar{x}^i_k, \hat{x}^i_m) = \frac{c^i_{j,k,m}(G^i_j(x^i_j, t), G^i_k(x^i_k, t), G^i_m(x^i_m, t) | \hat{P}) \hat{f}^i_j(x^i_j, t)}{c^i_{k,m,n}(G^i_k(x^i_k, t), G^i_m(x^i_m, t) | \hat{P})},
\]

(23)

\[
f^i_{j,k,m,n}(x^i_j, t | \bar{x}^i_k, x^i_m, x^i_n) = \frac{c^i_{j,k,m,n}(G^i_j(x^i_j, t), G^i_k(x^i_k, t), G^i_m(x^i_m, t), G^i_n(x^i_n, t) | \hat{P}) \hat{f}^i_j(x^i_j, t)}{c^i_{k,m,n}(G^i_k(x^i_k, t), G^i_m(x^i_m, t), G^i_n(x^i_n, t) | \hat{P})},
\]

(24)

\[
f^i_{j,k,m,n,s}(x^i_j, t | \bar{x}^i_k, x^i_m, x^i_n, x^i_s) = \frac{c^i_{j,k,m,n,s}(G^i_j(x^i_j, t), G^i_k(x^i_k, t), G^i_m(x^i_m, t), G^i_n(x^i_n, t), G^i_s(x^i_s, t) | \hat{P}) \hat{f}^i_j(x^i_j, t)}{c^i_{k,m,n,s}(G^i_k(x^i_k, t), G^i_m(x^i_m, t), G^i_n(x^i_n, t), G^i_s(x^i_s, t) | \hat{P})}.
\]

(25)

\[
f^{t}_{j,k,m,n,s} = \frac{\hat{f}^{t}_{j,k,m,n,s}}{c^{t}_{k,m,n,s}}.
\]

(26)

The Shannon differential entropy of the probability density function \( f^j(x^i_j, t | \hat{\theta}^j, \hat{\phi}^j) \), \( j \in \{1, 2, 3, 4, 5\} \), \( i = 1, \ldots, 5 \), for random process \( \{X^i_j(t) \in [t_0, T]\} \), is written as:

\[
H(X^i_j, t) = - \int f^j(x^i_j, t | \hat{\theta}^j, \hat{\phi}^j) \log_2(f^j(x^i_j, t | \hat{\theta}^j, \hat{\phi}^j)) dx = -E(\log_2(f^j(x^i_j, t | \hat{\theta}^j, \hat{\phi}^j))).
\]

(27)
where $Y_j$ is a support of the density function, $Y_j = [y_j, +\infty)$, $j = 1, \ldots, 4$, and $Y_j = (-\infty, +\infty)$, $j = 5$.

Therefore, the estimated entropies for marginal, five-dimensional, and conditional probability density functions (6)–(10) and (22)–(26) are defined as:

$$ H(X_j) = \frac{1}{N} \sum_{i=1}^{M} \sum_{j=1}^{n_3} \log_2 \left( f_i \left( x_{j}^{i}, t_{i}^{j} \big| \hat{P}_{j}, \hat{P}_{k} \right) \right), \quad j \in \{1, 2, 3, 4, 5\}, \quad (28) $$

$$ H(X_1, X_2, X_3, X_4, X_5) = \frac{1}{N} \sum_{i=1}^{M} \log_2 \left( f_i \left( x_{12345}^{i}, t_{i} \big| \hat{P}, \hat{P}_{1}, \hat{P}_{2}, \hat{P}_{3}, \hat{P}_{4}, \hat{P}_{5} \right) \right) \quad (29) $$

$$ H(X_j \mid X_k) = \frac{1}{N} \sum_{i=1}^{M} \sum_{j=1}^{n_3} \log_2 \left( f_{j|k} \left( x_{j}^{i} \big| t_{i} \big| \hat{P}_{j}, \hat{P}_{k} \right) \right), \quad k \in \{1, 2, 3, 4, 5\} \backslash j, \quad (30) $$

$$ H(X_j \mid X_k, X_m) = \frac{1}{N} \sum_{i=1}^{M} \log_2 \left( f_{j|km} \left( x_{j}^{i} \big| t_{i} \big| \hat{P}_{j}, \hat{P}_{k}, \hat{P}_{m} \right) \right), \quad j, k, m \in \{1, 2, 3, 4, 5\}, \quad (31) $$

$$ H(X_j \mid X_k, X_m, X_n) = \frac{1}{N} \sum_{i=1}^{M} \log_2 \left( f_{j|kmn} \left( x_{j}^{i} \big| t_{i} \big| \hat{P}_{j}, \hat{P}_{k}, \hat{P}_{m}, \hat{P}_{n} \right) \right), \quad j, k, m, n \in \{1, 2, 3, 4, 5\} \backslash j, \quad (32) $$

where $N = \sum_{i=1}^{M} n_3$.

Generally, mutual information, $I(X,Y)$, measures the information shared between two random variables $X$ and $Y$, and reveals the extent to which knowing one of these variables reduces uncertainty about the other. Mutual information is a fundamental measure of dependence expressed as the joint distribution of $X$ and $Y$ under the assumption of independence defined by:

$$ I(X,Y) = H(X) + H(Y) - H(X,Y) = H(X) - H(X/Y). \quad (34) $$

It is obvious that mutual information $I(X,Y) = 0$ if and only if $X$ and $Y$ are independent random variables. Regarding the relationship (34), it is advisable to define normalized information measures $\mathcal{R}_{j(\cdot), \ldots, (\cdot)} \in (0; 1)$ by using all developed conditional probability density functions, defined by Equations (23)–(26), in the form:

$$ \mathcal{R}_{j/k} = \frac{H(X_j) - H(X_j \mid X_k)}{\sqrt{H(X_j) H(X_j \mid X_k)}}, \quad k \in \{1, 2, 3, 4, 5\} \backslash j, \quad (35) $$

$$ \mathcal{R}_{j/k,m} = \frac{H(X_j) - H(X_j \mid X_k, X_m)}{\sqrt{H(X_j) H(X_j \mid X_k, X_m)}}, \quad k, m \in \{1, 2, 3, 4, 5\} \backslash j, \quad (36) $$

$$ \mathcal{R}_{j/k,m,n} = \frac{H(X_j) - H(X_j \mid X_k, X_m, X_n)}{\sqrt{H(X_j) H(X_j \mid X_k, X_m, X_n)}}, \quad k, m, n \in \{1, 2, 3, 4, 5\} \backslash j, \quad (37) $$

$$ \mathcal{R}_{j/k,m,n,s} = \frac{H(X_j) - H(X_j \mid X_k, X_m, X_n, X_s)}{\sqrt{H(X_j) H(X_j \mid X_k, X_m, X_n, X_s)}}, \quad k, m, n, s \in \{1, 2, 3, 4, 5\} \backslash j. \quad (38) $$

2.5. Trajectories of Tree Size Variable

As mentioned in the previous Section 2.2 of this work, the main feature of the newly developed model is that an object (for example, a tree or a stand) is understood as a mechanistic system of related size variables. The general course of tree development can be explained from the growth of these elementary size variables and their interactions. Using the probability density functions (6)–(10) and conditional probability density functions (23)–(26), we define the dynamics of the mean $m_j(t)$, and the conditional means
where the fixed-effect parameter estimates from Tables 2 and 3 are taken and random factor estimates are calculated using Equation (12); \( Y_j \) is a support of the density function, \( Y_j = [Y_j, +\infty) \), \( j = 1, \ldots, 4 \), and \( Y_j = (-\infty, +\infty) \), \( j = 5 \).

3. Results and Discussion

3.1. Estimating Results

In this study, the maximum likelihood methodology is presented within the framework of discretely observed stochastic processes. For all five stochastic growth processes \( \{X_j(t) \mid t \in [t_0, T)\} \), the parameters were estimated using Equations (12)–(14) (see Table 1: for the diameter and potentially available area growth, we used the dataset from the first data level; for the height growth, we used the dataset from the second data level; for the crown area and crown base height growth, we used the dataset from the third data level), and the results are presented in Table 2. The approximate asymptotic standard errors of the fixed effect parameters were defined by the diagonal elements of the inverse observed Fisher information matrix [60]. All parameter estimates are statistically significant \( (p < 0.05) \).

The estimation of the correlation matrix \( P \) by the maximum log-likelihood copula function defined by Equation (18) was divided into three steps. The dependence parameter, \( \rho_{12} \), between the diameter and potentially available area growth processes was estimated using the dataset from the first data level \( \{ (x_{1t}^{11}, x_{1t}^{21}), (x_{1t}^{12}, x_{1t}^{22}), \ldots, (x_{1n_1}^{11}, x_{1n_1}^{21}) \} \) at discrete times \( \{ t_1^{11}, t_1^{12}, \ldots, t_1^{1n_1} \} \); the dependence parameters, \( \rho_{13} \) and \( \rho_{23} \), between the diameter and height growth processes, and between the potentially available area and height growth processes, were estimated using the dataset from the second data level \( \{ (x_{2t}^{11}, x_{2t}^{21}, x_{2t}^{31}), (x_{2t}^{12}, x_{2t}^{22}, x_{2t}^{32}), \ldots, (x_{2n_2}^{11}, x_{2n_2}^{21}, x_{2n_2}^{31}) \} \) at discrete times \( \{ t_1^{21}, t_1^{22}, \ldots, t_1^{2n_2} \} \); finally, the remaining seven parameters of the correlation matrix, \( P \), were estimated using the third-level three-sample dataset of discrete measurements \( \{ (x_{3t}^{11}, x_{3t}^{21}, x_{3t}^{31}, x_{3t}^{41}, x_{3t}^{51}), (x_{3t}^{12}, x_{3t}^{22}, x_{3t}^{32}, x_{3t}^{42}, x_{3t}^{52}), \ldots, (x_{3n_3}^{11}, x_{3n_3}^{21}, x_{3n_3}^{31}, x_{3n_3}^{41}, x_{3n_3}^{51}) \} \) at discrete times \( \{ t_1^{31}, t_1^{32}, \ldots, t_1^{3n_3} \} \), \( i = 1, \ldots, M \) for all tree species (pine, spruce, and birch). The results are presented in Table 3.
3.2. Causal Effects along Different Explanatory Variables Pathways

The normalized information measures defined by Equations (35)–(38) are somewhat similar to the correlation coefficient. The results of calculated normalized information measures defined by Equations (35)–(38) for pine, spruce, and birch trees are presented in Tables 4–6.

Table 4. Normalized information measures $R_{j/k}$, $j = 1, \ldots, 5$, defined by Equations (35)–(38), for the pine species trees.

<table>
<thead>
<tr>
<th></th>
<th>Diameter</th>
<th>Potentially Available Area</th>
<th>Height</th>
<th>Crown Area</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{1/k}$</td>
<td>$R_{2/k}$</td>
<td>$R_{3/k}$</td>
<td>$R_{4/k}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$R_{1/k}$</td>
<td>$R_{2/k}$</td>
<td>$R_{3/k}$</td>
<td>$R_{4/k}$</td>
</tr>
<tr>
<td>$k, m$</td>
<td>0.0005</td>
<td>0.0050</td>
<td>0.002</td>
<td>0.007</td>
</tr>
<tr>
<td>$k, m, n, s$</td>
<td>0.4922</td>
<td>0.3751</td>
<td>0.5205</td>
<td>0.3915</td>
</tr>
<tr>
<td></td>
<td>0.0143</td>
<td>0.3903</td>
<td>0.5235</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>0.0148</td>
<td>0.5109</td>
<td>0.5235</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>0.1385 *</td>
<td>0.4941</td>
<td>0.4615</td>
<td>0.3915</td>
</tr>
<tr>
<td></td>
<td>0.5349</td>
<td>0.6017</td>
<td>0.4200</td>
<td>0.5282</td>
</tr>
<tr>
<td></td>
<td>0.5350</td>
<td>0.5350</td>
<td>0.4353</td>
<td>0.4829</td>
</tr>
<tr>
<td></td>
<td>0.6024</td>
<td>0.6024</td>
<td>0.6024</td>
<td>0.5335</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4. Cont.

<table>
<thead>
<tr>
<th>Crown Base Height</th>
<th>k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{5/k}$</td>
<td>0.0202</td>
<td>0.0003</td>
<td>0.0957</td>
<td>0.0017</td>
<td></td>
</tr>
<tr>
<td>$k, m$</td>
<td>1, 2</td>
<td>1, 3</td>
<td>1, 4</td>
<td>2, 3</td>
<td>2, 4</td>
</tr>
<tr>
<td>$R_{5/k,m}$</td>
<td>0.5262</td>
<td>0.5188</td>
<td>0.5233</td>
<td>0.4250</td>
<td>0.3852</td>
</tr>
<tr>
<td>$k, m, n$</td>
<td>1, 2, 3</td>
<td>1, 2, 4</td>
<td>1, 3, 4</td>
<td>2, 3, 4</td>
<td></td>
</tr>
<tr>
<td>$R_{5/k,m,n}$</td>
<td>0.4786</td>
<td>0.3998</td>
<td>0.4837</td>
<td>0.4831</td>
<td></td>
</tr>
<tr>
<td>$k, m, n, s$</td>
<td>1, 2, 3, 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{5/k,m,n,s}$</td>
<td>0.4839</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* the best value of the normalized information measure in the group with one, two, and three explanatory variables—in bold.

Table 5. Normalized information measures $R_{j/\cdot, j = 1, \ldots, 5}$, defined by Equations (35)–(38), for the spruce species trees.

<table>
<thead>
<tr>
<th>Diameter</th>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{1/k}$</td>
<td>0.0119</td>
<td>0.2750 *</td>
<td>0.0293</td>
<td>0.1171</td>
<td></td>
</tr>
<tr>
<td>$k, m$</td>
<td>2, 3</td>
<td>2, 4</td>
<td>2, 5</td>
<td>3, 4</td>
<td>3, 5</td>
</tr>
<tr>
<td>$R_{1/k,m}$</td>
<td>0.6600</td>
<td>0.4146</td>
<td>0.4888</td>
<td>0.6580</td>
<td>0.7024</td>
</tr>
<tr>
<td>$k, m, n$</td>
<td>2, 3, 4</td>
<td>2, 3, 5</td>
<td>2, 4, 5</td>
<td>3, 4, 5</td>
<td></td>
</tr>
<tr>
<td>$R_{1/k,m,n}$</td>
<td>0.6646</td>
<td>0.7043</td>
<td>0.5263</td>
<td>0.7034</td>
<td></td>
</tr>
<tr>
<td>$k, m, n, s$</td>
<td>2, 3, 4, 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{1/k,m,n,s}$</td>
<td>0.7050</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Potentially Available Area</th>
<th>k</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{2/k}$</td>
<td>0.0117</td>
<td>0.0062</td>
<td>0.0000</td>
<td>0.0173</td>
<td></td>
</tr>
<tr>
<td>$k, m$</td>
<td>1, 3</td>
<td>1, 4</td>
<td>1, 5</td>
<td>3, 4</td>
<td>3, 5</td>
</tr>
<tr>
<td>$R_{2/k,m}$</td>
<td>0.3810</td>
<td>0.3839</td>
<td>0.3866</td>
<td>0.3791</td>
<td>0.3855</td>
</tr>
<tr>
<td>$k, m, n$</td>
<td>1, 3, 4</td>
<td>1, 3, 5</td>
<td>1, 4, 5</td>
<td>3, 4, 5</td>
<td></td>
</tr>
<tr>
<td>$R_{2/k,m,n}$</td>
<td>0.3838</td>
<td>0.3869</td>
<td>0.3885</td>
<td>0.3873</td>
<td></td>
</tr>
<tr>
<td>$k, m, n, s$</td>
<td>1, 3, 4, 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{2/k,m,n,s}$</td>
<td>0.3885</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Height</th>
<th>k</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{3/k}$</td>
<td>0.2905</td>
<td>0.0067</td>
<td>0.0540</td>
<td>0.0823</td>
<td></td>
</tr>
<tr>
<td>$k, m$</td>
<td>1, 2</td>
<td>1, 4</td>
<td>1, 5</td>
<td>2, 4</td>
<td>2, 5</td>
</tr>
<tr>
<td>$R_{3/k,m}$</td>
<td>0.6686</td>
<td>0.7069</td>
<td>0.6686</td>
<td>0.4357</td>
<td>0.4518</td>
</tr>
<tr>
<td>$k, m, n$</td>
<td>1, 2, 4</td>
<td>1, 2, 5</td>
<td>1, 4, 5</td>
<td>2, 4, 5</td>
<td></td>
</tr>
<tr>
<td>$R_{3/k,m,n}$</td>
<td>0.7068</td>
<td>0.6690</td>
<td>0.7071</td>
<td>0.5201</td>
<td></td>
</tr>
<tr>
<td>$k, m, n, s$</td>
<td>1, 2, 4, 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{3/k,m,n,s}$</td>
<td>0.7070</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 5. Cont.

<table>
<thead>
<tr>
<th>Crown Area</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{4/k}$</td>
<td>0.1254</td>
<td>0.0188</td>
<td>0.0837</td>
<td>0.0045</td>
<td></td>
</tr>
<tr>
<td>$R_{4/k,m}$</td>
<td>0.5043</td>
<td>0.4974</td>
<td>0.5042</td>
<td>0.4666</td>
<td>0.3933</td>
</tr>
<tr>
<td>$R_{4/k,m,n}$</td>
<td>0.5046</td>
<td>0.5100</td>
<td>0.5044</td>
<td>0.4752</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crown Base Height</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{5/k}$</td>
<td>0.0381</td>
<td>0.0000</td>
<td>0.0671</td>
<td>0.0055</td>
<td></td>
</tr>
<tr>
<td>$R_{5/k,m}$</td>
<td>0.4120</td>
<td>0.4435</td>
<td>0.4156</td>
<td>0.4429</td>
<td>0.3751</td>
</tr>
<tr>
<td>$R_{5/k,m,n}$</td>
<td>0.4475</td>
<td>0.4182</td>
<td>0.4515</td>
<td>0.4531</td>
<td></td>
</tr>
<tr>
<td>$R_{5/k,m,n,s}$</td>
<td>0.4838</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* the best value of the normalized information measure in the group with one, two, and three explanatory variables—in bold.

Table 6. Normalized information measures $R_{j/k}$, $j = 1, \ldots, 5$, defined by Equations (35)–(38), for the birch species trees.

<table>
<thead>
<tr>
<th>Diameter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{1/k}$</td>
<td>0.0095</td>
<td>0.2365 *</td>
<td>0.0722</td>
<td>0.1126</td>
<td></td>
</tr>
<tr>
<td>$R_{1/k,m}$</td>
<td>0.6213</td>
<td>0.4494</td>
<td>0.4886</td>
<td>0.6126</td>
<td>0.6440</td>
</tr>
<tr>
<td>$R_{1/k,m,n}$</td>
<td>0.6227</td>
<td>0.6516</td>
<td>0.5469</td>
<td>0.6441</td>
<td></td>
</tr>
<tr>
<td>$R_{1/k,m,n,s}$</td>
<td>0.6518</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Potentially Available Area</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{2/k}$</td>
<td>0.0096</td>
<td>0.0036</td>
<td>0.0027</td>
<td>0.0052</td>
</tr>
<tr>
<td>$R_{2/k,m}$</td>
<td>0.3798</td>
<td>0.3781</td>
<td>0.3783</td>
<td>0.3722</td>
</tr>
<tr>
<td>$R_{2/k,m,n}$</td>
<td>0.3801</td>
<td>0.3799</td>
<td>0.3781</td>
<td>0.3744</td>
</tr>
<tr>
<td>$R_{2/k,m,n,s}$</td>
<td>0.3802</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In this article, the concept of causality is understood as the ability of one tree size variable to influence another. Namely, as observed by the Nobel laureate Wiener [67], for two variables measured simultaneously, “if we can predict the first variable better by using the past information from the second one than by using the information without it, then we call the second variable causal to the first one”. We consider that the normalized information measures defined by Equations (35)–(38) show causal relationships between different sets of tree size variables. Our main goal is to investigate the influence of a given set of tree size variables on the target variable.

Looking at the causality measures, expressed by normalized information measures and presented in Tables 4–6, we can assert that no significant differences in the causality between the pine, spruce, and birch tree species are observed. Substantial differences in causality appear when we analyze the link between a target tree size variable and other size variables. Let us begin by discussing the causality of the pine tree size variables, taking into account other size variables.

Table 4 shows that the tree diameter at breast height is most influenced by the tree crown area and, conversely, is least influenced by the potentially available area. The

<table>
<thead>
<tr>
<th>Height</th>
<th>k</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{3/k}$</td>
<td>0.2756</td>
<td>0.0041</td>
<td>0.1394</td>
<td>0.1008</td>
<td></td>
</tr>
<tr>
<td>$R_{3/k,m}$</td>
<td>0.6545</td>
<td>0.7263</td>
<td>0.6547</td>
<td>0.5124</td>
<td>0.4720</td>
</tr>
<tr>
<td>$R_{3/k,m,n}$</td>
<td>0.7295</td>
<td>0.6570</td>
<td>0.7349</td>
<td>0.6084</td>
<td></td>
</tr>
<tr>
<td>$R_{3/k,m,n,s}$</td>
<td>0.7384</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crown Area</th>
<th>k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{4/k}$</td>
<td>0.1127</td>
<td>0.0051</td>
<td>0.0878</td>
<td>0.0216</td>
<td></td>
</tr>
<tr>
<td>$R_{4/k,m}$</td>
<td>0.4838</td>
<td>0.4856</td>
<td>0.4855</td>
<td>0.4603</td>
<td>0.3939</td>
</tr>
<tr>
<td>$R_{4/k,m,n}$</td>
<td>0.4856</td>
<td>0.4855</td>
<td>0.4913</td>
<td>0.4669</td>
<td></td>
</tr>
<tr>
<td>$R_{4/k,m,n,s}$</td>
<td>0.4914</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crown Base Height</th>
<th>k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{5/k}$</td>
<td>0.0811</td>
<td>0.0030</td>
<td><strong>0.1362</strong></td>
<td>0.0242</td>
<td></td>
</tr>
<tr>
<td>$R_{5/k,m}$</td>
<td>0.4510</td>
<td>0.5090</td>
<td>0.4531</td>
<td>0.5079</td>
<td>0.3940</td>
</tr>
<tr>
<td>$R_{5/k,m,n}$</td>
<td>0.5093</td>
<td>0.4528</td>
<td>0.5155</td>
<td><strong>0.5157</strong></td>
<td></td>
</tr>
<tr>
<td>$R_{5/k,m,n,s}$</td>
<td>0.5159</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* the best value of the normalized information measure in the group with one, two, and three explanatory variables—in bold.
causality values calculated in Table 4 for the tree diameter predictions show that the prediction of the target size variable increases significantly when the two size variables are introduced, and it is the tree height and crown area that provide the synergistic information, as their coalition specifies information about diameter that is not specified by any singular explanatory tree size variable. The additional introduction of a third variable of tree size, namely, the crown base height, slightly increases the synergistic information available for causal interpretation. Finally, we can argue that using all four remaining tree size variables to predict the tree diameter has no additional effect and is redundant.

As can be seen from Table 4, the tree potentially available area is most influenced by the crown area. To explain the tree potentially available area, we can derive the most information by examining the synergistic influence of the tree height and crown area. The introduction of a third and a fourth variable to the prediction of the potentially available area leads to no synergy, implying they are redundant.

The tree height is most highly influenced by the tree diameter and, conversely, is least influenced by the potentially available area (see Table 4). The causality values calculated in Table 4 for the tree height predictions show that the prediction of the target size variable increases significantly when the two size variables are introduced, and it is the tree diameter and crown base height that provide the most synergistic information. The additional introduction of a third tree size variable, namely, the potentially available area, slightly increases the synergistic information for the tree height prediction. Finally, we can state that using all four remaining tree size variables to predict the tree height has no additional effect and is redundant compared to the synergistic information of three tree size variables.

The tree crown base height is most influenced by the tree height and, conversely, is least influenced by the potentially available area (see Table 4). The causality values calculated in Table 4 for the tree crown base height predictions show that the prediction of the target size variable increases significantly when the two size variables are introduced, and it is the tree height and crown width that provide the most synergistic information. The additional introduction of a third variable of tree size, namely, the diameter, slightly improves the information for the crown base height prediction. Finally, we can state that using all four remaining tree size variables to predict the tree crown base height has minimal effect and could be considered redundant compared to the three size variables (diameter, height, and crown width).

Table 4 shows that the tree crown area is most influenced by the tree diameter and, conversely, is least influenced by the crown base height. The causality values calculated in Table 4 for the tree crown area predictions show that the prediction of the target size variable increases significantly when the two size variables are introduced, and it is the tree diameter and potentially available area that provide the most synergistic information. The additional introduction of a third variable of tree size, namely, the height, slightly increases the information available for the crown area prediction. Finally, we can state that using all four remaining tree size variables in tree crown area prediction gives additional synergistic effects compared to the three size variables (diameter, potentially available area, and height).

When studying the causal links between the size variables of spruce and birch trees, no significant differences can be seen from the causality analysis of pine trees. It should be noted that the development of the diameter of spruce and birch trees is mostly determined by the distinctiveness of tree height formation, while the diameter of pine trees is mostly influenced by their crown area.

The tree diameter has the greatest influence on the formation of the potentially available area for birch trees, while the tree crown area is the determining factor for the pine and spruce trees. For the formation of potentially available area, for trees of the birch species, diameter and height show the greatest synergistic information for the coalition of two size variables, and for the coalition of three size variables, the greatest synergistic information occurs between the diameter, height, and the crown base height, while for pine and spruce
trees, the greatest synergistic information is generated, respectively, by the height, crown base height, and crown area, and the diameter, crown base height, and crown area. The tree potentially available area, height, and crown area generate the greatest synergistic information for height development in spruce and birch tree species, while the diameter, potentially occupied area, and crown area show the greatest synergistic information for pine trees. The tree potentially available area, height, and crown area generate the greatest synergistic information for the crown base height development in spruce and birch tree species, while the diameter, height, and crown area show the greatest influence on the pine trees. The tree diameter, potentially available area, and crown base height generate the greatest synergistic information for crown area development in spruce tree species, while the diameter, crown base height, and crown area show the greatest influence on the birch trees’ crown area predictions.

3.3. Growth Model Pathways

In this paper, the methodology for modeling the growth of tree size variables is presented by deriving the exact expressions of the probability density functions (see Equations (6)–(10)). In this regard, we can calculate the probability of the event that the tree size variable will take a value from a certain interval. Figure 3 shows the dynamics of what part (percentage) of trees in the stand (species composition of the stand: pine—42.5%, spruce—51.1%, and birch—6.4%) have a diameter of at least 20 cm, a potentially occupied area of no less than 10 m², a height of at least 20 m, a crown base height no less than 10 m, or a crown area no less than 20 m². From Figure 3a, it can be seen that when ranking the tree species according to the tree diameter reach of 20 cm, the highest rank should be given to pines, followed by birch, and finally the lowest rank to spruce. Figure 3b shows that when ranking tree species according to their having a reach of occupied area no less than 10 m², the highest rank is assigned to birch, while the dynamics of the area occupied by pine and spruce trees do not differ significantly. From Figure 3c, it can be seen that when ranking the tree species according to the tree height reach of 20 m, the highest rank should be given to pines, followed by birch, and finally the lowest rank to spruce. In addition, we can emphasize that birch trees maintain greater heights than pine trees until approximately 40 years of age. Figure 3d shows that trees of the spruce species have significantly lower crown base heights. Finally, Figure 1e shows that pine trees maintain a slightly larger crown area compared to the spruce trees.

![Figure 3](image_url)

*Figure 3. Cont.*
Table 7. Coefficients of determination $R^2_f$, $R^2_{fi}$, $j = 1, \ldots, 5$ for the pine species trees.

<table>
<thead>
<tr>
<th>Diameter</th>
<th>$R^2_f$</th>
<th>$R^2_{1/k}$</th>
<th>$R^2_{1/k,m}$</th>
<th>$R^2_{1/k,m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.453</td>
<td>0.453</td>
<td>0.744</td>
<td>0.758</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2, 3</td>
<td>2, 3, 4</td>
<td>2, 3, 4, 5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2, 3, 4</td>
<td>2, 3, 4, 5</td>
<td>2, 3, 4, 5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2, 3, 4, 5</td>
<td>2, 3, 4, 5</td>
<td>2, 3, 4, 5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2, 3, 4, 5</td>
<td>2, 3, 4, 5</td>
<td>2, 3, 4, 5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Potentially Available Area</th>
<th>$R^2_f$</th>
<th>$R^2_{1/k}$</th>
<th>$R^2_{1/k,m}$</th>
<th>$R^2_{1/k,m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.343</td>
<td>0.367</td>
<td>0.373</td>
<td>0.376</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1, 3</td>
<td>1, 3, 4</td>
<td>1, 3, 4, 5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1, 3, 4</td>
<td>1, 3, 4, 5</td>
<td>1, 3, 4, 5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1, 3, 4, 5</td>
<td>1, 3, 4, 5</td>
<td>1, 3, 4, 5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1, 3, 4, 5</td>
<td>1, 3, 4, 5</td>
<td>1, 3, 4, 5</td>
</tr>
</tbody>
</table>

Figure 3. Probability (in percent) of the event that the tree size variable will take a value from a certain interval for pine trees (red color), spruce trees (green color), and birch trees (black color): (a) trees in the stand have a diameter of at least 20 cm; (b) trees in the stand have a potentially occupied area no less than 10 m²; (c) trees in the stand have a height of at least 20 m; (d) trees in the stand have a crown base height of no less than 10 cm; (e) trees in the stand have a crown area of at least 20 m².

For all studied species, the developed growth models of the tree size variables, defined by Equations (39)–(43), correspond well to the data, as shown in Tables 7 and 8, using the coefficient of determination:

$$R^2_f = \frac{\sum_{m=i}^{M} \sum_{l=1}^{L} \left( x_{jl} - m^f_l \right)^2}{\sum_{m=i}^{M} \sum_{l=1}^{L} \left( y_l - \bar{y} \right)^2},$$

$$R^2_{fi} = \frac{\sum_{m=i}^{M} \sum_{l=1}^{L} \left( x_{jl} - m^f_l \right)^2}{\sum_{m=i}^{M} \sum_{l=1}^{L} \left( y_l - \bar{y} \right)^2}.$$

(44)
Comparing the results of the accuracy of the predictions of tree size variables, which are presented in Tables 7–9 according to the coefficient of determination, with the causality measures for tree size variables (presented in Tables 4–6), we notice that the trend in the calculated values of the causality measures almost fully corresponds to the values of the coefficients of determination. The accuracy of the tree size variable predictions was additionally evaluated by other statistical measures such as root mean squared error, absolute bias, bias, and their percentage values, but the ranking of the models remained unchanged and is shown in Tables 7–9. It is particularly worth noting that the predictions for all analyzed growth models did not show any significant bias according to the Student’s test [68].

### Table 7. Cont.

<table>
<thead>
<tr>
<th>Height</th>
<th>( R^2 )</th>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^2_{/k} )</td>
<td>0.824</td>
<td>0.618</td>
<td>0.747</td>
<td>0.704</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k, m )</td>
<td>1, 2</td>
<td>1, 4</td>
<td>1, 5</td>
<td>2, 4</td>
<td>2, 5</td>
<td>4, 5</td>
</tr>
<tr>
<td>( R^2_{/k,m} )</td>
<td>0.825</td>
<td>0.873</td>
<td>0.830</td>
<td>0.751</td>
<td>0.706</td>
<td>0.808</td>
</tr>
<tr>
<td>( k, m, n )</td>
<td>1, 2, 4</td>
<td>1, 2, 5</td>
<td>1, 4, 5</td>
<td>2, 4, 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2_{/k,m,n} )</td>
<td>0.873</td>
<td>0.831</td>
<td>0.873</td>
<td>0.807</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k, m, n, s )</td>
<td>1, 2, 4, 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2_{/k,m,n,s} )</td>
<td>0.873</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crown Area</th>
<th>( R^2 )</th>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^2_{/k} )</td>
<td>0.662</td>
<td>0.362</td>
<td>0.427</td>
<td>0.305</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k, m )</td>
<td>1, 2</td>
<td>1, 3</td>
<td>1, 5</td>
<td>2, 3</td>
<td>2, 5</td>
<td>3, 5</td>
</tr>
<tr>
<td>( R^2_{/k,m} )</td>
<td>0.683</td>
<td>0.671</td>
<td>0.679</td>
<td>0.470</td>
<td>0.357</td>
<td>0.471</td>
</tr>
<tr>
<td>( k, m, n )</td>
<td>1, 2, 3</td>
<td>1, 2, 5</td>
<td>1, 3, 5</td>
<td>2, 3, 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2_{/k,m,n} )</td>
<td>0.690</td>
<td>0.698</td>
<td>0.680</td>
<td>0.509</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k, m, n, s )</td>
<td>1, 2, 3, 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2_{/k,m,n,s} )</td>
<td>0.699</td>
<td></td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Crown Base Height</th>
<th>( R^2 )</th>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^2_{/k} )</td>
<td>0.752</td>
<td>0.727</td>
<td>0.824</td>
<td>0.729</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k, m )</td>
<td>1, 2</td>
<td>1, 3</td>
<td>1, 4</td>
<td>2, 3</td>
<td>2, 4</td>
<td>3, 4</td>
</tr>
<tr>
<td>( R^2_{/k,m} )</td>
<td>0.755</td>
<td>0.833</td>
<td>0.763</td>
<td>0.826</td>
<td>0.731</td>
<td>0.835</td>
</tr>
<tr>
<td>( k, m, n )</td>
<td>1, 2, 3</td>
<td>1, 2, 4</td>
<td>1, 3, 4</td>
<td>2, 3, 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2_{/k,m,n} )</td>
<td>0.834</td>
<td>0.764</td>
<td>0.836</td>
<td>0.836</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k, m, n, s )</td>
<td>1, 2, 3, 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2_{/k,m,n,s} )</td>
<td>0.837</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* the best value of the normalized information measure in the group with one, two, and three explanatory variables—in bold.
Table 8. Coefficients of determination $R^2_j, R^2_{j/\cdot, j=1, \ldots, 5}$ for the spruce species trees.

<table>
<thead>
<tr>
<th>Diameter</th>
</tr>
</thead>
</table>
| $R^2_1$ | 0.455  
| $k$ | 2 3 4 5  
| $R^2_{1/k}$ | 0.497 0.891* 0.503 0.697  
| $k,m$ | 2, 3 2, 4 2, 5 3, 4 3, 5 4, 5  
| $R^2_{1/k,m}$ | 0.895 0.541 0.699 0.887 0.914 0.726  
| $k,m,n$ | 2, 3, 4 2, 3, 5 2, 4, 5 3, 4, 5  
| $R^2_{1/k,m,n}$ | 0.891 0.914 0.726 0.913  
| $R^2_{1/k,m,n,s}$ | 0.913  
| Potentially Available Area |  
| $R^2_2$ | 0.381  
| $k$ | 1 3 4 5  
| $R^2_{2/k}$ | 0.396 0.389 0.373 0.421  
| $k,m$ | 1, 3 1, 4 1, 5 3, 4 3, 5 4, 5  
| $R^2_{2/k,m}$ | 0.395 0.408 0.421 0.405 0.419 0.423  
| $k,m,n$ | 1, 3, 4 1, 3, 5 1, 4, 5 3, 4, 5  
| $R^2_{2/k,m,n}$ | 0.406 0.416 0.424 0.425  
| $R^2_{2/k,m,n,s}$ | 0.422  
| Height |  
| $R^2_3$ | 0.445  
| $k$ | 1 2 4 5  
| $R^2_{3/k}$ | 0.909 0.467 0.569 0.651  
| $k,m$ | 1, 2 1, 4 1, 5 2, 4 2, 5 4, 5  
| $R^2_{3/k,m}$ | 0.907 0.892 0.911 0.602 0.648 0.725  
| $k,m,n$ | 1, 2, 4 1, 2, 5 1, 4, 5 2, 4, 5  
| $R^2_{3/k,m,n}$ | 0.893 0.908 0.892 0.727  
| $R^2_{3/k,m,n,s}$ | 0.892  
| Crown Area |  
| $R^2_4$ | 0.396  
| $k$ | 1 2 3 5  
| $R^2_{4/k}$ | 0.676 0.458 0.585 0.403  
| $k,m$ | 1, 2 1, 3 1, 5 2, 3 2, 5 3, 5  
| $R^2_{4/k,m}$ | 0.691 0.679 0.683 0.616 0.466 0.596  
| $k,m,n$ | 1, 2, 3 1, 2, 5 1, 3, 5 2, 3, 5  
| $R^2_{4/k,m,n}$ | 0.692 0.695 0.498 0.622  
| $R^2_{4/k,m,n,s}$ | 0.693  
|  

### Table 8. Cont.

<table>
<thead>
<tr>
<th>Crown Base Height</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2_{3/k}$</td>
<td>0.550</td>
<td>0.460</td>
<td>0.607</td>
<td>0.476</td>
</tr>
<tr>
<td>k, m</td>
<td>1, 2</td>
<td>1, 3</td>
<td>1, 4</td>
<td>2, 3</td>
</tr>
<tr>
<td>$R^2_{3/k,m}$</td>
<td>0.559</td>
<td>0.618</td>
<td>0.566</td>
<td>0.616</td>
</tr>
<tr>
<td>k, m, n</td>
<td>1, 2, 3</td>
<td>1, 2, 4</td>
<td>1, 3, 4</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>$R^2_{3/k,m,n}$</td>
<td>0.624</td>
<td>0.571</td>
<td>0.630</td>
<td><strong>0.632</strong></td>
</tr>
<tr>
<td>k, m, n, s</td>
<td>1, 2, 3, 4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*the best value of the normalized information measure in the group with one, two, and three explanatory variables—in bold.*

### Table 9. Coefficients of determination $R^2_j, R^2_j/k, j = 1, \ldots, 5$ for the birch species trees.

<table>
<thead>
<tr>
<th>Diameter</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2_1$</td>
<td>0.687</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$R^2_{1/k}$</td>
<td>0.709</td>
<td><strong>0.900</strong></td>
<td>0.754</td>
<td>0.835</td>
</tr>
<tr>
<td>k, m</td>
<td>2, 3</td>
<td>2, 4</td>
<td>2, 5</td>
<td>3, 4</td>
</tr>
<tr>
<td>$R^2_{1/k,m}$</td>
<td>0.907</td>
<td>0.768</td>
<td>0.840</td>
<td>0.900</td>
</tr>
<tr>
<td>k, m, n</td>
<td>2, 3, 4</td>
<td>2, 3, 5</td>
<td>2, 4, 5</td>
<td>3, 4, 5</td>
</tr>
<tr>
<td>$R^2_{1/k,m,n}$</td>
<td>0.907</td>
<td><strong>0.921</strong></td>
<td>0.865</td>
<td>0.917</td>
</tr>
<tr>
<td>k, m, n, s</td>
<td>2, 3, 4, 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2_{1/k,m,n,s}$</td>
<td>0.922</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Potentially Available Area</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2_2$</td>
<td>0.345</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$R^2_{2/k}$</td>
<td><strong>0.382</strong></td>
<td>0.361</td>
<td>0.355</td>
<td>0.358</td>
</tr>
<tr>
<td>k, m</td>
<td>1, 3</td>
<td>1, 4</td>
<td>1, 5</td>
<td>3, 4</td>
</tr>
<tr>
<td>$R^2_{2/k,m}$</td>
<td><strong>0.386</strong></td>
<td>0.381</td>
<td>0.381</td>
<td>0.360</td>
</tr>
<tr>
<td>k, m, n</td>
<td>1, 3, 4</td>
<td>1, 3, 5</td>
<td>1, 4, 5</td>
<td>3, 4, 5</td>
</tr>
<tr>
<td>$R^2_{2/k,m,n}$</td>
<td>0.383</td>
<td><strong>0.386</strong></td>
<td>0.380</td>
<td>0.359</td>
</tr>
<tr>
<td>k, m, n, s</td>
<td>1, 3, 4, 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2_{2/k,m,n,s}$</td>
<td>0.381</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The dynamics of the growth processes of all tree size variables are shown in Figure 4 for one randomly selected plot with the following tree species distribution: pine—42.5%, spruce—51.1%, and birch—6.4%. As can be seen from Figure 4a-c,e, the probability distributions of tree diameter, potentially available area, height, and crown area values are characterized by asymmetry because, for all tree species, there is an obvious difference...
between the mean (solid line) and the mode (dashed line). The greatest asymmetry (the largest difference between the mean curve and mode curve) is characterized by the growth processes of the tree crown area and the potentially available area. A fundamental finding of this study on the distribution of tree size variables is that they are positively skewed (mode < mean) [69]. In the stand shown in Figure 4, the growth process of pine trees in terms of diameter, height, crown base height, and crown area is significantly faster compared to the growth process of other tree species. It can be seen from Figure 4d that the dynamics of the crown base height of the spruce trees show a clearly lower trajectory compared to trees of other species. Furthermore, we notice that the area between the lower and upper quantiles (dot–dashed curves) effectively covers the entire range of values of the measured tree size variables.

![Figure 4. Mean (solid line), median (dotted line), mode (dashed line), lower quantile (dot–dashed line), and upper quantile (dot–dashed line) dynamics for pine trees (red color), spruce trees (green color), and birch trees (black color): (a) growth process of tree diameter; (b) growth process of tree occupied area; (c) growth process of tree height; (d) growth process of tree crown base height; (e) growth process of tree crown area; observed values—circles.](image)

The structural complexity measures of a forest stand necessarily reflect interactions between many different variables, but their calculation requires sophisticated multivariate analysis. Various measures (indices) have been created with the aim of expressing structural complexity in a single number, thus making the comparison of stands easy [23,69–71]. Over time, the large number of forest management technologies in practice usually changes the structure of the stand, such as via changes in species composition, cross-sectional area, as well as diameter and height distributions due to the loss of large trees. The unavoidable change in measures of structural composition requires giving these measures a dynamic form and enabling the definition of changes over the age of the stand. Thanks to the fact that in this paper the variables characterizing the tree size are treated as random processes, the derived probability transition functions themselves depend on time, and at the same time
all measures of the structural complexity acquire a dynamic form. Next, we will discuss one of them, the live crown ratio, but we can also analyze other measures of structural complexity, namely, the slenderness, the tree diameter diversity, and others.

One measure that determines the vitality of a tree is the living crown ratio, which is defined as the ratio of crown length to total tree height. Crown length depends in part on the shade tolerance of the species. The live crown ratio for ith stand is defined as:

\[
CR^i(t) = \frac{\int_0^\infty \int_0^\infty \frac{x_3^i}{x_3^i} f^{i,\lambda}(x_0^i, x_1^i, x_2^i, x_3^i, t|\theta_0^i, \theta_2^i)}{\int_0^\infty \int_0^\infty f^{i,\lambda}(x_0^i, x_1^i, x_2^i, x_3^i, 0|\theta_0^i, \theta_2^i)} dx_0^i dx_2^i}
\]

(45)

As expected, Figure 5a–c show that spruce crowns were longer than those of pine and birch trees, due to greater shade tolerance properties. For spruce trees, the live crown ratio shows a decreasing trend over time (see Figure 5b), while for pine and birch trees, these dynamics show an increasing trend (see Figure 5a,c).

![Figure 5. Dynamic of live crown ratio for three randomly selected plots (the first plot—red color, species composition—P 42.5, E 51.1, B 6.4; the second plot—black color, species composition—P 49.1, E 28.1, B 22.8; the third plot—green color, species composition—P 88.9, E 0.0, B 11.1): (a) pine trees; (b) spruce trees; (c) birch trees; observed values—circles.](image)

4. Conclusions

Using an extensive data set covering a wide region of Lithuanian mixed-species forests, a very complex five-dimensional growth model was developed based on the diffusion process and the copula function, using tree age, diameter at breast height, potentially available area, height, crown base height, and crown area. This paper explains the causal relationships between tree size variables (diameter, potentially occupied area, height, crown base height, and crown area) using differential Shannon entropy and the mutual information measure. Newly developed models take into account time-varying parameters, non-linear growth patterns, and the influence of random factors on future forest stand states, all of which are critical to understanding and predicting forest ecosystem behavior. Since, in this paper, the methodology for modeling the growth of tree size variables involves deriving exact expressions of probability density functions, we can calculate the probability of the event that the tree size variable will take a value from a certain interval.

Mutual information within the confines of Shannon information theory is a fundamental concept related to complex systems that has received much attention in computational biology. This study shows a clear application of synergistic information in understanding that tree size variables are influenced not only by one variable, but also by interactions between variables.
Notably, the applicability of our model extends beyond tree size variable studies to a variety of stand attributes, such as stand basal area, volume, CO₂ absorption, and many more.

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